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Global Well-Posedness for 2D Nonhomogeneous Magneto-Micropolar Equations with Density-Dependent Viscosity

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Abstract

We study an initial boundary value problem of 2D nonhomogeneous magneto-micropolar equations with density-dependent viscosity in smooth bounded domains. When the initial density can contain vacuum states, we prove that there is a unique global strong solution for the system under the assumption that initial velocity is suitably small. In particular, the initial data can be arbitrarily large except the gradient of velocity. Finally, we obtain the exponential decay rates of strong solutions by using the energy method.

Keywords Magneto-micropolar equations · Global well-posedness · Exponential decay · Vacuum

Mathematics Subject Classification 35Q35 · 76D03

1 Introduction

In the paper, we first introduce the following standard 3D nonhomogeneous incompressible magneto-micropolar equations

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$$
\begin{cases}
\rho_t + \mathrm{div}(\rho u) = 0, \\
(\rho u)_t + \mathrm{div}(\rho u \otimes u) - \mathrm{div}((\mu(\rho) + \kappa)\nabla u) + \nabla P = 2\kappa \nabla \times w + b \cdot \nabla b, \\
(\rho w)_t + \mathrm{div}(\rho u \otimes w) + 4\kappa w - \gamma \triangle w - \lambda \nabla \mathrm{div} w = 2\kappa \nabla \times u, \\
b_t + \nu \triangle b + u \cdot \nabla b - b \cdot \nabla u = 0, \\
\mathrm{div} u = \mathrm{div} b = 0,
\end{cases}
\tag{1.1}
$$

where ρ, u, w, b and P are the density, velocity field, micro-rotational, magnetic field and pressure of the fluid, respectively. The positive constant γ and λ are the angular viscosities and κ is the micro-rotation viscosity, while $\nu > 0$ is the magnetic diffusive coefficient. The kinematic viscosity $\mu(\rho)$ satisfies the following hypothesis

$$
\mu(\cdot) \in C^1[0, \infty), \quad and \quad 0 < \underline{\mu} \leq \mu(\phi) \leq \bar{\mu} < \infty, \quad for \quad \forall \phi \in [0, \infty). \tag{1.2}
$$

where $\underline{\mu}$ and $\bar{\mu}$ are some positive constant. In the special case when

$$
\begin{cases}
\rho = \rho(x_1, x_2, t), \quad u = (u_1(x_1, x_2, t), u_2(x_1, x_2, t), 0), \\
b = (b_1(x_1, x_2, t), b_2(x_1, x_2, t), 0), \quad w = (0, 0, w(x_1, x_2, t)),
\end{cases}
\tag{1.3}
$$

the 3D micropolar equations reduce to the 2D micropolar equations

$$
\begin{cases}
\rho_t + u \cdot \nabla \rho = 0, \\
\rho u_t + \rho u \cdot \nabla u - \mathrm{div}((\mu(\rho) + \kappa)\nabla u) + \nabla P = -2\kappa \nabla^{\perp} w + b \cdot \nabla b, \\
\rho w_t + \rho u \cdot \nabla w + 4\kappa w - \gamma \triangle w = 2\kappa \nabla^{\perp} \cdot u, \\
b_t + \nu \triangle b + u \cdot \nabla b - b \cdot \nabla u = 0, \\
\mathrm{div} u = \mathrm{div} b = 0.
\end{cases}
\tag{1.4}
$$

Here $u = (u_1, u_2)$ is a vector with the corresponding scalar vorticity, and w is a scalar function in what follows,

$$
\nabla^{\perp} \cdot u = \partial_1 u_2 - \partial_2 u_1, \quad \nabla^{\perp} w = (-\partial_2 w, \partial_1 w). \tag{1.5}
$$

Let $\Omega \in \mathbb{R}^2$ be a bounded smooth domain, and we consider the initial boundary value problem of (1.4) with the initial condition and the Dirichlet boundary condition:

$$
\begin{cases}
(\rho, \rho u, \rho w, b)(x, 0) = (\rho_0, \rho_0 u_0, \rho_0 w_0, b_0)(x), \quad x \in \Omega, \\
(u, w, b)(x, t) = (0, 0, 0), \quad x \in \partial\Omega, \quad t > 0.
\end{cases}
\tag{1.6}
$$

The system (1.1) describes the motion of electrically conducting micropolar fluids in the presence of a magnetic field. The magneto-micropolar fluid model was first proposed by Ahmadi–Shahinpoor [1] in the 1970s, which extends the valid domain of MHD equations and accounts for microrotation effect. There are some literatures focused on the mathematical theory of the incompressible viscous magneto-micropolar system, in particular, studying well-posedness of solutions to the magneto-micropolar fluid equations, also refer to [2] for relevant background. However, if the initial density includes the vacuum state, it has a few results to the existence of solutions for this system [3–5]. When the fluid is homogeneous (i.e. $\rho = const$), the local existence and uniqueness of strong solutions were firstly established by Rojas-Medar [6] with using the Galerkin method. Yamazaki [7] studied the global regularity of the two-dimensional magneto-micropolar fluid system, and they showed that with zero angular viscosity the solution triple remains smooth for all time. Next, Shang–Zhao [8] proved the global regularity of classical solutions to 2D magneto-micropolar fluid equations with only micro-rotational velocity dissipation and magnetic diffusion. Recently, appealing to a refined pure energy method, Tan-Wu-Zhou [9] investigated the global existence and decay estimate of solutions to magneto-micropolar fluid equations by assuming that the H^3-norm of the initial data is small, but the higher order derivatives can be arbitrary large. Lin and Xiang [10] considered the global well-posedness for the 2D incompressible magneto-micropolar fluid system with partial viscosity. Yet for the density-dependent viscosity and the initial density allowing vacuum states, it seems to need solve much more difficult problems to the system (1.1).

When the system do not consider the magnetic field (i.e. $b = 0$), the system (1.1) reduces to the nonhomogeneous micropolar fluid equations. Eringen [11] first introduced the micropolar fluids, which accounts for micro-rotation effects and micro-rotation in a fluid motion system, and can be viewed as non-Newtonian fluids with nonsymmetric stress tensor. When the connected open set $\Omega \in \mathbb{R}^3$ replaces the whole space $\mathbb{R}^3$, and the solution vanishes on $\partial\Omega \times [0, T]$, Galdi–Rionero [12] showed existence and uniqueness of weak solutions to the initial boundary value problem for the micropolar system. Dong–Zhang [13] and Liu–Wang [14] proved the global regularity of smooth solutions to the 2D micropolar fluid with the micro-rotation viscosity $\gamma = 0$. Zhang–Zhu [15] studied the global strong and classical solution for the 3D micropolar equations with vacuum, which assumed $\|\rho_0\|_{L^\infty}$ or $\|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|\sqrt{\rho_0}w_0\|_{L^2}^2$ is small enough. Recently, Song [16] concerned the global well-posedness for the 3D compressible micropolar system in the critical Besov space, and proposed the linear system for the compressible micropolar equations could be decomposed into a compressible Navier–Stokes equation and an incompressible micropolar system. However, if the influence of magnetic field on the motion is considered, it will also bring some difficulties on a priori estimates of the system (1.1).

For the incompressible magneto-micropolar fluid model, many scholars has been attracted by its research significance in physics and mathematics. When the initial density allowing vacuum states and the density-dependent viscosity, Zhang–Zhu [5] established the global strong solutions for the 3D nonhomogeneous incompressible magneto-micropolar equations under the condition that the initial energy is small enough. It is worth noting that these results are valid under the following compatibility conditions

$$\begin{cases} -(\mu_1+\xi)\triangle u_0 + \nabla P_0 - 2\xi \mathrm{curl} w_0 - b_0\cdot\nabla b_0 = \sqrt{\rho_0}g_1, \\ -\mu_2\triangle w_0 - (\mu_2+\lambda)\nabla \mathrm{div} w_0 + 4\xi w_0 - 2\xi \mathrm{curl} u_0 = \sqrt{\rho_0}g_2, \end{cases} \tag{1.7}$$

for some $(\nabla P_0, g_1, g_2) \in L^2$. Meanwhile, they also obtain the algebraic decay rates of the solutions provided that the initial energy is small enough. Later on, Zhong [17] extended this result to the entire two-dimensional space, and showed the local existence of strong solutions to 2D nonhomogeneous magneto-micropolar fluid with vacuum as far field density. Then, he established the global existence and exponential decay of strong solutions of nonhomogeneous magneto-micropolar fluid equations with large initial data and vacuum in paper [18]. Recently, With the help of weighted function and the duality principle of BMO space and Hardy space, Zhong [19] investigated the global well-posedness to nonhomogeneous magneto-micropolar fluid equations with zero density at infinity in $\mathbb{R}^2$. Furthermore, for the homogeneous Dirichlet boundary conditions of the velocity and micro-rotational velocity and Navier-slip boundary condition of the magnetic field, he proved the initial boundary value problem of 3D nonhomogeneous magneto-micropolar fluid equations in [20]. The above results are all the density-independent viscosity. It will bring much more difficulties to estimate the $L^\infty(0, T; L^2)$-norm for the gradients of velocity because of the density-dependent viscosity. However, the paper [5] proved the global regularity of this system (1.1) in 3D space and only obtained the algebraic decay rate. It is worth noting that they require must be satisfied some small energy conditions. As a result of the standard Sobolev embedding theorem, a prior estimate for the 3D case cannot be applied to the 2D case. The purpose of this paper is to establish the global well-posedness of solutions for (1.4)–(1.6) with vacuum in smooth bounded domains. Especially, there is no need other small initial energy but only need the initial velocity is suitably small, and we also yielded the exponential decay rate of strong solutions.

Now, we go back to (1.1). Before stating the main results, we first explain the notations and conventions used throughout this paper. For $1 \leq r \leq \infty$ and $k \geq 0$, the standard Lebesgue and Sobolve spaces are defined as $L^r = L^r(\Omega)$, $W^{k,r} = W^{k,r}(\Omega)$, and $H^k(\Omega) = W^{k,2}(\Omega)$, $r = 2$. The space $H^1_{0,\sigma}$ represent the closure in H^1 of the space $C^\infty_{0,\sigma} \triangleq \{f \in C^\infty_0(\Omega) | \mathrm{div} f = 0\}$.

The following is our main result of the paper:

Theorem 1.1 *Let $q \in (2, \infty)$ be a fixed constant, assume that the initial data (ρ_0, u_0, w_0, b_0) satisfy*

$$0 \leq \rho_0 \in W^{1,q}, u_0 \in H^1_{0,\sigma}, w_0 \in H^1_0, b_0 \in H^1_{0,\sigma}. \tag{1.8}$$

Then there exist some small positive constant ε_0 depending only on q, κ, ν, Ω, $\underline{\mu}$, $\bar{\mu} \triangleq \sup\limits_{[0,\bar{\rho}]} \mu(\rho)$, $\bar{\rho}$, $\|\nabla u_0\|_{L^2}$, $\|\nabla w_0\|_{L^2}$ and $\|\nabla b_0\|_{L^2}$, such that if

$$\|\nabla u_0\|^2_{L^2} \leq \varepsilon_0, \tag{1.9}$$

there is a unique strong solution (ρ, u, w, b, P) satisfying that for any $0 < \tau < T < \infty$ and $2 < r < min\{q, 3\}$,

$$
\begin{cases}
0 \le \rho \in C\left([0,T]; W^{1,q}\right), \nabla\mu(\rho) \in C\left([0,T]; L^q\right), \\
\nabla u \in L^\infty\left(0,T; L^2\right) \cap L^\infty\left(\tau,T; H^1\right) \cap L^2\left(\tau,T; W^{1,r}\right), \\
\nabla w \in L^\infty\left(0,T; L^2\right) \cap L^\infty\left(\tau,T; H^1\right) \cap L^2\left(\tau,T; H^2\right), \\
\nabla b \in L^\infty\left(0,T; L^2\right) \cap L^\infty\left(0,T; H^1\right) \cap L^2\left(0,T; W^{1,r}\right), \\
P \in L^\infty(0,T; L^2) \cap L^\infty\left(\tau,T; H^1\right) \cap L^2\left(\tau,T; W^{1,r}\right), \\
t\sqrt{\rho}u_t, t\sqrt{\rho}w_t, tb_t \in L^\infty\left(0,T; L^2\right), \\
t\nabla u_t, t\nabla w_t, t\nabla b_t \in L^2\left(0,T; L^2\right), \\
e^{\frac{\sigma t}{2}}\nabla u, e^{\frac{\sigma t}{2}}\nabla w, e^{\frac{\sigma t}{2}}\nabla b, e^{\frac{\sigma t}{2}}\sqrt{\rho}u_t, e^{\frac{\sigma t}{2}}\sqrt{\rho}w_t, e^{\frac{\sigma t}{2}}\triangle b \in L^2\left(0,T; L^2\right),
\end{cases}
\tag{1.10}
$$

where $\sigma \triangleq \min\{\frac{\underline{\mu}}{C_p\bar{\rho}}, \frac{\kappa}{C_p\bar{\rho}}, \frac{\nu}{C_p\bar{\rho}}\}$ *with* C_p *being the constant of Poincaré's inequality. Moreover, there exists some positive constant* C *depending only on* Ω, q, κ, ν, $\underline{\mu}$, $\bar{\mu}$, $\bar{\rho}$, $\|\nabla u_0\|_{L^2}$, $\|\nabla w_0\|_{L^2}$ *and* $\|\nabla b_0\|_{L^2}$ *such that for all* $t \ge 1$,

$$
\begin{cases}
\|\nabla u(\cdot,t)\|_{H^1} + \|\nabla w(\cdot,t)\|_{H^1} + \|\nabla b(\cdot,t)\|_{H^1} + \|\nabla P(\cdot,t)\|_{L^2} \le Ce^{-\sigma t}, \\
\|\sqrt{\rho}u_t(\cdot,t)\|_{L^2}^2 + \|\sqrt{\rho}w_t(\cdot,t)\|_{L^2}^2 + \|b_t(\cdot,t)\|_{L^2}^2 \le Ce^{-\sigma t}.
\end{cases}
\tag{1.11}
$$

Remark 1.1 For the Theorem 1.1, it holds for any function $\mu(\rho)$ satisfying (1.2) and for arbitrarily large initial density, which can contain vacuum condition. The initial data is no need satisfy any compatibility conditions [21]

$$
-div((\mu(\rho_0) + \kappa)\nabla u_0) + \nabla P_0 = \rho_0^{\frac{1}{2}} g,
$$

for some $(P_0, g) \in H^1 \times L^2$.

Remark 1.2 It should be noted that we improved the results of [5], we consider the density-dependent viscosity and the initial data can be arbitrarily large except $\|\nabla u_0\|_{L^2}^2$ ((1.9)). Finally, we also obtain time-independent estimates and exponential decay rates for the solutions.

Now we simply present the main idea of the proof and give the main difficulty in this paper. The local existence and uniqueness of strong solutions to the systems (1.4)–(1.6) follow from the paper [17] (see Lemma 2.1). We need to deduce some global a priori estimates on strong solution to (1.4)–(1.6) in proper higher regularity, and then extend the local solution to the global solution. Due to we consider the initial density can contain vacuum states and even have compact support, in particular, the viscosity coefficient is also affected by the density. Hence it not only increases the difficulty of estimating the the $L^\infty(0,T; L^2(\Omega))$-norm of $\|\nabla u\|_{L^2}^2$, but also solves the strong coupling between $u \cdot \nabla u$, $u \cdot \nabla b$ and $b \cdot \nabla u$. Firstly, the key ingredient here is to get the time-independent $L^1(0,T; L^\infty(\Omega))$-norm of ∇u. we derive that the bound on $L^2(0,T; L^2(\Omega))$-norm of $e^{\frac{\sigma t}{2}}\nabla u$ by applying the upper bounds on the density (3.3) and the Poincaré inequality. And then the most important thing is to estimate $L^\infty(0,T;L^2(\Omega))$-norm of ∇u by the Lemma 2.3 and (3.1). Next, we will obtain a key estimate (3.28) by multiplying $(1.4)_4$ by $b|b|^2$ and Gronwall's inequality, which is used to deal with the strong coupling between $b\cdot\nabla u$. In addition, we need to define a function $\zeta(t)\triangleq\min\{1,t\}$ to get the estimates on $L^\infty(0,T;L^2(\Omega))$-norm of $t^{\frac{1}{2}}\sqrt{\rho}u_t$ and $L^\infty(\zeta(T),T;L^2(\Omega))$-norm of $e^{\frac{1}{2}\sigma t}\sqrt{\rho}u_t$, which avoids the singularity of $\|\sqrt{\rho}u_t\|_{L^2}^2$ at $t=0$. Because of the coupling between magnetic field and the gradient of velocity and magnetic field, it is important to obtain the estimates on $H^2(\Omega)$-norm of w and $L^2(\Omega)$-norm of b_t by using the standard $L^2(\Omega)$-estimates of the elliptic system, the Eq. $(1.4)_4$ and Sobolev's inequality, which is used to control the $L^\infty(\zeta(T),T;L^2(\Omega))$-norm of $e^{\frac{1}{2}\sigma t}\sqrt{\rho}w_t$ and $L^\infty(\zeta(T),T;L^2(\Omega))$-norm of $e^{\frac{1}{2}\sigma t}b_t$, further in order to get the $L^1(0,T;L^\infty(\Omega))$-norm of ∇u. Finally, with a priori estimates stated above, we are in a position to prove Proposition 3.1. Meanwhile, it can bound the estimation of the time derivatives for the solutions (ρ,u,w,b,P) to extend the local solution to all time, and thus claims the proof of Theorem 1.1.

The rest of this paper is organized as follows: we introduce some elementary facts and inequalities in Sect. 2. The Sect. 3 is devoted to a priori estimates. Finally, we will give the proof of Theorem 1.1 in Sect. 4.

2 Preliminaries

We will recall some known facts and elementary inequalities which will be used frequently later. The following local existence of strong solutions whose proof is similar to [17].

Lemma 2.1 *Assume that* (ρ_0,u_0,w_0,b_0) *satisfies* (1.8). *Then there exist a small time* $T>0$ *and a unique strong solution* (ρ,u,w,b,P) *to the problem* (1.4)–(1.6) *in* $\Omega\times(0,T)$ *satisfies* (1.10)–(1.11).

Lemma 2.2 (See [22]) (Gagliardo–Nirenberg) *Let* v *belongs to* $L^q(\Omega)$, *and its derivatives of order* m, $\nabla^m v$, *belong to* $L^r(\Omega)$, $1\le s,r\le\infty$. *Then for the derivatives* $\nabla^j v$, $0\le j<m$, *we have*

$$\|\nabla^j v\|_{L^s(\Omega)}\le\tilde{C}\|\nabla^m v\|_{L^r(\Omega)}^{\alpha}\|v\|_{L^s(\Omega)}^{1-\alpha},\tag{2.1}$$

where

$$\tfrac{1}{s}=\tfrac{j}{n}+\alpha\left(\tfrac{1}{r}-\tfrac{m}{n}\right)+(1-\alpha)\tfrac{1}{s},$$

for all a in the interval

$$\tfrac{j}{m}\le\alpha\le 1,$$

and the constant $\tilde{C}$ *depends only on* n, m, j, s, r, α.

Next, we need the following regularity on the Stokes equations to derive the estimates of the derivatives of the solutions, whose proof can be found in [23].

Lemma 2.3 *Assume that $\rho \in W^{1,q}$, $q \in (2,\infty)$, $0 \le \rho \le \bar{\rho}$ and $\mu(\rho)$ satisfies* (1.2) *on $[0,\bar{\rho}]$. Let $(u,P) \in H^1_{1,\sigma} \times L^2$ be the unique weak solution to the following boundary value problem*

$$-div(\mu(\rho)\nabla u) + \nabla P = F, \quad divu = 0 \;\; in \;\; \Omega, \quad \int P dx = 0. \tag{2.2}$$

Then we have the following regularity results:
(i) If $F \in L^2$, then $(u,P) \in H^2 \times H^1$ and

$$\begin{aligned} \|u\|_{H^2} &\le C\|F\|_{L^2}(1 + \|\nabla\mu(\rho)\|_{L^q})^{\frac{q}{q-2}}, \\ \|P\|_{H^1} &\le C\|F\|_{L^2}(1 + \|\nabla\mu(\rho)\|_{L^q})^{\frac{2q-2}{q-2}}. \end{aligned} \tag{2.3}$$

(ii) If $F \in L^r$ for some $r \in (2,q)$, then $(u,P) \in W^{2,r} \times W^{1,r}$ and

$$\begin{aligned} \|u\|_{W^{2,r}} &\le C\|F\|_{L^r}(1 + \|\nabla\mu(\rho)\|_{L^q})^{\frac{qr}{2(q-r)}}, \\ \|P\|_{W^{1,r}} &\le C\|F\|_{L^r}(1 + \|\nabla\mu(\rho)\|_{L^q})^{1+\frac{qr}{2(q-r)}}. \end{aligned} \tag{2.4}$$

Here the constant C depends on Ω, q, r, $\underline{\mu}$, $\bar{\mu}$.

3 A Priori Estimates

In this section, we first let $T > 0$ be a fixed time and (ρ, u, w, b, P) be the smooth solution to (1.4)–(1.6) on $\Omega \times (0,T]$ with smooth initial data (ρ_0, u_0, w_0) satisfying (1.8). Because of the Lemma 2.1, we will establish some necessary a priori bounds for strong solutions (ρ, u, w, b, P) to the initial boundary value problem (1.4)–(1.6) to extend the local strong solution. Firstly, we assume that the following a priori hypothesis holds:

Proposition 3.1 *There exists a small positive constant ε_0 depending only on Ω, q, κ, ν, $\underline{\mu}$, $\bar{\mu}$, $\bar{\rho}$, $\|\nabla u_0\|_{L^2}$, $\|\nabla w_0\|_{L^2}$ and $\|\nabla b_0\|_{L^2}$ such that if (ρ, u, w, b, P) is a smooth solution of* (1.4)–(1.6) *on $\Omega \times (0,T]$ satisfying*

$$\sup_{t\in[0,T]} \|\nabla\mu(\rho)\|_{L^q} \le 2M, \quad \sup_{t\in[0,T]} e^{\sigma t}\|\nabla u\|_{L^2}^2 \le 4\|\nabla u_0\|_{L^2}^2, \tag{3.1}$$

and the following estimate holds:

$$\sup_{t\in[0,T]} \|\nabla\mu(\rho)\|_{L^q} \le M, \quad \sup_{t\in[0,T]} e^{\sigma t}\|\nabla u\|_{L^2}^2 \le 2\|\nabla u_0\|_{L^2}^2, \tag{3.2}$$

provided that $\|\nabla u_0\|_{L^2}^2 \le \varepsilon_0$.

We begin with the following boundedness of density and elementary estimates.

Lemma 3.1 *It holds that*

$$0 \le \rho(x,t) \le \sup_{t\in[0,T]} \|\rho\|_{L^\infty} = \bar{\rho}, \tag{3.3}$$

$$\begin{aligned}&\sup_{t\in[0,T]} \left(\|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho}w\|_{L^2}^2 + \|b\|_{L^2}^2\right)\\ &\quad + \int_0^T \left(\underline{\mu}\|\nabla u\|_{L^2}^2 + \gamma\|\nabla w\|_{L^2}^2 + \nu\|\nabla b\|_{L^2}^2\right) dt\\ &\le \left(\|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|\sqrt{\rho}_0 w_0\|_{L^2}^2 + \|b_0\|_{L^2}^2\right),\end{aligned} \tag{3.4}$$

$$\begin{aligned}&\sup_{t\in[0,T]} e^{\sigma t} \left(\|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho}w\|_{L^2}^2 + \|b\|_{L^2}^2\right)\\ &\quad + \int_0^T e^{\sigma t}\left(\underline{\mu}\|\nabla u\|_{L^2}^2 + \gamma\|\nabla w\|_{L^2}^2 + \nu\|\nabla b\|_{L^2}^2\right) dt\\ &\le \left(\|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|\sqrt{\rho}_0 w_0\|_{L^2}^2 + \|b_0\|_{L^2}^2\right),\end{aligned} \tag{3.5}$$

where $\sigma := min\{\frac{\underline{\mu}}{C_p\bar{\rho}}, \frac{\gamma}{C_p\bar{\rho}}, \frac{\nu}{C_p\bar{\rho}}\}\}$ *with* C_p *being the constant of Poincaré's inequality.*

Proof It follows from the transport equation $(1.4)_1$ to get the (3.3) (see Lious [17]). Next, we prove the (3.4), adding $(1.4)_2 \times u$ to $(1.4)_2 \times w$ and integrating by parts leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho}w\|_{L^2}^2\right) + \left(\underline{\mu} + \kappa\right)\|\nabla u\|_{L^2}^2 + \gamma\|\nabla w\|_{L^2}^2 + 4\kappa\|w\|_{L^2}^2\\ &\quad \le 4\kappa \int w\nabla^\perp \cdot u dx \le 4\kappa\|w\|_{L^2}^2 + \kappa\|\nabla u\|_{L^2}^2,\end{aligned} \tag{3.6}$$

which gives

$$\frac{d}{dt}\left(\|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho}w\|_{L^2}^2\right) + 2\left(\underline{\mu}\|\nabla u\|_{L^2}^2 + \gamma\|\nabla w\|_{L^2}^2\right) \le 0. \tag{3.7}$$

Multiplying $(1.4)_4$ by b and integration by parts over Ω, we derive that

$$\frac{1}{2}\frac{d}{dt}\|b\|_{L^2}^2 + \nu\|\nabla b\|_{L^2}^2 = 0, \tag{3.8}$$

this combining with (3.7) and integrating the inequality in t gives (3.4). It follows from the Poincaré's inequality and (3.3) that

$$\begin{aligned}&\|\sqrt{\rho}u\|_{L^2}^2 \le \|\rho\|_{L^\infty}\|u\|_{L^2}^2 \le \frac{C_p\bar{\rho}}{\underline{\mu}}(\underline{\mu}\|\nabla u\|_{L^2}^2) \le \sigma^{-1}(\underline{\mu}\|\nabla u\|_{L^2}^2),\\ &\|\sqrt{\rho}w\|_{L^2}^2 \le \|\rho\|_{L^\infty}\|w\|_{L^2}^2 \le \frac{C_p\bar{\rho}}{\gamma}(\gamma\|\nabla w\|_{L^2}^2) \le \sigma^{-1}(\gamma\|\nabla w\|_{L^2}^2),\\ &\|b\|_{L^2}^2 \le \frac{C_p}{\nu}(\nu\|\nabla b\|_{L^2}^2) \le \sigma^{-1}(\nu\|\nabla b\|_{L^2}^2),\end{aligned} \tag{3.9}$$

where $\sigma \triangleq \min\{\frac{\mu}{C_p\bar{\rho}}, \frac{\gamma}{C_p\bar{\rho}}, \frac{\nu}{C_p}\}$ with C_p being the constant of Poincar$\acute{e}$'s inequality. Form (3.9), (3.9) and (3.7) yields that

$$\begin{aligned}\frac{d}{dt}\left(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}w\|_{L^2}^2+\|b\|_{L^2}^2\right)+\sigma\|\sqrt{\rho}u\|_{L^2}^2+\sigma\|\sqrt{\rho}w\|_{L^2}^2+\sigma\|b\|_{L^2}^2\\+\underline{\mu}\|\nabla u\|_{L^2}^2+\gamma\|\nabla w\|_{L^2}^2+\nu\|\nabla b\|_{L^2}^2\le 0,\end{aligned}\tag{3.10}$$

then this multiplies by $e^{\sigma t}$ we have

$$\begin{aligned}&\frac{d}{dt}[e^{\sigma t}\left(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}w\|_{L^2}^2+\|b\|_{L^2}^2\right)]\\&\quad+e^{\sigma t}\left(\underline{\mu}\|\nabla u\|_{L^2}^2+\gamma\|\nabla w\|_{L^2}^2+\nu\|\nabla b\|_{L^2}^2\right)\le 0.\end{aligned}\tag{3.11}$$

Integrating the above inequality in t leads to (3.6) and completes the Proof of Lemma 3.1.

Lemma 3.2 *Let (ρ, u, w, b, P) be a smooth solution to (1.4)–(1.6) satisfying (3.1). Then there exists some positive constant C depending only on Ω, q, κ, ν, $\underline{\mu}$, $\bar{\mu}$, $\bar{\rho}$, $\|\nabla u_0\|_{L^2}$, $\|\nabla w_0\|_{L^2}$ and $\|\nabla b_0\|_{L^2}$ such that*

$$\begin{aligned}&\sup_{t\in[0,T]}\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right)+\int_0^T\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2\right.\\&\quad\left.+\|\triangle b\|_{L^2}^2+\||b||\nabla b|\|_{L^2}^2\right)dt\le C\|\nabla u_0\|_{L^2}^2,\end{aligned}\tag{3.12}$$

$$\begin{aligned}&\sup_{t\in[0,T]}e^{\sigma t}\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right)+\int_0^T e^{\sigma t}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2\right.\\&\quad\left.+\|\triangle b\|_{L^2}^2+\||b||\nabla b|\|_{L^2}^2\right)dt\le C\|\nabla u_0\|_{L^2}^2.\end{aligned}\tag{3.13}$$

Proof We can get the following equation from $(1.4)_1$

$$\mu(\rho)_t+u\cdot\nabla\mu(\rho)=0.\tag{3.14}$$

Next, multiplying $(1.4)_2$ and $(1.4)_3$ by u_t and w_t respectively and integrating by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left((\mu(\rho)+\kappa)|\nabla u|^2+\gamma|\nabla w|^2+4\kappa w^2\right)dx+\int\rho|u_t|^2dx+\int\rho|w_t|^2dx\\&\quad=-\int\rho u\cdot\nabla u\cdot u_t dx-\int\rho u\cdot\nabla w\cdot w_t dx-\frac{1}{2}\int u\cdot\nabla\mu(\rho)|\nabla u|^2dx\\&\qquad-2\kappa\int\nabla^{\perp}wu_t dx+2\kappa\int\nabla^{\perp}\cdot uw_t dx+\int b\cdot\nabla u_t dx\triangleq\sum_{i=1}^{6}I_i.\end{aligned}\tag{3.15}$$

Now, we will estimate each term on the right hand of (3.15) as following. Firstly, Applying to the Hölder's, Sobolev's and Gagliardo–Nirenberg inequality along with (3.1) that

$$
\begin{aligned}
|I_1| + |I_2| &\leq |\int \rho u \cdot \nabla u \cdot u_t dx| + |\int \rho u \cdot \nabla w \cdot w_t dx| \\
&\leq C\bar{\rho}^{\frac{1}{2}} \|u\|_{L^\infty} (\|\sqrt{\rho} u_t\|_{L^2} \|\nabla u\|_{L^2} + \|\sqrt{\rho} w_t\|_{L^2} \|\nabla w\|_{L^2}) \\
&\leq C\bar{\rho}^{\frac{1}{2}} \|u\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} (\|\sqrt{\rho} u_t\|_{L^2} \|\nabla u\|_{L^2} + \|\sqrt{\rho} w_t\|_{L^2} \|\nabla w\|_{L^2}) \\
&\leq \frac{1}{2} (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\sqrt{\rho} w_t\|_{L^2}^2) + C(\|\nabla u\|_{L^2}^3 + \|\nabla w\|_{L^2}^2) \|\nabla u\|_{H^1}. \qquad (3.16)
\end{aligned}
$$

According to Hölder's inequality and (3.1), we have

$$
\begin{aligned}
|I_3| &\leq \frac{1}{2} |\int u \cdot \mu(\rho) |\nabla u|^2 dx| \\
&\leq C \|\nabla \mu(\rho)\|_{L^q} \|u\|_{L^{\frac{2q}{q-2}}} \|\nabla u\|_{L^4}^2 \\
&\leq C \|\nabla u\|_{L^2}^2 \|\nabla u\|_{H^1}. \qquad (3.17)
\end{aligned}
$$

Using integration by parts, we get

$$
\begin{aligned}
I_4 + I_5 &= -2\kappa \int \nabla^\perp w u_t dx + 2\kappa \int \nabla^\perp \cdot u w_t dx \\
&= 2\kappa \int \nabla^\perp \cdot u_t w dx + 2\kappa \int \nabla^\perp \cdot u w_t dx \\
&= 2\kappa \frac{d}{dt} \int \nabla^\perp \cdot u w dx. \qquad (3.18)
\end{aligned}
$$

It follows from integration by parts, Sobolev's inequality, $(1.4)_4$ and $(1.4)_5$ together with $b|_{\partial\Omega}$=0 that

$$
\begin{aligned}
I_6 &= -\frac{d}{dt} \int b \cdot \nabla u \cdot b dx + \int b_t \cdot \nabla u \cdot b dx + \int b \cdot \nabla u \cdot b_t dx \\
&= -\frac{d}{dt} \int b \cdot \nabla u \cdot b dx + \int (\nu \triangle b - u \cdot \nabla b + b \cdot \nabla u) \cdot \nabla u \cdot b dx \\
&\quad + \int b \cdot \nabla u \cdot (\nu \triangle b - u \cdot \nabla b + b \cdot \nabla u) dx \\
&\leq -\frac{d}{dt} \int b \cdot \nabla u \cdot b dx + 2\nu \int |\nabla u||b||\triangle b| dx + 2 \int |b|^2 |\nabla u|^2 dx \\
&\quad + 2 \int |u||\nabla u||b||\nabla b| dx \\
&\leq -\frac{d}{dt} \int b \cdot \nabla u \cdot b dx + \frac{\nu}{4} \|\triangle b\|_{L^2}^2 + C\|b\|_{L^6}^6 + C\|\nabla u\|_{L^3}^3
\end{aligned}
$$

$$
\begin{aligned}
&+ C\|u\|_{L^\infty}\|\nabla b\|_{L^2}\||b||\nabla u|\|_{L^2}\\
&\le -\frac{d}{dt}\int b\cdot\nabla u\cdot b dx + \frac{\nu}{4}\|\triangle b\|_{L^2}^2 + C\|b\|_{L^2}^2\|\nabla b\|_{L^2}^4 + C\|\nabla u\|_{L^2}^2\|\nabla^2\|_{L^2}\\
&+ C\|u\|_{L^4}^{\frac{1}{2}}\|\nabla u\|_{L^4}^{\frac{1}{2}}\|b\|_{L^4}\|\nabla b\|_{L^2}\|\nabla u\|_{L^4}\\
&\le -\frac{d}{dt}\int b\cdot\nabla u\cdot b dx + \frac{\nu}{4}\|\triangle b\|_{L^2}^2 + C\|b\|_{L^2}^2\|\nabla b\|_{L^2}^4 + C\|\nabla u\|_{L^2}^2\|\nabla^2\|_{L^2}\\
&+ C\|b\|_{L^4}\|\nabla b\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{H^1}. \qquad (3.19)
\end{aligned}
$$

Therefore, substituting (3.16)–(3.19) into (3.15), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\Big((\mu(\rho)+\kappa)\,|\nabla u|^2 + \gamma|\nabla w|^2 + 4\kappa w^2 - \nabla^\perp\cdot uw + b\cdot\nabla u\cdot b\Big)dx\\
&+ \|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2\\
&\le \frac{1}{2}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2) + C(\|\nabla u\|_{L^2}^3 + \|\nabla w\|_{L^2}^2)\|\nabla u\|_{H^1}\\
&+ \frac{\nu}{4}\|\triangle b\|_{L^2}^2 + C\|b\|_{L^2}^2\|\nabla b\|_{L^2}^4 + C\|b\|_{L^4}\|\nabla b\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{H^1}. \qquad (3.20)
\end{aligned}
$$

Multiplying $(1.4)_4$ by $\triangle b$ and integrating the resulting equality over Ω along with Hölder's and Gagliardo-Nirenberg inequalities that

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla b\|_{L^2}^2 + 2\nu\|\triangle b\|_{L^2}^2\\
&\quad\le \int|\nabla u||\nabla b|^2dx + \int|\nabla u||b||\triangle b|dx\\
&\quad\le C\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2} + C(1+\|b\|_{L^2}^2)\|\nabla b\|_{L^2}^4 + \frac{\nu}{4}\|\triangle b\|_{L^2}^2, \qquad (3.21)
\end{aligned}
$$

which combining with (3.20), we can directly yield that

$$
\begin{aligned}
&A'(t) + \|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2 + \nu\|\triangle b\|_{L^2}^2\\
&\quad\le C(\|\nabla u\|_{L^2}^3 + \|\nabla w\|_{L^2}^2)\|\nabla u\|_{H^1} + C\|\nabla b\|_{L^2}^4\\
&\quad+ C\|b\|_{L^4}\|\nabla b\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{H^1}, \qquad (3.22)
\end{aligned}
$$

where

$$
A(t) \triangleq \int((\mu(\rho)+\kappa)|\nabla u|^2 + \gamma|\nabla w|^2 + |\nabla b|^2 + 4\kappa w^2 - \nabla^\perp\cdot uw + b\cdot\nabla u\cdot b)dx, \qquad (3.23)
$$

and satisfies

$$
\frac{\mu}{2}\|\nabla u\|_{L^2}^2 + \gamma\|\nabla w\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 - C_1\|b\|_{L^4}^4
$$

$$\leq A(t) \leq C\|\nabla u\|_{L^2}^2 + C\|\nabla w\|_{L^2}^2 + C\|\nabla b\|_{L^2}^2. \tag{3.24}$$

According to Lemma 2.3 with $F = \rho u_t - \rho u \cdot \nabla u - 2\kappa \nabla^{\perp} w + b \cdot \nabla b$ and combining with (3.1) and (3.3), we obtain

$$\begin{aligned}
&\|u\|_{H^2} + \|P\|_{H^1} \\
&\quad \leq C(\|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} + C\|\nabla w\|_{L^2} + C\|b \cdot \nabla b\|_{L^2})(1 + \|\nabla \mu(\rho)\|_{L^q})^{\frac{q}{q-2}} \\
&\quad \leq C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho} u_t\|_{L^2} + C\bar{\rho}\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}} + C\|\nabla w\|_{L^2} + C\||b||\nabla b|\|_{L^2} \\
&\quad \leq C\|\sqrt{\rho} u_t\|_{L^2} + C\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\|u\|_{H^2} + C\|\nabla w\|_{L^2} + C\||b||\nabla b|\|_{L^2},
\end{aligned} \tag{3.25}$$

which directly yields that

$$\|u\|_{H^2} + \|P\|_{H^1} \leq C\|\sqrt{\rho} u_t\|_{L^2} + C\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}^2 + C\|\nabla w\|_{L^2} + C\||b||\nabla b|\|_{L^2}. \tag{3.26}$$

Next, multiplying $(1.4)_4$ by $b|b|^2$ and integrating the resulting equality over Ω and together with Gagliardo–Nirenberg inequalities, we derive

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\|b\|_{L^4}^4 + \nu\||b||\nabla b|\|_{L^2}^2 + \frac{\nu}{2}\|\nabla |b|^2\|_{L^2}^2 \\
&\quad \leq C\|\nabla u\|_{L^2}\||b|^2\|_{L^4}^2 \\
&\quad \leq C\|\nabla u\|_{L^2}\||b|^2\|_{L^2}\|\nabla |b|^2\|_{L^2} \\
&\quad \leq \frac{\nu}{4}\|\nabla |b|^2\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|b\|_{L^4}^4 \\
&\quad \leq \frac{\nu}{4}\|\nabla |b|^2\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 + C\|\nabla b\|_{L^2}^4,
\end{aligned} \tag{3.27}$$

this together with Gronwall's inequality and (3.4) implies

$$\sup_{t\in[0,T]} \|b\|_{L^4}^4 + \int_0^T \||b||\nabla b|\|_{L^2}^2 dt \leq C. \tag{3.28}$$

Putting (3.26) into (3.22) and along with the above inequality, Young's inequality and (3.1), we show that

$$\begin{aligned}
&A'(t) + \frac{1}{2}\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\sqrt{\rho} w_t\|_{L^2}^2 + \nu\|\triangle b\|_{L^2}^2 \\
&\quad \leq C\|\nabla u\|_{L^2}^6 + C\|\nabla w\|_{L^2}^4 + C\|\nabla b\|_{L^2}^4 + C\|\nabla w\|_{L^2}^2\|\nabla u\|_{L^2}^3 \\
&\qquad + C\|\nabla b\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}^3 \\
&\quad \leq C\|\nabla u\|_{L^2}^6 + C\|\nabla w\|_{L^2}^4 + C\|\nabla b\|_{L^2}^4 + \epsilon\||b||\nabla b|\|_{L^2}.
\end{aligned} \tag{3.29}$$

Then, adding (3.27) multiplied by $4(C_1+1)$ to (3.29) and choosing ϵ suitably small, it follows from (3.1)

$$\begin{aligned}&\frac{d}{dt}(A(t)+4(C_1+1)\|b\|_{L^4}^4)+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2+\nu\|\triangle b\|_{L^2}^2+\||b||\nabla b|\|_{L^2}\\ &\quad\le C\|\nabla u\|_{L^2}^6+C\|\nabla w\|_{L^2}^4+C\|\nabla b\|_{L^2}^4\\ &\quad\le C(\|\nabla u\|_{L^2}^4+\|\nabla w\|_{L^2}^2+\|\nabla b\|_{L^2}^2)(A(t)+4(C_1+1)\|b\|_{L^4}^4).\end{aligned}\tag{3.30}$$

By (3.1) and (3.4), noting that

$$\int_0^T\|\nabla u\|_{L^2}^4dt\le\frac{1}{e^{0\cdot\sigma}}\sup_{t\in[0,T]}e^{\sigma t}\|\nabla u\|_{L^2}^2\int_0^T\|\nabla u\|_{L^2}^2dt\le C\|\nabla u_0\|_{L^2}^2.\tag{3.31}$$

Hence, integrating (3.30) over [0, T] and together with Gӧnwall's inequality and (3.4) leads to (3.12). Then, multiplying (3.21) by $e^{\sigma t}$ and combining with (3.24) and (3.12) yields

$$\begin{aligned}&\frac{d}{dt}e^{\sigma t}(A(t)+4(C_1+1)\|b\|_{L^4}^4)+e^{\sigma t}(\|\sqrt{\rho}u_t\|_{L^2}^2\\ &\qquad+\|\sqrt{\rho}w_t\|_{L^2}^2+\nu\|\triangle b\|_{L^2}^2+\||b||\nabla b|\|_{L^2})\\ &\quad\le Ce^{\sigma t}(A(t)+4(C_1+1)\|b\|_{L^4}^4)(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\sigma e^{\sigma t}A(t)\\ &\qquad+4\sigma e^{\sigma t}(C_1+1)\|b\|_{L^4}^4,\end{aligned}\tag{3.32}$$

which applying to the Gönwall's inequality, we can deduce (3.13) from (3.1), (3.5) and (3.12). It finishes the proof of Lemma 3.2. □

Lemma 3.3 *Let (ρ,u,w,b,P) be a smooth solution to (1.4)–(1.6) satisfying (3.1). Then there exists some positive constant C depending only on Ω, q, κ, ν, $\underline{\mu}$, $\bar{\mu}$, $\bar{\rho}$, $\|\nabla u_0\|_{L^2}$, $\|\nabla w_0\|_{L^2}$ and $\|\nabla b_0\|_{L^2}$ such that*

$$\begin{aligned}&\sup_{t\in[0,T]}t(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2+\|b_t\|_{L^2}^2)+\int_0^{\zeta(T)}t(\|\nabla u_t\|_{L^2}^2\\ &\qquad+\|\nabla w_t\|_{L^2}^2+\|\nabla b_t\|_{L^2}^2)dt\le C\|\nabla u_0\|_{L^2}^2,\end{aligned}\tag{3.33}$$

$$\begin{aligned}&\sup_{t\in[\zeta(T),T]}e^{\sigma t}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2+\|b_t\|_{L^2})+\int_{\zeta(T)}^T e^{\sigma t}(\|\nabla u_t\|_{L^2}^2\\ &\qquad+\|\nabla w_t\|_{L^2}^2+\|\nabla b_t\|_{L^2}^2)dt\le C\|\nabla u_0\|_{L^2}^2.\end{aligned}\tag{3.34}$$

Here, $\zeta(T)$ is defined by $\zeta(t)\triangleq\min\{1,t\}$.

Proof Differentiating (1.4)$_2$ and (1.4)$_3$ with respect to t respectively yields that

$$\begin{aligned}&\rho u_{tt}+\rho u\cdot\nabla u_t-div((\mu(\rho)+\kappa)\nabla u_t)+\nabla P_t+div(\mu(\rho)_t\nabla u)\\&\quad=(u\cdot\nabla\rho)(u_t+u\cdot\nabla u)-\rho u_t\cdot\nabla u-2\kappa\nabla^{\perp}w_t+b_t\cdot\nabla b+b\cdot\nabla b_t,\end{aligned}\tag{3.35}$$

$$\begin{aligned}\rho w_{tt}+\rho u\cdot\nabla w_t+4\kappa w_t-\gamma\triangle w_t=(u\cdot\nabla\rho)(w_t+u\cdot\nabla w)-\rho u_t\cdot\nabla w\\+2\kappa\nabla^{\perp}\cdot u_t.\end{aligned}\tag{3.36}$$

Next, multiplying (3.35) by u_t and (3.36) by w_t, together with integration by parts and (1.4)$_1$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int(\rho|u_t|^2+\rho|w_t|^2)dx+\int((\mu(\rho)+\kappa)|\nabla u_t|^2+\gamma|\nabla w_t|^2+4\kappa w_t^2)dx\\&\quad=-\int((u\cdot\nabla\rho)(u_t+u\cdot\nabla u)+\rho u_t\cdot\nabla u)\cdot u_t dx\\&\qquad-\int((u\cdot\nabla\rho)(w_t+u\cdot\nabla w)+\rho u_t\cdot\nabla w)w_t dx\\&\qquad-2\kappa\int(\nabla^{\perp}w_t\cdot u_t+\nabla^{\perp}\cdot u_t w_t)dx+\int u\cdot\nabla\mu(\rho)\nabla u\cdot\nabla u_t dx\\&\qquad+\int b_t\cdot\nabla b\cdot u_t dx+\int b\cdot\nabla b_t\cdot u_t dx\triangleq\sum_{i=1}^{6}J_i.\end{aligned}\tag{3.37}$$

It follows from integration by parts, Hölder's, Gagliardo–Nirenberg and Sobolev's inequality together with (3.1) and (3.12) that

$$\begin{aligned}J_1&\le\int(\rho|u|(|u||u_t||\nabla^2u|+|u||\nabla u_t||\nabla u|+|u_t||\nabla u|^2+|u_t||\nabla u_t|)+\rho|u_t|^2|\nabla u|)dx\\&\le\bar\rho\|u\|_{L^6}^2\|u_t\|_{L^6}\|\nabla^2u\|_{L^2}+\bar\rho\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}\\&\quad+\bar\rho\|u_t\|_{L^6}\|u\|_{L^6}\|\nabla u\|_{L^2}\|\nabla u\|_{L^6}\\&\quad+2\bar\rho^{\frac12}\|u\|_{L^6}\|\sqrt\rho u_t\|_{L^3}\|\nabla u_t\|_{L^2}+\|\sqrt\rho u_t\|_{L^4}^2\|\nabla u\|_{L^2}\\&\le C\bar\rho\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}^2\|u\|_{H^2}+C\bar\rho^{\frac12}\|\nabla u\|_{L^2}\|\sqrt\rho u_t\|_{L^2}^{\frac12}\|\sqrt\rho u_t\|_{L^6}^{\frac12}\|\nabla u_t\|_{L^2}\\&\quad+C\|\nabla u\|_{L^2}\|\sqrt\rho u_t\|_{L^2}^{\frac12}\|\sqrt\rho u_t\|_{L^6}^{\frac32}\\&\le C\bar\rho\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}^2\|u\|_{H^2}+C\bar\rho^{\frac34}\|\nabla u\|_{L^2}\|\sqrt\rho u_t\|_{L^2}^{\frac12}\|\nabla u_t\|_{L^2}^{\frac32}\\&\le\frac{\mu}{8}\|\nabla u_t\|_{L^2}^2+C\|u\|_{H^2}^2+C\|\sqrt\rho u_t\|_{L^2}^2.\end{aligned}\tag{3.38}$$

Similarly, we can directly yield that

$$\begin{aligned}J_2\le C\int&(\rho|u|(|u||w_t||\nabla^2w|+|u||\nabla w_t||\nabla w|+|w_t||\nabla u||\nabla w|\\&+2|w_t||\nabla w_t|)+\rho|u_t||w_t||\nabla w|)dx\end{aligned}$$

$$
\begin{aligned}
&\leq \bar{\rho}\|u\|_{L^6}^2\|w_t\|_{L^6}\|\nabla^2 w\|_{L^2} + \bar{\rho}\|u\|_{L^6}^2\|\nabla w\|_{L^6}\|\nabla w_t\|_{L^2} \\
&\quad + \bar{\rho}\|u\|_{L^6}\|\nabla u\|_{L^2}\|\nabla w\|_{L^6}\|w_t\|_{L^6} \\
&\quad + 2\bar{\rho}^{\frac{1}{2}}\|u\|_{L^6}\|\sqrt{\rho}w_t\|_{L^3}\|\nabla w_t\|_{L^2} + \bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}w_t\|_{L^3}\|u_t\|_{L^6}\|\nabla w\|_{L^2} \\
&\leq C\bar{\rho}\|\nabla w_t\|_{L^2}\|\nabla u\|_{L^2}^2\|w\|_{H^2} + C\bar{\rho}^{\frac{1}{2}}\|\nabla u\|_{L^2}\|\sqrt{\rho}w_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}w_t\|_{L^6}^{\frac{1}{2}}\|\nabla w_t\|_{L^2} \\
&\quad + C\bar{\rho}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}\|\sqrt{\rho}w_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}w_t\|_{L^6}^{\frac{1}{2}}\|\nabla w\|_{L^2} \\
&\leq C\bar{\rho}\|\nabla w_t\|_{L^2}\|\nabla u\|_{L^2}^2\|w\|_{H^2} + C\bar{\rho}^{\frac{1}{2}}\|\nabla u\|_{L^2}\|\sqrt{\rho}w_t\|_{L^2}^{\frac{1}{2}}\|\nabla w_t\|_{L^2}^{\frac{3}{2}} \\
&\quad + C\bar{\rho}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}\|\sqrt{\rho}w_t\|_{L^2}^{\frac{1}{2}}\|\nabla w_t\|_{L^2}^{\frac{1}{2}}\|\nabla w\|_{L^2} \\
&\leq \frac{\mu}{8}\|\nabla u_t\|_{L^2}^2 + \frac{\gamma}{2}\|\nabla w_t\|_{L^2}^2 + C(\|\sqrt{\rho}w_t\|_{L^2}^2 + \|w\|_{H^2}^2).
\end{aligned}
\tag{3.39}
$$

It deduces from the integration by parts and Cauchy-Schwarz inequality that

$$
I_3 = 4\kappa\int \nabla^{\perp}\cdot u_t w_t dx \leq 4\kappa\|w_t\|_{L^2}^2 + \kappa\|\nabla u_t\|_{L^2}^2. \tag{3.40}
$$

With the help of (3.1) and Sobolev's inequality, we get

$$
\begin{aligned}
J_4 &\leq C\|u\|_{L^{\frac{4q}{q-2}}}\|\nabla\mu(\rho)\|_{L^q}\|\nabla u\|_{L^{\frac{4q}{q-2}}}\|\nabla u_t\|_{L^2} \\
&\leq C\|\nabla u\|_{L^2}\|\nabla u\|_{H^1}\|\nabla u_t\|_{L^2} \\
&\leq \frac{\mu}{8}\|\nabla u_t\|_{L^2}^2 + C\|u\|_{H^2}^2.
\end{aligned}
\tag{3.41}
$$

By Sobolev's inequality and (3.28), we have

$$
\begin{aligned}
J_5 + J_6 &\leq \|b_t\|_{L^4}\|\nabla u_t\|_{L^2}\|b\|_{L^4} \leq C\|b_t\|_{L^2}^{\frac{1}{2}}\|\nabla b_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2} \\
&\leq \frac{\mu}{8}\|\nabla u_t\|_{L^2}^2 + C(\varepsilon)\|b_t\|_{L^2}^2 + \varepsilon\|\nabla b_t\|_{L^2}^2.
\end{aligned}
\tag{3.42}
$$

Substituting (3.38) and (3.42) into (3.37), it shows

$$
\begin{aligned}
&\frac{d}{dt}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2) + \underline{\mu}\|\nabla u_t\|_{L^2}^2 + \gamma\|\nabla w_t\|_{L^2}^2 \\
&\quad \leq C(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2 + \|u\|_{H^2}^2 + \|w\|_{H^2}^2) + C(\varepsilon)\|b_t\|_{L^2}^2 + \varepsilon\|\nabla b_t\|_{L^2}^2.
\end{aligned}
\tag{3.43}
$$

Multiplying $(1.4)_3$ by w and integrating by parts yield

$$
\begin{aligned}
4\kappa\|w\|_{L^2}^2 + \gamma\|\nabla w\|_{L^2}^2 &\leq \|-\rho w_t - \rho u\cdot\nabla w - 2\kappa\nabla^{\perp}\cdot u\|_{L^2}\|w\|_{L^2} \\
&\leq \kappa\|w\|_{L^2}^2 + C(\|\rho w_t\|_{L^2}^2 + \|\rho u\cdot\nabla w\|_{L^2}^2 + \|\nabla u\|_{L^2}^2),
\end{aligned}
\tag{3.44}
$$

which implies that

$$\|w\|_{H^1} \le C(\|\rho w_t\|_{L^2} + \|\rho u \cdot \nabla w\|_{L^2} + \|\nabla u\|_{L^2}). \tag{3.45}$$

According to the standard L^2-estimates of the elliptic system (see [24]) and together with (3.1), (3.3), (3.18) and (3.45), we obtain

$$\begin{aligned}
\|w\|_{H^2} &= \|w\|_{H^1} + \|\nabla^2 w\|_{L^2} \\
&\le \|w\|_{H^1} + C\|\rho w_t - \rho u \cdot \nabla w - 4\kappa w - 2\kappa \nabla^\perp \cdot u\|_{L^2} \\
&\le C(\|\rho w_t\|_{L^2} + \|\rho u \cdot \nabla w\|_{L^2} + \|\nabla u\|_{L^2}) \\
&\le C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho} w_t\|_{L^2} + C\bar{\rho}\|u\|_{L^6}\|\nabla w\|_{L^3} + C\|\nabla u\|_{L^2} \\
&\le C\|\sqrt{\rho} w_t\|_{L^2} + C\bar{\rho}\|\nabla u\|_{L^2}\|\nabla w\|_{L^2}^{\frac{1}{2}}\|\nabla^2 w\|_{L^2}^{\frac{1}{2}} + C\|\nabla u\|_{L^2} \\
&\le C\|\sqrt{\rho} w_t\|_{L^2} + \frac{1}{2}\|w\|_{H^2} + C\|\nabla u\|_{L^2}^2\|\nabla w\|_{L^2} + C\|\nabla u\|_{L^2},
\end{aligned} \tag{3.46}$$

which gives

$$\|w\|_{H^2} \le C\|\sqrt{\rho} w_t\|_{L^2} + C\|\nabla w\|_{L^2} + C\|\nabla u\|_{L^2}. \tag{3.47}$$

Combining with $(1.4)_4$, (3.12), Gagliardo-Nirenberg and Sobolev's inequality leads to

$$\begin{aligned}
\|b_t\|_{L^2}^2 &\le C\|\triangle b\|_{L^2}^2 + C\|u\|_{L^\infty}^2\|\nabla b\|_{L^2}^2 + C\|b\|_{L^4}^2\|\nabla u\|_{L^4}^2 \\
&\le C\|\triangle b\|_{L^2}^2 + C\|u\|_{L^4}\|\nabla u\|_{L^4}\|\nabla b\|_{L^2} + C\|\nabla b\|_{L^2}^2\|\nabla u\|_{L^2}\|\nabla u\|_{H^1} \\
&\le C\|\triangle b\|_{L^2}^2 + \|\nabla u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{H^1}^{\frac{1}{2}}\|\nabla b\|_{L^2} + C\|\nabla b\|_{L^2}^2\|\nabla u\|_{L^2}\|\nabla u\|_{H^1} \\
&\le C\|\triangle b\|_{L^2}^2 + C\|\nabla u\|_{H^1}^2 + C\|\nabla u\|_{L^2}^2 + C\|\nabla b\|_{L^2}^2.
\end{aligned} \tag{3.48}$$

Then, we deduce from (3.22), (3.26), (3.47) and (3.27) that

$$\begin{aligned}
&\frac{d}{dt}(\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\sqrt{\rho} w_t\|_{L^2}^2) + \underline{\mu}\|\nabla u_t\|_{L^2}^2 + \gamma\|\nabla w_t\|_{L^2}^2 \\
&\quad \le C(\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\sqrt{\rho} w_t\|_{L^2}^2 + \|\triangle b\|_{L^2}^2 + \||b||\nabla b|\|_{L^2}^2) \\
&\qquad + C(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \varepsilon\|\nabla b_t\|_{L^2}^2.
\end{aligned} \tag{3.49}$$

Differentiating $(1.4)_4$ with respect to t, we get

$$b_{tt} - \nu\triangle b_t + u_t \cdot \nabla b + u \cdot \nabla b_t + b_t \cdot \nabla u + b \cdot \nabla u_t = 0, \tag{3.50}$$

which multiplying by b_t and along with integration by parts, Sobolev's inequality and (3.12), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|b_t\|_{L^2}^2+\nu\|\nabla b_t\|_{L^2}^2 &\le C(\||u_t||b|\|_{L^2}+\||u|b_t\|_{L^2})\|\nabla b_t\|_{L^2}\\ &\le C(\|u_t\|_{L^4}\|b\|_{L^4}+\|u\|_{L^4}\|b_t\|_{L^4})\|\nabla b_t\|_{L^2}\\ &\le C(\|\nabla u_t\|_{L^2}\|\nabla b\|_{L^2}+\|\nabla u\|_{L^2}\|b_t\|_{L^2}^{\frac{1}{2}}\|\nabla b_t\|_{L^2}^{\frac{1}{2}})\|\nabla b_t\|_{L^2}\\ &\le \frac{\nu}{2}\|\nabla b_t\|_{L^2}^2+C_2\|\nabla u_t\|_{L^2}^2+C\|b_t\|_{L^2}^2, \end{aligned}\tag{3.51}$$

where C_2 is a positive constant. Next, adding (3.49)$\times\frac{2C_2}{\underline{\mu}}$ to (3.51) and choosing $\varepsilon=\frac{\nu}{4C_2}$, it yields

$$\begin{aligned}&\frac{d}{dt}\frac{2C_2}{\underline{\mu}}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2+\|b_t\|_{L^2}^2)+C_2\|\nabla u_t\|_{L^2}^2+\gamma\|\nabla w_t\|_{L^2}^2+\frac{\nu}{2}\|\nabla b_t\|_{L^2}^2\\ &\quad\le C(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2+\|\triangle b\|_{L^2}^2+\||b||\nabla b|\|_{L^2}^2)\\ &\qquad+C(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|\nabla b\|_{L^2}^2). \end{aligned}\tag{3.52}$$

Multiplying (3.52) by t and integrating it over [0, T], then together with (3.5) and (3.13) leads to

$$\begin{aligned}&\sup_{t\in[0,T]} t(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2+\|b_t\|_{L^2}^2)\\ &\qquad+\int_0^T t(\|\nabla u_t\|_{L^2}^2+\|\nabla w_t\|_{L^2}^2+\|\nabla b_t\|_{L^2})dt\\ &\quad\le C\sup_{t\in[0,T]}(te^{-\sigma t})\int_0^T e^{\sigma t}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2+\|\triangle b\|_{L^2}^2+\||b||\nabla b|\|_{L^2}^2)dx\\ &\qquad+C\sup_{t\in[0,T]}(te^{-\sigma t})\int_0^T e^{\sigma t}(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|\nabla b\|_{L^2}^2)dt\\ &\quad\le C\|\nabla u_0\|_{L^2}^2. \end{aligned}\tag{3.53}$$

Multiplying (3.52) by $e^{\sigma t}$ and together with (3.48), we derive

$$\begin{aligned}&\frac{d}{dt}\frac{2C_2}{\underline{\mu}}e^{\sigma t}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2+\|b_t\|_{L^2}^2)\\ &\qquad+e^{\sigma t}(\|\nabla u_t\|_{L^2}^2+\|\nabla w_t\|_{L^2}^2+\|\nabla b_t\|_{L^2}^2)\\ &\quad\le Ce^{\sigma t}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2+\|\triangle b\|_{L^2}^2+\||b||\nabla b|\|_{L^2}^2)+\sigma Ce^{\sigma t}\|b_t\|_{L^2}^2\\ &\qquad+Ce^{\sigma t}(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\sigma Ce^{\sigma t}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2)\\ &\quad\le Ce^{\sigma t}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2+\|\triangle b\|_{L^2}^2+\||b||\nabla b|\|_{L^2}^2)\\ &\qquad+Ce^{\sigma t}(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|\nabla b\|_{L^2}^2). \end{aligned}\tag{3.54}$$

Finally, integrating (3.54) by t over $[\zeta(T), T]$, and it deduces from (3.5) and (3.13) to lead to (3.34). The proof of Lemma 3.3 is finished.

Lemma 3.4 *Let* (ρ, u, w, b, P) *be a smooth solution to* (1.4)–(1.6) *satisfying* (3.1). *Then there exists some positive constant* C *depending only on* Ω, q, κ, ν, $\underline{\mu}$, $\bar{\mu}$, $\bar{\rho}$, $\|\nabla u_0\|_{L^2}$, $\|\nabla w_0\|_{L^2}$ *and* $\|\nabla b_0\|_{L^2}$ *such that*

$$\int_0^T \|\nabla u\|_{L^\infty} dt \leq C\|\nabla u_0\|_{L^2}^2. \tag{3.55}$$

Proof First, it follows from Lemma 2.2, (3.3), (3.1), Hölder's, Sobolev's and Gagliardo-Nirenberg inequalities that for any $r \in (2, min(q, 3))$,

$$\begin{aligned}
\|\nabla u\|_{L^\infty} &\leq \|u\|_{W^{2,r}} \leq C(\|\rho u_t\|_{L^r} + \|\rho u \cdot \nabla u\|_{L^r} + \|\nabla w\|_{L^r} + \|b \cdot \nabla b\|_{L^r}) \\
&\quad \times (1 + \|\nabla \mu(\rho)\|_{L^q})^{\frac{qr}{2(q-r)}} \\
&\leq C\|\rho u_t\|_{L^3} + C\||u||\nabla u|\|_{L^3} + C\|\nabla w\|_{L^4} + C\|b\|_{L^\infty}\|\nabla b\|_{L^4} \\
&\leq C\bar{\rho}^{\frac{3}{4}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|u_t\|_{L^6}^{\frac{1}{2}} + C\|u\|_{L^6}\|\nabla u\|_{L^6} \\
&\quad + C\|\nabla^2 w\|_{L^2} + C\|b\|_{L^4}^{\frac{1}{2}}\|\nabla b\|_{L^4}^{\frac{1}{2}} \\
&\leq C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{1}{2}} + C\|u\|_{H^1} + C\|w\|_{H^1} + C\|\nabla b\|_{L^2}^2 + C\|\triangle b\|_{L^2}^2,
\end{aligned} \tag{3.56}$$

which together with (3.26) and (3.47), it shows

$$\begin{aligned}
\|\nabla u\|_{L^\infty} &\leq C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{1}{2}} + C(\|\sqrt{\rho}u_t\|_{L^2} + \|\sqrt{\rho}w_t\|_{L^2}) + C\|\nabla b\|_{L^2}^2 \\
&\quad + C(\|\nabla u\|_{L^2} + \|\nabla w\|_{L^2}) + C\||b||\nabla b|\|_{L^2}.
\end{aligned} \tag{3.57}$$

Hence, it follows from (3.1), (3.4), (3.33) and (3.34), that for $t \in [0, \zeta(T)]$,

$$\begin{aligned}
&\int_0^{\zeta(T)} \|\nabla u\|_{L^\infty} dt \\
&\leq C(\int_0^{\zeta(T)} t^{-\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{2}{3}} dt)^{\frac{3}{4}}(\int_0^{\zeta(T)} t^{\frac{3}{2}}\|\nabla u_t\|_{L^2}^2 dt)^{\frac{1}{4}} \\
&\quad + C\int_0^{\zeta(T)} t^{-\frac{1}{2}}(t^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2} + t^{\frac{1}{2}}\|\sqrt{\rho}w_t\|_{L^2})dt + \int_0^{\zeta(T)} \|\nabla b\|_{L^2}^2 dt \\
&\quad + C\int_0^{\zeta(T)} (\|\nabla u\|_{L^2} + \|\nabla w\|_{L^2})dt + C\int_0^{\zeta(T)} (e^{\frac{\sigma t}{2}}\||b||\nabla b|\|_{L^2})e^{-\frac{\sigma t}{2}} dt \\
&\leq C\sup_{t\in[0,\zeta(T)]} (t\|\sqrt{\rho}u_t\|_{L^2}^2)^{\frac{1}{4}}(\int_0^{\zeta(T)} t^{-\frac{1}{2}} \cdot t^{-\frac{1}{3}} dt)^{\frac{3}{4}}(\int_0^{\zeta(T)} t^{\frac{3}{2}}\|\nabla u_t\|_{L^2})^{\frac{1}{4}} \\
&\quad + C\int_0^{\zeta(T)} e^{\sigma t}\||b||\nabla b|\|_{L^2} dt \int_0^{\zeta(T)} e^{-\sigma t} dt \leq C\|\nabla u_0\|_{L^2}^2.
\end{aligned} \tag{3.58}$$

Similarly, it follows from (3.1), (3.5) and (3.34), that for $t \in [\zeta(T), T]$,

$$
\begin{aligned}
\int_{\zeta(T)}^{T} \|\nabla u\|_{L^\infty} dt &\leq C\|\nabla u_0\|_{L^2} + C\Big(\int_{\zeta(T)}^{T} e^{-\sigma t} dt\Big)^{\frac{1}{2}}\Big(\int_{\zeta(T)}^{T} e^{\sigma t}\|\nabla u_t\|_{L^2}^2 dt\Big)^{\frac{1}{2}} \\
&\leq C\|\nabla u_0\|_{L^2}^2,
\end{aligned} \tag{3.59}
$$

this together with (3.58) yields (3.55). This completes the proof of Lemma 3.4. □

With Lemmas 3.1–3.4 at hand, we are in a position to prove Proposition 3.1.

Proof of Proposition 3.1. First, it follows from $(1.4)_1$, multiplying (3.14) by $|\nabla\mu(\rho)|^{q-2}\partial_j\mu(\rho)$ and integrating the resulting equality by parts, we obtain that

$$
\frac{d}{dt}\|\nabla\mu(\rho)\|_{L^q} \leq C\|\nabla u\|_{L^\infty}\|\nabla\mu(\rho)\|_{L^q}, \tag{3.60}
$$

which together with Gronwall's inequality and (3.55) yields

$$
\begin{aligned}
\sup_{t\in[0,T]} \|\nabla\mu(\rho)\|_{L^q} &\leq \|\nabla\mu(\rho_0)\|_{L^q} \exp\{q\int_0^T \|\nabla u\|_{L^\infty} dt\} \\
&\leq \|\nabla\mu(\rho_0)\|_{L^q} \exp\{C_3\|\nabla u_0\|_{L^2}^2\} \\
&\leq 2\|\nabla\mu(\rho_0)\|_{L^q},
\end{aligned} \tag{3.61}
$$

where $\|\nabla u_0\|_{L^2}^2 \leq \varepsilon_1 \triangleq \min\{1, \frac{ln2}{C_3}\}$. Next, it deduces from (3.30), (3.5) and (3.12) gives

$$
\begin{aligned}
&\sup_{t\in[0,T]} e^{\sigma t}\|\nabla u\|_{L^2}^2 \\
&\leq \sup_{t\in[0,T]} (\|\nabla u\|_{L^2}^2)^2 \int_0^T e^{\sigma t}\|\nabla u\|_{L^2}^2 dt + \sup_{t\in[0,T]} \|\nabla w\|_{L^2}^2 \int_0^T e^{\sigma t}\|\nabla w\|_{L^2}^2 dt \\
&\quad + \sup_{t\in[0,T]} \|\nabla b\|_{L^2}^2 \int_0^T e^{\sigma t}\|\nabla b\|_{L^2}^2 dt \\
&\leq C(\|\nabla u_0\|_{L^2}^2)^3 \leq \|\nabla u_0\|_{L^2}^2,
\end{aligned} \tag{3.62}
$$

where $\|\nabla u_0\|_{L^2}^2 \leq \varepsilon_0 \triangleq \min\{\varepsilon_1, C^{-\frac{1}{4}}\}$. Thus, we gain the (3.1) from (3.61) and (3.62). It completes the proof of the Proposition 3.1. □

Lemma 3.5 *Let (ρ, u, w, b, P) be a smooth solution to (1.4)–(1.6) satisfying (3.1). Then there exists some positive constant C depending only on Ω, q, κ, ν, $\underline{\mu}$, $\bar{\mu}$, $\bar{\rho}$, $\|\nabla u_0\|_{L^2}$, $\|\nabla w_0\|_{L^2}$ and $\|\nabla b_0\|_{L^2}$ such that*

$$
\begin{aligned}
&\sup_{t\in[0,T]}\|\rho\|_{H^1\cap W^{1,q}}+\sup_{t\in[0,T]}t(\|u\|_{H^2}^2+\|P\|_{H^1}^2+\|w\|_{H^2}^2+\|b\|_{H^2}^2)\\
&\quad+\int_0^T\zeta e^{\sigma t}(\|u\|_{H^2}^2+\|P\|_{H^1}^2+\|w\|_{H^2}^2+\|b\|_{H^2}^2\\
&\quad+\|u\|_{W^{2,r}}^2+\|P\|_{W^{1,r}}^2+\|w\|_{W^{2,r}}^2+\|b\|_{W^{2,r}}^2)dt\le C.
\end{aligned}\tag{3.63}
$$

Proof It is easy to deduce from (3.60) and (3.61) that

$$
\|\nabla\rho\|_{L^2\cap L^q}\le 2\|\nabla\rho_0\|_{L^2\cap L^q}.\tag{3.64}
$$

We notice that $(1.4)_4$ combining with (3.4), (3.12), Gagliardo-Nirenberg and Sobolev's inequality that

$$
\begin{aligned}
\|b\|_{H^2}^2&\le C(\|b_t\|_{L^2}^2+\|u\cdot\nabla b\|_{L^2}^2+\|b\cdot\nabla u\|_{L^2}^2+\|b\|_{H^1}^2)\\
&\le C\|b_t\|_{L^2}^2+C\|u\|_{L^6}^2\|\nabla b\|_{L^3}^2+C\|b\|_{L^\infty}^2\|\nabla u\|_{L^2}^2+C\|\nabla b\|_{L^2}^2\\
&\le C\|b_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^2\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}+C\|b\|_{L^2}\|b\|_{H^2}\|\nabla u\|_{L^2}^2+C\|\nabla b\|_{L^2}^2\\
&\le C\|b_t\|_{L^2}^2+C\|\nabla b\|_{L^2}^2+\frac{1}{2}\|b\|_{H^2}^2,
\end{aligned}\tag{3.65}
$$

which combining with (3.25) and (3.47) shows

$$
\begin{aligned}
&\|u\|_{H^2}^2+\|P\|_{H^1}^2+\|w\|_{H^2}^2+\|b\|_{H^2}^2\\
&\quad\le C(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}+\|b_t\|_{L^2}^2)\\
&\qquad+C(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+C\||b||\nabla b|\|_{L^2}^2\\
&\quad\le C(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}+\|b_t\|_{L^2}^2)+C(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|\nabla b\|_{L^2}^2),
\end{aligned}\tag{3.66}
$$

where

$$
\||b||\nabla b|\|_{L^2}^2\le C\|b\|_{H^2}^2\|\nabla b\|_{L^2}^2\le C\|b_t\|_{L^2}^2+C\|\nabla b\|_{L^2}^2.\tag{3.67}
$$

Then, it follows from (3.5), (3.13), and (3.34) that

$$
\begin{aligned}
&\sup_{t\in[0,T]}t(\|u\|_{H^2}^2+\|P\|_{H^1}^2+\|w\|_{H^2}^2+\|b\|_{H^2}^2)\\
&\quad+\int_0^T\zeta e^{\sigma t}(\|u\|_{H^2}^2+\|P\|_{H^1}^2+\|w\|_{H^2}^2+\|b\|_{H^2}^2)dt\le C.
\end{aligned}\tag{3.68}
$$

It deduces from Lemma 2.3, Sobolev's inequality, (3.3), (3.12), (3.13), (3.66) and (3.68) that for any $r\in(2,q)$,

$$
\begin{aligned}
&\|u\|_{W^{2,r}}^2+\|P\|_{W^{1,r}}^2\\
&\quad\le C\left(\|\rho u_t\|_{L^r}^2+\|\rho u\cdot\nabla u\|_{L^r}^2+\|\nabla w\|_{L^r}^2+\|b\cdot\nabla b\|_{L^r}^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times (1+\|\nabla\mu(\rho)\|_{L^q})^{1+\frac{qr}{2(q-r)}} \\
&\le C\left(\bar\rho^2\|u_t\|_{L^r}^2+\bar\rho^2\|u\|^2_{L^{\frac{qr}{q-r}}}\|\nabla u\|^2_{L^q}+\|\nabla^2 w\|^2_{L^2}+\|b\|^2_{L^{\frac{qr}{q-r}}}\|\nabla b\|^2_{L^q}\right) \\
&\le C\left(\|\nabla u_t\|^2_{L^2}+\|\nabla u\|^2_{L^2}\|\nabla^2 u\|^2_{L^2}+\|\nabla^2 w\|^2_{L^2}+\|\nabla b\|^2_{L^2}\|\nabla^2 b\|^2_{L^2}\right) \\
&\le C\left(\|\nabla u_t\|^2_{L^2}+\|\triangle b\|^2_{L^2}\right)+C\left(\|\sqrt{\rho}u_t\|^2_{L^2}+\|\sqrt{\rho}w_t\|^2_{L^2}+\|b_t\|^2_{L^2}\right) \\
&\quad+C\left(\|\nabla u\|^2_{L^2}+\|\nabla w\|^2_{L^2}+\|\nabla b\|^2_{L^2}\right)+C\||b||\nabla b|\|^2_{L^2}. \qquad (3.69)
\end{aligned}
$$

We infer from (1.4)$_3$, regularity theory of elliptic equations, (3.3), (3.12), (3.13), (3.66) and (3.68) that for any $r\in(2,q)$,

$$
\begin{aligned}
\|w\|^2_{W^{2,r}} &\le C\left(\|\rho w_t\|^2_{L^r}+\|\rho u\cdot\nabla w\|^2_{L^r}+\|w\|^2_{L^r}+\|\nabla u\|^2_{L^r}\right) \\
&\le C\left(\bar\rho^2\|w_t\|^2_{L^r}+\bar\rho^2\|u\|^2_{L^{\frac{qr}{q-r}}}\|\nabla w\|^2_{L^q}+\|\nabla w\|^2_{L^2}+\|\nabla^2 w\|^2_{L^2}\right) \\
&\le C\left(\|\nabla w_t\|^2_{L^2}+\|\nabla u\|^2_{L^2}\|\nabla^2 w\|^2_{L^2}+\|\nabla w\|^2_{L^2}+\|\nabla^2 w\|^2_{L^2}\right) \\
&\le C\left(\|\nabla w_t\|^2_{L^2}+\|\sqrt{\rho}w_t\|_{L^2}\right)+C\left(\|\nabla u\|^2_{L^2}+\|\nabla w\|^2_{L^2}\right). \qquad (3.70)
\end{aligned}
$$

Similarly, we can obtain from (1.4)$_4$ and Sobolev's inequality that

$$
\begin{aligned}
\|b\|^2_{W^{2,r}} &\le C\left(\|b_t\|^2_{L^r}+\|u\cdot\nabla b\|^2_{L^r}+\|b\cdot\nabla u\|^2_{L^r}\right) \\
&\le C\|\nabla b_t\|^2_{L^2}+\|u\|^2_{L^{\frac{qr}{q-r}}}\|\nabla b\|^2_{L^q}+\|b\|^2_{L^{\frac{qr}{q-r}}}\|\nabla u\|^2_{L^q}) \\
&\le C\left(\|\nabla b_t\|^2_{L^2}+\|\nabla u\|^2_{L^2}\|\nabla^2 b\|^2_{L^2}+\|\nabla b\|^2_{L^2}\|\nabla^2 u\|^2_{L^2}\right) \\
&\le C(\|\nabla b_t\|^2_{L^2}+\|\sqrt{\rho}u_t\|^2_{L^2}+\|\triangle b\|^2_{L^2}+C\||b||\nabla b|\|^2_{L^2} \\
&\quad+C\left(\|\nabla u\|^2_{L^2}+\|\nabla w\|^2_{L^2}\right), \qquad (3.71)
\end{aligned}
$$

and this along with (3.69) and (3.70) by (3.5) (3.3), (3.12), (3.13) and (3.68) gives

$$
\int_0^T \zeta e^{\sigma t}\left(\|u\|^2_{W^{2,r}}+\|P\|^2_{W^{1,r}}+\|w\|^2_{W^{2,r}}+\|b\|^2_{W^{2,r}}\right)dt\le C. \qquad (3.72)
$$

This combines (3.64) and (3.68) by indicates (3.63) and the proof of Lemma 3.5 is finished. □

4 Proof of Theorem 1.1

With all the a priori estimates obtained in Sect. 3 at hand, we are now in a position to prove Theorem 1.1.

By Lemma 2.1, there exists a $T_* > 0$ such that the problem (1.4)–(1.6) has a unique local strong solution (ρ, u, w, b) on $\Omega \times (0, T_*)$. We plan to extend the local solution to all time.

Set

$$T^* = \sup\{\, T \mid (\rho, u, w, b)\ is\ a\ strong\ solution\ on \times (0, T]\}. \tag{4.1}$$

First, for any $0 < \tau < T_* < T \leq T^*$ with T finite, one deduces from (3.12), (3.33) and (3.63) that for any $q > 2$,

$$\nabla u, \nabla w, \nabla b \in C([\zeta(t), T]; L^q), \tag{4.2}$$

where one has used the standard embedding

$$L^\infty(\zeta(t), T; H^1) \cap H^1(\zeta(t), T; H^{-1}) \hookrightarrow C(\zeta(t), T; L^q)\ for\ any\ q \in (2, \infty).$$

Moreover, it deduces from (3.64), (3.3), (3.12), (3.13) and ([25], Lemma 2.3) that

$$\rho \in C([0, T]; W^{1,q}),\ \ \rho u \in C([0, T]; L^2),\ \ \rho w \in C([0, T]; L^2). \tag{4.3}$$

Finally, if $T_* < \infty$, it follows from (4.2), (4.3), (3.4) and (3.12) that

$$(\rho, u, w, b)(x, T^*) = \lim_{t \to T^*} (\rho, u, w, b)(x, t)$$

satisfies the initial condition (1.8) at $t = T^*$. Thus, taking $(\rho, u, w, b)(x, T^*)$ as the initial data, due to Lemma 2.1, it can extend the strong solutions beyond T^*. This contradicts the assumption of T^* in (4.1). The proof of Theorem 1.1 is completed.

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Declarations

Conflicts of interest The author declares no conflicts of interest regarding the publication of this paper.

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