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Nonexistence of Global Solutions for a Class of Nonlinear Parabolic Equations on Graphs

Yang Liu[1]

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Abstract
In this paper, we consider the nonlinear parabolic equation

$$\begin{cases} u_t(t, x) = \Delta u(t, x) + f(u(t, x)), & (t, x)\in(0, +\infty) \times \Omega^\circ, \\ u(t, x) = 0, & (t, x)\in[0, +\infty) \times \partial\Omega, \\ u(0, x) = u_0(x) \geq 0, & x\in\Omega, \end{cases}$$

on the connected locally finite graph $G = (V, E)$, where Δ is the μ-Laplacian, $\Omega \subset V$ is a bounded domain on graphs, and $u_0(x)$ is a nonnegative and nontrivial initial value, f is locally Lipschitz continuous on $\mathbb{R}$, $f(0) = 0$ and $f(u) > 0$ for all $u > 0$. Using the concavity method, we prove that when the nonlinear term f and the initial value $u_0(x)$ satisfy certain conditions, the above equation admits the blow-up solutions. Moreover, we extend the condition of f to p-Laplacian parabolic equation on locally finite graphs, and we also obtain the blow-up solutions for $p > 2$.

Keywords Blow-up solutions · Discrete parabolic equation · Banach fixed point theorem · Concavity method · Locally finite graph

Mathematics Subject Classification 35R02 · 35A01 · 35K91 · 35K92

1 Introduction

Let us start with blow-up phenomenon of parabolic equations on $\mathbb{R}^N$, which was discussed by Kaplan [17] and Fujita [5, 6]. Fujita [5] considered the following Cauchy

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✉ Yang Liu
dliuyang@ruc.edu.cn

1 Shool of Mathematics, Renmin University of China, Beijing 100872, People's Republic of China

problem

$$\begin{cases} u_t(t, x) = \Delta u(t, x) + u^q(t, x), & (t, x)\in(0, +\infty) \times \mathbb{R}^N, \\ u(0, x) = u_0(x), & x\in\mathbb{R}^N, \end{cases} \tag{1.1}$$

where $q > 1$. Fujita obtained the critical exponent $q^* = 1 + 2/N$ and showed that, if $1 < q < q^*$, then for any nonnegative and nontrivial initial value, the solution of Eq. (1.1) blows up in finite time. After that, Fujita [6] also considered the following nonlinear parabolic equation

$$\begin{cases} u_t(t, x) = \Delta u(t, x) + f(u(t, x)), & (t, x)\in(0, +\infty) \times \Omega, \\ u(t, x) = 0, & (t, x)\in[0, +\infty) \times \partial\Omega, \\ u(0, x) = a(x), & x\in\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^m$ and the initial value $a(x)$ is nonnegative and nontrivial. Fujita supposed that f satisfies the following conditions, i.e., f is locally Lipschitz continuous and convex in $[0, +\infty)$; $f(0) \geq 0$ and $f(r) > 0$ for $r > 0$; $1/f$ is integrable at $r = +\infty$. And he showed that if $\int_r^{+\infty} 1/f(\lambda)d\lambda = \mathrm{o}(r^{-\frac{2}{m}})$ as $r \to 0^+$, then the solutions of (1.2) blow up in finite time. Then, Meier [29] investigated the blow-up phenomenon for the parabolic equations with nonlinear source

$$\begin{cases} u_t(t, x) = \Delta u(t, x) + \psi(t) f(u(t, x)), & (t, x)\in(0, +\infty) \times \Omega, \\ u(t, x) = 0, & (t, x)\in(0, +\infty) \times \partial\Omega, \\ u(0, x) = u_0(x) \geq 0, & x\in\Omega, \end{cases} \tag{1.3}$$

where $\psi(t) \approx t^q$ or $\psi(t) \approx e^{\beta t}$ with $\beta > 0$. Nabongo et al. [30] proved that the solutions of (1.3) blow up in finite time if the initial data is sufficiently large.

Nowadays, there have been increasingly more studies about partial differential equations on graphs. For the discrete Laplacian case, in a series of works [9–11], by variational method, Grigor'yan, Lin and Yang solved several elliptic differential equations on graphs. For more studies in this direction, see for examples [12–14, 16, 26, 28, 33, 36, 37] and the references therein. Recently, many results also have been obtained for parabolic equations on graphs or networks, the blow-up phenomenon of the semilinear heat Eq. (1.1) was studied by Lin and Wu [22] on finite graphs and locally finite graphs, the discrete parabolic equations of (1.2) were discussed by Chung et al. [2] on networks, by Lin and Wu [23] on locally finite graphs, the critical exponent for the initial boundary value problem of Eq. (1.3) was investigated by Zhou et al. [41] on graphs when $\psi(t) = e^{\beta t}$ and $f(u) = u^q$ with $\beta > 0$, $q > 1$, by Chung et al. [4] on networks. For other related works, see for examples [1, 8, 18, 19, 21, 24, 25, 27, 38, 39] and references therein.

Motivated by Levine [20], Philippin et al. [32] studied the blow-up solutions of Eq. (1.2) by concavity method on $\mathbb{R}^N$, and they obtained that for constant $\varepsilon > 0$, if f satisfies

$$(2 + \varepsilon)F(u) \leq uf(u), \quad u > 0, \tag{1.4}$$

where $F(u) = \int_0^u f(s)ds$, and the initial value satisfies

$$-\frac{1}{2}\int_{\Omega} |\nabla u_0|^2(x)dx + \int_{\Omega} F(u_0(x))dx > 0,$$

then the solutions of Eq. (1.2) blow up in finite time. Recently, for the discrete ω-Laplacian case of Eq. (1.2) on networks, Chung et al. [2] develop a new condition of f depending on domain in place of (1.4), and they extended the results of Philippin et al. [32] to the case of discrete ω-Laplacian equations on networks and obtained blow-up solutions by concavity method. Following their works, in this paper, we consider the blow-up solutions of the discrete μ-Laplacian nonlinear parabolic equations on graphs.

For clarity, we review the basic settings on graphs. Let $G = (V, E)$ be a graph, where V denotes the vertex set and E denotes the edge set. Throughout this paper, we always assume that G satisfies the following conditions (a)–(e), and G is called a connected locally finite graph.

(a) (Simple) G contains neither loops nor multiple edges.
(b) (Locally finite) For any $x \in V$, there exist only finite vertices $y \in V$ such that $xy \in E$.
(c) (Connected) For any $x, y \in V$, there exist finite edges connecting x and y.
(d) (Symmetric) For any $(x, y) \in V^2$, let $\omega : V \times V \to \mathbb{R}^+$ be a positive symmetric weight such that $\omega_{xy} = \omega_{yx}$, where we write ω_{xy} for $\omega(x, y)$.
(e) (Positive finite measure) $\mu : V \to \mathbb{R}^+$ defines a positive finite measure on V and satisfies $\mu_0 = \inf_{x\in V} \mu(x) > 0$.

$\Omega \subset V$ is said to be a domain if it is a connected subset of V. We also always assume that Ω is a domain satisfying the conditions (f) and (g).

(f) (Bounded domain) Let $d(x, y)$ be the minimal number of edges which connect x and y. For any two vertices $x, y \in \Omega$, if $d(x, y)$ is uniformly bounded from above, we call Ω is a bounded domain. The boundary of Ω is defined by

$$\partial\Omega = \{x \in \Omega\ \exists\ y \notin \Omega \text{ such that } xy \in E\}$$

and the interior of Ω is denoted by $\Omega^\circ = \Omega\backslash\partial\Omega$. In fact, a bounded domain Ω contains only finitely many vertices. Locally finite graph G is unbounded, thus V contains infinitely many vertices.
(g) $(D_\mu < +\infty)$ Let $m(x) = \sum_{y\sim x} \omega_{xy}$, where $y \sim x$ means $xy \in E$. D_μ is defined by

$$D_\mu = \max_{x\in\Omega} \frac{m(x)}{\mu(x)},$$

it is obvious that $D_\mu < +\infty$ under the condition (f). The value of D_μ depends on the selection of the bounded domain Ω, and if Ω changes, the value of D_μ will also change. This is the fundamental difference between a finite graph and the bounded domains on a locally finite graph.

Let $C(\Omega)$ be the set of real functions on Ω, for any function $u \in C(\Omega)$, the μ-Laplacian of u is defined by

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x)). \tag{1.5}$$

The associated gradient form of two functions $u, v \in C(\Omega)$ is given as

$$\Gamma(u,\ v)(x) = \frac{1}{2\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x))(v(y) - v(x)). \tag{1.6}$$

We write $\Gamma(u)(x) = \Gamma(u,\ u)(x)$ and denote the length of the gradient by

$$|\nabla u|(x) = \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x))^2 \right)^{\frac{1}{2}}.$$

For any function $h \in C(\Omega)$, the integral of h on a bounded domain Ω reads

$$\int_{\Omega} h d\mu = \sum_{x \in \Omega} \mu(x) h(x).$$

In this paper, we consider the following nonlinear parabolic equation on locally finite graphs

$$\begin{cases} u_t(t,\ x) = \Delta u(t,\ x) + f(u(t,\ x)), & (t,\ x) \in (0,\ +\infty) \times \Omega^\circ, \\ u(t,\ x) = 0, & (t,\ x) \in [0,\ +\infty) \times \partial\Omega, \\ u(0,\ x) = u_0(x), & x \in \Omega, \end{cases} \tag{1.7}$$

where Δ is μ-Laplacian defined by (1.5) on Ω, $u_0(x)$ is a nonnegative and nontrivial initial value, f satisfies the following properties:
(H_1) f be locally Lipschitz continuous on $\mathbb{R}$, namely, for all $m > 0$, there exists a constant $L > 0$ such that

$$|f(a) - f(b)| \le L|a - b|, \quad \forall\, a,\ b \in [-m,\ m].$$

(H_2) $f(0) = 0$ and $f(u) > 0$ for all $u > 0$.

Firstly, we use Banach fixed point theorem to prove the short time existence and the uniqueness of the solution for Eq. (1.7) and obtain our first result.

Theorem 1.1 *Assume that f satisfies the conditions (H_1) and (H_2). If $t > 0$ is small enough such that $t < 1/(2D_\mu + L)$, then there exists a unique nonnegative solution $u(t,\ x)$ to the equation (1.7) in the time interval* $[0,\ t]$.

Next, we only care about the long time nonexistence of the solution, which means the solution will blow-up in a finite time. To proceed, we give the definition of blow-up solutions of Eq. (1.7) on locally finite graph.

Definition 1.2 If there exists $x_0 \in \Omega^\circ$ such that $|u(t, x_0)| \to +\infty$ as $t \to T^-$, then the solution $u(t, x)$ of Eq. (1.7) blows up in a finite time T.

It is well known that if $\int_m^{+\infty} 1/f(\lambda)d\lambda = +\infty$ for some $m > 0$, the solutions of Eq. (1.7) are global. In fact, Osaood [31] showed that if Eq. (1.7) has blow-up solutions, f must satisfy

$$\int_m^{+\infty} \frac{d\lambda}{f(\lambda)} < +\infty \tag{1.8}$$

for some $m > 0$. However, if f only satisfies (1.8), it does not guarantee that Eq. (1.7) has blow-up solutions. In order to get the blow-up solutions, we should strengthen the condition of f, i.e., for any two constants $\delta,\ \varepsilon > 0$, f satisfies

$$(H_3) \quad f(u) \geq \delta u^{1+\varepsilon}, \quad \forall\, u \geq m > 1.$$

Now, we are ready to state our second result.

Theorem 1.3 *Assume that f satisfies the conditions* $(H_1)-(H_3)$*. If the initial value* $u_0(x)$ *is sufficiently large satisfying* $\max_{x\in\Omega} u_0(x) > \max\{(D_\mu/\delta)^{\frac{1}{\varepsilon}},\ m\}$*, then the nonnegative solutions* $u(t, x)$ *of the equation* (1.7) *blow up in a finite time* T*.*

We can see the condition (H_3) is independent of the eigenvalue of $-\Delta$, which depends on the domain Ω. Inspired by Chung et al. [2], we extend their new condition from networks to graphs. Let $0 < \lambda_1(\Omega)$ be the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition, which is defined by

$$\lambda_1(\Omega) = \inf_{u\not\equiv 0, u|_{\partial\Omega}=0} \frac{\int_\Omega |\nabla u|^2 d\mu}{\int_\Omega u^2 d\mu}. \tag{1.9}$$

Then, we develop a new condition of f, namely,
(H_4) for any constant $\varepsilon > 0$, if there exist some positive constants α and β such that for all $u > 0$, there holds

$$(2+\varepsilon)F(u) \leq uf(u) + \alpha u^2 + \beta,$$

where $F(u) = \int_0^u f(s)ds$, $0 < \alpha \leq \varepsilon\lambda_1(\Omega)/2$, $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$.

Under the new condition (H_4), we deduce another main result.

Theorem 1.4 *Assume that f satisfies the conditions* (H_1), (H_2) *and* (H_4)*. If the initial value* $u_0(x)$ *satisfies*

$$-\frac{1}{2}\int_\Omega |\nabla u_0|^2(x)d\mu + \int_\Omega (F(u_0(x)) - \beta)d\mu > 0, \tag{1.10}$$

then the nonnegative solutions $u(t, x)$ *of the equation* (1.7) *blow up in a finite time* T*.*

Finally, we also consider a p-Laplacian parabolic equation on locally finite graphs

$$\begin{cases} u_t(t,x) = \Delta_p u(t,x) + f(u(t,x)), & (t,x)\in(0,+\infty)\times\Omega^\circ, \\ u(t,x) = 0, & (t,x)\in[0,+\infty)\times\partial\Omega, \\ u(0,x) = u_0(x), & x\in\Omega, \end{cases} \tag{1.11}$$

where p-Laplacian Δ_p of $u \in C(\Omega)$ is represented by

$$\Delta_p u(x) = \frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}(|\nabla u|^{p-2}(y) + |\nabla u|^{p-2}(x))(u(y)-u(x)), \quad \forall\, p>1. \tag{1.12}$$

The first eigenvalue of $-\Delta_p$ with the Dirichlet boundary condition is given as

$$\lambda_1(\Omega) = \inf_{u\not\equiv 0, u|_{\partial\Omega}=0} \frac{\int_\Omega |\nabla u|^p d\mu}{\int_\Omega |u|^p d\mu}.$$

It is easy to see that p-Laplacian is μ-Laplacian when $p = 2$, but p-Laplacian is nonlinear operator since $\Delta_p(u+v)(x) \neq \Delta_p u(x) + \Delta_p v(x)$ by (1.12). In particular, there were many interesting works for p-Laplacian equations on graphs. Han et al. [12] studied a nonlinear p-Laplacian Schrödinger equation with $p > 2$ on locally finite graphs. After that, Shao et al. [34] made essential improvements to $p > 1$, and they obtained the existence and convergence of solutions under more general conditions. To further explore this field, refer to studies such as [7, 15, 35] and their respective references. What makes Theorem 1.4 interesting is that it can be nontrivially generalized to p-Laplacian parabolic Eq. (1.11). When $p > 2$, we have the following theorem.

Theorem 1.5 *Assume that f satisfies the conditions (H_1), (H_2) and*

$$(H_5)\ (p+\varepsilon)F(u) \leq uf(u) + \alpha u^p + \beta, \quad u > 0,$$

for any $\varepsilon > 0$ and some constants $\alpha, \beta > 0$ satisfying $0 < \alpha \leq \varepsilon\lambda_1(\Omega)/p$ with $p > 2$, where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta_p$ with the Dirichlet boundary condition. If the nonnegative and nontrivial initial value $u_0(x)$ satisfies

$$-\frac{1}{p}\int_\Omega |\nabla u_0|^p(x)d\mu + \int_\Omega (F(u_0(x)) - \beta)d\mu > 0, \tag{1.13}$$

then the nonnegative solutions $u(t,x)$ of Eq. (1.11) blow up in a finite time T.

Although we study the same type of equation as Lin and Wu [23] on graphs, they use the method of heat kernel estimate, and we propose a different method to prove the nonexistence of global solutions for Eq. (1.7). Following the lines of Chung et al. [2], we prove Theorems 1.4 and 1.5 by concavity method. Compared with Chung et al. [2], we extend their results to locally finite graphs. In addition, this paper also studies the p-Laplacian parabolic equation on graphs, which is discussed by Chung et al. [3] on Euclidean space $\mathbb{R}^N$.

The remaining parts of this paper are organized as follow: in Sect. 2, we introduce formulas of integration by parts about μ-Laplacian and p-Laplacian on graphs, and then, we introduce two important comparison principles on locally finite graphs and deduce that the solutions $u(t, x)$ of Eq. (1.7) are nonnegative. In Sect. 3, by Banach fixed point theorem, we prove Theorem 1.1 and obtain the short time existence and the uniqueness of a solution for (1.7). In Sect. 4, we consider the maximal existence time of the solutions to (1.7) and, respectively, prove Theorems 1.3 and 1.4, then we prove the blow-up solutions of Eq. (1.11) and complete the proof of Theorem 1.5.

2 Preliminary Analysis

2.1 Formula of Integration by Parts

The set of all functions with compact support is denoted by

$$C_c(\Omega) = \{u \in C(\Omega) : \{x \in \Omega : u(x) \neq 0\} \text{ is of finite cardinality}\},$$

where $C(\Omega)$ is the set of all real functions on Ω. For any $p > 1$, let $W_0^{1,p}(\Omega)$ be the completion of $C_c(\Omega)$, with respect to the norm

$$\|u\|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega (|\nabla u|^p + u^p) d\mu \right)^{\frac{1}{p}}.$$

Since the bounded domain Ω only contains finite vertices, $W_0^{1,p}(\Omega)$ is exactly a finite dimensional linear function space $\mathbb{R}^{|\Omega^\circ|}$, where $|\Omega^\circ|$ is the number of vertices in Ω°.

Now, we introduce two important conclusions. Lemma 2.1 comes from Zhang and Zhao [40] directly, and we omit this part of the proof.

Lemma 2.1 (Formula of integration by parts 1 [40]) *Suppose that $u \in W_0^{1,2}(\Omega)$ and Δu is well defined as (1.5). Let $v \in C_c(\Omega)$, where $\Omega \subset V$ is a bounded domain. Then, we have*

$$\int_\Omega \Gamma(u, v) d\mu = -\int_\Omega \Delta u v d\mu.$$

Lemma 2.2 (Formula of integration by parts 2) *Suppose that $u \in W_0^{1,p}(\Omega)$ and $\Delta_p u$ is well defined as (1.12) with $p > 1$. Let $v \in C_c(\Omega)$, where $\Omega \subset V$ is a bounded domain. Then, we have*

$$\int_\Omega |\nabla u|^{p-2} \Gamma(u, v) d\mu = -\int_\Omega \Delta_p u v d\mu.$$

Proof Inspired by Zhang et al. [40], together with the definition of associated gradient in (1.6) and p-Laplacian in (1.12), we have

$$\begin{aligned}\int_\Omega \Delta_p uvd\mu &= \sum_{x\in\Omega} \mu(x)\Delta_p u(x)v(x)\\ &= \frac{1}{2}\sum_{x\in\Omega}\sum_{y\sim x}\omega_{xy}(|\nabla u|^{p-2}(y)+|\nabla u|^{p-2}(x))(u(y)-u(x))v(x)\\ &= -\frac{1}{2}\sum_{y\in\Omega}\sum_{x\sim y}\omega_{xy}|\nabla u|^{p-2}(y)(u(x)-u(y))v(x)\\ &\quad -\frac{1}{2}\sum_{x\in\Omega}\sum_{y\sim x}\omega_{xy}|\nabla u|^{p-2}(x)(u(y)-u(x))(-v(x))\\ &= -\frac{1}{2}\sum_{x\in\Omega}\sum_{y\sim x}\omega_{xy}|\nabla u|^{p-2}(x)(u(y)-u(x))(v(y)-v(x))\\ &= -\int_\Omega |\nabla u|^{p-2}\Gamma(u,v)d\mu.\end{aligned}$$

This ends the proof of Lemma 2.2. □

2.2 Comparison Principle

Now, we introduce the following comparison principles on locally finite graphs, which were studied by Chung et al. [2] on networks. We extend their results to the bounded domain on locally finite graphs, which provides a new proof method of comparison principle discussed by Lin and Wu [23].

Lemma 2.3 (Comparison principle) *Let f satisfies (H_1). For any $T > 0$ (T may be $+\infty$), we assume that $u(t,x)$ and $v(t,x)$ are continuous and differentiable with respect to t in $(0,T)\times\Omega$, and satisfy*

$$\begin{cases} u_t(t,x)-\Delta u(t,x)-f(u(t,x))\ge v_t(t,x)-\Delta v(t,x)-f(v(t,x)), & (t,x)\in(0,T)\times\Omega^\circ,\\ u(t,x)\ge v(t,x), & (t,x)\in[0,T)\times\partial\Omega,\\ u(0,x)\ge v(0,x), & x\in\Omega. \end{cases} \tag{2.1}$$

Then, $u(t,x)\ge v(t,x)$ for any $(t,x)\in[0,T)\times\Omega$.

Proof For any $0<T'<T$, since f be locally Lipschitz continuous on $\mathbb{R}$, then there exists a constant $L>0$ such that

$$|f(a)-f(b)|\le L|a-b|,\quad \forall\, a,\ b\in[-m,m], \tag{2.2}$$

where $m=\max_{x\in\Omega^\circ}\max_{t\in(0,T']}\{|u(t,x)|,\ |v(t,x)|\}$.

For any $(t,x)\in[0,T']\times\Omega$, let $\tau(t,x)=u(t,x)-v(t,x)$, it follows from (2.1) that

$$\tau_t(t,x)-\Delta\tau(t,x)-[f(u(t,x))-f(v(t,x))]\ge 0. \tag{2.3}$$

In fact, μ-Laplacian Δ is a linear operator, which ensures (2.3) holds. For any $(t, x) \in [0, T'] \times \Omega$, we consider

$$\widetilde{\tau}(t, x) = e^{-2Lt}\tau(t, x). \tag{2.4}$$

Inserting (2.4) into (2.3), we have

$$\widetilde{\tau}_t(t, x) - \Delta\widetilde{\tau}(t, x) + 2L\widetilde{\tau}(t\, x) - e^{-2Lt}\left[f(u(t, x)) - f(v(t, x))\right] \geq 0. \tag{2.5}$$

Since $u(t, x)$, $v(t, x)$ are continuous with respect to t and Ω is a bounded domain, we can always find $(t_0, x_0) \in [0, T'] \times \Omega$ such that

$$\widetilde{\tau}(t_0, x_0) = \min_{x\in\Omega} \min_{t\in[0, T']} \widetilde{\tau}(t, x). \tag{2.6}$$

Then, the conclusion of Lemma 2.3 is equivalent to

$$\widetilde{\tau}(t_0, x_0) \geq 0 \tag{2.7}$$

for all $(t, x) \in [0, T'] \times \Omega$.

Next, we prove (2.7) by contradiction. We suppose $\widetilde{\tau}(t_0, x_0) < 0$. It follows from (2.1) that $\widetilde{\tau}(t, x) \geq 0$ in $[0, T'] \times \partial\Omega$ and $\widetilde{\tau}(0, x) \geq 0$ in Ω, then we deduce that $(t_0, x_0) \in (0, T'] \times \Omega^\circ$. In view of (2.6), we fix t_0 and get

$$\widetilde{\tau}(t_0, y) \geq \widetilde{\tau}(t_0, x_0) \tag{2.8}$$

for any $y \in \Omega$. Then, (2.8) implies

$$\Delta\widetilde{\tau}(t_0, x_0) = \frac{1}{\mu(x_0)} \sum_{y\sim x_0} w_{x_0 y}\left(\widetilde{\tau}(t_0, y) - \widetilde{\tau}(t_0, x_0)\right) \geq 0. \tag{2.9}$$

Fix $x_0 \in \Omega^\circ$, since $\widetilde{\tau}(t, x_0)$ is differentiable with respect to t in $(0, T']$, then it yields

$$\widetilde{\tau}_t(t_0, x_0) \leq 0. \tag{2.10}$$

By (2.2) and the assumption $\widetilde{\tau}(t_0, x_0) < 0$, we have

$$\begin{aligned} &2L\widetilde{\tau}(t_0\, x_0) - e^{-2Lt_0}\left[f(u(t_0, x_0)) - f(v(t_0, x_0))\right] \\ &\quad \leq 2L\widetilde{\tau}(t_0, x_0) + Le^{-2Lt_0}|u(t_0, x_0) - v(t_0, x_0)| \\ &\quad = 2L\widetilde{\tau}(t_0, x_0) + L|\widetilde{\tau}(t_0, x_0)| \\ &\quad = L\widetilde{\tau}(t_0\, , x_0) < 0. \end{aligned} \tag{2.11}$$

In view of (2.9)–(2.11), there holds

$$\widetilde{\tau}_t(t_0, x_0) - \Delta\widetilde{\tau}(t_0, x_0) + 2L\widetilde{\tau}(t_0, x_0) - e^{-2Lt_0}\left[f(u(t_0, x_0)) - f(v(t_0, x_0))\right] < 0,$$

which contradicts (2.5). Hence, $\widetilde{\tau}(t, x) \geq 0$ for any $(t, x) \in (0, T'] \times \Omega^\circ$, then we have $\tau(t, x) \geq 0$ for any $(t, x) \in [0, T'] \times \Omega$.

Finally, since T' is arbitrary, we get the desired conclusion. □

Lemma 2.4 *(Strong comparison principle) Suppose that f satisfies (H_1). For any $T > 0$ (T may be $+\infty$), we assume that $u(t, x)$ and $v(t, x)$ are continuous and differentiable with respect to t in $(0, T) \times \Omega$, and satisfy*

$$\begin{cases} u_t(t, x) - \Delta u(t, x) - f(u(t, x)) \geq v_t(t, x) - \Delta v(t, x) - f(v(t, x)), & (t, x) \in (0, T) \times \Omega^\circ, \\ u(t, x) \geq v(t, x), & (t, x) \in [0, T) \times \partial\Omega, \\ u(0, x) \geq v(0, x), & x \in \Omega. \end{cases} \tag{2.12}$$

If there exists a vertex $x^* \in \Omega^\circ$ such that $u(0, x^*) > v(0, x^*)$, then $u(t, x) > v(t, x)$ for any $(t, x) \in (0, T) \times \Omega^\circ$.

Proof By Lemma 2.3, we deduce that $u(t, x) \geq v(t, x)$ for any $(t, x) \in [0, T) \times \Omega$. Furthermore, f satisfies (2.2) since f be locally Lipschitz continuous on $\mathbb{R}$. Take $\tau(t, x)$ as the same as Lemma 2.3. Then, for any $0 < T' < T$, we get $\tau(t, x) \geq 0$ for all $(t, x) \in [0, T'] \times \Omega$. This together with condition (g) give

$$\begin{aligned} \Delta\tau(t, x^*) &= \frac{1}{\mu(x^*)} \sum_{y \sim x^*} w_{x^* y}(\tau(t, y) - \tau(t, x^*)) \\ &\geq -\frac{1}{\mu(x^*)} \sum_{y \sim x^*} w_{x^* y}\tau(t, x^*) \\ &\geq -D_\mu \tau(t, x^*). \end{aligned} \tag{2.13}$$

Then, from (2.12) we deduce that

$$\tau_t(t, x^*) - \Delta\tau(t, x^*) - \left[f(u(t, x^*)) - f(v(t, x^*))\right] \geq 0 \tag{2.14}$$

for any $t \in (0, T']$. Taking into account (2.2) and the fact $\tau(t, x) \geq 0$ for all $(t, x) \in [0, T'] \times \Omega$, combining (2.13) and (2.14), we have

$$\begin{aligned} \tau_t(t, x^*) &\geq \Delta\tau(t, x^*) + \left[f(u(t, x^*)) - f(v(t, x^*))\right] \\ &\geq -D_\mu \tau(t, x^*) - L|\tau(t, x^*)| \\ &\geq -(D_\mu + L)\tau(t, x^*). \end{aligned}$$

By calculating directly, we deduce that for any $t \in (0, T']$

$$\tau(t, x^*) \geq \tau(0, x^*) e^{-(D_\mu + L)t} > 0 \tag{2.15}$$

since $\tau(0, x^*) > 0$.

Now, we prove Lemma 2.4 by contradiction, we suppose that there exists $(t_0, x_0) \in (0, T'] \times \Omega^\circ$ such that

$$\tau(t_0, x_0) = \min_{x \in \Omega^\circ} \min_{t \in (0, T']} \tau(t, x) = 0.$$

Then, it follows that

$$\tau_t(t_0, x_0) \le 0, \quad \Delta\tau(t_0, x_0) \ge 0.$$

By (2.14), we have

$$0 \le \tau_t(t_0, x_0) - \Delta\tau(t_0, x_0) \le 0.$$

Hence, we get

$$\Delta\tau(t_0, x_0) = 0.$$

This leads to $u(t_0, y) = u(t_0, x_0) = 0$ for all $y \sim x_0$, and $u(t_0, x) = 0$ for all $x \in \Omega$ since Ω is a bounded and connected domain. This contradicts (2.15), therefore, for any $(t, x) \in (0, T'] \times \Omega^\circ$, we have $u(t, x) > v(t, x)$.

Finally, since T' is arbitrary, we complete the proof of Lemma 2.4. □

Let $T = +\infty$ and $v(t, x) \equiv 0$ for all $(t, x) \in [0, +\infty) \times \Omega$, it follows from Lemma 2.4 that the solutions $u(t, x)$ of Eq. (1.7) are nonnegative for all $(t, x) \in [0, +\infty) \times \Omega$ since $f(0) = 0$. Furthermore, by Lemma 2.4, we can get a much stronger conclusion. Since $u_0(x)$ is a nonnegative and nontrivial initial value, we deduce that the solutions $u(t, x)$ of Eq. (1.7) are positive for all $(t, x) \in (0, +\infty) \times \Omega^\circ$. As for the p-Laplacian parabolic Eq. (1.11), we can also obtain the comparison principle, and the strong comparison principle under $p > 2$, hence the similar conclusions also hold.

3 Short Time Existence

To begin with, we define a Banach space

$$X_{t_0} = \{u : [0, t_0] \times \Omega \to \mathbb{R} \mid u|_{\partial\Omega} = 0, \ u(\cdot, x) \in C[0, t_0] \text{ for each } x \in \Omega\},$$

with the norm

$$\|u\|_{X_{t_0}} = \max_{x\in\Omega} \max_{t\in[0, t_0]} |u(t, x)|,$$

where $t_0 > 0$ is a fixed constant. Then, we consider the operator $D : X_{t_0} \to X_{t_0}$ defined by

$$D[u](t, x) = \begin{cases} u_0(x) + \int_0^t \Delta u(s, x)ds + \int_0^t f(u(s, x))ds, & (t, x)\in[0, t_0] \times \Omega^\circ, \\ 0, & (t, x)\in[0, t_0] \times \partial\Omega. \end{cases}$$

It is easy to prove the operator D is well-defined, namely, D maps X_{t_0} to X_{t_0}. We omit this part of the proof. Next in order to use Banach fixed point theorem, we prove an important lemma.

Lemma 3.1 *Assume that f satisfies* (H_1). *If* t_0 *is sufficiently small, then the operator D is a strict contraction in the ball*

$$B(u_0,\ 2\|u_0\|_{X_{t_0}}) = \{u \in X_{t_0} \mid \|u - u_0\|_{X_{t_0}} \le 2\|u_0\|_{X_{t_0}}\}.$$

Proof Let $u,\ v \in B(u_0,\ 2\|u_0\|_{X_{t_0}})$. Since f satisfies (H_1), there exists a constant $L > 0$ such that

$$|f(a) - f(b)| \le L|a - b|, \quad \forall\, a,\ b \in [-m,\ m],$$

where $m = 3\|u_0\|_{X_{t_0}}$. Then, for any $(t,\ x) \in [0,\ t_0] \times \Omega$, we have

$$\begin{aligned}\left|\int_0^t \Delta u(s,\ x)ds\right| &\le \int_0^t \frac{1}{\mu(x)} \sum_{y\sim x} \omega_{xy}|u(s,\ y) - u(s,\ x)|ds \\ &\le 2\|u\|_{X_{t_0}} D_\mu t,\end{aligned}$$

then it implies that

$$\begin{aligned}|D[u](t,\ x) - D[v](t,\ x)| &= \left|\int_0^t \Delta(u - v)(s,\ x)ds + \int_0^t [f(u(s,\ x)) - f(v(s,\ x))]ds\right| \\ &\le 2\|u - v\|_{X_{t_0}} D_\mu t + L\|u - v\|_{X_{t_0}} t \\ &\le Ct_0\|u - v\|_{X_{t_0}}, \end{aligned} \tag{3.1}$$

where $C = 2D_\mu + L$. It follows from (3.1) that

$$\|D[u] - D[v]\|_{X_{t_0}} \le Ct_0\|u - v\|_{X_{t_0}}.$$

Hence, if t_0 is small enough such that $Ct_0 < 1$, we deduce that the operator D is a strict contraction in the ball $B(u_0,\ 2\|u_0\|_{X_{t_0}})$. □

Finally, by Banach fixed point theorem, we obtain the existence and uniqueness of solutions to Eq. (1.7) in the time interval $[0, t_0]$ if $t_0 < 1/(2D_\mu + L)$. This completes the proof of Theorem 1.1.

4 Blow-Up Solutions

In this section, we consider the maximal existence time of the solutions to Eq. (1.7) and prove Theorems 1.3 and 1.4. Then, we shall prove the nonexistence of global solutions to Eq. (1.11) and complete the proof of Theorem 1.5.

Proof of Theorem 1.3 To begin with, let us recall the condition (H_3) of f, for any constants $\delta,\ \varepsilon > 0$,

$$(H_3)\ f(u) \ge \delta u^{1+\varepsilon}, \quad u \ge m,$$

where $m > 1$.

We assume that f satisfies the conditions $(H_1)-(H_3)$ and $u_1 > \max\{(D_\mu/\delta)^{\frac{1}{\varepsilon}}, m\}$, where $u_1 = \max_{x\in\Omega} u_0(x)$. Let $u(t, x)$ is the solution of (1.7) in $[0, +\infty) \times \Omega$, and let $x_t \in \Omega$ be a vertex such that $u(t, x_t) = \max_{x\in\Omega} u(t, x)$, and then, we just have to prove that there exists $T > 0$ such that

$$u(t, x_t) \to +\infty$$

as $t \to T^-$.

By Lemma 2.3, we have $u(t, x) \geq 0$ for all $(t, x) \in [0, +\infty) \times \Omega$. Noting that $u(t, x_t)$ is continuous for all $t > 0$ and differentiable for almost all $t > 0$, we deduce that

$$\begin{aligned} u_t(s, x_s) &= \Delta u(s, x_s) + f(u(s, x_s)) \\ &= \frac{1}{\mu(x_s)} \sum_{y\sim x_s} \omega_{x_s y} (u(s, y) - u(s, x_s)) + f(u(s, x_s)) \\ &\geq -D_\mu u(s, x_s) + \delta u^{1+\varepsilon}(s, x_s) \end{aligned} \tag{4.1}$$

for almost all $s > 0$ and $u(s, x_s) \geq m$.

We now claim that for all $t > 0$,

$$u(t, x_t) > u_1$$

always holds. It follows from (4.1) and the fact $u_1 > (D_\mu/\delta)^{\frac{1}{\varepsilon}}$ that

$$\begin{aligned} \lim_{s\to 0^+} u_t(s, x_s) &\geq \lim_{s\to 0^+} [-D_\mu u(s, x_s) + \delta u^{1+\varepsilon}(s, x_s)] \\ &= -D_\mu u_1 + \delta u_1^{1+\varepsilon} > 0, \end{aligned}$$

then there exists a constant $s_1 > 0$ such that $u(s, x_s)$ is increasing in $(0, s_1)$. Hence, we get $u(t, x_t) > u_1$ in $(0, s_1)$. By contradiction, we suppose there exists $s > 0$ such that $u(s, x_s) \leq u_1$. Let $(0, s_1)$ be maximal on which $u(t, x_t) > u_1$ and $u(s, x_s) = u_1$, $s \in (0, s_1)$. Then, there exists $s^* \in (0, s_1)$ such that $u_t(s^*, x_{s^*}) < 0$ and $u(s^*, x_{s^*}) > u_1$, however,

$$0 > u_t(s^*, x_{s^*}) \geq -D_\mu u(s^*, x_{s^*}) + \delta u^{1+\varepsilon}(s^*, x_{s^*}) > 0,$$

which is a contradiction. Consequently, for all $t > 0$, we have $u(t, x_t) > u_1$, this leads to (4.1) holds for almost all $s > 0$.

Let $F : [u_1, +\infty) \to (0, F(u_1)]$ be a function defined by

$$F(u) = \int_u^{+\infty} \frac{ds}{-D_\mu s + \delta s^{1+\varepsilon}} < +\infty, \quad u \geq u_1. \tag{4.2}$$

Since $u_1 > m > 1$ and $\varepsilon > 0$, the anomalous integral of (4.2) converges for all $u \geq u_1$. Furthermore, we have $-D_\mu s + \delta s^{1+\varepsilon} > 0$ for all $s \geq u_1$ due to $u_1 > (D_\mu/\delta)^{\frac{1}{\varepsilon}}$. As a consequence, F is a decreasing continuous function from $[u_1, +\infty)$ onto $(0, F(u_1)]$, its inverse function is denotes by G. In view of (4.1), there holds

$$t \leq \int_0^t \frac{u_t(s, x_s)}{-D_\mu u(s, x_s) + \delta u^{1+\varepsilon}(s, x_s)} du = \int_{u_1}^{u(t, x_t)} \frac{ds}{-D_\mu s + \delta s^{1+\varepsilon}},$$

then it implies

$$F(u(t, x_t)) \leq F(u_1) - t. \tag{4.3}$$

Let inverse function G operate the two sides of (4.3), we deduce that

$$u(t, x_t) \geq G(F(u_1) - t),$$

it follows that $u(t, x_t) \to +\infty$ as $t \to F(u_1)^-$, and the finite blow-up time is

$$F(u_1) = \int_{u_1}^{+\infty} \frac{ds}{-D_\mu s + \delta s^{1+\varepsilon}} < +\infty.$$

This ends the proof of Theorem 1.3. □

Proof of Theorem 1.4 Since the condition (H_3) is strong and independent of the eigenvalue of $-\Delta$, inspired by Chung et al. [2], we develop a new condition of f as follows, for any $\varepsilon > 0$,

$$(H_4) \ (2 + \varepsilon)F(u) \leq uf(u) + \alpha u^2 + \beta, \quad u > 0,$$

where $F(u) = \int_0^u f(s)ds$, $0 < \alpha \leq \varepsilon\lambda_1(\Omega)/2$, α and β are positive constants, $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition.

Now, we prove Theorem 1.4 by concavity method, which was studied by Levine [20]. It is obvious that $u(t, x) \geq 0$ for all $(t, x) \in [0, +\infty) \times \Omega$ by Lemma 2.3. Let

$$I(t) = \int_\Omega u^2(t, x)d\mu, \quad \forall\, t \geq 0$$

then by formula of integration by parts from Lemma 2.1 and (1.7), we have

$$\begin{aligned} \frac{d}{dt} I(t) &= 2\int_\Omega u(t, x)u_t(t, x)d\mu \\ &= 2\int_\Omega u(t, x)\left(\Delta u(t, x) + f(u(t, x))\right)d\mu \\ &= -2\int_\Omega |\nabla u|^2(t, x)d\mu + 2\int_\Omega u(t, x)f(u(t, x))d\mu. \end{aligned} \tag{4.4}$$

We define a functional $J(t)$ by

$$J(t) = -\frac{1}{2}\int_{\Omega} |\nabla u|^2(t, x)d\mu + \int_{\Omega} (F(u(t, x)) - \beta)d\mu, \quad \forall\, t \geq 0.$$

Since the initial value $u_0(x)$ satisfies (1.10), we deduce that $J(0) > 0$. Taking the derivative of $J(t)$ with respect to t, we have

$$\begin{aligned}\frac{d}{dt}J(t) &= \int_{\Omega} \Delta u(t, x)u_t(t, x)d\mu + \int_{\Omega} f(u(t, x))u_t(t, x)d\mu \\ &= \int_{\Omega} (\Delta u(t, x) + f(u(t, x)))\, u_t(t, x)d\mu \\ &= \int_{\Omega} u_t^2(t, x)d\mu \geq 0, \end{aligned} \tag{4.5}$$

which implies that $J(t)$ is a nondecreasing function, and thus, $J(t) \geq J(0) > 0$ for all $t \geq 0$. Integrating (4.5) from 0 to t, we have

$$J(t) = \int_0^t \int_{\Omega} u_t^2(s, x)d\mu ds + J(0). \tag{4.6}$$

Let

$$K(t) = \int_0^t I(s)ds + M, \quad \forall\, t \geq 0, \tag{4.7}$$

where $M > 0$ is a sufficiently large constant to be determined later. Taking the derivative of $K(t)$ with respect to t, we have by (4.4),

$$\frac{d}{dt}K(t) = I(t) = 2\int_0^t \int_{\Omega} u(s, x)u_t(s, x)d\mu ds + \int_{\Omega} u_0^2(x)d\mu. \tag{4.8}$$

Noting that $0 < \lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition, by (1.9) we obtain that

$$\int_{\Omega} |\nabla u|^2(t, x)d\mu \geq \lambda_1(\Omega) \int_{\Omega} u^2(t, x)d\mu.$$

This together with condition (H_4) and (4.4), (4.6), taking the second derivative of $K(t)$, we have

$$\begin{aligned}
\frac{1}{2}\frac{d^2}{dt^2}K(t) &= -\int_\Omega |\nabla u|^2(t,\ x)d\mu + \int_\Omega u(t,\ x)f(u(t,\ x))d\mu \\
&\geq -\int_\Omega |\nabla u|^2(t,\ x)d\mu + \int_\Omega \Big((2+\varepsilon)F(u(t,\ x)) - \alpha u^2(t,\ x) - \beta\Big)\,d\mu \\
&\geq \left(-\frac{2+\varepsilon}{2} + \frac{\varepsilon}{2}\right)\int_\Omega |\nabla u|^2(t,\ x)d\mu \\
&\quad + \int_\Omega \Big((2+\varepsilon)F(u(t,\ x)) - \alpha u^2(t,\ x) - (2+\varepsilon)\beta\Big)\,d\mu \\
&\geq (2+\varepsilon)\left(-\frac{1}{2}\int_\Omega |\nabla u|^2(t,\ x)d\mu + \int_\Omega (F(u(t,\ x)) - \beta)\,d\mu\right) \\
&\quad + \frac{\varepsilon}{2}\left(\int_\Omega |\nabla u|^2(t,\ x)d\mu - \lambda_1(\Omega)\int_\Omega u^2(t,\ x)d\mu\right) \\
&\geq (2+\varepsilon)\left(\int_0^t\int_\Omega u_t^2(s,\ x)d\mu ds + J(0)\right). \qquad (4.9)
\end{aligned}$$

Using the Schwarz inequality and the Hölder inequality to (4.8), we have that for any $\rho > 0$,

$$\begin{aligned}
\left(\frac{d}{dt}K(t)\right)^2 &\leq 4(1+\rho)\left(\int_0^t\int_\Omega u(s,\ x)u_t(s,\ x)d\mu ds\right)^2 + \left(1+\frac{1}{\rho}\right)\left(\int_\Omega u_0^2(x)d\mu\right)^2 \\
&= 4(1+\rho)\left(\sum_{x\in\Omega}\mu(x)\int_0^t u(s,\ x)u_t(s,\ x)ds\right)^2 + \left(1+\frac{1}{\rho}\right)\left(\int_\Omega u_0^2(x)d\mu\right)^2 \\
&\leq 4(1+\rho)\left(\sum_{x\in\Omega}\left(\mu(x)\int_0^t u^2(s,\ x)ds\right)^{\frac{1}{2}}\left(\mu(x)\int_0^t u_t^2(s,\ x)ds\right)^{\frac{1}{2}}\right)^2 \\
&\quad + \left(1+\frac{1}{\rho}\right)\left(\int_\Omega u_0^2(x)d\mu\right)^2 \\
&\leq 4(1+\rho)\left(\sum_{x\in\Omega}\mu(x)\int_0^t u^2(s,\ x)ds\right)\left(\sum_{x\in\Omega}\mu(x)\int_0^t u_t^2(s,\ x)ds\right) \\
&\quad + \left(1+\frac{1}{\rho}\right)\left(\int_\Omega u_0^2(x)d\mu\right)^2 \\
&= 4(1+\rho)\left(\int_0^t\int_\Omega u^2(s,\ x)d\mu ds\right)\left(\int_0^t\int_\Omega u_t^2(s,\ x)d\mu ds\right) \\
&\quad + \left(1+\frac{1}{\rho}\right)\left(\int_\Omega u_0^2(x)d\mu\right)^2. \qquad (4.10)
\end{aligned}$$

In view of (4.7), (4.9) and (4.10), there holds for any $\xi > 0$,

$$\begin{aligned}
&\frac{d^2K(t)}{dt^2}K(t) - (1+\xi)\left(\frac{d}{dt}K(t)\right)^2 \\
&\geq 2(2+\varepsilon)\left(\int_0^t\int_\Omega u_t^2(s,\,x)d\mu ds + J(0)\right)\left(\int_0^t\int_\Omega u^2(s,\,x)d\mu ds + M\right) \\
&\quad - 4(1+\rho)(1+\xi)\left(\int_0^t\int_\Omega u^2(s,\,x)d\mu ds\right)\left(\int_0^t\int_\Omega u_t^2(s,\,x)d\mu ds\right) \\
&\quad - \left(1+\frac{1}{\rho}\right)(1+\xi)\left(\int_\Omega u_0^2(x)d\mu\right)^2. \qquad (4.11)
\end{aligned}$$

Choosing $\xi = \rho = \sqrt{(2+\varepsilon)/2} - 1 > 0$, and we obtain from (4.11) that

$$\begin{aligned}
&\frac{d^2K(t)}{dt^2}K(t) - (1+\xi)\left(\frac{d}{dt}K(t)\right)^2 \\
&\geq 2(2+\varepsilon)J(0)M - \frac{(2+\varepsilon)\left(\sqrt{2(2+\varepsilon)}+2\right)}{2\varepsilon}\left(\int_\Omega u_0^2(x)d\mu\right)^2. \qquad (4.12)
\end{aligned}$$

Since $J(0) > 0$, if we choose $M > 0$ sufficiently large, say

$$M = \frac{\left(\sqrt{2(2+\varepsilon)}+2\right)\left(\sum_{x\in\Omega}\mu(x)u_0^2(x)\right)^2}{4J(0)\varepsilon},$$

then it follows from (4.12) that

$$\frac{d^2K(t)}{dt^2}K(t) - (1+\xi)\left(\frac{d}{dt}K(t)\right)^2 > 0.$$

As a consequence,

$$\frac{d}{dt}\frac{K'(t)}{K^{1+\xi}(t)} > 0.$$

Hence, we get an ordinary differential equation

$$\begin{cases} K'(t) \geq \frac{K'(0)}{K^{1+\xi}(0)}K^{1+\xi}(t), & t > 0, \\ K(0) = M. \end{cases}$$

By a straightforward calculation, we get

$$K(t) \geq \left(\frac{1}{M^\xi} - \frac{\xi\sum_{x\in\Omega}\mu(x)u_0^2(x)}{M^{1+\xi}}t\right)^{-\frac{1}{\xi}}.$$

This leads to $K(t) \to +\infty$ as

$$t \to \frac{\left(\sqrt{2(2+\varepsilon)}+2\right)^2 \left(\sum_{x\in\Omega} \mu(x)u_0^2(x)\right)}{4J(0)\varepsilon^2},$$

which implies $u(t, x)$ blows up in a finite time, and we complete the proof of Theorem 1.4. □

Finally, we consider the blow-up solutions of the p-Laplacian parabolic Eq. (1.11) and complete the proof of Theorem 1.5 when $p > 2$.

Proof of Theorem 1.5 By comparison principle, we have $u(t, x) \geq 0$ for all $(t, x) \in [0, +\infty) \times \Omega$. Then, by formula of integration by parts from Lemma 2.2 and (1.11), we deduce that

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^2(t, x)d\mu &= 2\int_\Omega u(t, x)u_t(t, x)d\mu \\ &= 2\int_\Omega u(t, x)\left(\Delta_p u(t, x) + f(u(t, x))\right)d\mu \\ &= -2\int_\Omega |\nabla u|^p(t, x)d\mu + 2\int_\Omega u(t, x)f(u(t, x))d\mu. \end{aligned} \tag{4.13}$$

The functional $J_p(t)$ is defined by

$$J_p(t) = -\frac{1}{p}\int_\Omega |\nabla u|^p(t, x)d\mu + \int_\Omega (F(u(t, x)) - \beta)\,d\mu, \quad t \geq 0.$$

Taking into account (1.13), we have $J_p(0) > 0$. Then, taking the derivative of $J_p(t)$, by Lemma 2.2 we have

$$\begin{aligned} J_p'(t) &= \int_\Omega |\nabla u|^{p-2}(t, x)\Gamma\left(u(t, x), u_t(t, x)\right)d\mu + \int_\Omega f(u(t, x))u_t(t, x)d\mu \\ &= \int_\Omega \left(\Delta_p u(t, x) + f(u(t, x))\right)u_t(t, x)d\mu \\ &= \int_\Omega u_t^2(t, x)d\mu \geq 0, \end{aligned} \tag{4.14}$$

which implies that $J_p(t) > 0$ for all $t \geq 0$. Integrating (4.14) from 0 to t, we get

$$J_p(t) = \int_0^t \int_\Omega u_t^2(s, x)d\mu ds + J_p(0). \tag{4.15}$$

Let

$$R(t) = \int_0^t \int_\Omega u^2(s, x)d\mu ds + N, \quad t \geq 0, \tag{4.16}$$

where $N > 0$ is a sufficiently large constant. Then, it follows from (4.13) that

$$R'(t) = \int_\Omega u^2(t, x)d\mu = 2\int_0^t \int_\Omega u(s, x)u_t(s, x)d\mu ds + \int_\Omega u_0^2(x)d\mu.$$

Furthermore, in view of the condition (H_5), together with (4.15), we deduce that

$$\begin{aligned}
\frac{1}{2}R''(t) &= -\int_\Omega |\nabla u|^p(t, x)d\mu + \int_\Omega u(t, x)f(u(t, x))d\mu \\
&\geq \left(-\frac{(p+\varepsilon)}{p} + \frac{\varepsilon}{p}\right)\int_\Omega |\nabla u|^p(t, x)d\mu \\
&\quad + \int_\Omega \big((p+\varepsilon)F(u(t, x)) - \alpha u^p(t, x) - \beta\big)\, d\mu \\
&\geq (p+\varepsilon)\left(-\frac{1}{p}\int_\Omega |\nabla u|^p(t, x)d\mu + \int_\Omega (F(u(t, x)) - \beta)\, d\mu\right) \\
&\quad + \frac{\varepsilon}{p}\left(\int_\Omega |\nabla u|^p(t, x)d\mu - \lambda_1(\Omega)\int_\Omega u^p(t, x)d\mu\right) \\
&\geq (p+\varepsilon)\left(\int_0^t \int_\Omega u_t^2(s, x)d\mu ds + J_p(0)\right).
\end{aligned} \tag{4.17}$$

By the same method as (4.10), for any $\rho > 0$, it follows that

$$\begin{aligned}
R'(t)^2 &\leq 4(1+\rho)\left(\int_0^t \int_\Omega u^2(s, x)d\mu ds\right)\left(\int_0^t \int_\Omega u_t^2(s, x)d\mu ds\right) \\
&\quad + \left(1 + \frac{1}{\rho}\right)\left(\int_\Omega u_0^2(x)d\mu\right)^2.
\end{aligned} \tag{4.18}$$

Together with (4.16)–(4.18), we have for any $\xi > 0$,

$$\begin{aligned}
R''(t)R(t) - (1+\xi)R'(t)^2 \geq\ & 2(p+\varepsilon)\left(\int_0^t \int_\Omega u_t^2(s, x)d\mu ds + J_p(0)\right) \\
& \left(\int_0^t \int_\Omega u^2(s, x)d\mu ds + N\right) \\
& - 4(1+\rho)(1+\xi)\left(\int_0^t \int_\Omega u^2(s, x)d\mu ds\right) \\
& \left(\int_0^t \int_\Omega u_t^2(s, x)d\mu ds\right) \\
& - \left(1 + \frac{1}{\rho}\right)(1+\xi)\left(\int_\Omega u_0^2(x)d\mu\right)^2.
\end{aligned} \tag{4.19}$$

Noting that $p > 2$, we choose $\xi = \rho = \sqrt{(p+\varepsilon)/2} - 1 > 0$, then it follows from (4.19) that

$$R''(t)R(t) - (1+\xi)R'(t)^2 \geq 2(p+\varepsilon)J_p(0)N - \frac{(p+\varepsilon)\left(\sqrt{2(p+\varepsilon)}+2\right)}{2(p-2+\varepsilon)}\left(\int_\Omega u_0^2(x)d\mu\right)^2.$$

Since $J_p(0) > 0$, if we choose $N > 0$ sufficiently large such that

$$N = \frac{\left(\sqrt{2(p+\varepsilon)}+2\right)\left(\sum_{x\in\Omega}\mu(x)u_0^2(x)\right)^2}{4J_p(0)(p-2+\varepsilon)}.$$

As a consequence, we have

$$\frac{d}{dt}\frac{R'(t)}{R^{1+\xi}(t)} > 0.$$

Hence, by calculating directly we get

$$R(t) \geq \left(\frac{1}{N^\xi} - \frac{\xi\sum_{x\in\Omega}\mu(x)u_0^2(x)}{N^{1+\xi}}t\right)^{-\frac{1}{\xi}},$$

which yields that $u(t, x)$ blows up in a finite time

$$T = \frac{\left(\sqrt{2(p+\varepsilon)}+2\right)^2\left(\sum_{x\in\Omega}\mu(x)u_0^2(x)\right)}{4J_p(0)(p-2+\varepsilon)^2},$$

and we complete the proof of Theorem 1.5. □

Remark 4.1 In fact, Theorem 1.5 is equivalent to Theorem 1.4 when $p = 2$. Therefore, Theorem 1.5 generalizes the conclusion of Theorem 1.4 to the case $p > 2$. However, the case $1 < p < 2$ has not been solved.

Declarations

Conflict of interest The author declares that there are no conflicts of interests regarding the publication of this paper.

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