Bull. Malays. Math. Sci. Soc. (2023) 46:182
https://doi.org/10.1007/s40840-023-01576-1

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Atomic Decomposition of Weighted Multi-parameter Mixed Hardy Spaces

Wei Ding[1] **· Fangli Zou**[1]

Received: 21 February 2023 / Revised: 24 June 2023 / Accepted: 21 August 2023 /
Published online: 6 September 2023

Abstract
It is well-known that atomic decomposition is an important tool to study the boundedness of some singular integral operators on Hardy spaces. Moreover, to study the boundedness of an operator in the Journé class, Fefferman R. builded a criterion by considering its action on rectangle atoms only. In this paper, we mainly establish atomic decomposition of multi-parameter mixed Hardy space which has been developed recently.

Keywords Mixed Hardy space · Weighted · Atomic decomposition

Mathematics Subject Classification 42B35 · 42B30 · 42B25 · 42B20

1 Introduction

Multi-parameter harmonic analysis containing multi-parameter function spaces and boundedness of operators have been extensively studied over the past decades. We refer readers to the work in [1, 2, 7–10, 12, 13, 15, 21–23, 25, 27, 29–32, 34–37, 39–47, 49–52].

The product Hardy space was first introduced in [25, 38]. Immediately after, Chang and Fefferman R. developed this theory in [4–6]. At the same time, Fefferman and

---

Communicated by Rosihan M. Ali.

Supported by NNSF of China Grants (12271322) .

✉ Wei Ding
dingwei@ntu.edu.cn

Fangli Zou
1950586160@qq.com

1 School of Sciences, Nantong University, Nantong 226007, People's Republic of China

Stein studied product convolution singular integral operators which satisfy analogous conditions of double Hilbert transform [18]. In [33], Journé generalized this result to product non-convolution singular integral operators and proved the $L^\infty \to BMO$ boundedness for such operators, which opened the door to prove the product H^p boundedness of operators in the Journé's class. Besides that, authors in [11, 37, 48] studied weighted multi-parameter Hardy spaces. For more results about multi-parameter Hardy spaces, we refer readers to [1–3, 11, 12, 20, 21, 25, 26, 28–31, 35–37, 46, 48].

Recently, the theory of multi-parameter mixed Hardy space has been developed in [14]. To be more precise, let $\psi_0^{(1)}, \psi^{(1)} \in \mathcal{S}(\mathbb{R}^{n_1})$ with

$$\operatorname{supp}\widehat{\psi_0^{(1)}} \subseteq \{\xi \in \mathbb{R}^{n_1} : |\xi| \le 2\}; \ \widehat{\psi_0^{(1)}}(\xi) = 1, \text{ if } |\xi| \le 1, \tag{1.1}$$

and

$$\operatorname{supp}\widehat{\psi^{(1)}} \subseteq \{\xi \in \mathbb{R}^{n_1} : \frac{1}{2} \le |\xi| \le 2\}, \tag{1.2}$$

and

$$|\widehat{\psi_0^{(1)}}(\xi)|^2 + \sum_{j=1}^{\infty} |\widehat{\psi^{(1)}}(2^{-j}\xi)|^2 = 1, \quad \text{for all } \xi \in \mathbb{R}^{n_1}. \tag{1.3}$$

Let $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^{n_2})$ with

$$\operatorname{supp}\widehat{\psi^{(2)}} \subseteq \{\xi \in \mathbb{R}^{n_2} : \frac{1}{2} \le |\xi| \le 2\}, \tag{1.4}$$

and

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi^{(2)}}(2^{-j}\xi)|^2 = 1, \quad \text{for all } \xi \in \mathbb{R}^{n_2} \setminus \{0\}. \tag{1.5}$$

Then, for $j, k \in \mathbb{Z}, j \ge 1$, set that $\psi_j^{(1)}(x) = 2^{jn_1}\psi^{(1)}(2^j x)$, $\psi_k^{(2)}(x) = 2^{kn_2}\psi^{(2)}(2^k x)$ and that $\psi_{j,k}(x, y) = \psi_j^{(1)}(x)\psi_k^{(2)}(y)$, $\psi_{0,k}(x, y) = \psi_0^{(1)}(x)\psi_k^{(2)}(y)$.

Denote that $\mathcal{S}_0(\mathbb{R}^{n_1+n_2}) = \{f \in \mathcal{S}(\mathbb{R}^{n_1+n_2}) : \int_{\mathbb{R}^{n_2}} f(x_1, x_2)x_2^\alpha dx_2 = 0, \ \forall\, |\alpha| \ge 0, \ \forall\, x_1 \in \mathbb{R}^{n_1}\}$. For $i = 1, 2$ and any $j \in \mathbb{Z}$, denote that $\Pi_j^{n_i} = \{I : I$ are dyadic cubes in $\mathbb{R}^{n_i}$ with the side length $l(I) = 2^{-j}$, and the left lower corners of I are $x_I = 2^{-j}\ell$, $\ell \in \mathbb{Z}^{n_i}\}$, $\Pi_{j,k} = \Pi_j^{n_1} \times \Pi_k^{n_2}$, and that $\Pi = \cup_{j,k\in\mathbb{Z}}\Pi_{j,k}$.

The following discrete multi-parameter Calderón's reproducing formula was obtained in [16]:

Theorem A Suppose that $\psi_0^{(1)}, \psi^{(1)} \in \mathcal{S}(\mathbb{R}^{n_1})$ and $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^{n_2})$ satisfy conditions in (1.1)–(1.3) and (1.4)–(1.5), respectively. Then

$$f(x_1, x_2) = \sum_{j\in\mathbb{N}, k\in\mathbb{Z}} \sum_{I\times J\in\Pi_{j,k}} |I||J|(\psi_{j,k} * f)(x_I, x_J) \times \psi_{j,k}(x_1 - x_I, x_2 - x_J), \quad (1.6)$$

where the series converges in $L^2(\mathbb{R}^{n_1+n_2})$, $\mathcal{S}_0(\mathbb{R}^{n_1+n_2})$ and $\mathcal{S}_0'(\mathbb{R}^{n_1+n_2})$, the dual space of $\mathcal{S}_0(\mathbb{R}^{n_1+n_2})$.

We recall some definitions of product weights in two parameter setting [24]. For $1 < p < \infty$, a nonnegative locally integrable function $\omega \in A_p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ if there exists a constant $C > 0$ such that

$$\Big(\frac{1}{|R|}\int_R \omega(x)\mathrm{d}x\Big)\Big(\frac{1}{|R|}\int_R \omega(x)^{-1/(p-1)}\mathrm{d}x\Big)^{p-1} < C$$

for any dyadic rectangle R, that is $R \in \Pi$. We say that $\omega \in A_1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ if there exists a constant $C > 0$ such that

$$M_s\omega(x) \le C\omega(x), \text{ a.e. } x \in \mathbb{R}^{n_1+n_2},$$

where M_s is the strong maximal function defined by

$$M_s f(x) = \sup_{x\in R\in\Pi} \frac{1}{|R|}\int_R |f(y)|\mathrm{d}y.$$

Finally, define that $\omega \in A_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ by

$$A_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \bigcup_{1\le p<\infty} A_p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}).$$

In this paper, the classical Muckenhoupt's weights on $\mathbb{R}^n$ is denoted by $A_p(\mathbb{R}^n)$.

Given a weight ω on $\mathbb{R}^n$, for $0 < r < \infty$, $L^r_\omega(\mathbb{R}^n)$ is defined by

$$L^r_\omega(\mathbb{R}^n) = \{f : \int_{\mathbb{R}^n} |f(x)|^r\omega(x)\mathrm{d}x < \infty\}.$$

Based on the discrete Calderón's identity (1.6), weighted mixed Hardy spaces are introduced in [14].

Definition 1.1 Let $0 < p < \infty$ and $\omega \in A_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Suppose that $\psi_0^{(1)}, \psi^{(1)} \in \mathcal{S}(\mathbb{R}^{n_1})$ and $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^{n_2})$ satisfy conditions in (1.1)–(1.3) and (1.4)–(1.5), respectively. The weighted multi-parameter mixed Hardy space $H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is defined to be the set of $f \in \mathcal{S}_0'(\mathbb{R}^{n_1+n_2})$ such that $\|f\|_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} = \|S(f)(x)\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})} < \infty$, where

$$S(f)(x) = \Big(\sum_{j\in\mathbb{N},k\in\mathbb{Z}} \sum_{I\times J\in\Pi_j^{n_1}\times\Pi_k^{n_2}} |\psi_{j,k} * f(x_I, x_J)|^2 \chi_I(x_1)\chi_J(x_2)\Big)^{\frac{1}{2}}. \quad (1.7)$$

It is well-known that atomic decomposition plays an important role in studying the boundedness of singular operators, and it is much more complicated in multi-parameter setting. In the present paper, we consider the atomic decomposition of $H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. The atoms of $H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ are defined as follows.

Definition 1.2 Let $0 < p \le 1$. A function $a(x_1, x_2)$ is said to be an atom for $H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ if it satisfies the following properties:

(1) $a(x_1, x_2)$ is supported in an open set $\Omega \subseteq \mathbb{R}^{n_1+n_2}$ with finite measure.
(2) $\|a\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})} \le \omega(\Omega)^{\frac{1}{2}-\frac{1}{p}}$.
(3) a can be further decomposed as

$$a = \sum_{R\subset\mathcal{M}(\Omega)} a_R$$

with

$$\sum_{R\subset\mathcal{M}(\Omega)} \|a_R\|^2_{L^2_\omega(\mathbb{R}^{n_1+n_2})} \lesssim \omega(\Omega)^{1-\frac{2}{p}}, \quad (1.8)$$

where a_R are named as rectangle-atoms associated with the dyadic rectangles $R = I \times J$, and supported in τR for some positive integer $\tau > 1$ independent of a and a_R, and $\mathcal{M}(\Omega)$ is the set of all maximal dyadic rectangles in Ω. Furthermore, a_R has the following vanishing moment conditions:

(i) in the x_2 direction,

$$\text{for a.e. } x_1 \in \tau I, \int a_R(x_1, x_2)x_2^\alpha \mathrm{d}x_2 = 0, 0 \le |\alpha| \le N_p^2 = \left[\frac{2n_2}{p} - n_2\right].$$

(ii) in the x_1 direction, there exists a positive constant ϱ, when $\ell(I) < \varrho$,

$$\text{for a.e. } x_2 \in \tau J, \int a_R(x_1, x_2)x_1^\beta \mathrm{d}x_1 = 0, 0 \le |\beta| \le N_p^1 = \left[\frac{2n_1}{p} - n_1\right].$$

Theorem 1.1 *Let $0 < p \le 1$, $\omega \in A_2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Then there exists a constant $c_{p,n_1,n_2,\omega}$, such that for all $H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ atoms a,*

$$\|a\|_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \le c_{p,n_1,n_2,\omega}.$$

It has been proved in [14] that $L^2_\omega(\mathbb{R}^{n_1+n_2}) \cap H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is dense in $H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ for all $0 < p < \infty$. Once we obtain the atomic decomposition in this dense set, it is easy to generalize to whole space.

Theorem 1.2 *Let* $0 < p \leq 1$ *and* $\omega \in A_2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. *Then* $f \in H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ *if and only if there exist a sequence* $\{a_k\}$ *of* $H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ *atoms and a sequence* $\{\lambda_k\}$ *of real numbers satisfying* $\sum_{k\in\mathbb{Z}} |\lambda_k|^p \leq C\|f\|^p_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$ *such that*

$$f(x) = \sum_{k\in\mathbb{Z}} \lambda_k a_k(x), \text{ in } H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}),$$

where the series also converges to f *in* $L^2_\omega(\mathbb{R}^{n_1+n_2})$.

One will see that the above atomic decomposition theorem is very useful to prove the boundedness of some kinds of singular operators, such as multi-pseudodifferential operators, and those in mixed Journé class. It will be exhibited in our following papers. As a direct application of Theorem 1.1 and Theorem 1.2, we have the following results.

Theorem 1.3 *Let* $0 < p \leq 1$ *and* $\omega \in A_2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. *Assume that* T *is a linear singular integral operator bounded on* $L^2_\omega(\mathbb{R}^{n_1+n_2})$. *Then* T *is bounded on* $H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ *if and only if*

$$\sup\{\|T(a)\|_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} : a \text{ is any atom of } H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})\} < \infty. \quad (1.9)$$

We now describe the strategy of this paper. In Section 2, we mainly prove the uniform boundedness of atoms on $H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Comparison with the unweighted case, the uniform boundedness of atoms on weighted mixed Hardy spaces is much more involved. In Section 3, we establish atomic decomposition on $H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. The proof of Theorem 1.3 is also placed in this section.

2 The Uniform Boundedness of Atoms

In this section, we mainly discuss the uniform boundedness of atoms on $H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ when $0 < p \leq 1$. First of all, let's recall a key theorem discovered by Journé. For this, given any open set $\Omega \subseteq \mathbb{R}^{n_1+n_2}$, denote that $\mathcal{M}_1(\Omega)$ the collection of all dyadic rectangles $R = I \times J \subseteq \Omega$, $R \in \Pi$, which are maximal in the x_1 direction. Define $\mathcal{M}_2(\Omega)$ similarly. It is easy to see that $\mathcal{M}(\Omega) \subseteq \mathcal{M}_1(\Omega)$ and $\mathcal{M}(\Omega) \subseteq \mathcal{M}_2(\Omega)$.

Define that

$$\widetilde{\Omega} = \left\{x \in \mathbb{R}^{n_1+n_2} : M^\omega_s(\chi_\Omega)(x) > \frac{1}{(10\tau)^{n_1+n_2}}\right\},$$

and similarly for $\widetilde{\widetilde{\Omega}}$, $\widetilde{\widetilde{\widetilde{\Omega}}}$. Here, $M^\omega_s(g)(x)$ is a weighted strong Maximal function defined by

$$M^\omega_s(g)(x) = \sup_{x\in R} \frac{1}{\omega(R)} \int_R |g(y)|\omega(y)\mathrm{d}y.$$

Obviously, $\Omega \subseteq \widetilde{\Omega} \subseteq \widetilde{\widetilde{\Omega}} \subseteq \widetilde{\widetilde{\widetilde{\Omega}}}$. By the weighted strong maximal theorem (see [17]), one has $\omega(\widetilde{\widetilde{\widetilde{\Omega}}}) \lesssim \omega(\Omega)$.

For any $R \in \mathcal{M}(\Omega)$, let $\tilde{I} \subseteq \mathbb{R}^{n_1}$ be the largest dyadic cube containing I such that $\tilde{R} = \tilde{I} \times J \subseteq \widetilde{\Omega}$, and $\tilde{J}$ be the largest dyadic cube containing J such that $\tilde{\tilde{R}} = \tilde{I} \times \tilde{J} \subseteq \widetilde{\widetilde{\Omega}}$. Define $\gamma = \gamma(R) = \frac{\ell(\tilde{I})}{\ell(I)}$ and $\gamma' = \gamma'(R) = \frac{\ell(\tilde{J})}{\ell(J)}$. The following weighted version of Journé's lemma is in [19].

Lemma 2.1 *Let Ω be an open set in $\mathbb{R}^{n_1+n_2}$. If $\omega \in A_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then for any $\eta > 0$*

$$\sum_{R \in \mathcal{M}_2(\Omega)} \omega(R)\gamma(R)^{-\eta} \leq C_\eta \omega(\Omega).$$

Proof of Theorem 1.1: Let a be any $H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ atom supported on an open set $\Omega \subseteq \mathbb{R}^{n_1+n_2}$ with $\omega(\Omega) < \infty$.

Firstly, by L^2_ω boundedness of operator S and condition 2 in the Definition 1.2, it is easy to have that

$$\begin{aligned}\int_{\widetilde{\widetilde{\widetilde{\Omega}}}} S(a)(x)^p \omega(x)\mathrm{d}x &\leq \omega(\widetilde{\widetilde{\widetilde{\Omega}}})^{1-p/2} \|S(a)(\cdot)\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})} \\ &\leq C\omega(\Omega)^{1-p/2}\|a\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})} \leq C.\end{aligned}$$

Then this theorem will be proved if we obtain that

$$\int_{\widetilde{\widetilde{\widetilde{\Omega}}}^c} S(a)(x)^p \omega(x)\mathrm{d}x < C, \tag{2.1}$$

where C is a positive constant independent of a. According to the Definition 1.2, there are some rectangle atoms a_R such that $a = \sum_{R \in \mathcal{M}(\Omega)} a_R$. Then (2.1) follows from

$$\sum_{R \subset \mathcal{M}(\Omega)} \int_{\widetilde{\widetilde{\widetilde{\Omega}}}^c} S(a_R)(x)^p \omega(x)\mathrm{d}x \leq C. \tag{2.2}$$

For any $R \in \mathcal{M}(\Omega)$ centered at (x_0, y_0), we now estimate $\int_{\widetilde{\widetilde{\widetilde{\Omega}}}^c} S(a_R)(x)^p \omega(x)\mathrm{d}x$. Note that $\tilde{R} \in \mathcal{M}_1(\widetilde{\Omega})$, $\tilde{\tilde{R}} \in \mathcal{M}_2(\widetilde{\widetilde{\Omega}})$ and $10\tau\tilde{\tilde{R}} \subseteq \widetilde{\widetilde{\widetilde{\Omega}}}$. Hence one has that

$$\begin{aligned}\int_{\widetilde{\widetilde{\widetilde{\Omega}}}^c} S(a_R)(x)^p \omega(x)\mathrm{d}x &\leq \int_{(10\tau\tilde{I})^c \times \mathbb{R}^{n_2}} S(a_R)(x)^p \omega(x)\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^{n_1} \times (10\tau\tilde{J})^c} S(a_R)(x)^p \omega(x)\mathrm{d}x \\ &= I + II.\end{aligned}$$

To estimate the first term, we split it into two terms as follows:

$$\begin{aligned} I &= \int_{(10\tau\tilde{I})^c\times(10\tau J)^c} S(a_R)(x)^p\omega(x)\mathrm{d}x + \int_{(10\tau\tilde{I})^c\times(10\tau J)} S(a_R)(x)^p\omega(x)\mathrm{d}x \\ &= U(R)+V(R). \end{aligned}$$

Recall that $S(a_R)(x_1,x_2) = \Big(\sum_{j\in\mathbb{N},k\in\mathbb{Z}} \sum_{I'\times J'\in\Pi_j^{n_1}\times\Pi_k^{n_2}} |\psi_{j,k} * a_R(x_{I'},x_{J'})|^2 \chi_{I'}(x_1)\chi_{J'}(x_2)\Big)^{\frac{1}{2}}$. Note that, for any fixed j,k, the rectangles in $\Pi_j^{n_1}\times\Pi_k^{n_2}$ are disjoint. Hence one can rewrite $S(a_R)$ as follows:

$$S(a_R)(x_1,x_2) = \left(\sum_{j\in\mathbb{N},k\in\mathbb{Z}} |\sum_{I'\times J'\in\Pi_j^{n_1}\times\Pi_k^{n_2}} \psi_{j,k} * a_R(x_{I'},x_{J'})\chi_{I'}(x_1)\chi_{J'}(x_2)|^2\right)^{\frac{1}{2}}.$$

According to side length of I, there are two cases: $\ell(I)\geq\varrho$ and $\ell(I)<\varrho$.

We now discuss the first case: $\ell(I)\geq\varrho$. For $U(R)$, using the cancellation condition of a_R in the x_2 direction and the Taylor's Theorem, one has that

$$\begin{aligned} |\psi_{j,k} * a_R(x_{I'},x_{J'})| &= |\int \psi_j^{(1)}(x_{I'}-y_1)[\psi_k(x_{J'}-y_2) \\ &\quad - \sum_{|\alpha|\leq N_p^2} 2^{k|\alpha|}(y_0-y_2)^\alpha (D^\alpha\psi^{(2)})_k \\ &\quad (x_{J'}-y_0)]a_R(y)\mathrm{d}y| \\ &= |\int \psi_j^{(1)}(x_{I'}-y_1) \\ &\quad \sum_{|\alpha|=N_p^2+1} 2^{k|\alpha|}(y_0-y_2)^\alpha (D^\alpha\psi^{(2)})_k \\ &\quad (x_{J'}-y_0-\theta(y_2-y_0)]a_R(y)\mathrm{d}y| \end{aligned}$$

for some $\theta\in(0,1)$. Set $N=N_p^2+1$. It implies that

$$\begin{aligned} &|\psi_{j,k} * a_R(x_{I'},x_{J'})| \lesssim \\ &\int \frac{2^{jn_1}}{(1+2^j|x_{I'}-y_1|)^{n_1+L}} \frac{2^{k(n_2+N)}\ell(J)^N}{(1+2^k|x_{J'}-y_0-\theta(y_2-y_0)|)^{n_2+L}} |a_R(y)|\mathrm{d}y \end{aligned}$$

for any positive integer L. Note that $|x_{J'}-x_2|\leq 2^{-k}$ if $x_2\in J'$, which yields that $1+2^k|x_{J'}-y_0-\theta(y_2-y_0)| \approx 1+2^k|x_2-y_0-\theta(y_2-y_0)| \approx 1+2^k|x_2-y_0|$ since $x_2\in(10\tau J)^c$. Similarly, $1+2^j|x_{I'}-y_1|\approx 1+2^j|x_1-x_0|$. Hence when $(x_1,x_2)\in(10\tau\tilde{I})^c\times(10\tau J)^c$,

$$\begin{aligned}
&|\psi_{j,k} * a_R(x_{I'}, x_{J'})| \\
&\lesssim \frac{2^{jn_1}}{(1+2^j|x_1-x_0|)^{n_1+L}} \frac{2^{k(n_2+N)}\ell(J)^N}{(1+2^k|x_2-y_0|)^{n_2+L}} \int |a_R(y)|\mathrm{d}y \qquad (2.3)\\
&\le \frac{2^{-jL}}{|x_1-x_0|^{n_1+L}} \frac{2^{k(n_2+N)}\ell(J)^N}{(1+2^k|x_2-y_0|)^{n_2+L}} \|a_R\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})} (\omega^{-1}(R))^{\frac{1}{2}}
\end{aligned}$$

by Hölder's inequality. Moreover, by a standard estimate, if $L > N$, one has that

$$\sum_{k\in\mathbb{Z}} \frac{2^{k(n_2+N)}}{(1+2^k|x_2-y_0|)^{n_2+L}} \lesssim \frac{1}{|x_2-y_0|^{n_2+N}}.$$

Therefore,

$$\begin{aligned}
U(R) &= \int_{(10\tau\tilde{I})^c\times(10\tau J)^c} \Big(\sum_{j\in\mathbb{N},k\in\mathbb{Z}} \sum_{I'\times J'\in\Pi_j^{n_1}\times\Pi_k^{n_2}} |\psi_{j,k} * a_R(x_{I'}, x_{J'})|^2 \chi_{I'}(x_1)\chi_{J'}(x_2)\Big)^{\frac{p}{2}} \omega(x)\mathrm{d}x \\
&\lesssim \int_{(\frac{\ell(\tilde{I})}{\ell(I)}I)^c\times(10\tau J)^c} \frac{1}{|x_1-x_0|^{(n_1+L)p}} \frac{\ell(J)^{Np}}{|x_2-y_0|^{(n_2+N)p}} \omega(x_1,x_2)\mathrm{d}x \\
&\quad \cdot \|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})} (\omega^{-1}(R))^{\frac{p}{2}} \\
&\le \sum_{i=0}^\infty \sum_{s=0}^\infty \int_{|x_1-x_0|\approx 2^i\ell(\tilde{I})} \int_{|x_2-y_0|\approx 2^s\ell(J)} \frac{1}{|x_1-x_0|^{(n_1+L)p}} \\
&\quad \frac{\ell(J)^{Np}}{|x_2-y_0|^{(n_2+N)p}} \omega(x_1,x_2)\mathrm{d}x \cdot \|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})} (\omega^{-1}(R))^{\frac{p}{2}} \\
&\approx \sum_{i=0}^\infty \sum_{s=0}^\infty (2^i\ell(\tilde{I}))^{-(n_1+L)p} (2^s\ell(J))^{-(n_2+N)p} \ell(J)^{Np} \|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})} (\omega^{-1}(R))^{\frac{p}{2}} \\
&\quad \cdot\omega(2^i \frac{\ell(\tilde{I})}{\ell(I)} I \times 2^s J) \\
&\lesssim \sum_{i=0}^\infty \sum_{s=0}^\infty (2^i\ell(\tilde{I}))^{-(n_1+L)p} (2^s\ell(J))^{-(n_2+N)p} \ell(J)^{Np} \|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})} (\omega^{-1}(R))^{\frac{p}{2}} \\
&\quad \cdot 2^{2in_1} 2^{2sn_2} (\frac{\ell(\tilde{I})}{\ell(I)})^{2n_1} \omega(I\times J) \\
&\lesssim \ell(\tilde{I})^{-(n_1+L)p} \ell(J)^{-n_2 p} \|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})} (\omega^{-1}(R))^{\frac{p}{2}} \left(\frac{\ell(\tilde{I})}{\ell(I)}\right)^{2n_1} \omega(I\times J)
\end{aligned}$$

by choosing L such that $(n_1+L)p - 2n_1 > 0$. Hence,

$$U(R) \lesssim (\frac{\ell(\tilde{I})}{\ell(I)})^{-[(n_1+L)p-2n_1]} |R|^{-p} \|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})} (\omega^{-1}(R))^{\frac{p}{2}} \omega(I\times J)$$

since $\ell(I) \geq \varrho$. At last, since $\omega^{-1}(R) \lesssim |R|^2\omega(R)^{-1}$ when $\omega \in A_2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, we have that

$$U(R) \lesssim \gamma(R)^{-[(n_1+L)p-2n_1]}\omega(R)^{1-\frac{p}{2}}\|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})},$$

where $\gamma(R) = \frac{\ell(\tilde{I})}{\ell(I)}$.

We now discuss $V(R)$. In a fashion similar to obtain (2.3), one has that

$$\begin{aligned}|\psi_{j,k} * a_R(x_{I'}, x_{J'})| &\leq \int |\psi_j(x_{I'} - y_1)||\int \psi_k(x_{J'} - y_2)a_R(y_1, y_2)\mathrm{d}y_2|\mathrm{d}y_1\\ &\lesssim \int \frac{2^{jn_1}}{(1+2^j|x_{I'} - y_1|)^{n_1+L}}|\int \psi_k(x_{J'} - y_2)a_R(y_1, y_2)\mathrm{d}y_2|\mathrm{d}y_1\\ &\lesssim \frac{2^{jn_1}}{(1+2^j|x_1 - x_0|)^{n_1+L}}\int_{\tau I}|\int \psi_k(x_{J'} - y_2)a_R(y_1, y_2)\mathrm{d}y_2|\mathrm{d}y_1.\end{aligned}$$

It yields that

$$\begin{aligned}&|\sum_{I'\times J'\in\Pi_j^{n_1}\times\Pi_k^{n_2}} \psi_{j,k} * a(x_{I'}, x_{J'})\chi_{I'}(x_1)\chi_{J'}(x_2)|\\ &\lesssim \frac{2^{jn_1}}{(1+2^j|x_1 - x_0|)^{n_1+L}}\int_{\tau I}\sum_{J'\in\Pi_k^{n_2}}|\int \psi_k(x_{J'} - y_2)a_R(y_1, y_2)\mathrm{d}y_2|\chi_{J'}(x_2)\mathrm{d}y_1.\end{aligned}$$

Hence

$$\begin{aligned}&\sum_{j\in\mathbb{N},k\in\mathbb{Z}}|\sum_{I'\times J'\in\Pi_j^{n_1}\times\Pi_k^{n_2}} \psi_{j,k} * f(x_{I'}, x_{J'})\chi_{I'}(x_1)\chi_{J'}(x_2)|^2\\ &\lesssim \sum_{j\in\mathbb{N},k\in\mathbb{Z}}\\ &\left(\frac{2^{jn_1}}{(1+2^j|x_1-x_0|)^{n_1+L}}\right)^2\left(\int_{\tau I}\sum_{J'\in\Pi_k^{n_2}}|\int \psi_k(x_{J'} - y_2)a_R(y_1, y_2)\mathrm{d}y_2|\chi_{J'}(x_2)\mathrm{d}y_1\right)^2.\end{aligned}$$

Then,

$$\begin{aligned}V(R) &\lesssim \int_{(10\tau\tilde{I})^c\times(10\tau J)}\Bigg(\sum_{j\in\mathbb{N},k\in\mathbb{Z}}(\frac{2^{jn_1}}{(1+2^j|x_1 - x_0|)^{n_1+L}})^2\\ &\quad\times\left(\int_{\tau I}\sum_{J'\in\Pi_k^{n_2}}|\int \psi_k(x_{J'} - y_2)a_R(y_1, y_2)\mathrm{d}y_2|\chi_{J'}(x_2)\mathrm{d}y_1)^2\right)^{p/2}\omega(x_1, x_2)\mathrm{d}x\\ &\lesssim \sum_{i=0}^\infty\int_{|x_1-x_0|\approx 2^i\ell(\tilde{I})}\int_{10\tau J}\left(\frac{1}{|x_1 - x_0|^{(n_1+L)}}\right)^p\end{aligned}$$

$$\times\left(\sum_{k\in\mathbb{Z}}\left(\int_{\tau I}\sum_{J'\in\Pi_k^{n_2}}|\int\psi_k(x_{J'}-y_2)a_R(y_1,y_2)\mathrm{d}y_2|\chi_{J'}(x_2)\mathrm{d}y_1\right)^2\right)^{p/2}\omega(x_1,x_2)\mathrm{d}x$$

$$\le\frac{1}{\ell(\tilde{I})^{Lp}}\sum_{i=0}^{\infty}2^{-Lpi}\int_{|x_1-x_0|\approx 2^i\ell(\tilde{I})}\int_{10\tau J}$$

$$\times\left(\sum_{k\in\mathbb{Z}}(\frac{1}{|x_1-x_0|^{n_1}}\int_{\tau I}\sum_{J'\in\Pi_k^{n_2}}|\int\psi_k(x_{J'}-y_2)a_R(y_1,y_2)\mathrm{d}y_2|\chi_{J'}(x_2)\mathrm{d}y_1)^2\right)^{p/2}$$

$$\omega(x_1,x_2)\mathrm{d}x$$

$$\le\frac{1}{\ell(\tilde{I})^{Lp}}\sum_{i=0}^{\infty}2^{-Lpi}\int_{|x_1-x_0|\approx 2^i\ell(\tilde{I})}\int_{10\tau J}$$

$$\times\left(\sum_{k\in\mathbb{Z}}\big(\mathcal{M}^{(1)}(\sum_{J'\in\Pi_k^{n_2}}|\int\psi_k(x_{J'}-y_2)a_R(\cdot,y_2)\mathrm{d}y_2|\chi_{J'}(x_2)\big)^2(x_1)\right)^{p/2}\omega(x_1,x_2)\mathrm{d}x,$$

where $\mathcal{M}^{(1)}$ is the Hardy-Littlewood maximal operator associated with the first direction x_1. By Hölder's inequality and L^2 boundedness of $\mathcal{M}^{(1)}$, one has that

$$\int_{|x_1-x_0|\approx 2^i\ell(\tilde{I})}\int_{10\tau J}\left(\sum_{k\in\mathbb{Z}}\big(\mathcal{M}^{(1)}(\sum_{J'\in\Pi_k^{n_2}}|\int\psi_k(x_{J'}-y_2)a_R\right.$$

$$\left.(\cdot,y_2)\mathrm{d}y_2|\chi_{J'}(x_2)\big)^2(x_1)\right)^{p/2}\cdot\omega(x_1,x_2)\mathrm{d}x$$

$$\le\omega\left(\frac{2^i\ell(\tilde{I})}{\ell(I)}I,10\tau J\right)^{1-\frac{p}{2}}$$

$$\left(\int\int\sum_{k\in\mathbb{Z}}(\mathcal{M}^{(1)}\left(\sum_{J'\in\Pi_k^{n_2}}|\int\psi_k(x_{J'}-y_2)a_R\right.\right.$$

$$\left.\left.(\cdot,y_2)\mathrm{d}y_2|\chi_{J'}(x_2)\right)^2(x_1)\omega(x_1,x_2)\mathrm{d}x\right)^{p/2}$$

$$\lesssim\omega(\frac{2^i\ell(\tilde{I})}{\ell(I)}I,10\tau J)^{1-\frac{p}{2}}$$

$$\left(\int\int\sum_{k\in\mathbb{Z}}\left(\sum_{J'\in\Pi_k^{n_2}}|\int\psi_k(x_{J'}-y_2)a_R(x_1,y_2)\mathrm{d}y_2|\chi_{J'}(x_2)\right)^2\omega(x_1,x_2)\mathrm{d}x\right)^{p/2}$$

$$\lesssim\omega\left(\frac{2^i\ell(\tilde{I})}{\ell(I)}I,10\tau J\right)^{1-\frac{p}{2}}\|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})}$$

$$\lesssim\left(\frac{2^i\ell(\tilde{I})}{\ell(I)}\right)^{2n_1(1-\frac{p}{2})}\omega(R)^{1-\frac{p}{2}}\|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})}.$$

It gives that

$$V(R) \lesssim \frac{1}{\ell(\tilde{I})^{Lp}} \sum_{i=0}^{\infty} 2^{-Lpi} \left(\frac{2^i \ell\left(\tilde{I}\right)}{\ell(I)}\right)^{2n_1(1-\frac{p}{2})} \omega(R)^{1-\frac{p}{2}} \|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})}$$
$$\lesssim \gamma(R)^{-(n_1+L)p+2n_1} \omega(R)^{1-\frac{p}{2}} \|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})},$$

since $\ell(I) \geq \varrho$.

For the second case $\ell(I) < \varrho$, in a fashion similar to case one, using the cancellation condition of a_R in two directions and Taylor's Theorem, one has that

$$|\psi_{j,k} * a_R(x_{I'}, x_{J'})|$$
$$\lesssim \frac{2^{j(n_1+N')}\ell(I)^{N'}}{(1+2^j|x_1-x_0|)^{n_1+L}} \frac{2^{k(n_2+N)}\ell(J)^N}{(1+2^k|x_2-y_0|)^{n_2+L}} \|a_R\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})} (\omega^{-1}(R))^{\frac{1}{2}},$$

where $N' = N_p^1 + 1$. It follows that

$$U(R) \lesssim \int_{(\frac{\ell(\tilde{I})}{\ell(I)} I)^c \times (10\tau J)^c} \frac{\ell(I)^{N'p}}{|x_1-x_0|^{(n_1+N')p}} \frac{\ell(J)^{Np}}{|x_2-y_0|^{(n_2+N)p}} \omega(x_1, x_2) \mathrm{d}x$$
$$\cdot \|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})} (\omega^{-1}(R))^{\frac{p}{2}}$$
$$\leq \sum_{i=0}^{\infty}\sum_{s=0}^{\infty} \int_{|x_1-x_0|\approx 2^i \ell(\tilde{I})} \int_{|x_2-y_0|\approx 2^s \ell(J)} \frac{\ell(I)^{N'p}}{|x_1-x_0|^{(n_1+N')p}} \frac{\ell(J)^{Np}}{|x_2-y_0|^{(n_2+N)p}} \omega(x_1, x_2) \mathrm{d}x$$
$$\cdot \|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})} (\omega^{-1}(R))^{\frac{p}{2}}$$
$$\lesssim \sum_{i=0}^{\infty}\sum_{s=0}^{\infty} (2^i \ell(\tilde{I}))^{-(n_1+N')p} \ell(I)^{N'p} (2^s \ell(J))^{-(n_2+N)p} \ell(J)^{Np} \|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})} (\omega^{-1}(R))^{\frac{p}{2}}$$
$$\cdot 2^{2in_1} 2^{2sn_2} \left(\frac{\ell(\tilde{I})}{\ell(I)}\right)^{2n_1} \omega(I \times J)$$
$$\lesssim \gamma(R)^{-[(n_1+N')p-2n_1]} \omega(R)^{1-\frac{p}{2}} \|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})},$$

since $(n_1 + N')p - 2n_1 > 0$.

For $V(R)$, let $x_1 \in (10\tau\tilde{I})^c$. Similarly, for any positive integer L,

$$|\psi_{j,k} * a_R(x_{I'}, x_{J'})| \lesssim \frac{2^{j(n_1+N)}\ell(I)^N}{(1+2^j|x_1-x_0|)^{n_1+L}} \int_{\tau I} \left| \int \psi_k(x_{J'} - y_2) a_R(y_1, y_2) \mathrm{d}y_2 \right| \mathrm{d}y_1.$$

It yields that

$$
\begin{aligned}
&|\sum_{I'\times J'\in\Pi_j^{n_1}\times\Pi_k^{n_2}}\psi_{j,k}*a(x_{I'},x_{J'})\chi_{I'}(x_1)\chi_{J'}(x_2)|\\
&\quad\lesssim\frac{2^{j(n_1+N)}\ell(I)^N}{(1+2^j|x_1-x_0|)^{n_1+L}}\int_{\tau I}\sum_{J'\in\Pi_k^{n_2}}|\int\psi_k(x_{J'}-y_2)a_R(y_1,y_2)\mathrm{d}y_2|\chi_{J'}(x_2)\mathrm{d}y_1.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sum_{j\in\mathbb{N},k\in\mathbb{Z}}|\sum_{I'\times J'\in\Pi_j^{n_1}\times\Pi_k^{n_2}}\psi_{j,k}*f(x_{I'},x_{J'})\chi_{I'}(x_1)\chi_{J'}(x_2)|^2\\
&\quad\lesssim\sum_{j\in\mathbb{N},k\in\mathbb{Z}}(\frac{2^{j(n_1+N)}\ell(I)^N}{(1+2^j|x_1-x_0|)^{n_1+L}})^2(\int_{\tau I}\sum_{J'\in\Pi_k^{n_2}}|\int\psi_k(x_{J'}-y_2)a_R(y_1,y_2)\mathrm{d}y_2|\chi_{J'}(x_2)\mathrm{d}y_1)^2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&V(R)\\
&\quad\lesssim\int_{(10\tau\tilde{I})^c\times(10\tau J)}\left(\sum_{j\in\mathbb{N},k\in\mathbb{Z}}\left(\frac{2^{j(n_1+N)}\ell(I)^N}{(1+2^j|x_1-x_0|)^{n_1+L}}\right)^2\right.\\
&\qquad\left.\left(\int_{\tau I}\sum_{J'\in\Pi_k^{n_2}}|\int\psi_k(x_{J'}-y_2)a_R(y_1,y_2)\mathrm{d}y_2|\chi_{J'}(x_2)\mathrm{d}y_1\right)^2\right)^{p/2}\omega(x_1,x_2)\mathrm{d}x\\
&\quad\lesssim\sum_{i=0}^{\infty}\int_{|x_1-x_0|\approx 2^i\ell(\tilde{I})}\int_{10\tau J}\left(\frac{\ell(I)^N}{|x_1-x_0|^{(n_1+N)}}\right)^p\\
&\qquad\left(\sum_{k\in\mathbb{Z}}\left(\int_{\tau I}\sum_{J'\in\Pi_k^{n_2}}|\int\psi_k(x_{J'}-y_2)a_R(y_1,y_2)\mathrm{d}y_2|\chi_{J'}(x_2)\mathrm{d}y_1\right)^2\right)^{p/2}\omega(x_1,x_2)\mathrm{d}x\\
&\quad\le\left(\frac{\ell(I)^N}{\ell(\tilde{I})^N}\right)^p\sum_{i=0}^{\infty}2^{-Npi}\int_{|x_1-x_0|\approx 2^i\ell(\tilde{I})}\int_{10\tau J}\\
&\qquad\left(\sum_{k\in\mathbb{Z}}(\mathcal{M}^{(1)}\left(\sum_{J'\in\Pi_k^{n_2}}|\int\psi_k(x_{J'}-y_2)a_R(\cdot,y_2)\mathrm{d}y_2|\chi_{J'}(x_2)\right)^2(x_1)\right)^{p/2}\omega(x_1,x_2)\mathrm{d}x\\
&\quad\lesssim\left(\frac{\ell(I)^N}{\ell(\tilde{I})^N}\right)^p\sum_{i=0}^{\infty}2^{-Npi}\omega\left(\frac{2^i\ell(\tilde{I})}{\ell(I)}I,10\tau J\right)^{1-\frac{p}{2}}\|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})}\\
&\quad\lesssim\gamma(R)^{-(n_1+N)p+2n_1}\omega(R)^{1-\frac{p}{2}}\|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})}.
\end{aligned}
$$

Hence, we obtain that

$$
I\lesssim(\frac{\ell(\tilde{I})}{\ell(I)})^{-(n_1+N')p+2n_1}\omega(R)^{1-\frac{p}{2}}\|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})}.
$$

Similarly,

$$II \lesssim \gamma'(R)^{-(n_2+N)p+2n_2}\omega(R)^{1-\frac{p}{2}}\|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})}.$$

By weighted version of Journé's Lemma 2.1, one has that

$$\begin{aligned}
&\sum_{R\in\mathcal{M}(\Omega)}\int_{\widetilde{\widetilde{\Omega}}^c}|S(a_R)(x)|^p\omega(x)\mathrm{d}x\\
&\lesssim \sum_{R\in\mathcal{M}(\Omega)}\gamma(R)^{-(n_1p+N'p-2n_1)}\omega(R)^{1-\frac{p}{2}}\|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})}\\
&\quad+\sum_{R\in\mathcal{M}(\Omega)}\gamma'(R)^{-(n_2p+Np-2n_2)}\omega(R)^{1-\frac{p}{2}}\|a_R\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})}\\
&\le \Big(\sum_{R\in\mathcal{M}_1(\Omega)}\gamma(R)^{-(n_1p+N'p-2n_1)p'}\omega(R)\Big)^{1/p'}\left(\sum_{R\in\mathcal{M}(\Omega)}\|a_R\|^2_{L^2_\omega(\mathbb{R}^{n_1+n_2})}\right)^{p/2}\\
&+\Big(\sum_{R\in\mathcal{M}_2(\Omega)}\gamma'(R)^{-(n_2p+Np-2n_2)}\omega(R)\Big)^{1/p'}\left(\sum_{R\in\mathcal{M}(\Omega)}\|a_R\|^2_{L^2_\omega(\mathbb{R}^{n_1+n_2})}\right)^{p/2}\\
&\lesssim \omega(\Omega)^{1-p/2}\left(\sum_{R\in\mathcal{M}(\Omega)}\|a_R\|^2_{L^2_\omega(\mathbb{R}^{n_1+n_2})}\right)^{p/2}\le C,
\end{aligned}$$

where $p'=\frac{2}{2-p}$ satisfying $\frac{p}{2}+\frac{1}{p'}=1$. Thus we obtain (2.2) and then finish the proof. □

3 Atomic Decomposition

To discuss the atomic decomposition of $H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, we need a new discrete Calderón-type identity composed by some test functions with compact supports which was obtained in [14]. To do this, given a positive integer M large enough, let $\phi_0^{(1)},\phi^{(1)}\in\mathcal{S}(\mathbb{R}^{n_1})$ satisfy that

$$\mathrm{supp}\phi_0^{(1)}\subseteq\{x\in\mathbb{R}^{n_1}:|x|\le 1\};\ \int\phi_0^{(1)}=1, \tag{3.1}$$

and

$$\mathrm{supp}\phi^{(1)}\subseteq\{x\in\mathbb{R}^{n_1}:|x|\le 1\};\ \int\phi^{(1)}(x)x^\alpha\mathrm{d}x=0,\ \text{for all }|\alpha|\le M, \tag{3.2}$$

and

$$|\widehat{\phi_0^{(1)}}(\xi)|^2 + \sum_{j=1}^{\infty} |\widehat{\phi^{(1)}}(2^{-j}\xi)|^2 = 1, \quad \text{for all } \xi \in \mathbb{R}^{n_1}. \tag{3.3}$$

Let $\phi^{(2)} \in \mathcal{S}(\mathbb{R}^{n_2})$ with

$$\operatorname{supp}\phi^{(2)} \subseteq \{x \in \mathbb{R}^{n_2} : |x| \le 1\}; \int \phi^{(2)}(x) x^\alpha \mathrm{d}x = 0, \text{ for all } |\alpha| \le M, \tag{3.4}$$

and

$$\sum_{j\in\mathbb{Z}} |\widehat{\phi^{(2)}}(2^{-j}\xi)|^2 = 1, \quad \text{for all } \xi \in \mathbb{R}^{n_2} \setminus \{0\}. \tag{3.5}$$

Theorem 3.1 *For $0 < p \le 1$, let $\phi_0^{(1)}$, $\phi^{(1)} \in \mathcal{S}(\mathbb{R}^{n_1})$ and $\phi^{(2)} \in \mathcal{S}(\mathbb{R}^{n_2})$ satisfy conditions (3.1)–(3.3) and (3.4)–(3.5), respectively. Suppose that $\omega \in A_2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Then for any $f \in L^2_\omega(\mathbb{R}^{n_1+n_2}) \cap H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, there exists $\tilde{f} \in L^2_\omega(\mathbb{R}^{n_1+n_2}) \cap H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ such that*

$$f(x) = \sum_{j\in\mathbb{N}, k\in\mathbb{Z}} \sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} |I||J|(\phi_{j,k} * \tilde{f})(x_I, x_J) \times \phi_{j,k}(x_1 - x_I, x_2 - x_J), \tag{3.6}$$

where the series converges in $L^2_\omega(\mathbb{R}^{n_1+n_2})$ and $H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, and N is some large positive integer independent of f. Moreover,

$$\|\tilde{f}\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})} \approx \|f\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})}$$

and

$$\|\tilde{f}\|_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \approx \|f\|_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.$$

Proof of Theorem 1.2: For $f \in L^2_\omega(\mathbb{R}^{n_1+n_2}) \cap H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, let $\tilde{S}(f)(x) = \Big(\sum_{j\in\mathbb{N},k\in\mathbb{Z}} \sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} |(\phi_{j,k} * \tilde{f})(x_I, x_J|^2 \chi_I \chi_J\Big)^{1/2}$, where $\phi_{j,k}$ satisfies the conditions of Theorem 3.1. Firstly, rewrite (3.6)as follows:

$$\begin{aligned} f(x) &= \sum_{j=0,k\in\mathbb{Z}} \sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} |I||J|(\phi_{j,k} * \tilde{f})(x_I, x_J) \times \phi_{j,k}(x_1 - x_I, x_2 - x_J) \\ &\quad + \sum_{j>1,k\in\mathbb{Z}} \sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} |I||J|(\phi_{j,k} * \tilde{f})(x_I, x_J) \times \phi_{j,k}(x_1 - x_I, x_2 - x_J) \\ &= f_1(x) + f_2(x). \end{aligned}$$

We now decompose f_1 into atoms. For any $i \in \mathbb{Z}$, set that

$$\Omega_i = \{x \in \mathbb{R}^{n_1+n_2} : \tilde{S}(f)(x) > 2^i\}.$$
$$\mathcal{B}_i = \{R : R \in \Pi^{n_1}_{j+N} \times \Pi^{n_2}_{k+N}, j = 0, k \in \mathbb{Z},$$
$$\omega(R \cap \Omega_i) > \frac{1}{2}\omega(R), \quad \omega(R \cap \Omega_{i+1}) \leq \frac{1}{2}\omega(R)\},$$

and

$$\tilde{\Omega}_i = \{x \in \mathbb{R}^{n_1+n_2} : M^{\omega}_s(\chi_{\Omega_i}) > \frac{1}{10^{N(n_1+n_2)}}\}.$$

Obviously, $\cup_{R\in\mathcal{B}_i} R \subseteq \tilde{\Omega}_i$. For $R \in \Pi^{n_1}_{j+N} \times \Pi^{n_2}_{k+N}$, denote $\phi_R = \phi_{j,k}$. Then one can rewrite f_1 as following

$$f_1(x) = \sum_{i=-\infty}^{+\infty} \sum_{R\in\mathcal{B}_i} |R|(\phi_R * \tilde{f})(x_I, x_J) \times \phi_R(x_1 - x_I, x_2 - x_J) = \sum_i \lambda_i a_i(x).$$

Set

$$\tilde{\Omega}^{\prime}_i = \left\{x \in \mathbb{R}^{n_1+n_2} : M_s(\chi_{\tilde{\Omega}_i}) > \frac{1}{10^{N(n_1+n_2)}}\right\}.$$

Note that if $(x_1, x_2) \in \operatorname{supp} \phi_R(\cdot - x_I, \cdot - x_J)$, one has that $|x_1 - x_I| \leq 2^{-j}$, $|x_2 - x_J| \leq 2^{-k}$, which implies that $\operatorname{supp} \phi_R(\cdot - x_I, \cdot - x_J) \subseteq 10^N R \subseteq \tilde{\Omega}^{\prime}_i$. By the weighted boundedness of M_s, one has that $\omega(\tilde{\Omega}^{\prime}_i) \approx \omega(\tilde{\Omega}_i)$. Denote that

$$\lambda_i = C\omega(\tilde{\Omega}^{\prime}_i)^{\frac{1}{p}-\frac{1}{2}} \|\Big(\sum_{R\in\mathcal{B}_i} |\phi_R * \tilde{f}(x_I, x_J)\chi_R(x)|^2\Big)^{\frac{1}{2}}\|_{L^2_{\omega}(\mathbb{R}^{n_1+n_2})},$$

and

$$a_i(x) = \frac{1}{\lambda_i} \sum_{R\in\mathcal{B}_i} |R|(\phi_R * \tilde{f})(x_I, x_J)\phi_R(x_1 - x_I, x_2 - x_J).$$

Then one has that

$$f_1(x) = \sum_i \lambda_i a_i(x).$$

We now check that every a_i is an atom $H^p_{mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Firstly, $\operatorname{supp} a_i \subseteq \tilde{\Omega}^{\prime}_i$. Moreover, by the duality argument,

$$\|a_i\|_{L^2_{\omega}(\mathbb{R}^{n_1+n_2})} = \frac{1}{\lambda_i} \sup_{\|g\|_{L^2_{\omega^{-1}}(\mathbb{R}^{n_1+n_2})}\leq 1} |\int (\sum_{R\in\mathcal{B}_i} |R| \Big(\phi_R * \tilde{f}\Big)$$

$$
\begin{aligned}
&(x_I, x_J)\phi_R(x_1 - x_I, x_2 - x_J))g(x)\mathrm{d}x| \\
&= \frac{1}{\lambda_i} \sup_{\|g\|_{L^2_{\omega^{-1}}(\mathbb{R}^{n_1+n_2})} \le 1} |\int \sum_{R\in\mathcal{B}_i} (\phi_R * \tilde{f})(x_I, x_J) \times \tilde{\phi}_R * g(x_I, x_J)\chi_R(x)\mathrm{d}x| \\
&\le \frac{1}{\lambda_i} \sup_{\|g\|_{L^2_{\omega^{-1}}(\mathbb{R}^{n_1+n_2})} \le 1} \int \Big(\sum_{R\in\mathcal{B}_i} |(\phi_R * \tilde{f})(x_I, x_J)|^2 \chi_R(x)\omega(x)\Big)^{1/2} \\
&\quad \Big(\sum_{R\in\mathcal{B}_i} |(\tilde{\phi}_R * g)(x_I, x_J)|^2 \chi_R(x)\omega^{-1}(x)\Big)^{1/2} \mathrm{d}x,
\end{aligned}
$$

which yields that $\|a_i\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})} \le \omega(\tilde{\Omega}_i')^{\frac{1}{2}-\frac{1}{p}}$.

Furthermore, for $Q \in \mathcal{M}(\Omega_i)$, if we set that

$$
a_{i,Q}(x) = \frac{1}{\lambda_i} \sum_{R\in\mathcal{B}_i, R\subseteq Q} |Q|(\phi_R * \tilde{f})(x_I, x_J)\phi_R(x_1 - x_I, x_2 - x_J),
$$

then $a_i = \sum_{Q\in\mathcal{M}(\Omega_i)} a_{i,Q}$, and $\operatorname{supp} a_{i,Q} \subseteq 2^{N+4}Q$. Moreover, the side length of Q in the first direction is 2^{-N} denoted by ϱ since $Q \in \mathcal{B}_i$, and there is no any vanishing moment in this direction. While $a_{i,Q}$ satisfies vanishing moment in x_2 direction. On the other hand, by the duality argument again,

$$
\|a_{i,Q}\|^2_{L^2_\omega(\mathbb{R}^{n_1+n_2})} \le \frac{1}{\lambda_i} \int \sum_{R\in\mathcal{B}_i, R\subseteq Q} |(\phi_R * \tilde{f})(x_I, x_J)|^2 \chi_R(x)\omega(x)\mathrm{d}x.
$$

It gives that

$$
\begin{aligned}
\sum_{Q\in\mathcal{M}(\Omega_i)} \|a_{i,Q}\|^2_{L^2_\omega(\mathbb{R}^{n_1+n_2})} &\lesssim \frac{1}{\lambda_i^2} \int \sum_{R\in\mathcal{B}_i} |(\phi_R * \tilde{f})(x_I, x_J)|^2 \chi_R(x)\omega(x)\mathrm{d}x \\
&\lesssim \omega(\Omega_i')^{1-\frac{2}{p}}.
\end{aligned}
$$

For $\sum_i \lambda_i^p$, using $\omega(R \cap \tilde{\Omega}_i \backslash \Omega_{i+1}) > \frac{1}{2}\omega(R)$ when $R \in B_i$, one has that

$$
\begin{aligned}
&\|\Big(\sum_{R\in\mathcal{B}_i} |\phi_R * \tilde{f}(x_I, x_J)\chi_R(x)|^2 \Big)^{\frac{1}{2}} \|^2_{L^2_\omega(\mathbb{R}^{n_1+n_2})} \\
&= \sum_{R\in\mathcal{B}_i} \omega(R)|(\phi_R * \tilde{f})(x_I, x_J)|^2 \\
&\le 2 \sum_{R\in\mathcal{B}_i} \omega(R \cap \tilde{\Omega}_i \backslash \Omega_{i+1})|(\phi_R * \tilde{f})(x_I, x_J)|^2 \\
&= 2 \int_{\tilde{\Omega}_i \backslash \Omega_{i+1}} \sum_{R\in\mathcal{B}_i} |(\phi_R * \tilde{f})(x_I, x_J)|^2 \chi_R(x)\omega(x)\mathrm{d}x
\end{aligned}
$$

$$\leq 2^{2(i+1)+1}\omega(\tilde{\Omega}_i).$$

Hence

$$\sum_i \lambda_i^p \leq C\sum_i 2^{pi}\omega(\Omega_i) \leq C\|\tilde{S}(f)\|^p_{L^p_\omega(\mathbb{R}^{n_1+n_2})} \leq C\|f\|^p_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \tag{3.7}$$

with a constant C independent of f.

Similarly, one can obtain the atomic decompositions of f_2. Different from the above, the rectangle atoms decomposed from f_2 have desired vanishing moment both in two directions. The results that $\sum_i \lambda_i a_i(x) \to f$ in $H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ and in $L^2_\omega(\mathbb{R}^{n_1+n_2})$ are followed from (3.7) and duality argument, respectively.

For $f\in H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, by the density, there are $\{f_i\}_{i\geq 0}\subseteq L^2_\omega(\mathbb{R}^{n_1+n_2})\cap H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ such that $\|f_i\|_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\leq 2^{-i+1}\|f\|_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$, and $f(x)=\sum_{i\geq 0} f_i(x)$ in $H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. Since $f_i\in L^2(\mathbb{R}^{n_1+n_2})\cap H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, we can decompose f_i into atoms to obtain that

$$f_i(x)=\sum_k \lambda_k^{(i)}a_k^{(i)}(x),\ \text{ in } H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}),\quad\text{and}$$
$$\sum_k |\lambda_k^{(i)}|^p \leq C\|f_i\|^p_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})},$$

where C is a absolute constant, which yields that

$$f(x)=\sum_{i\geq 0}\sum_k \lambda_k^{(i)}a_k^{(i)}(x).$$

Moreover,

$$\sum_{i\geq 0}\sum_k |\lambda_k^{(i)}|^p \leq \sum_{i\geq 0} C\|f_i\|^p_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$$
$$\leq C\sum_{i\geq 0}2^{-i+1}\|f\|^p_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \lesssim \|f\|^p_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.$$

The converse is obvious by Theorem 1.1.

This completes the proof. □

Proof of Theorem 1.3: If T is bounded on $H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, then (1.9) is obtained by Theorem 1.1 directly. For the converse, let $f\in L^2_\omega(\mathbb{R}^{n_1+n_2})\cap H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, then by Theorem 1.2, there exists a sequence atoms $\{a_k\}$ such that $f(x)=\sum_{k\in\mathbb{Z}}\lambda_k a_k(x)$ in $L^2_\omega(\mathbb{R}^{n_1+n_2})$, where the real numbers sequence $\{\lambda_k\}$ satisfies $\sum_{k\in\mathbb{Z}}|\lambda_k|^p \lesssim \|f\|^p_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$. Since T is bounded on $L^2_\omega(\mathbb{R}^{n_1+n_2})$, one has $T(f)(x)=\sum\lambda_k T(a_k)(x)$ in $L^2_\omega(\mathbb{R}^{n_1+n_2})$, which implies that this series (subsequence) converges almost everywhere. Hence,

$$\|T(f)\|^p_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \le \sum |\lambda_k|^p \|T(a_k)\|^p_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \lesssim \|f\|^p_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.$$

by (1.9).

This completes the proof. □

Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

References

1. Carbery, A., Seeger, A.: H^p and L^p-variants of multiparameter Calderón–Zygmund theory. Trans. Am. Math. Soc. **334**(2), 719–747 (1992)
2. Carleson, L.: A counterexample for measures bounded on H^p for the bidisc. Mittag-Leffler Report, No. 7 (1974)
3. Chang, S.-Y.A.: Carleson measures on the bi-disc. Ann. Math. **109**, 613–620 (1979)
4. Chang, S.-Y.A., Fefferman, R.: A continuous version of duality of H^1 with BMO on the bidisc. Ann. Math. **112**, 179–201 (1980)
5. Chang, S.-Y.A., Fefferman, R.: The Calderón–Zygmund decomposition on product domains. Am. J. Math. **104**, 455–468 (1982)
6. Chang, S.-Y.A., Fefferman, R.: Some recent developments in Fourier Analysis and H^p theory on product domains. Bull. Am. Math. Soc. **12**, 1–43 (1985)
7. Chen, J., Ding, W., Lu, G.: Pseudodifferential operators on multi-parameter local hardy spaces. Forum Math. **32**(4), 91–936 (2020)
8. Chen, J., Lu, G.: Hörmander type theorems for multi-linear and multi-parameter Fourier multiplier operators with limited smoothness. Nonlinear Anal. **101**, 98–112 (2014)
9. Chen, J., Lu, G.: Hömander type theorem on bi-parameter Hardy spaces for Fourier multipliers with optimal smoothness. Rev. Mat. Iberoam. **34**(4), 1541–1561 (2018)
10. Dai, W., Lu, G.: L^p estimates for multi-linear and multi-parameter pseudo-differential operators. Bull. Math. Sci. France **143**(3), 567–597 (2015)
11. Ding, Y., Han, Y., Lu, G., et al.: Boundedness of singular integrals on multiparameter weighted Hardy spaces $H^p_w(R^n \times R^m)$. Potential Anal. **37**, 31–56 (2012)
12. Ding, W., Lu, G., Zhu, Y.: Multi-parameter local Hardy spaces. Nonlinear Anal. **184**, 352–380 (2019)
13. Ding, W., Lu, G.: Duality of multi-parameter Triebel–Lizorkin spaces associated with the composition of two singular integral operators. Trans. Am. Math. Soc. **368**(10), 7119–7152 (2016)
14. Ding, W., Xu, Y., Zhu, Y.: Weighted multi-parameter mixed Hardy spaces and their applications. Czechoslovak Math. J. **72**(147), 709–734 (2022)
15. Ding, W., Lu, G., Zhu, Y.: Discrete Littlewood–Paley characterization of multi parameter local hardy spaces. Forum Math. **31**(6), 1467–1488 (2019)
16. Ding, W., Zhu, Y.: Mixed Hardy spaces and their applications. Acta Math. Sci. **40B**(4), 945–969 (2020)
17. Fefferman, R.: Strong differentiation with respect to measures. Am. J. Math. **103**, 33–40 (1981)
18. Fefferman, R., Stein, E.M.: Singular integrals on product spaces. Adv. Math. **45**, 117–143 (1982)
19. Fefferman, R.: Calderón–Zygmund theory for product domains-H^p spaces. Proc. Natl. Acad. Sci. **83**, 840–843 (1986)
20. Fefferman, R.: Harmonic analysis on product spaces. Ann. Math. **126**, 109–130 (1987)
21. Fefferman, R.: A^p weight and singular integrals. Am. J. Math. **110**(5), 975–987 (1988)
22. Fefferman, R., Pipher, J.: Multiparameter operators and sharp weighted inequalities. Am. J. Math. **119**(2), 337–369 (1997)
23. Ferguson, S., Lacey, M.: A characterization of product BMO by commutators. Acta Math. **189**(2), 143–160 (2002)

24. Garcia Cuerva, J.: Rubio de Francia. Weighted norm inequalities and related topics. North Holland, Amsterdam, J. L. (1985)
25. Gundy, R., Stein, E.M.: H^p theory for the polydisk. Proc. Natl. Acad. Sci. **76**, 1026–1029 (1979)
26. Han, Y., Lee, M., Lin, C., et al.: Calderón–Zygmund operators on product Hardy spaces. J. Funct. Anal. **258**, 2834–2861 (2010)
27. Han, Y., Li, J., Lu, G., et al.: $H^p \to H^p$ boundedness implies $H^p \to L^p$ boundedness. Forum Math. **23**(4), 729–756 (2011)
28. Han, Y., Lin, C., Lu, G., et al.: Hardy spaces associated with different homogeneities and boundedness of composition operators. Rev. Mat. Iberoam. **29**(4), 1127–1157 (2013)
29. Han, Y., Lu, G., Ruan, Z.: Boundedness criterion of Journé's class of singular integrals on multiparameter Hardy spaces. J. Funct. Anal. **264**(5), 1238–1268 (2013)
30. Han, Y., Lu, G., Ruan, Z.: Boundedness of singular Integrals in Journé's class on weighted multiparameter Hardy spaces. J. Geom. Anal. **24**(4), 2186–2228 (2014)
31. Han, Y., Lu, G., Sawyer, E.: Flag Hardy spaces and Marcinkiewicz multipliers on the Heisenberg group. Anal. PDE **7**(7), 1465–1534 (2014)
32. Hong, Q., Lu, G.: Weighted L^p estimates for rough bi-parameter Fourier integral operators. J. Differ. Equ. **265**(3), 1097–1127 (2018)
33. Journé, J.L.: Calderón–Zygmund operators on product spaces. Rev. Mat. Iberoamericana **1**, 55–91 (1985)
34. Journé, J.L.: Two problems of Calderón–Zygmund theory on product spaces. Ann. Inst. Fourier (Grenoble) **38**, 111–132 (1988)
35. Lu, G., Ruan, Z.: Duality theory of weighted Hardy spaces with arbitrary number of parameters. Forum Math. **26**(5), 1429–1457 (2014)
36. Lu, G., Xiao, Y.: Dual spaces of weighted multi-parameter Hardy spaces associated with the Zygmund dilation. Adv. Nonlinear Stud. **12**(3), 533–553 (2012)
37. Lu, G., Zhu, Y., Singular integrals and weighted Triebel–Lizorkin and Besov spaces of arbitrary number of parameters. Acta Math. Sin. (Engl. Ser.,) 29(1), 39-52 (2013)
38. Malliavin, M.P., Malliavin, P.: Intégrales de Lusin–Calderón pour les fonctions biharmoniques. Bull. Sci. Math. **101**, 357–384 (1977)
39. Müller, D., Ricci, F., Stein, E.: Marcinkiewicz multipliers and multi-parameter structure on Heisenberg (-type) groups I. Invent. Math. **119**(2), 199–233 (1995)
40. Müller, D., Ricci, F., Stein, E.: Marcinkiewicz multipliers and multi-parameter structure on Heisenberg (-type) groups II. Math. Z. **221**(2), 267–291 (1996)
41. Muscalu, C., Pipher, J., Tao, T., et al.: Bi-parameter paraproducts. Acta Math. **193**(2), 269–296 (2004)
42. Muscalu, C., Pipher, J., Tao, T., et al.: Multi-parameter paraproducts. Rev. Mat. Iberoam. **22**(3), 963–976 (2006)
43. Nagel, A., Ricci, F., Stein, E.: Singular integrals with flag kernels and analysis on quadratic CR manifolds. J. Funct. Anal. **181**(1), 29–118 (2001)
44. Nagel, A., Ricci, F., Stein, E., et al.: Singular integrals with flag kernels on homogeneous groups. I. Rev. Mat. Iberoam. **28**(3), 631–722 (2012)
45. Nagel, A., Ricci, R., Stein, E.,et al.: Algebras of singular integral operators with kernels controlled by multiple norms. Mem. Am. Math. Soc., **256**(1230) (2018)
46. Pipher, J.: Journe's covering lemma and its extension to higher dimensions. Duke Math. J. **53**, 683–690 (1986)
47. Ricci, F., Stein, E.M.: Multiparameter singular integrals and maximal functions. Ann. Inst. Fourier (Grenoble) **42**(3), 637–670 (1992)
48. Ruan, Z.: Weighted Hardy spaces in three-parameter case. J. Math. Anal. Appl. **367**(2), 625–639 (2010)
49. Stein, E., Street, B.: Stein, Multi-parameter singular Radon transforms II: the L^p theory. Adv. Math. **248**, 736–783 (2013)
50. Stein, E., Street, B.: Multi-parameter singular Radon transforms III: real analytic surfaces. Adv. Math. **229**(4), 2210–2238 (2012)
51. Street, B.: Multi-parameter singular Radon transforms I: the L^2 theory. J. Anal. Math. **116**, 83–162 (2012)
52. Street, B.: Multi-parameter Singular Integrals, Annals of Mathematics Studies, 189. Princeton University Press, Princeton (2014)