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Notes on Anisotropic Liouville-type Theorems for 3D Stationary Nematic Liquid Crystal Equations

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Abstract
In this note, we study the steady incompressible nematic liquid crystal flows in $\mathbb{R}^3$. We establish the anisotropic Liouville-type theorems when a smooth solution (u, d) satisfies some suitable conditions.

Keywords Liouville-type theorem · Stationary incompressible nematic liquid crystal equations

Mathematics Subject Classification 35Q35 · 35B53

1 Introduction

We study the Liouville-type problem for the following simplified version of Ericksen–Leslie system modeling the hydrodynamic flow of stationary incompressible nematic liquid crystals in $\mathbb{R}^3$ [7, 14]

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$$\begin{cases} u\cdot\nabla u-\Delta u+\nabla P=-\mathrm{div}(\nabla d\odot\nabla d),\\ u\cdot\nabla d=\Delta d+|\nabla d|^2d,\\ \mathrm{div}u=0,\quad |d|=1, \end{cases} \tag{1.1}$$

where $u(x)$ is the velocity field, $P(x)$ is the pressure function and $d(x)$ is the macroscopic average of the nematic liquid crystal orientation field. The nonlinear super-critical term $\nabla_i d\odot\nabla_j d=\sum_{k=1}^3\frac{\partial d_k}{\partial x_i}\cdot\frac{\partial d_k}{\partial x_j}$ and the additional uniform decays condition as follows

$$|u(x)|+|\nabla d(x)|\to 0,\ as\ |x|\to\infty. \tag{1.2}$$

There are lots of studies on the global well-posedness and regularity criterion for nematic liquid crystal equations in [15, 16], and the system (1.1) can be regarded as the Navier–Stokes equations coupled with the transported harmonic map heat flows. When $\nabla d=0$, the system (1.1) reduces to the stationary incompressible Navier–Stokes equations. About the Liouville-type theorems for the stationary incompressible Navier-Stokes equations, there is a well-known result that is given in the book [8] of Galdi who has proved that a smooth solution $u=0$ if $u\in L^{\frac{9}{2}}(\mathbb{R}^3)$. Later, Chae [2] improved the condition that as long as $\Delta u\in L^{\frac{6}{5}}(\mathbb{R}^3)$. And Seregin [17] also obtained an improved condition $u\in L^6(\mathbb{R}^3)\cap BMO^{-1}$. Chae-Wolf [3] showed a logarithmic improvement of Galdi's work, assuming that

$$N(u):=\int_{\mathbb{R}^3}|u|^{\frac{9}{2}}\{\ln(2+1/|u|)\}^{-1}\mathrm{d}<\infty.$$

Recently, Chae [5] first proved that the smooth solution is trivial by anisotropic integrability condition on the components of the velocity. Relative to previous isotropic results, anisotropic integrability condition means that the condition itself changes when the coordinate axis is rotated with respect to the origin. For stationary incompressible nematic liquid crystal equations, Hao et al. [9] obtained the Liouville-type theorems for incompressible liquid crystal flow when u and ∇d satisfy the Galdi's [8] condition, or some decay conditions. And Jarrin in article [10] extended the Liouville results to the local Morrey spaces. For more interesting works on Liouville-type problems, we refer to [1, 3, 4, 13] and the references therein.

We will introduce some notations.

$$\tilde{x}_1\triangleq(x_2,x_3),\ \ \tilde{x}_2\triangleq(x_3,x_1)\ \tilde{x}_3\triangleq(x_1,x_2)$$

and for a region $\Omega\subseteq\mathbb{R}^2$ we represent

$$\int_\Omega g\mathrm{d}_{\tilde{x}_1}=\int_\Omega g\mathrm{d}_{x_2}\mathrm{d}_{x_3},\ \int_\Omega g\mathrm{d}_{\tilde{x}_2}=\int_\Omega g\mathrm{d}_{x_3}\mathrm{d}_{x_1},\ \int_\Omega g\mathrm{d}_{\tilde{x}_3}=\int_\Omega g\mathrm{d}_{x_1}\mathrm{d}_{x_2}.$$

Given $0 < p, q \leq \infty$, $i \in \{1, 2, 3\}$, we write $g \in L^p_{x_i} L^q_{\tilde{x}_i}$ when

$$\|g\|_{L^p_{x_i} L^q_{\tilde{x}_i}} \triangleq \left\{ \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |g|^q \mathrm{d}_{\tilde{x}_i} \right)^{\frac{p}{q}} \mathrm{d}x_i \right)^{\frac{1}{p}} < \infty$$

for $0 < p, q < \infty$ with obvious extensions to the case $p = +\infty$ or $q = +\infty$.

Now, for incompressible nematic liquid crystal equations in $\mathbb{R}^3$, we address sufficient conditions to prove a Liouville theorem and state the main results as follows:

Theorem 1.1 *Let (u, d) be a smooth solution of the stationary nematic liquid crystal equations (1.1). Suppose*

$$(u, \nabla d) \in L^6(\mathbb{R}^3) \cap L^r(\mathbb{R}^3), \tag{1.3}$$

and

$$u_i \in L^{\frac{r}{r-2}}_{x_i} L^s_{\tilde{x}_i}(\mathbb{R} \times \mathbb{R}^2), \ \forall i = 1, 2, 3 \tag{1.4}$$

with r, s satisfying

$$\frac{2}{r} + \frac{1}{s} \geq \frac{1}{2}, \ 2 < r < \infty, 1 \leq s \leq \infty \tag{1.5}$$

then $u = 0$ and d satisfies the harmonic map equation of

$$\Delta d + |\nabla d|^2 d = 0. \tag{1.6}$$

Theorem 1.2 *Let d: $\mathbb{R}^3 \to \mathbb{S}^2$ be a solution of (1.6) and $\nabla d \in L^r(\mathbb{R}^3)$, then d is a constant map on $\mathbb{R}^3$.*

Remark 1.1 For the harmonic maps equation in (1.6), many scholars studied it from geometric perspective, e.g. Choi [6] proved Liouville theorem under the image of the harmonic map d lies in a hemisphere. Schoen and Uhlenbeck [18] established Liouville theorems for minimizing harmonic maps into the Euclidean sphere. Jin [12] proved that d is a constant map if d approaches a constant at infinity, which inspired us that ∇d has a faster decay at infinity, d may be a constant map.

Remark 1.2 In particular case that $r = s = 6$, the assumption of the above u_i reduces to

$$u_i \in (L^{\frac{3}{2}}_{x_i} \cap L^6_{x_i}) L^6_{\tilde{x}_i}(\mathbb{R} \times \mathbb{R}^2), \ \forall i = 1, 2, 3$$

which means a mild decay in horizontal direction combined with the faster decay in vertical direction, then the smooth solution is a trivial solution.

Remark 1.3 The smooth solution can be replaced by weak solution in hypothesis of Theorems 1.1 and 1.2. O.Jarrin showed in [11] that the weak solution $(u, \nabla d) \in L^p$ of (1.1) becomes smooth solution if $p > 3$.

Remark 1.4 Compared with Liouville theorem to incompressible Navier–Stokes equations in [5], the study of incompressible nematic liquid crystal equations is much more complicated due to the transported harmonic map heat flows. Our results in this note improve the Chae's results in [5] when $\nabla d = 0$ in (1.1).

2 The Proofs of Theorems

Proof of Theorem 1.1 We employ the index sets as follows:

$$\mathfrak{I}_1 = \{2, 3\},\ \mathfrak{I}_2 = \{1, 3\},\ \mathfrak{I}_3 = \{1, 2\}.$$

We choose a smooth non-increasing real valued function ϕ satisfying

$$\phi(s) = \begin{cases} 1,\ 0 \le s \le 1, \\ 0, \qquad s \ge 4, \end{cases}$$

and $\phi(s) \in [0, 1]$ for $s \ge 0$. For each $R > 0$, we define radial cut-off functions

$$\eta_R(x) = \prod_{j=1}^{3} \phi(\frac{x_j^2}{R^2}),$$

$$\bar{\eta}_{i,R}(x) = \prod_{j \in \mathfrak{I}_i} \phi(\frac{x_j^2}{R^2}),\ i = \{1, 2, 3\},$$

and

$$\tilde{D}_i \triangleq \{\tilde{x}_i \in \mathbb{R}^2 | |x_j| < 2R,\ \forall j \in \mathfrak{I}_i\},\ i = \{1, 2, 3\}.$$

Multiplying $(1.1)_1$, $(1.1)_2$ by $u\eta_R(x)$, $(\Delta d + |\nabla d|^2 d)\eta_R(x)$ respectively, and integrating over $\mathbb{R}^3$, we get

$$\begin{aligned} &\int_{\mathbb{R}^3} u \cdot \nabla u \cdot u\eta_R(x)\mathrm{d}x - \int_{\mathbb{R}^3} \Delta u \cdot u\eta_R(x)\mathrm{d}x + \int_{\mathbb{R}^3} \nabla P \cdot u\eta_R(x)\mathrm{d}x \\ &+ \int_{\mathbb{R}^3} \mathrm{div}(\nabla d \odot \nabla d)u\eta_R(x)\mathrm{d}x \\ &+ \int_{\mathbb{R}^3} |\Delta d + |\nabla d|^2 d|^2 \eta_R(x)\mathrm{d}x - \int_{\mathbb{R}^3} (u \cdot \nabla d)(\Delta d + |\nabla d|^2 d)\eta_R(x)\mathrm{d}x = 0. \end{aligned} \tag{2.1}$$

Using (A), (B), (C), (D), (E) and (F) to represent the terms on the left-hand side of (2.1), we will estimate them as follows.

2.1. Estimate of (A)
By integrating by parts and $(1.1)_3$, we can get

$$
\begin{aligned}
(A) &= \int_{\mathbb{R}^3} u \cdot \nabla \frac{|u|^2}{2} \eta_R(x) \mathrm{d}x \\
&= -\frac{1}{2} \int_{\mathbb{R}^3} |u|^2 \cdot \operatorname{div}(u \eta_R(x)) \mathrm{d}x \\
&= -\frac{1}{2} \int_{\mathbb{R}^3} |u|^2 u \cdot \nabla \eta_R(x) \mathrm{d}x.
\end{aligned}
$$

2.2. Estimate of (B)
By integrating by parts and $(1.1)_3$, we compute that

$$
\begin{aligned}
(B) &= \int_{\mathbb{R}^3} \nabla u : \nabla(u \eta_R(x)) \mathrm{d}x \\
&= \int_{\mathbb{R}^3} |\nabla u|^2 \cdot \eta_R(x) \mathrm{d}x + \int_{\mathbb{R}^3} \nabla u : (u \otimes \nabla \eta_R(x)) \mathrm{d}x \\
&=: (B_1) - \int_{\mathbb{R}^3} u \cdot \operatorname{div}(u \otimes \nabla \eta_R(x)) \mathrm{d}x \\
&=: (B_1) - \int_{\mathbb{R}^3} u \cdot \nabla u \cdot \nabla \eta_R(x)) \mathrm{d}x - \int_{\mathbb{R}^3} |u|^2 \Delta \eta_R(x)) \mathrm{d}x \\
&=: (B_1) - \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 \cdot \Delta \eta_R(x) \mathrm{d}x \\
&=: (B_1) + (B_2).
\end{aligned}
$$

2.3. Estimate of (C)
By integrating by parts and $(1.1)_3$, we show that

$$
\begin{aligned}
(C) &= -\int_{\mathbb{R}^3} P \operatorname{div}(u \eta_R(x)) \mathrm{d}x \\
&= -\int_{\mathbb{R}^3} P u \cdot \nabla \eta_R(x) \mathrm{d}x.
\end{aligned}
$$

2.4. Estimate of (D)
By integrating by parts and $(1.1)_3$, we obtain that

$$
\begin{aligned}
(D) &= \int_{\mathbb{R}^3} u_i \nabla_j (\nabla_i d \cdot \nabla_j d) \eta_R(x) \mathrm{d}x \\
&= \int_{\mathbb{R}^3} u_i (\nabla_i \nabla_j d \cdot \nabla_j d + \nabla_i d \cdot \nabla_j^2 d) \eta_R(x) \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \frac{1}{2} u_i \nabla_i |\nabla d|^2 \eta_R(x) + (u \cdot \nabla d) \cdot \Delta d \eta_R(x) \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= - \int_{\mathbb{R}^3} \frac{1}{2} |\nabla d|^2 u \cdot \nabla \eta_R(x) \mathrm{d}x + \int_{\mathbb{R}^3} (u \cdot \nabla d) \cdot \Delta d \eta_R(x) \mathrm{d}x \\
&=: (D_1) + (D_2).
\end{aligned}
$$

2.5. Estimate of (F)

Recalling that $|d| = 1$, we can easily get that

$$
\begin{aligned}
(F) &= - \int_{\mathbb{R}^3} u \cdot \nabla d \cdot \Delta d \eta_R(x) \mathrm{d}x - \int_{\mathbb{R}^3} u \cdot \nabla d \cdot |\nabla d|^2 d \eta_R(x) \mathrm{d}x \\
&= - \int_{\mathbb{R}^3} u \cdot \nabla d \cdot \Delta d \eta_R(x) \mathrm{d}x - \int_{\mathbb{R}^3} u \cdot \nabla \frac{|d|^2}{2} \cdot |\nabla d|^2 \eta_R(x) \mathrm{d}x \\
&= - \int_{\mathbb{R}^3} u \cdot \nabla d \cdot \Delta d \eta_R(x) \mathrm{d}x.
\end{aligned}
$$

It is obvious that $(D_2) + (F) = 0$. Putting the above estimates of (A), (B), (C), (D), (E) and (F) together in (2.1), we have

$$
\begin{aligned}
&- \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 u \cdot \nabla \eta_R(x) \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla u|^2 \eta_R(x) \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 \cdot \Delta \eta_R(x) \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} P u \cdot \nabla \eta_R(x) \mathrm{d}x - \int_{\mathbb{R}^3} \frac{1}{2} |\nabla d|^2 u \cdot \nabla \eta_R(x) \mathrm{d}x + \int_{\mathbb{R}^3} |\Delta d + |\nabla d|^2 d|^2 \eta_R(x) \mathrm{d}x = 0.
\end{aligned}
$$

Furthermore, we can write

$$
\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla u|^2 \eta_R(x) \mathrm{d}x + \int_{\mathbb{R}^3} |\Delta d + |\nabla d|^2 d|^2 \eta_R(x) \mathrm{d}x \\
&\quad = \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 u \cdot \nabla \eta_R(x) \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 \cdot \Delta \eta_R(x) \mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^3} P u \cdot \nabla \eta_R(x) \mathrm{d}x + \int_{\mathbb{R}^3} \frac{1}{2} |\nabla d|^2 u \cdot \nabla \eta_R(x) \mathrm{d}x \\
&\quad = \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 \cdot \Delta \eta_R(x) \mathrm{d}x + \int_{\mathbb{R}^3} Q u \cdot \nabla \eta_R(x) \mathrm{d}x \\
&\triangleq (G_1) + (G_2),
\end{aligned}
\tag{2.2}
$$

where Q is denoted by

$$
Q = \frac{1}{2} |u|^2 + \frac{1}{2} |\nabla d|^2 + P. \tag{2.3}
$$

We will estimate (G_1) and (G_2) separately in the following subsections.

2.6. Estimate of (G_1)

$$
\begin{aligned}
(G_1) &= \frac{1}{2}\sum_{i=1}^{3}\int_{\{R\le|x_i|\le 2R\}}\int_{\tilde{D}_i}|u|^2\bar{\eta}_{i,R}\{\frac{2}{R^2}\phi'(\frac{x_i^2}{R^2})+\frac{4x_i^2}{R^4}\phi''(\frac{x_i^2}{R^2})\}\mathrm{d}_{\tilde{x}_i}\mathrm{d}_{x_i}\\
&\le \frac{C}{R^2}\sum_{i=1}^{3}(\int_{\{R\le|x_i|\le 2R\}}\int_{\tilde{D}_i}|u|^6\mathrm{d}x)^{\frac{1}{3}}(\int_{\{R\le|x_i|\le 2R\}}\int_{\tilde{D}_i}1\mathrm{d}_{\tilde{x}_i}\mathrm{d}_{x_i})^{\frac{2}{3}}\\
&\le C\sum_{i=1}^{3}(\int_{\{R\le|x_i|\le 2R\}}\int_{\mathbb{R}^2}|u|^6\mathrm{d}x)^{\frac{1}{3}}.
\end{aligned}
$$

By letting $R\to+\infty$, we obtain

$$
\lim_{R\to+\infty}(G_1)=0. \tag{2.4}
$$

2.7. Estimate of (G_2)
Taking the divergence to the equation $(1.1)_1$, we have

$$
\Delta P=-\sum_{i,j=1}^{3}\partial_i\partial_j(u_iu_j+\partial_id^k\partial_jd^k),
$$

which can be written

$$
P=-\sum_{i,j=1}^{3}R_iR_j(u_iu_j+\partial_id^k\partial_jd^k)+g
$$

where $R_j=\frac{\partial_j}{\sqrt{-\Delta}}$ is the $j-th$ Riesz transform and $\Delta g=0$ in $\mathbb{R}^3$. Because of the continuity of the operator R_j and the condition $(u,\nabla d)\in L^6(\mathbb{R}^3)$, it holds that

$$
g=0,\ \ \|P\|_{L^r}\le\|u\|_{L^{2r}}^2+\|\nabla d\|_{L^{2r}}^2,\forall r\in(1,+\infty),
$$

which easily to get the estimate of Q that

$$
\|Q\|_{L^r}\le\|u\|_{L^{2r}}^2+\|\nabla d\|_{L^{2r}}^2. \tag{2.5}
$$

By employing the anisotropic condition to (G_2),

$$
\begin{aligned}
(G_2) &= \sum_{i=1}^{3}\frac{2}{R^2}|\int_{\mathbb{R}^3}x_i\bar{\eta}_{i,R}Qu_i\cdot\phi'\left(\frac{x_i^2}{R^2}\right)\mathrm{d}x|\\
&\le\frac{C}{R}\sum_{i=1}^{3}\int_{\{R<|x_i|<2R\}}\int_{\tilde{D}_i}|Q||u_i|\mathrm{d}\tilde{x}_i\mathrm{d}x_i
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{C}{R}\sum_{i=1}^{3}\int_{\{R<|x_i|<2R\}}\left(\int_{\tilde{D}_i}|Q|^{\frac{r}{2}}\mathrm{d}\tilde{x}_i\right)^{\frac{2}{r}}\left(\int_{\tilde{D}_i}|u_i|^{s}\mathrm{d}\tilde{x}_i\right)^{\frac{1}{s}}\left(\int_{\tilde{D}_i}1\mathrm{d}\tilde{x}_i\right)^{\frac{rs-r-2s}{rs}}\mathrm{d}x_i\\
&\leq CR^{\frac{2(rs-r-2s)}{rs}-1}\sum_{i=1}^{3}\int_{\{R<|x_i|<2R\}}\left(\int_{\tilde{D}_i}|Q|^{\frac{r}{2}}\mathrm{d}\tilde{x}_i\right)^{\frac{2}{r}}\left(\int_{\tilde{D}_i}|u_i|^{s}\mathrm{d}\tilde{x}_i\right)^{\frac{1}{s}}\mathrm{d}x_i\\
&\leq CR^{\frac{2(rs-r-2s)}{rs}-1}\\
&\times\sum_{i=1}^{3}\left(\int_{\{R<|x_i|<2R\}}\int_{\tilde{D}_i}|Q|^{\frac{r}{2}}\mathrm{d}\tilde{x}_i\mathrm{d}x_i\right)^{\frac{2}{r}}\left\{\int_{\{R<|x_i|<2R\}}\left(\int_{\tilde{D}_i}|u_i|^{s}\mathrm{d}\tilde{x}_i\right)^{\frac{r}{s(r-2)}}\mathrm{d}x_i\right\}^{\frac{r-2}{r}}\\
&\leq CR^{\frac{2(rs-r-2s)}{rs}-1}\left(\int_{\mathbb{R}^3}|Q|^{\frac{r}{2}}\mathrm{d}x\right)^{\frac{2}{r}}\sum_{i=1}^{3}\left\{\int_{\{R<|x_i|<2R\}}\left(\int_{\tilde{D}_i}|u_i|^{s}\mathrm{d}\tilde{x}_i\right)^{\frac{r}{s(r-2)}}\mathrm{d}x_i\right\}^{\frac{r-2}{r}}\\
&\leq CR^{\frac{2(rs-r-2s)}{rs}-1}\left\{\left(\int_{\mathbb{R}^3}|u|^{r}\mathrm{d}x\right)^{\frac{2}{r}}+\left(\int_{\mathbb{R}^3}|\nabla d|^{r}\mathrm{d}x\right)^{\frac{2}{r}}\right\}\\
&\quad\times\sum_{i=1}^{3}\left\{\int_{\{R<|x_i|<2R\}}\left(\int_{\mathbb{R}^2}|u_i|^{s}\mathrm{d}\tilde{x}_i\right)^{\frac{r}{s(r-2)}}\mathrm{d}x_i\right\}^{\frac{r-2}{r}},
\end{aligned}
$$

where we have employed the estimate of Q in the last inequality. Noting that the condition (1.5) holds for

$$\frac{2(rs-r-2s)}{rs}-1\leq 0.$$

Hence

$$\lim_{R\to+\infty}(G_2)=0. \tag{2.6}$$

Putting the estimates (G_1) and (G_2) into (2.2), we have

$$\int_{\mathbb{R}^3}|\nabla u|^2\eta_R(x)\mathrm{d}x+\int_{\mathbb{R}^3}|\Delta d+|\nabla d|^2 d|^2\eta_R(x)\mathrm{d}x=0,\ as\ R\to+\infty.$$

Hence, it holds that

$$\nabla u=0,$$

and

$$\Delta d+|\nabla d|^2 d=0. \tag{2.7}$$

Since $u\in L^6(\mathbb{R}^3)$, we have $u=0$ and finish the proof of Theorem 1.1. □

Proof of Theorem 1.2 Taking the inner product of (2.7) with $x \cdot \nabla d$ over $\mathbb{R}^3$, it leads to

$$\int_{\mathbb{R}^3} (x \cdot \nabla) d \cdot (\Delta d + |\nabla d|^2 d) \mathrm{d}x = 0. \tag{2.8}$$

By the fact

$$\int_{\mathbb{R}^3} (x \cdot \nabla) d |\nabla d|^2 d \mathrm{d}x = \int_{\mathbb{R}^3} \frac{1}{2} |\nabla d|^2 x \cdot \nabla |d|^2 \mathrm{d}x = 0,$$

which implies

$$\int_{\mathbb{R}^3} (x \cdot \nabla) d \Delta d \mathrm{d}x = \int_{\mathbb{R}^3} x_j \partial_j d_i \partial_k \partial_k d_i \mathrm{d}x = 0. \tag{2.9}$$

However, from the second equality in (2.9), it gets

$$\begin{aligned}
&\int_{\mathbb{R}^3} x_j \partial_j d_i \partial_k \partial_k d_i \mathrm{d}x \\
=& -\int_{\mathbb{R}^3} (\epsilon_{jk} \partial_j d_i \cdot \partial_k d_i + x_j \partial_k \partial_j d_i \partial_k d_i) \mathrm{d}x \\
=& -\int_{\mathbb{R}^3} |\nabla d|^2 \mathrm{d}x - \int_{\mathbb{R}^3} x_j \partial_j (\frac{1}{2} |\nabla d|^2) \mathrm{d}x \\
=& -\int_{\mathbb{R}^3} |\nabla d|^2 \mathrm{d}x + \frac{3}{2} \int_{\mathbb{R}^3} |\nabla d|^2 \mathrm{d}x \\
=& \frac{1}{2} \int_{\mathbb{R}^3} |\nabla d|^2 \mathrm{d}x.
\end{aligned} \tag{2.10}$$

where

$$\epsilon_{jk} = \begin{cases} 1, & x_j = x_k, \\ 0, & x_j \neq x_k, \end{cases}$$

Combining the (2.9) and (2.10), implies

$$\int_{\mathbb{R}^3} |\nabla d|^2 \mathrm{d}x = 0.$$

Hence $\nabla d \equiv 0$ and d is a constant map. This completes the proof of Theorem 1.2. □

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