Bull. Malays. Math. Sci. Soc. (2024) 47:4
https://doi.org/10.1007/s40840-023-01596-x

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Regularity of 3-Path Ideals of Trees and Unicyclic Graphs

Rajiv Kumar[1] · Rajib Sarkar[2]

Received: 14 August 2023 / Revised: 7 October 2023 / Accepted: 15 October 2023 /
Published online: 8 November 2023

Abstract

Let G be a simple graph and $I_3(G)$ be its 3-path ideal in the corresponding polynomial ring R. In this article, we prove that for an arbitrary graph G, $\operatorname{reg}(R/I_3(G))$ is bounded below by $2\nu_3(G)$, where $\nu_3(G)$ denotes the 3-path induced matching number of G. We give a class of graphs, namely trees for which the lower bound is attained. Also, for a unicyclic graph G, we show that $\operatorname{reg}(R/I_3(G)) \leq 2\nu_3(G) + 2$ and provide an example that shows that the given upper bound is sharp.

Keywords t-Path ideal · Trees and unicyclic graphs · Castelnuovo–Mumford regularity

Mathematics Subject Classification 13D02 · 13C13 · 05E40

1 Introduction

Let G be a finite simple graph with the vertex set $V(G)$ and the edge set $E(G)$. Let $R = \mathbb{K}[x : x \in V(G)]$ be the polynomial ring, where $\mathbb{K}$ is an arbitrary field. In [6], the notion of path ideals has been introduced. For a graph G, a square-free monomial ideal,

$$I_t(G) := \langle x_1 \cdots x_t : \text{ where } P : x_1, \ldots, x_t \text{ is a } t\text{-path in } G \rangle \subseteq R$$

Communicated by Rosihan M. Ali.

✉ Rajib Sarkar
rajib.sarkar63@gmail.com

Rajiv Kumar
gargrajiv00@gmail.com; rajiv.kumar@iitjammu.ac.in

1 Department of Mathematics, Indian Institute of Technology Jammu, Jammu, JK 181221, India

2 Stat-Math Unit, Indian Statistical Institute Kolkata, 203 B.T. Road, Kolkata 700108, India

is called *t-path ideal* of G. For the last few decades, researchers have been trying to establish the connection between the combinatorial invariants of graphs and the algebraic invariants of t-path ideal, see [1, 2, 4, 5, 7, 12, 13]. The Castelnuovo–Mumford regularity of an ideal is an important algebraic invariant that measures the complexity of the module. The *Castelnuovo–Mumford regularity* (or simply regularity) of a finitely generated graded R-module M, written $\operatorname{reg}(M)$ is defined as $\operatorname{reg}(M) := \max\{j-i : \operatorname{Tor}_i(M, \mathbb{K})_j \neq 0\}$. There are a few classes of graphs for which the explicit formula of the regularity of t-path ideal is known. In [5], Bouchat et al. studied the t-path ideal of rooted trees, and they provided a recursive formula for computing the graded Betti numbers of t-path ideals. Also, they gave a general bound for the regularity of t-path ideal of a rooted tree. In particular, for a rooted tree G, they proved that $\operatorname{reg}(R/I_t(G)) \leq (t-1)[l_t(G) + p_t(G)]$, where $l_t(G)$ denotes the number of leaves in G, where level is at least $t-1$ and $p_t(G)$ denotes the maximal number of pairwise disjoint paths of length t in G. In [1] and [2], Alilooee and Faridi computed the regularity of t-path ideal of lines and cycles in terms of the number of vertices, respectively. Banerjee [3] studied the regularity of t-path ideal of gap free graphs and proved that the t-path ideals of gap free, claw free and whiskered-K_4 free graphs have linear minimal free resolutions for all $t \geq 3$. In this article, we restrict ourselves to $t = 3$ and study the regularity of $I_3(G)$. It is important to note that for a 3-path ideal of gap free graph G, Banerjee proved that $\operatorname{reg}(R/I_3(G)) \leq \max\{\operatorname{reg}(R/I_2(G)), 2\}$, where $I_2(G)$ is the 2-path ideal or monomial edge ideal of G, see [3, Theorem 3.3]. In [11], Katzman proved that for any graph G, $\operatorname{reg}(R/I_2(G)) \geq \nu(G)$, where $\nu(G)$ denotes the induced matching number of G. Motivated from the definition of $\nu(G)$, in this article, we define in an obvious way the 3-path induced matching number, denoted by $\nu_3(G)$ (see Sect 2 for the definition). Next, we prove that a similar lower bound can be obtained for the regularity of $I_3(G)$. More precisely, we prove $\operatorname{reg}(R/I_3(G)) \geq 2\nu_3(G)$. We also observe that this is a sharp lower bound for $\operatorname{reg}(R/I_3(G))$. In fact, we show that if G is a tree, then $\operatorname{reg}(R/I_3(G)) = 2\nu_3(G)$, see Theorem 4.4. It is desirable to answer the following problem:

Problem 1.1 Classify the classes of graphs G that satisfy the property

$$\operatorname{reg}(R/I_3(G)) = 2\nu_3(G).$$

Theorem 4.4 gives a class of graphs that satisfies the property $\operatorname{reg}(R/I_3(G)) = 2\nu_3(G)$. On the other hand, Example 4.7 shows that some of the unicyclic graphs satisfy the desired property but not the whole class. More concretely, in Theorem 4.6, we prove that if G is a unicyclic graph, then $\operatorname{reg}(R/I_3(G)) \leq 2\nu_3(G) + 2$. Moreover, in Example 4.7, we give examples of unicyclic graphs showing that the regularity of 3-path ideal of a unicyclic graph can attain any of the values between the lower and upper bound of the regularity.

2 Preliminaries

In this section, we recall all necessary definitions which will be used throughout the article.

Definition 2.1 Let G be a graph with the vertex set $V(G) = \{x_1, \dots, x_n\}$ and the edge set $E(G)$.

(i) A subgraph H of a graph G is called an *induced subgraph* if for all $x_i, x_j \in V(H)$ such that $\{x_i, x_j\} \in E(G)$ implies that $\{x_i, x_j\} \in E(H)$. For a vertex $x \in V(G)$, let $G \setminus \{x\}$ denote the induced subgraph on the vertex set $V(G) \setminus \{x\}$.

(ii) A *path* on n vertices is a graph whose vertices can be listed in the order $x_1, \dots, x_n$ such that the edge set is $\{\{x_i, x_{i+1}\} : 1 \leq i \leq n\}$ and it is denoted by P_n. For $t \geq 2$, a path of length t in G is called a *t-path*.

(iii) A *cycle* on n vertices $\{x_1, \dots, x_n\}$, denoted by C_n, is the graph with the edge set $E(P_n) \cup \{\{x_1, x_n\}\}$. A graph G is said to be *tree* if it does not contain any cycle. A disconnected tree is called a *forest*. A graph G is called *unicyclic* if G contains only one cycle.

(iv) For a vertex $x \in V(G)$, the set $\{y \in V(G) : \{x, y\} \in E(G)\}$ is called the *neighborhood* of x in G and it is denoted by $N_G(x)$. The set $N_G[x]$ denotes $N_G(x) \cup \{x\}$. For an edge $e = \{x, y\} \in E(G)$, the *neighborhood* of e is defined as

$$N_G(e) := (N_G(x) \setminus \{y\}) \cup (N_G(y) \setminus \{x\}).$$

$N_G[e]$ denotes the set $N_G(e) \cup \{x, y\}$, i.e., $N_G[e] = N_G[x] \cup N_G[y]$.

v) For a vertex $x \in V(G)$, the set $\{\{y, z\} \in E(G) : \{x, y, z\}$ is a 3-path in $G\}$ is called the *neighborhood edge set* of x in G and it is denoted by $N_G^{edge}(x)$.

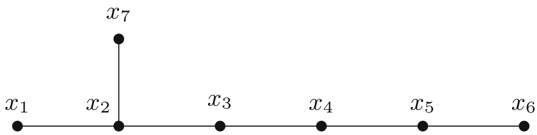
(vi) A *3-path matching* in a graph G is a subgraph consisting pairwise disjoint 3-paths. If the subgraph is induced, then 3-path matching is said to be a 3-*path induced matching* of G. The largest size of a 3-path induced matching is called the 3-*path induced matching number*, and it is denoted by $\nu_3(G)$.

The following remark shows the relationship between the 3-path induced matching number of a graph and its induced subgraph (Fig. 1).

Remark 2.2 One can observe that for an induced subgraph H of G, we have $\nu_3(H) \leq \nu_3(G)$.

Example 2.3 Note that $\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}$ is a 3-path matching but not a 3-path induced matching in G. Here, $\{x_1, x_2, x_7\}, \{x_4, x_5, x_6\}$ is a 3-path induced matching in G and $\nu_3(G) = 2$.

Fig. 1 G



3 A Lower Bound for the Regularity

For an arbitrary graph G, we first give a general lower bound for the regularity of 3-path ideal in terms of the regularity of 3-path ideal of its induced subgraph, and as a consequence, we prove that the regularity of $R/I_3(G)$ is bounded below by $2\nu_3(G)$.

Proposition 3.1 *Let H be an induced subgraph of G. Then,*

$$\beta_{i,j}(R_H/I_3(H)) \leq \beta_{i,j}(R/I_3(G))$$

where $R_H = \mathbb{K}[x : x \in V(H)]$. Moreover, $\operatorname{reg}(R_H/I_3(H)) \leq \operatorname{reg}(R/I_3(G))$.

Proof We prove that $R_H/I_3(H)$ is an algebra retract of $R/I_3(G)$. We first show that $I_3(H) = I_3(G) \cap R_H$. It is clear that $I_3(H) \subseteq I_3(G) \cap R_H$. For the converse part, let $f \in I_3(G) \cap R_H$. Suppose $f = \sum gh$ for $g \in R$ and $h \in I_3(G)$. We consider the mapping $\varphi : R \longrightarrow R_H$ by defining $\varphi(x) = x$ if $x \in V(H)$, otherwise $\varphi(x) = 0$. Therefore,

$$\begin{aligned}\varphi(f) &= \sum \varphi(g)\varphi(h) \\ &= \sum \varphi(g)h, \text{ where } h \in I_3(H).\end{aligned}$$

This implies that $f \in I_3(H)$, and so $I_3(H) = I_3(G) \cap R_H$. Thus, $R_H/I_3(H)$ is a $\mathbb{K}$-subalgebra of $R/I_3(G)$. Now consider $R_H/I_3(H) \overset{i}{\hookrightarrow} R/I_3(G) \overset{\bar{\varphi}}{\to} R_H/I_3(H)$, where $\bar{\varphi}$ is the map induced by φ. It can be observed that $\bar{\varphi} \circ i$ is the identity map on $R_H/I_3(H)$. Hence, $R_H/I_3(H)$ is an algebra retract of $R/I_3(G)$. Now, the assertion follows from [14, Corollary 2.5]. □

As an immediate consequence, we have a lower bound of the regularity of 3-path ideal of any graph G in terms of the combinatorial invariants of G.

Corollary 3.2 *Let G be a simple graph and $I_3(G)$ be its 3-path ideal. Then,*

$$\operatorname{reg}(R/I_3(G)) \geq 2\nu_3(G).$$

Proof For simplicity of notation, let $s = \nu_3(G)$. Suppose that $\{P_1, P_2, \ldots, P_s\}$ is a 3-path induced matching in G. Let H be the induced subgraph of G on the vertices $\cup_{i=1}^{s} V(P_i)$. Then, $I_3(H)$ is a complete intersection. Thus, by the Koszul complex, $\operatorname{reg}(R_H/I_3(H)) = 2\nu_3(G)$. Hence, the assertion follows from Proposition 3.1. □

4 Path Ideals of Trees and Unicyclic Graphs

In this section, we consider the 3-path ideal of trees and unicyclic graphs. In fact, we compute the exact regularity of $I_3(G)$ when G is a tree, and for unicyclic graphs, we give a sharp upper bound of $\operatorname{reg}(I_3(G))$. We first prove some technical lemmas which will be needed to prove the main results.

Lemma 4.1 *Let G be a simple graph and $I_3(G)$ be its 3-path ideal. Let $e = \{x, y\} \in E(G)$. Then, we have the followings:*

(1) $I_3(G) : xy = L + J$, *where* $L = \langle N_G(e)\rangle$ *and* $J = I_3(G\backslash N_G[e])$.
(2) $(I_3(G) + \langle xy\rangle) : x = \langle y\rangle + I_2(H) + I_3(G\backslash N_G[x])$, *where H is the union of* $N_G^{edge}(x)$ *and the complete graph on the vertex set* $N_G(x) \setminus \{y\}$.

Proof (1): Clearly, $L + J \subset I_3(G) : xy$. On the other side, let $u \in I_3(G) : xy$. This implies that $uxy \in I_3(G)$, and hence, there exists a minimal monomial generator $v \in I_3(G)$ such that $v \mid uxy$. If $v \nmid u$, then $\gcd(v, xy) \neq 1$. This forces that $\operatorname{supp}(u) \cap N_G(e) \neq \emptyset$. This gives $u \in L$, and hence, $I_3(G) : xy = L + J$.

(2): It can be easily seen that $\langle y\rangle + I_2(H) + I_3(G\backslash N_G[x]) \subset (I_3(G) + \langle xy\rangle) : x$. For the converse part, let $u \in (I_3(G) + \langle xy\rangle) : x$. Then, $ux \in I_3(G) + \langle xy\rangle$. Assume that $y \nmid u$. Then, $ux \in I_3(G)$, and hence, there exists a minimal monomial generator $v \in I_3(G)$ such that $v \mid ux$. If $v \nmid u$, then $x \mid v$. Let $v = xv_1$. Since v is 3-path in G, $x \in N_G(v_1)$. Now, if v_1 is an edge in G, then $v_1 \in N_G^{\text{edge}}(x)$, on the other hand $\operatorname{supp}(v_1) \subset N_G(x)$ which further implies that $u \in I_2(H)$. This yields that $(I_3(G) + \langle xy\rangle) : x \subset \langle y\rangle + I_2(H) + I_3(G\backslash N_G[x])$, and hence,

$$(I_3(G) + \langle xy\rangle) : x = \langle y\rangle + I_2(H) + I_3(G \setminus N_G[x])$$

which completes the proof. □

We recall a property satisfied by a tree from [10, Proposition 4.1].

Remark 4.2 If G is a tree containing a vertex of degree at least two, then there exists a vertex $v \in V(G)$ with $N_G(v) = \{v_1, \ldots, v_r\}$, where $r \geq 2$ and $\deg_G(v_i) = 1$ for $i < r$.

Lemma 4.3 *Let G be a tree and $v \in V(G)$ with the notation as in Remark 4.2. Then,*

$$\nu_3(G \setminus N_G[e]) \leq \nu_3(G) - 1, \ \textit{where } e = \{v, v_r\}$$

Proof Let $\nu_3(G\backslash N_G[e]) = s$ and $P_1, \ldots, P_s$ be a 3-path induced matching in $G\backslash N_G[e]$. Since $\deg_G(v_1) = 1$, $N_G(v_1) \cap V(P_i) = \emptyset$ for $1 \leq i \leq s$. Also, note that for $1 \leq i \leq s$, $N_G(v) \cap V(P_i) = \emptyset = N_G(v_r) \cap V(P_i)$. Therefore, $\{v_1, v, v_r\}, P_1, \ldots, P_s$ is a 3-path induced matching in G, and hence, $\nu_3(G) \geq s + 1$. □

Now, we are ready to prove the exact regularity formula for the 3-path ideals of trees.

Theorem 4.4 *Let G be a tree and $I_3(G)$ be its 3-path ideal. Then,* $\operatorname{reg}(R/I_3(G)) = 2\nu_3(G)$.

Proof By Corollary 3.2, it is enough to prove that $\operatorname{reg}(R/I_3(G)) \leq 2\nu_3(G)$.

We proceed by induction on $|V(G)|$. By Remark 4.2, let v be such a vertex of G such that $N_G(v) = \{v_1, \ldots, v_r\}$ and $\deg_G(v_i) = 1$ for $i < r$, where $r \geq 2$. Set $u = v_r$, and let $N_G(u) = \{u_1 = v, u_2, \ldots, u_s\}$ for $s \geq 1$. Consider the following short exact sequence:

$$0 \longrightarrow \frac{R}{I_3(G) : uv}(-2) \xrightarrow{\cdot uv} \frac{R}{I_3(G)} \longrightarrow \frac{R}{\langle uv, I_3(G)\rangle} \longrightarrow 0. \tag{1}$$

Using Lemma 4.1(1), we have

$$I_3(G) : uv = \langle v_i : 1 \leq i \leq r-1\rangle + \langle u_j : 2 \leq j \leq s\rangle + I_3(G \setminus N_G[e]),$$
$$\text{where } e = \{u, v\}.$$

By virtue of Lemma 4.3, we have $\nu_3(G \setminus N_G[e]) \leq \nu_3(G) - 1$. Therefore, by inductive hypothesis, $\operatorname{reg}(R/(I_3(G) : uv)) \leq 2\nu_3(G \setminus N_G[e]) \leq 2\nu_3(G) - 2$. Now, set $J = \langle uv, I_3(G)\rangle$ and consider the following short exact sequence:

$$0 \longrightarrow \frac{R}{J : u}(-1) \xrightarrow{\cdot u} \frac{R}{J} \longrightarrow \frac{R}{\langle u, J\rangle} \longrightarrow 0. \tag{2}$$

Observe that, $\langle u, J\rangle = \langle u\rangle + I_3(G \setminus \{u\})$. By Remark 2.2, $\nu_3(G \setminus \{u\}) \leq \nu_3(G)$. Thus, it follows from inductive hypothesis that $\operatorname{reg}(R/\langle u, J\rangle) \leq 2\nu_3(G \setminus \{u\}) \leq 2\nu_3(G)$. On the other hand, by Lemma 4.1(2), $J : u = \langle v\rangle + I_2(H) + I_3(G \setminus N_G[u])$, where H is the union of $N_G^{edge}(u)$ and the complete graph on the vertex set $N_G(u) \setminus \{v\}$.

Case-I: If $s = 1$, then $N_G^{edge}(u) = \{\{v, v_i\} : 1 \leq i \leq r-1\}$. Also, note that the graph $G \setminus N_G[u]$ does not have an edge. This implies that $J : u = \langle v\rangle$, and hence $\operatorname{reg}(S/(J : u)) = 0 \leq 2\nu_3(G) - 1$.

Case-II: Suppose $s \geq 2$ and $\deg(u_i) = 1$ for $2 \leq i \leq s$. Then, H is the union of $N_G^{edge}(u) = \{\{v, v_i\} : 1 \leq i \leq r-1\}$ and the complete graph on the vertex set $\{u_2, \ldots, u_s\}$. In this case, the graph $G \setminus N_G[u]$ has no edge. Thus, $J : u = \langle v\rangle + I_2(H)$. Since H is co-chordal, by [8, Theorem 1], $I_2(H)$ has a linear resolution. Hence, we get $\operatorname{reg}(R/(J : u)) = 1 \leq 2\nu_3(G) - 1$.

Therefore, it follows from [15, Corollary 18.7] applying to the short exact sequences (1) and (2) that

$$\operatorname{reg}(R/I_3(G)) \leq \max\{\operatorname{reg}(R/(I_3(G) : uv)) + 2, \operatorname{reg}(R/(J : u)) + 1, \operatorname{reg}(R/\langle u, J\rangle)\}.$$

Hence, $\operatorname{reg}(R/J) \leq 2\nu_3(G)$.

Case-III: Suppose now $s \geq 2$ and $\deg_G(u_i) \geq 2$ for some $2 \leq i \leq s$. Without loss of generality, we assume that $\deg_G(u_i) \geq 2$ for all $2 \leq i \leq t$ and $\deg(u_i) = 1$ for all $t + 1 \leq i \leq s$. In this case, H is a union of the edges $N_G^{\text{edge}}(u)$ and the complete graph on the vertex set $\{u_2, \ldots, u_s\}$, i.e.,

$$J : u = \langle v\rangle + \langle u_i u_j : 2 \leq i < j \leq s\rangle + \sum_{i=2}^{t} \langle u_i x : x \in N_G(u_i) \setminus \{u\}\rangle + I_3\left(G \setminus N_G[u]\right).$$

Set $J_1 = J : u$ and $J_i = J_1 + \langle u_2, \ldots, u_i \rangle$ for $2 \le i \le t$. For $2 \le i \le t$, consider the following short exact sequence:

$$0 \longrightarrow \frac{R}{J_{i-1} : u_i}(-1) \xrightarrow{\cdot u_i} \frac{R}{J_{i-1}} \longrightarrow \frac{R}{J_i} \longrightarrow 0. \tag{3}$$

By Lemma 4.1, $J : uu_i = \langle u_j : 1 \le j \le s \text{ and } j \ne i \rangle + \langle w : w \in N_G(u_i) \backslash \{u\} \rangle + I_3(G \backslash N_G[\{u, u_i\}]$. Therefore,

$$\begin{aligned} J_{i-1} : u_i &= (J_1 + \langle u_2, \ldots, u_{i-1} \rangle) : u_i = (J : uu_i) + \langle u_2, \ldots, u_{i-1} \rangle \\ &= \langle u_j : 1 \le j \le s,\ j \ne i \rangle + \langle w : w \in N_G(u_i) \setminus \{u\} \rangle + I_3(G \setminus N_G[\{u, u_i\}]. \end{aligned}$$

Further, we can write

$$\begin{aligned} J_{i-1} : u_i = \langle u_j : 1 \le j \le s,\ j \ne i \rangle + \langle w : w \in N_G(u_i) \setminus \{u\} \rangle \\ + I_3(G \setminus \{N_G[v] \cup N_G[u] \cup N_G[u_i]\}) \end{aligned}$$

for $2 \le i \le t$. Also, $J_t = \langle v, u_2, \ldots, u_t \rangle + I_3(G \backslash \{N_G[v] \cup N_G[u]\})$. Now, it follows from [15, Corollary 18.7] applying to the short exact sequence (3) that

$$\operatorname{reg}(R/J_1) \le \max\{\operatorname{reg}(R/(J_{i-1} : u_i)) + 1, \operatorname{reg}(R/J_t) : 2 \le i \le t\}.$$

Since $G \setminus \{N_G[v] \cup N_G[u] \cup N_G[u_i]\}$ is an induced subgraph of $G \setminus N_G[e]$, by Remark 2.2 and Lemma 4.3, we have $\nu_3(G \backslash \{N_G[v] \cup N_G[u] \cup N_G[u_i]\}) \le \nu_3(G \backslash \{N_G[v] \cup N_G[u]\}) \le \nu_3(G) - 1$.

By inductive hypothesis, $\operatorname{reg}(R/(J_{i-1} : u_i)) \le 2\nu_3(G \backslash \{N_G[v] \cup N_G[u] \cup N_G[u_i]\}) \le 2(\nu_3(G) - 1)$ and $\operatorname{reg}(R/J_t) \le 2\nu_3(G \backslash \{N_G[v] \cup N_G[u]\}) \le 2\nu_3(G) - 2$. By applying [15, Corollary 18.7] to the above short exact sequences, we have

$$\begin{aligned} \operatorname{reg}(R/I_3(G)) &\le \max\{\operatorname{reg}(R/(I_3(G) : uv)) + 2, \operatorname{reg}(R/J)\} \\ &\le \max\{\operatorname{reg}(R/(I_3(G) : uv)) + 2, \operatorname{reg}(R/\langle u, J \rangle), \operatorname{reg}(R/J_1) + 1\} \\ &\le \max\{\operatorname{reg}(R/(I_3(G) : uv)) + 2, \operatorname{reg}(R/\langle u, J \rangle), \\ &\quad \operatorname{reg}(R/(J_1 : u_2)) + 2, \operatorname{reg}(R/J_2) + 1\} \\ &\le \max_{2 \le i \le t} \{\operatorname{reg}(R/(I_3(G) : uv)) + 2, \operatorname{reg}(R/\langle u, J \rangle), \\ &\quad \operatorname{reg}(R/(J_{i-1} : u_i)) + 2, \operatorname{reg}(R/J_t) + 1\}. \end{aligned}$$

Hence, the assertion follows. □

Now, we proceed to study the regularity of 3-path ideal of unicyclic graphs. If G is a cycle, then the regularity of $R/I_3(G)$ has been computed in [2]. So, we assume that G is not a cycle. We give a sharp upper bound for the regularity of $R/I_3(G)$. The idea of the proof is kind of similar to the proof of Theorem 4.4. We fix the following notation for unicyclic graphs.

Notation 4.5 Let G be a unicyclic graph with the induced cycle C. Then, trees are attached to at least one vertex of C, say $u \in V(C)$. Let $v \in N_G(u)\backslash V(C)$ and $e = \{u, v\}$. Clearly, $N_G(u) \setminus \{v\}$ contains at least 2 vertices and set $N_G(u) = \{u_1 = v, u_2, \ldots, u_t\}$ for $t \geq 3$.

Theorem 4.6 *Let G be a unicyclic graph and $I_3(G)$ be its 3-path ideal. Then,*

$$2\nu_3(G) \leq \operatorname{reg}(R/I_3(G)) \leq 2\nu_3(G) + 2.$$

Proof Let G be a unicyclic graph with the notation as in Notation 4.5. The lower bound for $\operatorname{reg}(R/I_3(G))$ follows from Corollary 3.2. So, here we only establish the upper bound. Consider the short exact sequence (1). By Lemma 4.1(1), $I_3(G) : uv = \langle N_G(e)\rangle + I_3(G \setminus N_G[e])$, where $e = \{u, v\}$. Since $G\backslash N_G[e]$ is an induced subgraph of G, by Remark 2.2, $\nu_3(G\backslash N_G[e]) \leq \nu_3(G)$. Note that $G \setminus N_G[e]$ is a tree. Thus, it follows from Theorem 4.4 that

$$\operatorname{reg}(R/(I_3(G) : uv)) = 2\nu_3(G \setminus N_G[e]) \leq 2\nu_3(G).$$

Now, set $J = \langle uv, I_3(G)\rangle$ and we consider the short exact sequence (2), where $J : u = \langle v\rangle + I_2(H) + I_3(G\backslash\{N_G[u]\})$, where H is the union of $N_G^{edge}(u)$ and the complete graph on the vertex set $N_G(u)\backslash\{v\}$. Also, $\langle u, J\rangle = \langle u, I_3(G\backslash\{u\})\rangle$. Since $G\backslash\{u\}$ is a tree, by Theorem 4.4, we have

$$\operatorname{reg}(R/\langle u, J\rangle) = 2\nu_3(G \setminus \{u\}) \leq 2\nu_3(G).$$

Set $J_1 = J : u$ and $J_i = J_1 + \langle u_2, \ldots, u_i\rangle$, where $N_G(u) = \{v, u_2, \ldots, u_t\}$ for $2 \leq i \leq t$ and consider short exact sequences (3).

It can be observed that $J_{i-1} : u_i = \langle u_j : 1 \leq j \leq t,\ j \neq i\rangle + \langle w : w \in N_G(u_i)\backslash\{u\}\rangle + I_3(G\backslash\{N_G[u] \cup N_G[u_i]\})$ for $2 \leq i \leq t$ and $J_t = \langle v, u_2, \ldots, u_t\rangle + I_3(G \setminus \{N_G[u]\})$. Now it follows from [15, Corollary 18.7] applying to the short exact sequence (3) that

$$\operatorname{reg}(R/J_1) \leq \max\{\operatorname{reg}(R/(J_{i-1} : u_i)) + 1, \operatorname{reg}(R/J_t) : 2 \leq i \leq t\}.$$

Now, $\nu_3(G \setminus \{N_G[u] \cup N_G[w_i]\}) \leq \nu_3(G)$ and $\nu_3(G \setminus \{N_G[u]\}) \leq \nu_3(G)$ follow from Remark 2.2. Since $G\backslash\{N_G[u] \cup N_G[w_i]\}$ and $G\backslash\{N_G[u]\}$ are trees, by Theorem 4.4, $\operatorname{reg}(R/(J_i : w_i)) = 2\nu_3(G\backslash\{N_G[u] \cup N_G[w_i]\}) \leq 2\nu_3(G)$ and $\operatorname{reg}(R/J_t) = 2\nu_3(G \setminus \{N_G[u]\}) \leq 2\nu_3(G)$. Therefore, it follows from applying [15, Corollary 18.7] to short exact sequences 1, 2 and 3 that $\operatorname{reg}(R/I_3(G)) \leq 2\nu_3(G) + 2$. □

We now show by examples that all the three possibilities for the regularity of $R/I_3(G)$, namely $2\nu_3(G)$, $2\nu_3(G) + 1$ and $2\nu_3(G) + 2$, indeed occur for unicyclic graphs.

Example 4.7 Consider graphs G_1, G_2 and G_3 as in Fig. 2. Then, using Macaulay 2 ([9]), it can be computed that $\operatorname{reg}(R/I_3(G_1)) = 2$, $\operatorname{reg}(R/I_3(G_2)) = 3$ and $\operatorname{reg}(R/I_3(G_3)) = 6$. Note that $\nu_3(G_1) = 1 = \nu_3(G_2)$ and $\nu_3(G_3) = 2$.

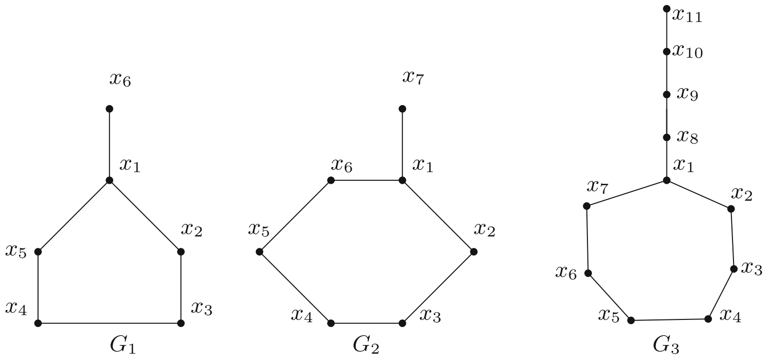


Fig. 2 Unicyclic graphs

Acknowledgements The second author thanks the National Board for Higher Mathematics (NBHM), Department of Atomic Energy, Government of India, for financial support through Postdoctoral Fellowship.

Data Availability Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Alilooee, A., Faridi, S.: Graded Betti numbers of path ideals of cycles and lines. J. Algebra Appl. **17**(1), 1850011, 17 (2018)
2. Alilooee, A., Faridi, S.: On the resolution of path ideals of cycles. Commun. Algebra **43**(12), 5413–5433 (2015)
3. Banerjee, A.: Regularity of path ideals of gap free graphs. J. Pure Appl. Algebra **221**(10), 2409–2419 (2017)
4. Bouchat, R.R., Brown, T.M.: Multi-graded Betti numbers of path ideals of trees. J. Algebra Appl. **16**(1), 1750018, 20 (2017)
5. Bouchat, R.R., Hà, H.T., O'Keefe, A.: Path ideals of rooted trees and their graded Betti numbers. J. Combin. Theory Ser. A **118**(8), 2411–2425 (2011)
6. Conca, A., De Negri, E.: M-sequences, graph ideals, and ladder ideals of linear type. J. Algebra **211**(2), 599–624 (1999)
7. Erey, N.: Multigraded Betti numbers of some path ideals. In: Combinatorial Structures in Algebra and Geometry, volume 331 of Springer Proc. Math. Stat., Springer, Cham, [2020], pp. 51–65 (2020)
8. Fröberg, R.: On Stanley-Reisner rings. In: Topics in algebra, Part 2 (Warsaw, 1988), volume 26 of Banach Center Publ., PWN, Warsaw, pp. 57–70 (1990)
9. Grayson, D.R., Stillman, M.E.: Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/
10. Jacques, S., Katzman, M.: The Betti numbers of forests. arXiv Mathematics e-prints, page arXiv:math/0501226, (January 2005)
11. Katzman, M.: Characteristic-independence of Betti numbers of graph ideals. J. Combin. Theory Ser. A **113**(3), 435–454 (2006)
12. Kiani, D., Madani, S.S.: Betti numbers of path ideals of trees. Commun. Algebra **44**(12), 5376–5394 (2016)

13. Kubitzke, M., Olteanu, A.: Algebraic properties of classes of path ideals of posets. J. Pure Appl. Algebra **218**(6), 1012–1033 (2014)
14. Ohsugi, H., Herzog, J., Hibi, T.: Combinatorial pure subrings. Osaka J. Math. **37**(3), 745–757 (2000)
15. Peeva, I.: Graded syzygies. Algebra and Applications, vol. 14. Springer-Verlag, London Ltd, London (2011)