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The Second Hankel Determinant of Logarithmic Coefficients for Strongly Ozaki Close-to-Convex Functions

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Abstract

The aim of this paper is to determine sharp bound for the second Hankel determinant of logarithmic coefficients $H_{2,1}(F_f/2)$ of strongly Ozaki close-to-convex functions in the open unit disk. Furthermore, sharp bound of $H_{2,1}(F_{f^{-1}}/2)$, where f^{-1} is the inverse function of f, is also computed. The results show an invariance property of the second Hankel determinants of logarithmic coefficients $H_{2,1}(F_f/2)$ and $H_{2,1}(F_{f^{-1}}/2)$ for strongly convex functions.

Keywords Logarithmic coefficient · Hankel determinant · Strongly Ozaki close-to-convex

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1 Introduction

Let $\mathscr{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad (z \in \mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}), \tag{1.1}$$

and let $\mathscr{S}$ be the class of functions in $\mathscr{A}$ which are univalent in $\mathbb{U}$.

A function f of the form (1.1) is said to be *starlike of order* α, $(0 \leq \alpha < 1)$, in $\mathbb{U}$ if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha \qquad (z \in \mathbb{U}).$$

The set of all such functions is denoted by $\mathscr{S}^*(\alpha)$.

By $\mathscr{K}(\alpha)$, we denote the class of *convex functions of order* α $(\alpha < 1)$, in $\mathbb{U}$ that satisfy the following inequality:

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \qquad (z \in \mathbb{U}).$$

For $\alpha := 0$, these classes reduce to the well-known classes $\mathscr{S}^*$ and $\mathscr{K}$, the class of *starlike functions* and the class of *convex functions*, respectively.

Moreover, a function f of the form (1.1) is said to be *strongly convex of order* α, $(0 < \alpha \leq 1)$, in $\mathbb{U}$ if

$$\left| \arg \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \right| < \frac{\pi\alpha}{2} \qquad (z \in \mathbb{U}).$$

The set of all such functions is denoted by $\mathscr{K}_c(\alpha)$.

A function $f \in \mathscr{A}$ belongs to $\mathscr{C}$, the class of *close-to-convex functions* in $\mathbb{U}$, if and only if there exists $g \in \mathscr{S}^*$ and $\theta \in (-\pi/2, \pi/2)$ such that

$$\operatorname{Re} \left\{ e^{i\theta} \frac{zf'(z)}{g(z)} \right\} > 0 \qquad (z \in \mathbb{U}).$$

Geometrically, f is close-to-convex if and only if the image of $C_R := \{z \in \mathbb{C} : |z| = R\}$ for every $R \in (0, 1)$, has no "hairpin turns"; that is, there are no sections of the curve $f(C_R)$ in which the tangent vector turns backward through an angle $\geq \pi$.

Although the class of close-to-convex functions was introduced by Kaplan [12] in 1952, in 1935 Ozaki [21, 22] had already considered the functions in $\mathscr{A}$ satisfying the following condition:

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > -\frac{1}{2}, \qquad (z \in \mathbb{U}). \tag{1.2}$$

Functions satisfying the inequality (1.2) are close-to-convex, and therefore, they are in $\mathcal{S}$ by the definition of Kaplan [12].

Recently, Kargar and Ebadian [13] generalized Ozaki's condition as follows:

Definition 1 [13] Let $\mathcal{F}(\lambda)$ for $-1/2 < \lambda \leq 1$, denote the class of locally univalent normalized analytic functions f in the unit disk satisfying the condition

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2}-\lambda, \qquad (z \in \mathbb{U}).$$

When $1/2 \leq \lambda \leq 1$, the functions in $\mathcal{F}(\lambda)$ are called *Ozaki close-to-convex*. The class $\mathcal{F}(1)$ was studied by Ponnusamy et al. [23]. Also, $\mathcal{F}(1/2) = \mathcal{K}$. Clearly, $\mathcal{F}(\lambda) \subset \mathcal{K} \subset \mathcal{S}^*$ for all $\lambda \in (-1/2, 1/2)$.

Recently, Allu et al. extended the class $\mathcal{F}(\lambda)$ as follows:

Definition 2 [3, 31] Let $0 < \alpha \leq 1$ and $1/2 \leq \lambda \leq 1$. Then $f \in \mathcal{A}$ is called *strongly Ozaki-close-to-convex* if and only if

$$\left|\arg\left\{\frac{2\lambda-1}{2\lambda+1}+\frac{2}{2\lambda+1}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}\right| < \frac{\alpha\pi}{2}, \qquad (z \in \mathbb{U}). \tag{1.3}$$

This class is denoted by $\mathcal{F}_O(\lambda, \alpha)$.

The class $\mathcal{F}_O(\lambda, \alpha)$ is the subclass of $\mathcal{S}$, and it is obvious that $\mathcal{F}_O(1/2, \alpha) = \mathcal{K}_c(\alpha)$ (see [3]).

Associated with each $f \in \mathcal{S}$ is a function

$$F_f(z) := \log\frac{f(z)}{z} = 2\sum_{k=1}^{\infty}\gamma_k z^k, \qquad (z \in \mathbb{U}). \tag{1.4}$$

The numbers γ_k are called the *logarithmic coefficients of* f. It is well known that the logarithmic coefficients play a crucial role in Milin conjecture (cf. [20], see also [9, p. 155]). It is surprising that for the class $\mathcal{S}$ the sharp estimates of single logarithmic coefficients are known only for two initial ones, namely

$$|\gamma_1| \leq 1 \qquad \text{and} \qquad |\gamma_2| \leq \frac{1}{2}+\frac{1}{e^2} = 0.6353\ldots$$

and are unknown for $k \geq 3$. Recently, logarithmic coefficients have been studied by many researches and upper bounds of logarithmic coefficients of functions in various subclasses of $\mathcal{S}$ have been obtained (e.g., [1, 2, 6, 17, 30, 34]). For a summary of some of the significant results concerning the logarithmic coefficients for univalent functions, we refer to [32].

Since each class $\mathcal{F}_O(\lambda, \alpha)$ is compact and $f(0) = f'(0) - 1 = 0$ for every $f \in \mathcal{F}_O(\lambda, \alpha)$, there exists $r_0 \in (0, 1)$ such that $\mathbb{U}_{r_0} := \{z \in \mathbb{C} : |z| < r_0\} \subset f(\mathbb{U})$

for every $f \in \mathscr{F}_O(\lambda, \alpha)$. Thus, every function in $\mathscr{F}_O(\lambda, \alpha)$ is invertible and

$$\begin{aligned} f^{-1}(w) &= w + \sum_{k=2}^{\infty} \delta_k w^k \\ &= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots, \\ &\quad (w \in \mathbb{U}_{r_0}), \end{aligned} \tag{1.5}$$

in $\mathbb{U}_{r_0}$ (see, e.g., [10, pp. 56-57]). Therefore for each $f \in \mathscr{F}_O(\lambda, \alpha)$ we can define

$$F_{f^{-1}}(w) := \log \frac{f^{-1}(w)}{w} = 2\sum_{k=1}^{\infty} \Gamma_k w^k, \qquad (w \in \mathbb{U}_{r_0}). \tag{1.6}$$

The numbers Γ_k can be called as the *logarithmic coefficients of the inverse function of* f.

For $q, n \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ of $f \in \mathscr{A}$ of form (1.1) is defined as

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} \cdots & a_{n+q} \\ \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} \cdots & a_{n+2(q-1)} \end{vmatrix}. \tag{1.7}$$

The Hankel determinant $H_{2,1}(f) = a_3 - a_2^2$ is the well-known Fekete–Szegö functional. The second Hankel determinant $H_{2,2}(f)$ is given by $H_{2,2}(f) = a_2 a_4 - a_3^2$.

The problem of computing the upper bound of $|H_{q,n}(f)|$ over various subfamilies of $\mathscr{A}$ is interesting and widely studied in Geometric Function Theory. Sharp upper bounds of $|H_{2,2}(f)|$ and $|H_{3,1}(f)|$ for subclasses of analytic functions were obtained by various authors [7, 11, 16, 18, 19, 25–27].

Very recently, Kowalczyk and Lecko [14] introduced the Hankel determinant $H_{q,n}(F_f/2)$, which entries are logarithmic coefficients of f, i.e., $H_{q,n}(F_f/2)$ is of the form (1.7) with a_n replaced by γ_n. Similarly, we can define the determinant $H_{q,n}(F_{f^{-1}}/2)$ by replacing a_n by Γ_n in (1.7).

For a function $f \in \mathscr{S}$ given in (1.1), by differentiating (1.4), one can obtain

$$\gamma_1 = \frac{1}{2}a_2, \quad \gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \quad \gamma_3 = \frac{1}{2}\left(a_4 - a_2 a_3 + \frac{1}{3}a_2^3\right).$$

Therefore,

$$H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{4}\left(a_2 a_4 - a_3^2 + \frac{1}{12}a_2^4\right). \tag{1.8}$$

Furthermore, if $f \in \mathscr{S}$, then for $f_\theta \in \mathscr{S}$, $\theta \in \mathbb{R}$, defined as

$$f_\theta(z) := \mathrm{e}^{-\mathrm{i}\theta} f(\mathrm{e}^{\mathrm{i}\theta} z) \qquad (z \in \mathbb{U}),$$

we find that (see [15])

$$H_{2,1}\left(\frac{1}{2}F_{f_\theta}\right) = \mathrm{e}^{4\mathrm{i}\theta} H_{2,1}\left(\frac{1}{2}F_f\right).$$

Kowalczyk and Lecko [15] obtained sharp bounds for $|H_{2,1}(F_f/2)|$ for the classes of starlike and convex functions of order α. The problem of computing the sharp bounds of $|H_{2,1}(F_f/2)|$ for strongly starlike and strongly convex functions has been considered by Sümer Eker et. al. [29]. Furthermore, upper bounds for the second Hankel determinant of logarithmic coefficients for some different subclasses of class $\mathscr{S}$ have been obtained by Srivastava et al. [28] and Allu and Arora [4].

For a function $f \in \mathscr{S}$ given in (1.1), by differentiating (1.6) together with (1.5), one can obtain

$$\Gamma_1 = -\frac{1}{2}a_2, \quad \Gamma_2 = -\frac{1}{2}a_3 + \frac{3}{4}a_2^2, \quad \Gamma_3 = -\frac{1}{2}a_4 + 2a_2a_3 - \frac{5}{3}a_2^3.$$

Therefore,

$$H_{2,1}(F_{f^{-1}}/2) = \Gamma_1\Gamma_3 - \Gamma_2^2 = \frac{1}{4}\left(a_2a_4 - a_3^2 - a_2^2a_3 + \frac{13}{12}a_2^4\right). \tag{1.9}$$

The aim of this paper is to give the sharp bounds for $|H_{2,1}(F_f/2)|$ and $|H_{2,1}(F_{f^{-1}}/2)|$ for the class of strongly Ozaki close-to-convex functions.

Let $\mathscr{P}$ denote the class of analytic functions p in $\mathbb{U}$ satisfying $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{U}$. Thus, every $p \in \mathscr{P}$ can be represented as

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \qquad (z \in \mathbb{U}). \tag{1.10}$$

Elements of $\mathscr{P}$ are called Carathéodory functions.

To establish our main results, we will require the following lemmas.

Lemma 1 ([5] (see also [15])) *If $p \in \mathscr{P}$ is of the form (1.10) with $c_1 \geq 0$, then*

$$\begin{aligned} c_1 &= 2d_1, \\ c_2 &= 2d_1^2 + 2(1-d_1^2)d_2, \\ c_3 &= 2d_1^3 + 4(1-d_1^2)d_1d_2 - 2(1-d_1^2)d_1d_2^2 + 2(1-d_1^2)(1-|d_2|^2)d_3 \end{aligned} \tag{1.11}$$

for some $d_1 \in [0, 1]$ and $d_2, d_3 \in \overline{\mathbb{U}} := \{z \in \mathbb{C} : |z| \leq 1\}$.

For $d_1 \in \mathbb{U}$ and $d_2 \in \partial\mathbb{U} := \{z \in \mathbb{C} : |z| = 1\}$, there is a unique function $p \in \mathscr{P}$ with c_1 and c_2 as in (1.11), namely

$$p(z) = \frac{1 + (\overline{d_1}d_2 + d_1)z + d_2z^2}{1 + (\overline{d_1}d_2 - d_1)z - d_2z^2}, \qquad (z \in \mathbb{U}).$$

Lemma 2 [8] *Given real numbers A, B, C, let*

$$Y(A,B,C) := \max\left\{\left|A+Bz+Cz^2\right| + 1 - |z|^2 : z \in \overline{\mathbb{U}}\right\}.$$

I. *If $AC \geq 0$, then*

$$Y(A,B,C) = \begin{cases} |A|+|B|+|C|, & |B| \geq 2(1-|C|), \\ 1+|A|+\dfrac{B^2}{4(1-|C|)}, & |B| < 2(1-|C|). \end{cases}$$

II. *If $AC < 0$, then*

$$Y(A,B,C) = \begin{cases} 1-|A|+\dfrac{B^2}{4(1-|C|)}, & -4AC(C^{-2}-1) \leq B^2 \wedge |B| < 2(1-|C|), \\ 1+|A|+\dfrac{B^2}{4(1+|C|)}, & B^2 < \min\{4(1+|C|)^2, -4AC(C^{-2}-1)\} \\ R(A,B,C), & \text{otherwise.} \end{cases}$$

where

$$R(A,B,C) := \begin{cases} |A|+|B|-|C|, & |C|(|B|+4|A|) \leq |AB|, \\ -|A|+|B|+|C|, & |AB| \leq |C|(|B|-4|A|), \\ (|A|+|C|)\sqrt{1-\dfrac{B^2}{4AC}}, & \text{otherwise.} \end{cases}$$

2 Second Hankel Determinant of Logarithmic Coefficients for Strongly Ozaki Close-to-Convex Functions

Theorem 1 *Let $\alpha \in (0,1]$ and $\lambda \in [1/2,1]$. If $f \in \mathscr{F}_O(\lambda,\alpha)$, then*

$$\left|H_{2,1}(F_f/2)\right| \leq \begin{cases} \dfrac{\alpha^2(1+2\lambda)^2}{144}, & F \leq 2, \\ \dfrac{\alpha^2(1+2\lambda)^2}{576}\left(4+\dfrac{(F-2)^2}{16+4F-E}\right), & F > 2, \end{cases} \tag{2.1}$$

where $E := \alpha^2(4\lambda^2-4\lambda-3)$ and $F := \alpha(5+2\lambda)$. The inequalities in (2.1) are sharp.

Proof Let $\alpha \in (0,1]$, $\lambda \in [1/2,1]$ and $f \in \mathscr{F}_O(\lambda,\alpha)$ be of the form (1.1). Then by (1.3), we have

$$\frac{2\lambda-1}{2\lambda+1} + \frac{2}{2\lambda+1}\left(1+\frac{zf''(z)}{f'(z)}\right) = (p(z))^\alpha, \qquad (z \in \mathbb{U}), \tag{2.2}$$

for some function $p \in \mathscr{P}$ of the form (1.10). So equating coefficients we obtain

$$\begin{aligned} a_2 &= \frac{\alpha(1+2\lambda)}{4}c_1, \\ a_3 &= \frac{\alpha(1+2\lambda)}{24}\left(2c_2 + (2\alpha + 2\alpha\lambda - 1)c_1^2\right), \\ a_4 &= \frac{\alpha(1+2\lambda)}{576}\Big((8 - 21\alpha + 16\alpha^2 - 18\alpha\lambda + 30\alpha^2\lambda + 12\alpha^2\lambda^2)c_1^3 \\ &\quad - 6(4 - 7\alpha - 6\alpha\lambda)c_1c_2 + 24c_3\Big). \end{aligned} \tag{2.3}$$

Since the class $\mathscr{F}_O(\lambda, \alpha)$ and $|H_{2,1}(F_f/2)|$ are rotationally invariant, without loss of generality we may assume that $a_2 \geq 0$, so $c := c_1 \in [0, 2]$ (i.e., in view of (1.11) that $d_1 \in [0, 1]$). By using (1.8), (2.3) and (1.11), we obtain

$$\begin{aligned} \gamma_1\gamma_3 - \gamma_2^2 &= \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right) \\ &= \frac{\alpha^2(1+2\lambda)^2}{2304}\Big[(8-E)d_1^4 + 4F(1-d_1^2)d_1^2d_2 \\ &\quad -8(1-d_1^2)(d_1^2+2)d_2^2 + 24(1-d_1^2)(1-|d_2|^2)d_1d_3\Big], \end{aligned} \tag{2.4}$$

where $E = \alpha^2(4\lambda^2 - 4\lambda - 3)$ and $F = \alpha(5 + 2\lambda)$.

Now, we may have the following cases on d_1.

Case 1. Suppose that $d_1 = 1$. Then by (2.4) we obtain

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| = \frac{\alpha^2(1+2\lambda)^2}{2304}(8-E)$$

Case 2. Suppose that $d_1 = 0$. Then by (2.4) we obtain

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| = \frac{\alpha^2(1+2\lambda)^2}{144}|d_2|^2 \leq \frac{\alpha^2(1+2\lambda)^2}{144}.$$

Case 3. Suppose that $d_1 \in (0, 1)$. By the fact that $|d_3| \leq 1$, applying the triangle inequality to (2.4) we can write

$$\begin{aligned} |\gamma_1\gamma_3 - \gamma_2^2| &= \Bigg|\frac{\alpha^2(1+2\lambda)^2}{2304}\big[(8-E)d_1^4 + 4F(1-d_1^2)d_1^2d_2 \\ &\quad -8(1-d_1^2)(d_1^2+2)d_2^2 + 24(1-d_1^2)(1-|d_2|^2)d_1d_3\big]\Bigg| \\ &\leq \frac{\alpha^2(1+2\lambda)^2d_1(1-d_1^2)}{96}\left[\left|\frac{8-E}{24(1-d_1^2)}d_1^3 + \frac{F}{6}d_1d_2 - \frac{d_1^2+2}{3d_1}d_2^2\right| + 1 - |d_2|^2\right] \\ &= \frac{\alpha^2(1+2\lambda)^2d_1(1-d_1^2)}{96}\left[\left|A + Bd_2 + Cd_2^2\right| + 1 - |d_2|^2\right] \end{aligned} \tag{2.5}$$

where

$$A := \frac{8-E}{24(1-d_1^2)} d_1^3, \quad B := \frac{F}{6} d_1 \quad \text{and} \quad C := -\frac{d_1^2+2}{3d_1}.$$

Since $AC < 0$, we apply Lemma 2 only for the case II.

We consider the following sub-cases.

3(a) Note that

$$|B| - 2(1-|C|) = \frac{1}{6d_1}\left[4(1-d_1)(2-d_1) + Fd_1^2\right] > 0.$$

Therefore, $|B| < 2(1-|C|)$ does not hold for $d_1 \in (0, 1)$, $\lambda \in [1/2, 1]$ and $\alpha \in (0, 1]$.

3(b) We can easily see that

$$4\big(1+|C|\big)^2 > 0.$$

Furthermore, since $AC < 0$ and

$$\frac{1}{C^2} - 1 = -\frac{(1-d_1^2)(4-d_1^2)}{(d_1^2+2)^2} < 0,$$

the inequality

$$B^2 < \min\left\{4(1+|C|)^2, -4AC\left(\frac{1}{C^2}-1\right)\right\}$$

is false for $d_1 \in (0, 1)$, $\lambda \in [1/2, 1]$ and $\alpha \in (0, 1]$.

3(c) Since $0 < F \leq 7$, we obtain

$$4|C| - |B| = \frac{1}{6d_1}\Big((8-F)d_1^2 + 16\Big) > 0,$$

and this implies

$$|C|(|B| + 4|A|) - |AB| = |BC| + |A|(4|C| - |B|) > 0.$$

Consequently, the inequality $|C|(|B| + 4|A|) \leq |AB|$ does not hold for $d_1 \in (0, 1)$, $\lambda \in [1/2, 1]$ and $\alpha \in (0, 1]$.

3(d) We can write

$$\begin{aligned} |AB| - |C|(|B| - 4|A|) &= \frac{(8-E)F}{144(1-d_1^2)} d_1^4 - \frac{d_1^2+2}{3d_1}\left(\frac{F}{6}d_1 - \frac{8-E}{6(1-d_1^2)}d_1^3\right) \\ &= \frac{1}{144(1-t)}(Kt^2 + Lt + M), \end{aligned}$$

where $t := d_1^2 \in (0, 1)$ and

$$K := 64 + 16F - 8E - EF, \quad L := 128 + 8F - 16E, \quad M := -16F.$$

Since $-4 \le E < 0$ and $0 < F \le 7$, it is easy to see that $K > 0$, $L > 0$ and $M < 0$ for $\lambda \in [1/2, 1]$ and $\alpha \in (0, 1]$.

For the equation $Kt^2 + Lt + M = 0$, we have $\Delta > 0$. Since

$$\frac{M}{K} < 0 \quad \text{and} \quad K + L + M > 0,$$

for $\lambda \in [1/2, 1]$ and $\alpha \in (0, 1]$, the equation $Kt^2 + Lt + M = 0$ has a unique positive root $t_1 < 1$. Thus, the inequality $|AB| - |C|\,(|B| - 4|A|) \le 0$ holds for $(0, d_1^*]$, where $d_1^* = \sqrt{t_1}$. So we can write from (2.5) and Lemma 2,

$$\begin{aligned}\left|\gamma_1\gamma_3 - \gamma_2^2\right| &\le \frac{\alpha^2(1+2\lambda)^2 d_1(1-d_1^2)}{96}\,(-|A| + |B| + |C|) \\ &= \frac{\alpha^2(1+2\lambda)^2}{2304}\Phi(d_1),\end{aligned}$$

where

$$\Phi(x) := (E - 4F - 16)x^4 + 4(F-2)x^2 + 16, \quad x \in [0, d_1^*]. \tag{2.6}$$

We note that $\Phi'(x) = 0$ for $x \in (0, d_1^*)$ holds only for

$$x = \sqrt{\frac{2(F-2)}{16 + 4F - E}} =: \xi, \tag{2.7}$$

in the case when $F - 2 > 0$. Clearly $\xi > 0$. Now, we will show that $0 < \xi < d_1^*$. Since $-4 \le E < 0$ and $0 < F \le 7$, we obtain

$$\begin{aligned}K\xi^4 + L\xi^2 + M = -\frac{4}{(16+4F-E)^2}\Big[&F^3(E+32) + 8F^2(E+28) \\ &+4F(208 - 9E - E^2) + 16E^2 - 352E + 1792\Big] < 0,\end{aligned}$$

which confirms that $0 < \xi < d_1^*$. Moreover, the function Φ attains its maximum value at ξ on $[0, d_1^*]$. Thus for $F - 2 > 0$, we obtain

$$\begin{aligned}\left|\gamma_1\gamma_3 - \gamma_2^2\right| &\le \frac{\alpha^2(1+2\lambda)^2}{2304}\Phi(\xi) \\ &= \frac{\alpha^2(1+2\lambda)^2}{576}\left(4 + \frac{(F-2)^2}{16+4F-E}\right).\end{aligned}$$

Furthermore, if $F - 2 \le 0$, then the function Φ is decreasing on $[0, d_1^*]$. Thus, we have

$$\begin{aligned}\left|\gamma_1\gamma_3 - \gamma_2^2\right| &\le \frac{\alpha^2(1+2\lambda)^2}{2304}\Phi(d_1) \\ &\le \frac{\alpha^2(1+2\lambda)^2}{2304}\Phi(0) = \frac{\alpha^2(1+2\lambda)^2}{144}.\end{aligned}$$

3(e) Next consider the case $d_1 \in [d_1^*, 1)$. Using the last case of the Lemma 2,

$$\begin{aligned}\left|\gamma_1\gamma_3 - \gamma_2^2\right| &\le \frac{\alpha^2(1+2\lambda)^2 d_1(1-d_1^2)}{96}\left((|A|+|C|)\sqrt{1-\frac{B^2}{4AC}}\right)\\ &= \frac{\alpha^2(1+2\lambda)^2}{2304}\Psi(d_1),\end{aligned}$$

where

$$\Psi(x) := (16 - 8x^2 - Ex^4)\sqrt{1 + \frac{F^2(1-x^2)}{2(8-E)(x^2+2)}}, \quad x \in [d_1^*, 1].$$

For $x \in [d_1^*, 1]$, we have

$$\begin{aligned}\Psi'(x) =& (-16x - 4Ex^3)\sqrt{1 + \frac{F^2(1-x^2)}{2(8-E)(x^2+2)}}\\ &+ (Ex^4 + 8x^2 - 16)\frac{3F^2x}{2(8-E)(x^2+2)^2\sqrt{1 + \frac{F^2(1-x^2)}{2(8-E)(x^2+2)}}}.\end{aligned}$$

Since for $-4 \le E < 0$,

$$-16x - 4Ex^3 < 0$$

and

$$Ex^4 + 8x^2 - 16 < 8(x^2 - 2) < 0$$

for $\alpha \in (0, 1]$ and $x \in [d_1^*, 1]$, we deduce that Ψ is a decreasing function. This implies that

$$\begin{aligned}\left|\gamma_1\gamma_3 - \gamma_2^2\right| &\le \frac{\alpha^2(1+2\lambda)^2}{2304}\Psi(d_1)\\ &\le \frac{\alpha^2(1+2\lambda)^2}{2304}\Psi(d_1^*)\\ &= \frac{\alpha^2(1+2\lambda)^2}{2304}\Phi(d_1^*),\end{aligned}$$

where Φ is given in (2.6).

Summarizing parts from Case 1-3, it follows the inequalities (2.1).

To show the sharpness for the case $F - 2 \le 0$, consider the function

$$p(z) := \frac{1-z^2}{1+z^2}, \qquad (z \in \mathbb{U}).$$

It is obvious that the function p is in $\mathscr{P}$ with $c_1 = c_3 = 0$ and $c_2 = -2$. The corresponding function $f \in \mathscr{F}_O(\lambda, \alpha)$ is described by (2.2). Hence by (2.3) it follows that $a_2 = a_4 = 0$ and $a_3 = -\alpha(1+2\lambda)/6$. From (2.4), we obtain

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| = \frac{\alpha^2(1+2\lambda)^2}{144}.$$

For the case $F - 2 > 0$, consider the function

$$p(z) := \frac{1-z^2}{1-2\xi z + z^2}, \qquad (z \in \mathbb{U}),$$

where ξ is given by (2.7). From Lemma 1, it follows that $p \in \mathscr{P}$. The corresponding function $f \in \mathscr{F}_O(\lambda, \alpha)$ is described by (2.2) and has the following coefficients

$$\begin{aligned} a_2 =& \frac{1}{2}\alpha\xi(1+2\lambda), \\ a_3 =& \frac{1}{6}\alpha(1+2\lambda)\big[-1+\xi^2(1+2\alpha+2\alpha\lambda)\big], \\ a_4 =& \frac{1}{72}\alpha(1+2\lambda)\big[-3\xi(2+7\alpha+6\alpha\lambda) \\ &+ \xi^3(8+21\alpha+18\alpha\lambda+16\alpha^2+30\alpha^2\lambda+12\alpha^2\lambda^2)\big]. \end{aligned}$$

Hence, from (2.4) we obtain

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| = \frac{\alpha^2(1+2\lambda)^2}{576}\left(4 + \frac{(F-2)^2}{16+4F-E}\right).$$

This completes the proof.

For $\lambda = 1/2$, we get the bounds for the class $\mathscr{K}_c(\alpha)$ given in [29].

Corollary 1 *Let $\alpha \in (0, 1]$. If $f \in \mathscr{K}_c(\alpha)$, then*

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \le \begin{cases} \dfrac{\alpha^2}{36}, & 0 < \alpha \le \dfrac{1}{3}, \\ \dfrac{\alpha^2(13\alpha^2+18\alpha+17)}{144(\alpha^2+6\alpha+4)}, & \dfrac{1}{3} < \alpha \le 1. \end{cases}$$

The inequalities are sharp.

3 Second Hankel Determinant of Logarithmic Coefficients for Inverse Functions

The following lemma will be used in the proof of the main result of this section.

Lemma 3 *Let $T \in (2, 11]$ and $S \in (4, 39]$. Define $H : [0, 1] \to \mathbb{R}$ by*

$$H(x) := h_1(x)\sqrt{h_2(x)},$$

where for $x \in [0, 1]$,

$$h_1(x) := Sx^2 - 8x + 16 \quad and \quad h_2(x) := 1 + \frac{T^2(1-x)}{2(S+8)(x+2)}.$$

Then H *is a convex function.*

Proof To prove the lemma, we will use the same method as in [24, p. 2524]. By differentiating H twice, we obtain

$$\begin{aligned}(h_2(x))^{3/2}H''(x) =& h_1''(x)\big(h_2(x)\big)^2 + h_1'(x)h_2(x)h_2'(x)\\ &+ \frac{1}{2}h_1(x)h_2(x)h_2''(x) - \frac{1}{4}h_1(x)\big(h_2'(x)\big)^2\\ =& \frac{G(x)}{16(S+8)^2(x+2)^4},\end{aligned}$$

where for $x \in [0, 1]$,

$$\begin{aligned}G(x) :=& \big[(8x^4 + 28x^3 + 3x^2 - 80x + 32)S - 312x + 240\big]T^4\\ &+ \big[\big(-32x^4 - 184x^3 - 336x^2 - 64x + 256\big)S^2\\ &+ \big(-256x^4 - 1472x^3 - 2688x^2 + 256x + 3584\big)S + 6144x + 12288\big]T^2\\ &+ \big(32x^4 + 256x^3 + 768x^2 + 1024x + 512\big)S^3\\ &+ \big(512x^4 + 4096x^3 + 12288x^2 + 16384x + 8192\big)S^2\\ &+ \big(2048x^4 + 16384x^3 + 49152x^2 + 65536x + 32768\big)S.\end{aligned}$$

We show that our assertion is true by proving that $G(x) \geq 0$ for $x \in [0, 1]$. For $x \in [0, 1]$ and $S \in (4, 39]$, define

$$J(u) := A_0 + A_1 u + A_2 u^2$$

where

$$\begin{aligned}A_0 :=& \left(32x^4 + 256x^3 + 768x^2 + 1024x + 512\right) S^3\\ &+ \left(512x^4 + 4096x^3 + 12288x^2 + 16384x + 8192\right) S^2\\ &+ \left(2048x^4 + 16384x^3 + 49152x^2 + 65536x + 32768\right) S\\ A_1 :=& \left(-32x^4 - 184x^3 - 336x^2 - 64x + 256\right) S^2\\ &+ \left(-256x^4 - 1472x^3 - 2688x^2 + 256x + 3584\right) S\\ &+ 6144x + 12288\\ A_2 :=& \left(8x^4 + 28x^3 + 3x^2 - 80x + 32\right) S - 312x + 240.\end{aligned}$$

I. Consider the first case $A_2 \leq 0$. Then

$$\begin{aligned} J(4) &= \left(32x^4 + 256x^3 + 768x^2 + 1024x + 512\right) S^3 \\ &\quad + \left(384x^4 + 3360x^3 + 10944x^2 + 16128x + 9216\right) S^2 \\ &\quad + \left(1152x^4 + 10944x^3 + 38448x^2 + 65280x + 47616\right) S + 19584x + 52992 > 0 \end{aligned}$$

for $x \in [0, 1]$. Furthermore,

$$J(121) = k_0 + k_1 S + k_2 S^2 + k_3 S^3,$$

where for $x \in [0, 1]$,

$$\begin{aligned} k_0 &:= -3824568x + 5000688 \\ k_1 &:= 88200x^4 + 248220x^3 - 232173x^2 - 1074768x + 934944 \\ k_2 &:= -3360x^4 - 18168x^3 - 28368x^2 + 8640x + 39168 \\ k_3 &:= 32x^4 + 256x^3 + 768x^2 + 1024x + 512. \end{aligned}$$

Since $k_3 > 0$ for $x \in [0, 1]$ and $S > 4$, we see that

$$J(121) \geq k_0 + k_1 S + (k_2 + 4k_3) S^2,$$

and

$$k_2 + 4k_3 = -3232x^4 - 17144x^3 - 25296x^2 + 12736x + 41216 > 0.$$

Hence and by the fact that $S > 4$, we obtain

$$J(121) \geq k_0 + (k_1 + 4k_2 + 16k_3)\, S.$$

Thus, since $k_0 > 0$, if $k_1 + 4k_2 + 16k_3 \geq 0$, then $J(121) > 0$.
If $k_1 + 4k_2 + 16k_3 < 0$, then

$$(k_1 + 4k_2 + 16k_3)\, S > (k_1 + 4k_2 + 16k_3)\, 39,$$

and therefore for $x \in [0, 1]$,

$$\begin{aligned} J(121) &\geq k_0 + (k_1 + 4k_2 + 16k_3)\, 39 \\ &= -504192x^4 - 2674464x^3 - 3946176x^2 - 147171336x + 201456528 > 0. \end{aligned}$$

Thus, since $A_2 \leq 0$, we deduce that

$$J(u) \geq \min\{J(4), J(121)\} > 0, \quad u \in [4, 121].$$

II. Next we consider the case $A_2 > 0$. Then

$$J(u) = A_0 + A_1 u + A_2 u^2 \geq A_0 + \big(A_1 + 4A_2\big)u =: \tilde{J}(u).$$

We can easily see that

$$\begin{aligned}\tilde{J}(4) &= \Big(32x^4 + 256x^3 + 768x^2 + 1024x + 512\Big)\, S^3 \\ &\quad + \Big(384x^4 + 3360x^3 + 10944x^2 + 16128x + 9216\Big)\, S^2 \\ &\quad + \Big(1152x^4 + 10944x^3 + 38448x^2 + 65280x + 47616\Big)\, S \\ &\quad + 19584x + 52992 > 0.\end{aligned}$$

Furthermore,

$$\tilde{J}(121) = q_0 + q_1 S + q_2 S^2 + q_3 S^3,$$

where

$$\begin{aligned} q_0 &:= 592416x + 1603008 \\ q_1 &:= -25056x^4 - 148176x^3 - 274644x^2 + 57792x + 481920 \\ q_2 &:= -3360x^4 - 18168x^3 - 28368x^2 + 8640x + 39168 \\ q_3 &:= 32x^4 + 256x^3 + 768x^2 + 1024x + 512. \end{aligned}$$

Since $S > 4$, we obtain

$$\tilde{J}(121) > q_0 + q_1 S + (q_2 + 4q_3)S^2.$$

Since $q_0 > 0$, $q_1 > 0$ and

$$q_2 + 4q_3 = -3232x^4 - 17144x^3 - 25296x^2 + 12736x + 41216 > 0,$$

it follows that $\tilde{J}(121) > 0$.

Hence, since the function $\tilde{J}$ is linear with respect to u, $\tilde{J}(4) > 0$ and $\tilde{J}(121) > 0$, we deduce that

$$J(u) \geq \tilde{J}(u) \geq \min\{\tilde{J}(4), \tilde{J}(121)\} > 0, \quad u \in [4, 121].$$

Finally, note that the cases **I** and **II** imply that $J(u) > 0$ for $u \in (4, 121]$, $S \in [4, 39]$ and $x \in [0, 1]$, which shows that $G(x) \geq 0$. This completes the proof of Lemma 3.

Theorem 2 *Let* $\alpha \in (0,1]$ *and* $\lambda \in [1/2,1]$. *If* $f \in \mathscr{F}_O(\lambda,\alpha)$, *then*

$$\left|H_{2,1}(F_{f^{-1}}/2)\right| \leq \begin{cases} \dfrac{\alpha^2(1+2\lambda)^2}{144}, & T \leq 2, \\ \dfrac{\alpha^2(1+2\lambda)^2}{576}\left(4+\dfrac{(T-2)^2}{16+4T+S}\right), & T>2,\ S \leq 2\left(\sqrt{2T^2+8T+40}-T-2\right), \\ \dfrac{\alpha^2(1+2\lambda)^2}{2304}(8+S), & T>2,\ S>2\left(\sqrt{2T^2+8T+40}-T-2\right), \end{cases} \tag{3.1}$$

where $T := \alpha(1+10\lambda)$ *and* $S := \alpha^2(44\lambda^2+4\lambda-9)$.

The inequalities in (3.1) *are sharp.*

Proof Let $\alpha \in (0,1]$, $\lambda \in [1/2,1]$ and $f \in \mathscr{F}_O(\lambda,\alpha)$ be of the form (1.1). Then, (2.2) holds for some function $p \in \mathscr{P}$ of the form (1.10). Since the class $\mathscr{F}_O(\lambda,\alpha)$ and $|H_{2,1}(F_{f^{-1}}/2)|$ are rotationally invariant, without loss of generality we may assume that $a_2 \geq 0$. Thus, by (2.3) we assume that $c := c_1 \in [0,2]$, i.e., in view of (1.11) that $d_1 \in [0,1]$. By (1.9), (2.3) and (1.11), we get

$$\begin{aligned} H_{2,1}(F_{f^{-1}}/2) &= \Gamma_1\Gamma_3 - \Gamma_2^2 \\ &= \frac{1}{4}\left(a_2a_4 - a_3^2 - a_2^2a_3 + \frac{13}{12}a_2^4\right) \\ &= \frac{\alpha^2(1+2\lambda)^2}{2304}\Theta, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} \Theta :=& -8(2+d_1^2)(1-d_1^2)d_2^2 - 4T(1-d_1^2)d_1^2d_2 \\ &+ 24(1-d_1^2)(1-|d_2|^2)d_1d_3 + (8+S)d_1^4 \end{aligned} \tag{3.3}$$

for some $d_1 \in [0,1]$ and $d_2, d_3 \in \overline{\mathbb{U}}$.

I. Assume first that $T = \alpha(1+10\lambda) \leq 2$. Then, by applying the triangle inequality to (3.3) and by the fact that $|d_2| \leq 1$ and $|d_3| \leq 1$, we obtain

$$|\Theta| \leq (S-4T)d_1^4 + 4(T-2)d_1^2 + 16. \tag{3.4}$$

Since $S - 4T < 0$ and $T \leq 2$, by (3.4) we have

$$|\Theta| \leq 16,$$

which together with (3.2) shows the first inequality in (3.1).

II. Next assume that $T = \alpha(1+10\lambda) > 2$.

Case 1. Suppose that $d_1 = 1$. Then by (3.2) and (3.3), we obtain

$$\left|\Gamma_1\Gamma_3 - \Gamma_2^2\right| = \frac{\alpha^2(1+2\lambda)^2}{2304}(8+S).$$

Case 2. Suppose that $d_1 = 0$. Then by (3.2) and (3.3), we obtain

$$\left|\Gamma_1\Gamma_3 - \Gamma_2^2\right| = \frac{\alpha^2(1+2\lambda)^2}{144}|d_2|^2 \leq \frac{\alpha^2(1+2\lambda)^2}{144}.$$

Case 3. Suppose that $d_1 \in (0, 1)$. Since $|d_3| \leq 1$, by applying the triangle inequality to (3.2) we can write

$$\left|\Gamma_1\Gamma_3 - \Gamma_2^2\right| \leq \frac{\alpha^2(1+2\lambda)^2 d_1(1-d_1^2)}{96}\left[\left|A + Bd_2 + Cd_2^2\right| + 1 - |d_2|^2\right], \quad (3.5)$$

where

$$A := -\frac{8+S}{24(1-d_1^2)}d_1^3, \quad B := \frac{T}{6}d_1, \quad C := \frac{d_1^2+2}{3d_1}.$$

Since $AC < 0$, we apply Lemma 2 only for the case II.

We have

$$2 < T = \alpha(1+10\lambda) \leq 11,$$

and therefore

$$\frac{4}{9} < S = \alpha^2(44\lambda^2 + 4\lambda - 9) \leq 39$$

for $\lambda \in [1/2, 1]$ and $\alpha \in (0, 1]$.

We consider the following sub-cases.

3(a)

Note that

$$\begin{aligned}
|B| - 2(1 - |C|) &= \frac{T}{6}d_1 - 2\left(1 - \frac{d_1^2+2}{3d_1^2}\right) \\
&= \frac{1}{6d_1}\left(4d_1^2 - 12d_1 + Td_1^2 + 8\right) \\
&= \frac{1}{6d_1}\left(4(1-d_1)(2-d_1) + Td_1^2\right) > 0.
\end{aligned}$$

Therefore, $|B| < 2(1-|C|)$ does not hold for $d_1 \in (0, 1)$, $\lambda \in [1/2, 1]$ and $\alpha \in (0, 1]$.

3(b) We can easily see that

$$\begin{aligned}
\frac{1}{C^2} - 1 &= \frac{9d_1^2}{(d_1^2+2)^2} - 1 \\
&= -\frac{1}{(d_1^2+2)^2}(d_1^4 - 5d_1^2 + 4) \\
&= -\frac{1}{(d_1^2+2)^2}(1-d_1^2)(4-d_1^2) < 0,
\end{aligned}$$

which yields

$$-4AC\left(\frac{1}{C^2}-1\right)<0.$$

Therefore, the inequality

$$B^2<\min\left\{4(1+|C|)^2,-4AC\left(\frac{1}{C^2}-1\right)\right\}$$

is false for $d_1\in(0,1)$, $\lambda\in[1/2,1]$ and $\alpha\in(0,1]$.

3(c) Since

$$\begin{aligned}4|C|-|B| &= 4\frac{d_1^2+2}{3d_1}-\frac{T}{6}d_1\\ &=\frac{1}{6}((8-T)d_1^2+16)\\ &\geq\frac{1}{6}(16-3d_1^2)>0,\end{aligned}$$

we have

$$|C|(|B|+4|A|)-|AB|=|BC|+|A|(4|C|-|B|)>0.$$

Consequently, the inequality $|C|(|B|+4|A|)\leq|AB|$ does not hold for $d_1\in(0,1)$, $\lambda\in[1/2,1]$ and $\alpha\in(0,1]$.

3(d) We can write

$$\begin{aligned}|AB|-|C|(|B|-4|A|) &= \frac{(8+S)T}{144(1-d_1^2)}d_1^4-\frac{d_1^2+2}{3d_1}\left(\frac{T}{6}d_1-\frac{4(8+S)}{24(1-d_1^2)}d_1^3\right)\\ &=\frac{1}{144(1-t)}(Pt^2+Qt+R),\end{aligned}$$

where $t:=d_1^2\in(0,1)$ and

$$P:=64+8S+16T+ST,\quad Q:=128+8T+16S,\quad R:=-16T.$$

It is easy to see that $P>0$, $Q>0$ and $R<0$ for $\lambda\in[1/2,1]$ and $\alpha\in(0,1]$.

For the equation $Pt^2+Qt+R=0$, we have $\Delta>0$. Since

$$\frac{R}{P}<0\quad\text{and}\quad P+Q+R>0,$$

for $\lambda\in[1/2,1]$ and $\alpha\in(0,1]$, the equation $Pt^2+Qt+R=0$ has a unique positive root $t_1<1$. Thus, the inequality $|AB|-|C|(|B|-4|A|)\leq0$ holds for $(0,d_1^{**}]$, where $d_1^{**}:=\sqrt{t_1}$. So we can write from (3.5) and Lemma 2,

$$\begin{aligned}\left|\Gamma_1\Gamma_3-\Gamma_2^2\right| &\leq\frac{\alpha^2(1+2\lambda)^2d_1(1-d_1^2)}{96}(-|A|+|B|+|C|)\\ &=\frac{\alpha^2(1+2\lambda)^2}{2304}\Phi(d_1),\end{aligned}$$

where

$$\Phi(x) := (-16 - 4T - S)x^4 + 4(T - 2)x^2 + 16, \quad x \in [0, 1].$$

We note that $\Phi'(x) = 4(-16-4T-S)x^3 + 8(T-2)x = 0$ for $x \in (0, 1)$ holds only for

$$x = \sqrt{\frac{2(T-2)}{16 + 4T + S}} =: \xi, \tag{3.6}$$

in the case when $T > 2$, i.e., for $\alpha(1 + 10\lambda) > 2$. Clearly $0 < \xi < 1$ and the function Φ attains at ξ its maximum value on [0, 1]. Therefore, in the case when $0 < \xi \le d_1^{**}$ we have

$$\begin{aligned}\left|\Gamma_1\Gamma_3 - \Gamma_2^2\right| &\le \frac{\alpha^2(1+2\lambda)^2}{2304}\Phi(\xi) \\ &= \frac{\alpha^2(1+2\lambda)^2}{2304}\left[(-16 - 4T - S)\xi^4 + 4(T-2)\xi^2 + 16\right] \\ &= \frac{\alpha^2(1+2\lambda)^2}{576}\left(4 + \frac{(T-2)^2}{16 + 4T + S}\right).\end{aligned}$$

3(e) Next consider the case $x \in [d_1^{**}, 1)$. Using the last case of the Lemma 2,

$$\begin{aligned}\left|\Gamma_1\Gamma_3 - \Gamma_2^2\right| &\le \frac{\alpha^2(1+2\lambda)^2 d_1(1-d_1^2)}{96}\left((|A| + |C|)\sqrt{1 - \frac{B^2}{4AC}}\right) \\ &= \frac{\alpha^2(1+2\lambda)^2}{2304} h_1(d_1^2)\sqrt{h_2(d_1^2)},\end{aligned}$$

where for $x \in [0, 1]$,

$$h_1(x) := Sx^2 - 8x + 16 \quad \text{and} \quad h_2(x) := 1 + \frac{T^2(1-x)}{2(S+8)(x+2)}.$$

It is easy to see that h_2 is a positive decreasing function in $[d_1^{**}, 1)$.

i) If $S \le 4$, then h_1 is a positive decreasing function in $[d_1^{**}, 1)$. Hence,

$$\begin{aligned}P\xi^4 + Q\xi^2 + R =& P\left(\frac{2(T-2)}{16+4T+S}\right)^2 + Q\left(\frac{2(T-2)}{16+4T+S}\right) + R \\ =& \frac{4}{(16+4T+S)^2}\Big[T^3(S-32) + 8T^2(S-28) + 4T(S^2 - 9S - 208) \\ & -16S^2 - 352S - 1792\Big] < 0.\end{aligned}$$

Therefore by Part 3(d), it follows that $0 < \xi < d_1^{**}$. Since $h_1\sqrt{h_2}$ is decreasing in $[d_1^{**}, 1)$ and as easy to check $h_1(d_1^{**})\sqrt{h_2(d_1^{**})} = \Phi(d_1^{**})$, we get

$$\begin{aligned} \left|\Gamma_1\Gamma_3 - \Gamma_2^2\right| &\leq \frac{\alpha^2(1+2\lambda)^2}{2304} h_1(d_1^{**})\sqrt{h_2(d_1^{**})} \\ &= \frac{\alpha^2(1+2\lambda)^2}{2304}\Phi(d_1^{**}) \\ &\leq \frac{\alpha^2(1+2\lambda)^2}{2304}\Phi(\xi). \end{aligned}$$

ii) When $4 < S \leq 39$ and $2 < T \leq 11$, we can write

$$H(d_1^2) = h_1(d_1^2)\sqrt{h_2(d_1^2)},$$

where H is the function defined in Lemma 3. Since by Lemma 3 the function H is convex, we deduce that

$$H(d_1^2) \leq \max\{H(d_1^{**}), H(1)\} = \max\{H(d_1^{**}), 8 + S\}.$$

Suppose that $S \leq 2(\sqrt{2T^2 + 8T + 40} - T - 2)$, i.e., that $S^2 \leq -4ST - 8S + 4T^2 + 16T + 144$. Then

$$\begin{aligned} P\xi^4 + Q\xi^2 + R =& \frac{4}{(16+4T+S)^2}\Big[4S^2T - 16S^2 + ST^3 + 8ST^2 - 36ST \\ & -352S - 32T^3 - 224T^2 - 832T - 1792\Big] \\ \leq& \frac{4}{(16+4T+S)^2}\Big[4T(-4ST - 8S + 4T^2 + 16T + 144) \\ & - 16S^2 + ST^3 + 8ST^2 - 36ST - 352S - 32T^3 \\ & -224T^2 - 832T - 1792\Big] \\ =& \frac{4}{(16+4T+S)^2}\Big[-16S^2 + ST^3 - 8ST^2 - 68ST - 352S \\ & -16T^3 - 160T^2 - 256T - 1792\Big] < 0, \end{aligned}$$

which by Part 3(d) yields $0 < \xi < d_1^{**}$. Since then $\Phi(\xi) \geq 8 + S$, we get

$$H(d_1^2) \leq \max\{H(d_1^{**}), H(1)\} = \max\{H(d_1^{**}), 8 + S\} \leq \Phi(\xi).$$

Suppose that $S > 2(\sqrt{2T^2 + 8T + 40} - T - 2)$. Then,

$$H(d_1^{**}) = \Phi(d_1^{**}) \leq \Phi(\xi) \leq 8 + S$$

and hence

$$H(d_1^2) \leq \max\{H(d_1^{**}), H(1)\} = \max\{H(d_1^{**}), 8 + S\} = 8 + S.$$

Summarizing parts from Case 1–3, it follows (3.1).

In order to show that the inequalities are sharp, first let $f \in \mathscr{F}_O(\lambda, \alpha)$ be defined by (2.2) with

$$p(z) := \frac{1 + z^2}{1 - z^2} = 1 + \sum_{k=1}^{\infty} 2z^{2k}, \qquad (z \in \mathbb{U}),$$

Then, in view of (2.3) we have

$$a_2 = 0, \quad a_3 = \frac{\alpha(1 + 2\lambda)}{6}, \quad a_4 = 0.$$

Thus, from (3.2) we get

$$\begin{aligned} \left|H_{2,1}(F_{f^{-1}}/2)\right| &= \left|\frac{13}{48}a_2^4 - \frac{1}{4}a_2^2a_3 + \frac{1}{4}a_4a_2 - \frac{1}{4}a_3^2\right| \\ &= \frac{\alpha^2(1 + 2\lambda)^2}{144}, \end{aligned}$$

which shows that the first bound in (3.1) is sharp.

Next let $f \in \mathscr{F}_O(\lambda, \alpha)$ be defined by (2.2) with

$$p(z) := \frac{1 + 2\xi z + z^2}{1 - z^2}, \qquad (z \in \mathbb{U}),$$

where ξ given by (3.6). Then in view of (2.3) we have

$$\begin{aligned} a_2 =& \frac{\alpha(1 + 2\lambda)}{2}\xi, \\ a_3 =& \frac{\alpha(1 + 2\lambda)}{6}\left(1 + (2\alpha + 2\alpha\lambda - 1)\xi^2\right), \\ a_4 =& \frac{\alpha(1 + 2\lambda)\xi}{72}\left[(8 - 21\alpha + 16\alpha^2 - 18\alpha\lambda + 30\alpha^2\lambda + 12\alpha^2\lambda^2)\xi^2\right. \\ & \left. + 21\alpha + 18\alpha\lambda - 6\right]. \end{aligned}$$

Thus, from (3.2) we get

$$\begin{aligned} \left|H_{2,1}(F_{f^{-1}}/2)\right| &= \left|\frac{13}{48}a_2^4 - \frac{1}{4}a_2^2a_3 + \frac{1}{4}a_4a_2 - \frac{1}{4}a_3^2\right| \\ &= \frac{\alpha^2(1 + 2\lambda)^2}{576}\left(4 + \frac{(T - 2)^2}{16 + 4T + S}\right), \end{aligned}$$

which shows that the second bound in (3.1) is sharp.

Finally, let $f \in \mathscr{F}_O(\lambda, \alpha)$ be defined by (2.2) with

$$p(z) := \frac{1+2z+z^2}{1-z^2}, \qquad (z \in \mathbb{U}).$$

Then, in view of (2.3) we have

$$\begin{aligned} a_2 =& \frac{\alpha(1+2\lambda)}{2}, \quad a_3 = \frac{\alpha^2}{3}(1+\lambda)(1+2\lambda) \\ a_4 =& \frac{\alpha(1+2\lambda)}{36}(6\alpha^2\lambda^2 + 15\alpha^2\lambda + 8\alpha^2 + 1). \end{aligned}$$

Thus, from (3.2) we get

$$\begin{aligned} \left|H_{2,1}(F_{f^{-1}}/2)\right| &= \left|\frac{13}{48}a_2^4 - \frac{1}{4}a_2^2 a_3 + \frac{1}{4}a_4 a_2 - \frac{1}{4}a_3^2\right| \\ &= \frac{\alpha^2(1+2\lambda)^2}{2304}(8+S), \end{aligned}$$

which shows that the third bound in (3.1) is sharp.

In [33], it was shown that the bounds of $H_{2,2}(f)$ and $H_{2,2}(f^{-1})$ for the convex functions of order alpha were the same, reflecting other invariant properties related to the coefficients f and f^{-1}. Sim et al. [24] improved these bounds to achieve sharp bounds. The following result shows that for $\lambda = 1/2$, i.e., for the class $\mathscr{K}_c(\alpha)$, the sharp bounds for the second Hankel determinant of logarithmic coefficients $H_{2,1}(F_f/2)$, given in Corollary 1, and $H_{2,1}(F_{f^{-1}}/2)$ are also the same.

Corollary 2 *Let* $\alpha \in (0, 1]$. *If* $f \in \mathscr{K}_c(\alpha)$, *then*

$$\left|\Gamma_1\Gamma_3 - \Gamma_2^2\right| \le \begin{cases} \dfrac{\alpha^2}{36}, & 0 < \alpha \le \dfrac{1}{3}, \\ \dfrac{\alpha^2(13\alpha^2 + 18\alpha + 17)}{144(\alpha^2 + 6\alpha + 4)}, & \dfrac{1}{3} < \alpha \le 1. \end{cases}$$

The inequalities are sharp.

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