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Meromorphic Solutions of Nonlinear Systems of Fermat Type

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Abstract

We give the alternative proofs to consider Fermat systems of complex differential or difference or delay-differential equations. In addition, we also use value distribution of meromorphic functions to consider the existence of meromorphic solutions of complex differential or delay-differential systems.

Keywords Meromorphic functions · Fermat equations · Value distribution · Complex delay-differential equations

Mathematics Subject Classification 30D35 · 39A45

1 Introduction and Main Results

Let us observe Fermat-type functional equation

$$f(z)^2 + g(z)^2 = 1, \tag{1.1}$$

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where $f(z)$ and $g(z)$ are transcendental meromorphic functions. Iyer [12] and Gross [7] obtained that all entire solutions of (1.1) are $f(z) = \sin(h(z))$, $g(z) = \cos(h(z))$, where $h(z)$ is any entire function, and there are no other entire solutions. If $g(z)$ is a differential polynomial of $f(z)$ in (1.1), then this equation is always called Fermat-type differential equation. Using the above Iyer and Gross's result and a basic computation on

$$f(z)^2 + f'(z)^2 = 1, \tag{1.2}$$

we can obtain that all meromorphic solutions $f(z) = \sin(z+B)$, where B is a constant. Further researches on Fermat-type differential equations with different forms can be found in [25, 27] or [19, Chapter 6]. Some results on Fermat-type difference equations can be found in [18, 20, 21] and Fermat-type differential-difference equations or partial differential-difference equations can be found in [11, 18, 23, 24]. Of course, Eq. (1.1) can be seen as the functional version of Fermat equation $x^2 + y^2 = 1$, where x, y are rational functions, while the Fermat-type matrix equations $X^n + Y^n = Z^n$ are also considered in [1–3], where X, Y, Z are 2-by-2 rational or integer matrices and n is a positive integer. The elementary notations and results of Nevanlinna theory, such as the proximity function $m(r, f)$, the counting function $N(r, f)$, the reduced counting function $\overline{N}(r, f)$, the characteristic function $T(r, f)$, the order $\rho(f)$ and the hyper-order $\rho_2(f)$, can be found in [9, 26].

Recently, the present authors and Si [14, Theorem 2.1] considered Fermat-type matrix differential equation

$$\begin{pmatrix} f(z) & g(z) \\ h(z) & k(z) \end{pmatrix}^2 + \begin{pmatrix} f'(z) & g'(z) \\ h'(z) & k'(z) \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.3}$$

where at least one of $f(z)$, $g(z)$, $h(z)$, $k(z)$ is a non-constant meromorphic function. The properties on meromorphic matrix solutions are described in [14], where f and g may satisfy the following bi-Fermat-type differential equation

$$f(z)^2 + f'(z)^2 + g(z)^2 + g'(z)^2 = 1. \tag{1.4}$$

Hence, the characteristics of meromorphic solutions f and g of (1.4) are important for the meromorphic matrix solutions of (1.3). However, the meromorphic solutions or even entire solutions of (1.4) are difficult to obtain presently. We see that the equation (1.4) implies many questions indeed, such as the existence of meromorphic solutions on systems of complex differential equations with different types. Some discussions on

$$\begin{cases} f(z)^2 + g(z)^2 = \sin^2 h(z), \\ f'(z)^2 + g'(z)^2 = \cos^2 h(z), \end{cases}$$

have been considered in [14], where $h(z)$ is any entire function. In this paper, we observe the system

$$\begin{cases} f(z)^2 + g'(z)^2 = \dfrac{1}{2} + h(z), \\ f'(z)^2 + g(z)^2 = \dfrac{1}{2} - h(z). \end{cases} \tag{1.5}$$

It is obvious that all the meromorphic solutions of (1.5) solve the equation (1.4), where $h(z)$ is any meromorphic function. If $h(z)$ is entire in (1.5), then all meromorphic solutions $f(z)$ and $g(z)$ are also entire. For example, if $h(z) = \sin^2 z - \frac{1}{2}$, then $(f, g) = \left(\pm\frac{\sqrt{2}}{2}\sin z, \pm\frac{\sqrt{2}}{2}\cos z\right)$ solve (1.5). If $h(z)$ is meromorphic with at least a pole z_0, then the multiplicities of the pole z_0 of $f(z)$ and $g(z)$ must be equal. However, it is difficult to give all meromorphic solutions of (1.5) for the variability of $h(z)$. Without loss of generality, the right-hand sides of the system (1.5) are assumed to be the unit if $h(z) \equiv 0$; in this situation, all the meromorphic solutions can be given completely in Theorem 1.1.

Theorem 1.1 *All the transcendental meromorphic solutions of the system of complex differential equations*

$$\begin{cases} f(z)^2 + g'(z)^2 = 1, \\ f'(z)^2 + g(z)^2 = 1, \end{cases} \tag{1.6}$$

must satisfy one of the following cases:

(i) $f(z) = \sin(z + \alpha)$, $g(z) = \sin(z + \beta)$ *and* $\alpha - \beta = 2k\pi$;
(ii) $f(z) = \sin(-z + \alpha)$, $g(z) = \sin(-z + \beta)$ *and* $\alpha - \beta = 2k\pi + \pi$,

where k *is an integer,* α *and* β *are constants.*

Remark 1.2 (1) It is easy to see that $f(z) = \sin z$ and $g(z) = \sin(-z)$ solve (1.6) also. These solutions can be included in the case (ii) by taking $f(z) = \sin(-z + \pi)$ and $g(z) = \sin(-z)$.

(2) It remains open for us to describe all the meromorphic solutions of

$$\begin{cases} f(z)^2 + g^{(k)}(z)^2 = P_1(z)e^{\kappa_1(z)}, \\ f^{(k)}(z)^2 + g(z)^2 = P_2(z)e^{\kappa_2(z)}, \end{cases}$$

where $k \geq 2$ is a positive integer, $\kappa_1(z)$ and $\kappa_2(z)$ are any entire functions, and $P_1(z)$ and $P_2(z)$ are any polynomials.

(3) The authors did not find the relevant results on (1.6), although we felt that this system should have been studied, but we believe that the proof is different.

In addition, the present authors and Si [15] also considered Fermat-type matrix difference equation

$$\begin{pmatrix} f(z) & g(z) \\ h(z) & k(z) \end{pmatrix}^2 + \begin{pmatrix} f(z+c) & g(z+c) \\ h(z+c) & k(z+c) \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where at least one of $f(z)$, $g(z)$, $h(z)$, $k(z)$ is a non-constant meromorphic function and c is a nonzero constant. The following bi-Fermat-type difference equation

$$f(z)^2 + f(z+c)^2 + g(z)^2 + g(z+c)^2 = 1$$

are also presented in [15]. In this paper, we will obtain all the transcendental meromorphic solutions of the system of complex difference equations

$$\begin{cases} f(z)^2 + g(z+c)^2 = 1, \\ g(z)^2 + f(z+c)^2 = 1, \end{cases} \tag{1.7}$$

as follows.

Theorem 1.3 *Transcendental entire solutions of (1.7) are expressed by* $f(z) = \sin(h_1(z))$, $g(z) = \sin(h_2(z))$, *where* $h_1(z)$ *and* $h_2(z)$ *are non-constant entire functions satisfying one of the following cases:*

(1) $h_1(z+c) = h_2(z) + \frac{\pi}{2} + 2n\pi$ *and* $h_2(z+c) = h_1(z) + \frac{\pi}{2} + 2m\pi$;
(2) $h_1(z+c) = h_2(z) + \frac{\pi}{2} + 2n\pi$ *and* $h_2(z+c) = -h_1(z) + \frac{\pi}{2} + 2m\pi$;
(3) $h_1(z+c) = -h_2(z) + \frac{\pi}{2} + 2n\pi$ *and* $h_2(z+c) = h_1(z) + \frac{\pi}{2} + 2m\pi$;
(4) $h_1(z+c) = -h_2(z) + \frac{\pi}{2} + 2n\pi$ *and* $h_2(z+c) = -h_1(z) + \frac{\pi}{2} + 2m\pi$,

where m, n *are integers. Furthermore, if* f *and* g *are of finite order, then* $h_1(z)$ *and* $h_2(z)$ *satisfy one of the following two cases:*

(i) $h_1(z) = \mu z + \nu_1$ *and* $h_2(z) = \mu z + \nu_2$, *where* $\mu c + \nu_1 = \nu_2 + \frac{\pi}{2} + 2n\pi$ *and* $\mu c + \nu_2 = \nu_1 + \frac{\pi}{2} + 2m\pi$. *Then* $2\mu c = \pi + 2(n+m)\pi$;
(ii) $h_1(z) = \mu z + \nu_1$ *and* $h_2(z) = -\mu z + \nu_2$, *where* $\mu c + \nu_1 = -\nu_2 + \frac{\pi}{2} + 2n\pi$ *and* $-\mu c + \nu_2 = -\nu_1 + \frac{\pi}{2} + 2m\pi$. *Then* $2\mu c = 2(n-m)\pi$.

Transcendental meromorphic solutions of (1.7) are $f(z) = \frac{2\beta_1(z)}{1+\beta_1(z)^2}$ *and* $g(z) = \frac{1-\beta_2(z)^2}{1+\beta_2(z)^2}$, *where* $\beta_1(z)$ *and* $\beta_2(z)$ *are transcendental meromorphic functions satisfying one of the following three cases:*

(a) $\beta_1(z+c) = \beta_2(z)$, $\beta_1(z) = \beta_2(z+c)$, $\beta_1(z)$ *and* $\beta_2(z)$ *are periodic functions with period* $2c$;
(b) $\beta_1(z+c) = \beta_2(z)$, $\beta_1(z) = -\beta_2(z+c)$, $\beta_1(z)$ *and* $\beta_2(z)$ *are anti-periodic functions with period* $2c$;
(c) $\beta_1(z+c) \neq \beta_2(z)$, $\beta_1(z+c)\beta_2(z) = 1$, $\beta_2(z)^2\beta_2(z+2c)^2 = 1$ *and* $\beta_1(z)^2\beta_1(z+2c)^2 = 1$.

Remark 1.4 We give an example to show that all cases can occur in Theorem 1.3. Meromorphic functions f and g below solve the system

$$\begin{cases} f(z)^2 + g(z+\pi)^2 = 1, \\ g(z)^2 + f(z+\pi)^2 = 1. \end{cases}$$

(I) Taking the suitable integers m, n which may be different when they appear. $f(z) = \sin(e^{2iz} + \frac{z}{2} + 2n\pi)$ and $g(z) = \sin(e^{2iz} + \frac{z}{2} + 2m\pi)$, where $h_1(z)$ and $h_2(z)$ satisfy the case (1); $f(z) = \sin(e^{\frac{iz}{2}} + \frac{\pi}{2} + 2n\pi)$ and $g(z) = \sin(ie^{\frac{iz}{2}} + 2m\pi)$, where $h_1(z)$ and $h_2(z)$ satisfy the case (2); $f(z) = \sin(ie^{\frac{iz}{2}} + 2n\pi)$ and $g(z) = \sin(e^{\frac{iz}{2}} + \frac{\pi}{2} + 2m\pi)$, where $h_1(z)$ and $h_2(z)$ satisfy the case (3); $f(z) = \sin(e^{iz} - \frac{z}{2} + 2n\pi)$ and $g(z) = \sin(e^{iz} + \frac{z+2\pi}{2} + 2m\pi)$, where $h_1(z)$ and $h_2(z)$ satisfy the case (4).

(II) $f(z) = \sin z$ and $g(z) = \sin(z + \frac{\pi}{2})$, where $h_1(z)$ and $h_2(z)$ satisfy the case (i); $f(z) = \sin z$ and $g(z) = \sin(-z - \frac{\pi}{2})$, where $h_1(z)$ and $h_2(z)$ satisfy the case (ii).

(III) $f(z) = \frac{2\tan\frac{z}{2}}{1+(\tan\frac{z}{2})^2}$ and $g(z) = \frac{1-(-\cot\frac{z}{2})^2}{1+(-\cot\frac{z}{2})^2}$, where $h_1(z)$ and $h_2(z)$ satisfy the case (a); $f(z) = \frac{2\sin\frac{z}{2}}{1+(\sin\frac{z}{2})^2}$ and $g(z) = \frac{1-(\cos\frac{z}{2})^2}{1+(\cos\frac{z}{2})^2}$, where $h_1(z)$ and $h_2(z)$ satisfy the case (b); $f(z) = \frac{2e^{\cos\frac{z}{2}}}{1+(e^{\cos\frac{z}{2}})^2}$ and $g(z) = \frac{1-(e^{\sin\frac{z}{2}})^2}{1+(e^{\sin\frac{z}{2}})^2}$, where $h_1(z)$ and $h_2(z)$ satisfy the case (c).

Furthermore, we recall that Gao [4, Theorem 1.1] proved the transcendental entire solutions $f(z)$ and $g(z)$ with finite order of the following system of complex delay-differential equations

$$\begin{cases} f'(z)^2 + g(z+c)^2 = 1, \\ g'(z)^2 + f(z+c)^2 = 1, \end{cases} \tag{1.8}$$

are $(f(z), g(z)) = (\sin(z + b_1), \sin(z + b_2))$, where $c = k\pi$, k is an integer and b_1, b_2 are constants. In this paper, we will give a different proof and give the supplement on the entire solutions of (1.8).

Theorem 1.5 *The transcendental entire solutions of* (1.8) *are* $f(z) = \sin(h_2(z - c))$ *and* $g(z) = \sin(h_1(z - c))$, *where* $h_2'(z - c)^2 \equiv h_2'(z + c)^2$ *and* $h_2'(z - c)^2 h_1'(z)^2 \equiv 1$. *Specially, the transcendental entire solutions with finite order of* (1.8) *satisfy one of the following four cases:*

(i) $f(z) = \sin(z + b)$, $g(z) = \sin(z + b + k\pi + 2m\pi)$ *and* $c = k\pi$;
(ii) $f(z) = \sin(-z + b)$, $g(z) = \sin(-z + b + k\pi + m\pi)$ *and* $c = k\pi$;
(iii) $f(z) = \sin(-z + b)$, $g(z) = \sin(z - b + \frac{\pi}{2} + k\pi + 2m\pi)$ *and* $c = \frac{\pi}{2} + k\pi$;
(iv) $f(z) = \sin(z + b)$, $g(z) = \sin(-z - b + \frac{\pi}{2} + k\pi + m\pi)$ *and* $c = \frac{\pi}{2} + k\pi$,

where k, m *are any integers and* b *is a constant.*

In the following, we will consider the system (1.5) from the point of view of value distribution of meromorphic functions. Theorem 1.1 implies that both $f(z)^2 + g'(z)^2$ and $f'(z)^2 + g(z)^2$ can have no zeros. If $h(z)$ has no zeros or finitely many zeros in (1.5), then both $f(z)^2 + g'(z)^2 - \frac{1}{2}$ and $f'(z)^2 + g(z)^2 - \frac{1}{2}$ can have no zeros or have finitely many zeros. If $h(z)$ is a transcendental entire function in (1.5), then at least one of $\frac{1}{2} + h(z)$ and $\frac{1}{2} - h(z)$ admits infinitely many zeros by the second main theorem of Nevanlinna theory; in this case, it means that for the entire solutions $f(z)$ and $g(z)$ of (1.5), then at least one of $f(z)^2+g'(z)^2$ and $f'(z)^2+g(z)^2$ must have infinitely many zeros.

The above observations inspire us to consider the zeros of a pair of complex differential polynomials $f(z)^n+\beta_1(z)g^{(k)}(z)^m-\alpha_1(z)$ and $g(z)^n+\beta_2(z)f^{(k)}(z)^m-\alpha_2(z)$, where $\beta_1(z)$, $\beta_2(z)$, $\alpha_1(z)$ and $\alpha_2(z)$ are nonzero small functions with respect to $f(z)$ and $g(z)$ and k, m, n are positive integers. These considerations can be seen as the variations to the classical results given by Hayman [10, Theorems 8 and 9], namely the zero distribution of $f(z)^n f'(z)-a$ or $f(z)^n+bf'(z)-a$, where a, b are nonzero constants. We agree to say that a meromorphic function $f(z)$ in the complex plane is properly meromorphic if $f(z)$ has at least one pole.

Theorem 1.6 *Let $f(z)$ and $g(z)$ be transcendental meromorphic functions, $\beta_1(z)$, $\beta_2(z)$, $\alpha_1(z)$ and $\alpha_2(z)$ be nonzero small functions with respect to $f(z)$ and $g(z)$ and k, m, n be positive integers. If one of the following conditions is satisfied*

(*a*) *at least one of $f(z)$ and $g(z)$ is a transcendental properly meromorphic function and $n \geq 2m(k+1)+4$;*
(*b*) *$f(z)$ and $g(z)$ are all transcendental entire functions and $n \geq 2m+2$,*

then at least one of $f(z)^n+\beta_1(z)g^{(k)}(z)^m-\alpha_1(z)$ and $g(z)^n+\beta_2(z)f^{(k)}(z)^m-\alpha_2(z)$ have infinitely many zeros.

Remark 1.7 (1) In a recent paper, Gao and Liu [5, Theorem 1.1] considered the zero distribution of another paired complex differential polynomials $f(z)^n g^{(k)}(z)-a(z)$ and $g(z)^n f^{(k)}(z)-a(z)$, where $a(z)$ is a nonzero small function with respect to $f(z)$ and $g(z)$.

(2) Remark the following system

$$\begin{cases} f(z)^n+g^{(k)}(z)=P_1(z)e^{Q_1(z)}, \\ g(z)^n+f^{(k)}(z)=P_2(z)e^{Q_2(z)}, \end{cases} \tag{1.9}$$

where $P_1(z)$, $P_2(z)$, $Q_1(z)$ and $Q_2(z)$ are any non-constant polynomials. If $n=1$, then the system (1.9) changes into

$$F(z)+F^{(k)}(z)=P_1(z)e^{Q_1(z)}+P_2(z)e^{Q_2(z)}$$

by letting $F(z)=f+g$ and summing the two equations of (1.9). If $n \geq 2$, by the sum of two equations, then (1.9) changes into

$$f(z)^2+g(z)^2+(f(z)+g(z))^{(k)}=P_1(z)e^{Q_1(z)}+P_2(z)e^{Q_2(z)}.$$

A natural question is to describe all the entire or meromorphic solutions of the above equation which is similar as the differential equation of Tumura–Clunie type

$$f(z)^n+f^{(k)}(z)=P_1(z)e^{Q_1(z)}+P_2(z)e^{Q_2(z)}.$$

The above equation was considered frequently, such as [16, 17] and their references.

Combining the similar proofs of Theorem 1.6 with the important result [8, Lemma 8.3] in difference, we can obtain the following result without giving the proof details.

Theorem 1.8 *Let $f(z)$ and $g(z)$ be transcendental meromorphic functions with hyper-order less than one, $\beta_1(z)$, $\beta_2(z)$, $\alpha_1(z)$ and $\alpha_2(z)$ be nonzero small functions with respect to $f(z)$ and $g(z)$. If one of the following conditions is satisfied*

(*a*) $n \geq 2m(k+1)+4$ *and at least one of $f(z)$ and $g(z)$ is a transcendental properly meromorphic function;*
(*b*) $n \geq 2m+2$, *f and g are all transcendental entire functions,*

where n, k and m are positive integers, then at least one of $f(z)^n + \beta_1(z)g^{(k)}(z+c_1)^m - \alpha_1(z)$ and $g(z)^n + \beta_2(z)f^{(k)}(z+c_2)^m - \alpha_2(z)$ have infinitely many zeros.

Corollary 1.9 *Let $s_1(z)$ and $s_2(z)$ be nonzero polynomials, $t_1(z)$ and $t_2(z)$ be entire functions. If $n \geq 2m(k+1)+4$, then the system*

$$\begin{cases} f(z)^n + \beta_1(z)g^{(k)}(z)^m - \alpha_1(z) = s_1(z)e^{t_1(z)}, \\ g(z)^n + \beta_2(z)f^{(k)}(z)^m - \alpha_2(z) = s_2(z)e^{t_2(z)}, \end{cases} \tag{1.10}$$

has no any transcendental meromorphic solutions and

$$\begin{cases} f(z)^n + \beta_1(z)g^{(k)}(z+c_1)^m - \alpha_1(z) = s_1(z)e^{t_1(z)}, \\ g(z)^n + \beta_2(z)f^{(k)}(z+c_2)^m - \alpha_2(z) = s_2(z)e^{t_2(z)}, \end{cases} \tag{1.11}$$

has no any transcendental meromorphic solutions with hyper-order less than one.

2 Proofs of Theorems

Proof of Theorem 1.1 Firstly, we confirm that all meromorphic solutions $f(z)$ and $g(z)$ must be entire from (1.6). Otherwise, assume that z_0 is a pole of $f(z)$ with multiplicity m, then z_0 is a pole of $g(z)$ with multiplicity $m-1$ from the first equation of (1.6). From the second equation of (1.6), z_0 is a pole of $g(z)$ with multiplicity $m+1$, which is impossible. Combining the first equation of (1.6) with Iyer and Gross's result in the introduction, we assume that

$$\begin{cases} f(z) = \sin(h_1(z)), \\ g'(z) = \cos(h_1(z)), \end{cases} \tag{2.1}$$

where $h_1(z)$ is a non-constant entire function. By the second equation of (1.6), we assume that

$$\begin{cases} f'(z) = \cos(h_2(z)), \\ g(z) = \sin(h_2(z)). \end{cases} \tag{2.2}$$

Using the first equation of (2.1) and the first equation of (2.2), we have

$$h_1'(z)\cos(h_1(z)) = \cos(h_2(z)). \tag{2.3}$$

The second equation of (2.1) and the second equation of (2.2) imply that

$$h_2'(z)\cos(h_2(z)) = \cos(h_1(z)). \tag{2.4}$$

By (2.3) and (2.4), we conclude

$$h_1'(z)h_2'(z)\cos(h_1(z)) = \cos(h_1(z)),$$

which implies that $h_1'(z)h_2'(z) \equiv 1$. Since $h_1(z)$ and $h_2(z)$ are non-constant entire functions, then there are two possibilities only:

(1) $h_1'(z) = e^{s(z)}$ and $h_2'(z) = e^{-s(z)}$, where $s(z)$ is a non-constant entire function.
(2) $h_1(z) = \lambda z + \alpha$ and $h_2(z) = \frac{1}{\lambda}z + \beta$, where λ is a nonzero constant, α and β are any constants.

In the following, we will affirm that the case (1) cannot happen. Taking the first derivative of (2.3), we have

$$h_1''\cos h_1 - h_1'^2\sin h_1 = -h_2'\sin h_2. \tag{2.5}$$

Taking derivative again for (2.5), we have

$$h_1'''\cos h_1 - 3h_1''h_1'\sin h_1 - h_1'^3\cos h_1 = -h_2''\sin h_2 - h_2'^2\cos h_2. \tag{2.6}$$

Substitute $\cos h_2$ and $\sin h_2$ into (2.6), we conclude that

$$\begin{aligned}&\left((h_1'''h_2' - h_1'^3h_2' - h_2''h_1'' + h_2'^3h_1')^2 + (3h_1''h_1'h_2' - h_1'^2h_2'')^2\right)\cos^2 h_1\\ &= (3h_1''h_1'h_2' - h_1'^2h_2'')^2.\end{aligned} \tag{2.7}$$

By [26, Theorem 1.46] and (2.3), we have that h_1 and h_2 are small functions with respect to $\cos h_1$. Using the Valiron–Mohon'ko lemma [13, Theorem 2.2.5] to (2.7), then

$$3h_1''h_1'h_2' - h_1'^2h_2'' \equiv 0 \tag{2.8}$$

and

$$(h_1'''h_2' - h_1'^3h_2' - h_2''h_1'' + h_2'^3h_1')^2 + (3h_1''h_1'h_2' - h_1'^2h_2'')^2 \equiv 0,$$

otherwise $T(r, \cos h_1) = S(r, \cos h_1)$. Integrating (2.8), we have $\mu h_1'^3 = h_2'$, where μ is a nonzero constant. Combining the above with $h_1'(z)h_2'(z) \equiv 1$, we get that h_1'

and h_2' are constants, which is impossible in (1). Now, we will show that $\lambda = \pm 1$ within the case (2). From (2.5), we have

$$\lambda^2 \sin(\lambda z + \alpha) = \frac{1}{\lambda} \sin\left(\frac{1}{\lambda} z + \beta\right).$$

By a basic computation and (2.3), we have

$$\lambda^4 (1 - \cos^2(\lambda z + \alpha)) = \frac{1}{\lambda^2} - \cos^2(\lambda z + \alpha).$$

Hence, $\lambda^2 = 1$. From (2.3), if $\lambda = 1$, we have $\alpha - \beta = 2k\pi$, if $\lambda = -1$, we have $\alpha - \beta = 2k\pi + \pi$, where k is an integer. The proof of Theorem 1.1 is completed. □

Proof of Theorem 1.3 Firstly, suppose that $f(z)$ and $g(z)$ are transcendental entire functions. Combining the system (1.7) with Iyer and Gross's result again, we assume that

$$\begin{cases} f(z) = \sin(h_1(z)), \\ g(z+c) = \cos(h_1(z)), \end{cases} \tag{2.9}$$

and

$$\begin{cases} g(z) = \sin(h_2(z)), \\ f(z+c) = \cos(h_2(z)), \end{cases} \tag{2.10}$$

where $h_1(z)$ and $h_2(z)$ are non-constant entire functions. From the first equation of (2.9) and the second equation of (2.10), we have

$$f(z+c) = \sin h_1(z+c) = \cos h_2(z) = \cos(-h_2(z)) = \sin\left(\pm h_2(z) + \frac{\pi}{2}\right).$$

Hence,

$$h_1(z+c) = \pm h_2(z) + \frac{\pi}{2} + 2n\pi, \tag{2.11}$$

where n is an integer. From the second equation of (2.9) and the first equation of (2.10), we have

$$g(z+c) = \sin h_2(z+c) = \cos h_1(z) = \sin\left(\pm h_1(z) + \frac{\pi}{2}\right).$$

Hence,

$$h_2(z+c) = \pm h_1(z) + \frac{\pi}{2} + 2m\pi, \tag{2.12}$$

where m is an integer. There exist four possibilities as follows:

(1) If $h_1(z+c) = h_2(z) + \frac{\pi}{2} + 2n\pi$ and $h_2(z+c) = h_1(z) + \frac{\pi}{2} + 2m\pi$, then we have

$$h_1(z+c) = h_1(z-c) + \pi + 2(n+m)\pi,$$

and

$$h_2(z+c) = h_2(z-c) + \pi + 2(n+m)\pi.$$

If f and g are of finite order, then $h_1(z)$ and $h_2(z)$ are first polynomials by Pólya's theorem [22] or [9, Theorem 2.9], then the above two equations imply that $h_1(z)$ and $h_2(z)$ must be linear polynomials by a basic computation. From $h_1(z+c) = h_2(z) + \frac{\pi}{2} + 2n\pi$, we have $h_1(z) = \mu z + \nu_1$ and $h_2(z) = \mu z + \nu_2$, where $\mu c + \nu_1 = \nu_2 + \frac{\pi}{2} + 2n\pi$ and $\mu c + \nu_2 = \nu_1 + \frac{\pi}{2} + 2m\pi$. Then, $2\mu c = \pi + 2(n+m)\pi$.

(2) If $h_1(z+c) = h_2(z) + \frac{\pi}{2} + 2n\pi$ and $h_2(z+c) = -h_1(z) + \frac{\pi}{2} + 2m\pi$, then we have

$$h_1(z+c) = -h_1(z-c) + \pi + 2(n+m)\pi,$$

and

$$h_2(z+c) = -h_2(z-c) + 2(m-n)\pi.$$

Obviously, there are no polynomials $h_1(z)$ and $h_2(z)$ satisfying the above two equations for the reason that $h_1(z+c)$ and $-h_1(z-c)$ have the opposite coefficients of the highest degree. Thus, f and g cannot be entire functions with finite order in this case.

(3) If $h_1(z+c) = -h_2(z) + \frac{\pi}{2} + 2n\pi$ and $h_2(z+c) = h_1(z) + \frac{\pi}{2} + 2m\pi$, then we have

$$h_1(z+c) = -h_1(z-c) + 2(n-m)\pi,$$

and

$$h_2(z+c) = -h_2(z-c) + \pi + 2(n+m)\pi.$$

In this case, f and g cannot be entire functions with finite order by the same method used in (2).

(4) If $h_1(z+c) = -h_2(z) + \frac{\pi}{2} + 2n\pi$ and $h_2(z+c) = -h_1(z) + \frac{\pi}{2} + 2m\pi$, then we have

$$h_1(z+c) = h_1(z-c) + 2(n-m)\pi,$$

and

$$h_2(z+c) = h_2(z-c) + 2(m-n)\pi.$$

Using the similar method in (1). We can get that $h_1(z) = \mu z + \nu_1$ and $h_2(z) = -\mu z + \nu_2$, where $\mu c + \nu_1 = -\nu_2 + \frac{\pi}{2} + 2n\pi$ and $-\mu c + \nu_2 = -\nu_1 + \frac{\pi}{2} + 2m\pi$. Then, $2\mu c = 2(n-m)\pi$.

Secondly, suppose that $f(z)$ and $g(z)$ are transcendental meromorphic functions. By Gross's result in [6], we assume that

$$\begin{cases} f(z) = \dfrac{2\beta_1(z)}{1+\beta_1^2(z)}, \\ g(z+c) = \dfrac{1-\beta_1(z)^2}{1+\beta_1^2(z)}, \end{cases} \tag{2.13}$$

and

$$\begin{cases} f(z+c) = \dfrac{2\beta_2(z)}{1+\beta_2(z)^2}, \\ g(z) = \dfrac{1-\beta_2(z)^2}{1+\beta_2(z)^2}, \end{cases} \tag{2.14}$$

where $\beta_1(z)$ and $\beta_2(z)$ are non-rational meromorphic functions. From the first equation of (2.13) and the first equation of (2.14), we obtain

$$\beta_1(z+c) - \beta_2(z) = \beta_1(z+c)\beta_2(z)(\beta_1(z+c) - \beta_2(z)). \tag{2.15}$$

From the second equation of (2.13) and the second equation of (2.14), we have

$$\beta_1(z)^2 = \beta_2(z+c)^2. \tag{2.16}$$

The basic discussions are stated as follows:

(1) If $\beta_1(z+c) = \beta_2(z)$ and $\beta_1(z) = \beta_2(z+c)$, then $\beta_1(z) = \beta_1(z+2c)$ and $\beta_2(z) = \beta_2(z+2c)$.
(2) If $\beta_1(z+c) = \beta_2(z)$ and $\beta_1(z) = -\beta_2(z+c)$, then $\beta_1(z) + \beta_1(z+2c) = 0$ and $\beta_2(z) + \beta_2(z+2c) = 0$.
(3) If $\beta_1(z+c) \neq \beta_2(z)$, then $\beta_1(z+c)\beta_2(z) = 1$ follows by (2.15). Furthermore, $\beta_2(z)^2\beta_2(z+2c)^2 = 1$ and $\beta_1(z)^2\beta_1(z+2c)^2 = 1$ follows by (2.16). □

Proof of Theorem 1.5 Using the Iyer and Gross's result again, we assume that

$$\begin{cases} f'(z) = \cos(h_1(z)), \\ g(z+c) = \sin(h_1(z)), \end{cases} \tag{2.17}$$

and

$$\begin{cases} g'(z) = \cos(h_2(z)), \\ f(z+c) = \sin(h_2(z)), \end{cases} \tag{2.18}$$

where $h_1(z)$ and $h_2(z)$ are non-constant entire functions. By a basic computation from the above two systems, we obtain

$$\begin{cases} h_1'(z)\cos(h_1(z)) = \cos(h_2(z+c)), \\ h_2'(z)\cos(h_2(z)) = \cos(h_1(z+c)). \end{cases} \tag{2.19}$$

Shifting the first equation of (2.19), it follows

$$\begin{cases} h_1'(z+c)\cos(h_1(z+c)) = \cos(h_2(z+2c)), \\ h_2'(z)\cos(h_2(z)) = \cos(h_1(z+c)). \end{cases} \tag{2.20}$$

Hence, we have

$$h_1'(z+c)h_2'(z)\cos(h_2(z)) = \cos(h_2(z+2c)). \tag{2.21}$$

We will affirm that $h_1'(z+c)h_2'(z) \equiv \pm 1$. Let $A(z) = h_1'(z+c)h_2'(z)$. Taking the first derivative of (2.21), we have

$$A'(z)\cos(h_2(z)) - A(z)h_2'(z)\sin(h_2(z)) = -h_2'(z+2c)\sin(h_2(z+2c)). \tag{2.22}$$

Combining the square of (2.22) with (2.21), we have

$$\begin{aligned} &A'(z)^2\cos^2(h_2(z)) + A(z)^2h_2'(z)^2\sin^2(h_2(z)) \\ &\quad -2A(z)A'(z)h_2'(z)\sin(h_2(z))\cos(h_2(z)) \\ &= h_2'(z+2c)^2\sin^2(h_2(z+2c)) \\ &= h_2'(z+2c)^2(1 - A(z)^2\cos^2(h_2(z))). \end{aligned}$$

Hence, we have

$$\begin{aligned} &\big(A'(z)^2 - A(z)^2(h_2'(z)^2 - h_2'(z+2c)^2)\big)\cos^2(h_2(z)) + A(z)^2h_2'(z)^2 - h_2'(z+2c)^2 \\ &= 2A(z)A'(z)h_2'(z)\sin(h_2(z))\cos(h_2(z)). \end{aligned}$$

Taking the square of above equation, we obtain

$$\begin{aligned} &\Big((A'(z)^2 - A(z)^2(h_2'(z)^2 - h_2'(z+2c)^2))\cos^2(h_2(z)) + A(z)^2h_2'(z)^2 - h_2'(z+2c)^2\Big)^2 \\ &= \Big(2A(z)A'(z)h_2'(z)\sin(h_2(z))\cos(h_2(z))\Big)^2 \\ &= 4A(z)^2A'(z)^2h_2'(z)^2\cos^2(h_2(z))(1-\cos^2(h_2(z))). \end{aligned}$$

Observe the coefficients of $\cos^4(h_2(z))$ and $\cos^2(h_2(z))$, for avoiding a contradiction with Valiron–Mohon'ko lemma [13, Theorem 2.2.5], we have

$$\begin{cases} A(z)^2 h_2'(z)^2 - h_2'(z+2c)^2 \equiv 0, \\ 4A(z)^2 A'(z)^2 h_2'(z)^2 \equiv 0, \\ A'(z)^2 - A(z)^2 (h_2'(z)^2 - h_2'(z+2c)^2) \equiv 0. \end{cases} \tag{2.23}$$

Since $A(z)$ and $h_2'(z)$ are nonzero entire functions, then $A'(z) \equiv 0$ follows by the second equation of (2.23) and

$$h_2'(z)^2 - h_2'(z+2c)^2 \equiv 0$$

that is

$$h_2'(z-c)^2 - h_2'(z+c)^2 \equiv 0$$

follows by the third equation of (2.23). So $A(z)^2 \equiv 1$ that is

$$h_2'(z)^2 h_1'(z+c)^2 = h_2'(z-c)^2 h_1'(z)^2 \equiv 1. \tag{2.24}$$

Furthermore, if f and g are finite order, then h_1 and h_2 are polynomials. From (2.24), we can assume $h_1(z) = \alpha_1 z + \beta_1$ and $h_2(z) = \alpha_2 z + \beta_2$, where $(\alpha_1\alpha_2)^2 = 1$. Next, we will affirm $\alpha_1 = \pm 1$ and $\alpha_2 = \pm 1$. Taking the first derivative of the second equation of (2.19), we have

$$h_2''(z)\cos(h_2(z)) - h_2'(z)^2 \sin(h_2(z)) = -h_1'(z+c)\sin h_1(z+c).$$

Namely,

$$-\alpha_2^2 \sin(\alpha_2 z + \beta_2) = -\alpha_1 \sin(\alpha_1 z + \alpha_1 c + \beta_1).$$

Combining the square of above equation with the second equation of (2.19), we have

$$\alpha_2^4[1 - \cos^2(\alpha_2 z + \beta_2)] = \alpha_1^2[1 - \alpha_2^2 \cos^2(\alpha_2 z + \beta_2)].$$

From $(\alpha_1\alpha_2)^2 = 1$, we have

$$\alpha_2^4[1 - \cos^2(\alpha_2 z + \beta_2)] = \frac{1}{\alpha_2^2} - \cos^2(\alpha_2 z + \beta_2).$$

By a basic computation, we have $\alpha_2^2 = 1$ and $\alpha_1^2 = 1$. From (2.21) and (2.19), we have the following four cases:

(i) If $\alpha_1 = 1$ and $\alpha_2 = 1$, then $c = k\pi$ and $\beta_1 = \beta_2 + k\pi + 2m\pi$;
(ii) If $\alpha_1 = -1$ and $\alpha_2 = -1$, then $c = k\pi$ and $\beta_1 = \beta_2 + k\pi + m\pi$;
(iii) If $\alpha_1 = 1$ and $\alpha_2 = -1$, then $c = \frac{\pi}{2} + k\pi$ and $\beta_1 = -\beta_2 + \frac{\pi}{2} + k\pi + 2m\pi$;
(iv) If $\alpha_1 = -1$ and $\alpha_2 = 1$, then $c = \frac{\pi}{2} + k\pi$ and $\beta_1 = -\beta_2 + \frac{\pi}{2} + k\pi + m\pi$.

Then, the corresponding entire solutions with finite order of (1.8) are:

(i) $f(z)=\sin(z+b)$ and $g(z)=\sin(z+b+k\pi+2m\pi)$, $c=k\pi$;
(ii) $f(z)=\sin(-z+b)$ and $g(z)=\sin(-z+b+k\pi+m\pi)$, $c=k\pi$;
(iii) $f(z)=\sin(-z+b)$ and $g(z)=\sin(z-b+\frac{\pi}{2}+k\pi+2m\pi)$, $c=\frac{\pi}{2}+k\pi$;
(iv) $f(z)=\sin(z+b)$ and $g(z)=\sin(-z-b+\frac{\pi}{2}+k\pi+m\pi)$, $c=\frac{\pi}{2}+k\pi$,

where k, m are any integers and b is a constant. The proof of Theorem 1.5 is completed. □

Proof of Theorem 1.6 Let

$$\psi(z)=\frac{\beta_1(z)g^{(k)}(z)^m-\alpha_1(z)}{f(z)^n}+1$$

and

$$\phi(z)=\frac{\beta_2(z)f^{(k)}(z)^m-\alpha_2(z)}{g(z)^n}+1.$$

Then, we will discuss the following three cases.

(1) If $f(z)$ and $g(z)$ are all transcendental properly meromorphic functions, using the first main theorem of Nevanilnna theory and [9, Theorem 3.1], then we have

$$\begin{aligned} nT(r,f(z))=T(r,f^n(z)) &\le T\left(r,\frac{\psi(z)-1}{\beta_1(z)g^{(k)}(z)^m-\alpha_1(z)}\right)+O(1)\\ &\le T(r,\psi(z))+T\left(r,\frac{1}{\beta_1(z)g^{(k)}(z)^m-\alpha_1(z)}\right)+O(1)\\ &\le T(r,\psi(z))+T(r,g^{(k)}(z)^m)+S(r)\\ &\le T(r,\psi(z))+m(k+1)T(r,g(z))+S(r), \end{aligned} \tag{2.25}$$

where $S(r)=o(T(r))$ and $T(r)=\max\{T(r,f(z)),T(r,g(z))\}$. Hence, we conclude

$$T(r,\psi(z))\ge nT(r,f(z))-m(k+1)T(r,g(z))+S(r). \tag{2.26}$$

Similarly, we conclude

$$T(r,\phi(z))\ge nT(r,g(z))-m(k+1)T(r,f(z))+S(r). \tag{2.27}$$

In addition, we can obtain the following inequality by the expression of $\psi(z)$,

$$\begin{aligned} T(r,\psi(z)) &\le T(r,f(z)^n)+T(r,\beta_1(z)g^{(k)}(z)^m-\alpha_1(z))+O(1)\\ &\le T(r,f(z)^n)+T(r,g^{(k)}(z)^m)+S(r)\\ &\le nT(r,f(z))+m(k+1)T(r,g(z))+S(r). \end{aligned} \tag{2.28}$$

The next inequality can be proved as the above by the expression of $\phi(z)$

$$T(r,\phi(z))\le nT(r,g(z))+m(k+1)T(r,f(z))+S(r). \tag{2.29}$$

Thus, $S(r,\psi(z)) = o(T(r))$ and $S(r,\phi(z)) = o(T(r))$. Using the second main theorem of Nevanlinna theory and [9, Theorem 3.1], we obtain

$$\begin{aligned} T(r,\psi(z)) &\le \overline{N}(r,\psi(z)) + \overline{N}\left(r,\frac{1}{\psi(z)}\right) + \overline{N}\left(r,\frac{1}{\psi(z)-1}\right) + S(r,\psi(z)) \\ &\le \overline{N}\left(r,\frac{1}{f(z)}\right) + \overline{N}(r,g(z)) + \overline{N}\left(r,\frac{1}{f(z)^n+\beta_1(z)g^{(k)}(z)^m-\alpha_1(z)}\right) \\ &\quad + \overline{N}\left(r,\frac{1}{\beta_1(z)g^{(k)}(z)^m-\alpha_1(z)}\right) + \overline{N}(r,f(z)) + S(r,\psi) \\ &\le 2T(r,f(z)) + \overline{N}\left(r,\frac{1}{f(z)^n+\beta_1(z)g^{(k)}(z)^m-\alpha_1(z)}\right) \\ &\quad + (m(k+1)+1)T(r,g(z)) + S(r,\psi(z)). \end{aligned} \tag{2.30}$$

Similarly, we have

$$\begin{aligned} T(r,\phi(z)) \le 2T(r,g(z)) &+ \overline{N}\left(r,\frac{1}{g(z)^n+\beta_2(z)f^{(k)}(z)-\alpha_2(z)}\right) \\ &+ (m(k+1)+1)T(r,f(z)) + S(r,\phi(z)). \end{aligned} \tag{2.31}$$

Hence, if $f(z)^n + \beta_1(z)g^{(k)}(z)^m - \alpha_1(z)$ and $g(z)^n + \beta_2(z)f^{(k)}(z)^m - \alpha_2(z)$ have finitely many zeros, we have

$$(n - 2m(k+1) - 3)(T(r,f(z)) + T(r,g(z))) \le S(r),$$

which is impossible with $n \ge 2m(k+1)+4$, our conclusion is proved.

(2) If only one of $f(z)$ and $g(z)$ is transcendental properly meromorphic function, without loss of generality, let $f(z)$ be properly meromorphic and $g(z)$ be entire. By removing the counting function of poles of $g(z)$, then Eq. (2.27) remains true and Eq. (2.26) changes into

$$T(r,\psi(z)) \ge nT(r,f(z)) - mT(r,g(z)) + S(r), \tag{2.32}$$

and (2.30) changes into

$$\begin{aligned} T(r,\psi(z)) \le 2T(r,f(z)) &+ \overline{N}\left(r,\frac{1}{f(z)^n+\beta_1(z)g^{(k)}(z)^m-\alpha_1(z)}\right) \\ &+ mT(r,g(z)) + S(r,\psi(z)), \end{aligned} \tag{2.33}$$

and (2.31) changes into

$$\begin{aligned} T(r,\phi(z)) \le T(r,g(z)) &+ \overline{N}\left(r,\frac{1}{g(z)^n+\beta_2(z)f^{(k)}(z)-\alpha_2(z)}\right) \\ &+ (m(k+1)+1)T(r,f(z)) + S(r,\phi(z)). \end{aligned}$$

Hence, if $f(z)^n+\beta_1(z)g^{(k)}(z)^m-\alpha_1(z)$ and $g(z)^n+\beta_2(z)f^{(k)}(z)^m-\alpha_2(z)$ have finitely many zeros, we can get

$$(n-2m(k+1)-3)T(r,f(z))+(n-2m-1)T(r,g(z))\leq S(r),$$

which is impossible with $n\geq 2m(k+1)+4$.

(3) If $f(z)$ and $g(z)$ are all transcendental entire functions, then Eq. (2.27) changes into

$$T(r,\phi(z))\geq nT(r,g(z))-mT(r,f(z))+S(r),$$

and Eq. (2.26) changes into

$$T(r,\psi(z))\geq nT(r,f(z))-mT(r,g(z))+S(r),$$

and (2.30) changes into

$$\begin{aligned}T(r,\psi(z))\leq T(r,f(z))+\overline{N}\left(r,\frac{1}{f(z)^n+\beta_1(z)g^{(k)}(z)^m-\alpha_1(z)}\right)\\+mT(r,g(z))+S(r,\psi(z)),\end{aligned}$$

and (2.31) changes into

$$\begin{aligned}T(r,\phi(z))\leq T(r,g(z))+\overline{N}\left(r,\frac{1}{g(z)^n+\beta_2(z)f^{(k)}(z)-\alpha_2(z)}\right)\\+mT(r,f(z))+S(r,\phi(z)).\end{aligned}$$

Hence, if $f(z)^n+\beta_1(z)g^{(k)}(z)^m-\alpha_1(z)$ and $g(z)^n+\beta_2(z)f^{(k)}(z)^m-\alpha_2(z)$ have finitely many zeros, we can get

$$(n-2m-1)[T(r,f(z))+T(r,g(z))]\leq S(r),$$

which is impossible with $n\geq 2m+2$. The proof of Theorem 1.6 is completed. □

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Conflict of interest The authors declare that they have no conflict of interest.

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