Bull. Malays. Math. Sci. Soc. (2023) 46:177
https://doi.org/10.1007/s40840-023-01569-0

BULLETIN *of the* MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

On the Existence of *G*-Permutable Subgroups in Alternating Groups

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Received: 1 February 2023 / Revised: 26 July 2023 / Accepted: 28 July 2023 /
Published online: 21 August 2023

Abstract
Recall that a subgroup A of a group G is called G-permutable in G if for every subgroup B of G there exists an element $x \in G$ such that A and B^x commute. The following question was posed in the Kourovka Notebook: is there an integer n such that for all $m > n$ the alternating group A_m has no non-trivial A_m-permutable subgroups? We give a positive answer to this question. Moreover, in the case of prime p we prove that A_p has no non-trivial A_p-permutable subgroups except $p = 5$.

Keywords Finite group · Alternating group · G-permutable subgroup

Mathematics Subject Classification 20D05 · 20B05

1 Introduction

Recall that a subgroup A of a group G is *permutable with a subgroup* B if $AB = BA$. If A is permutable with each subgroup of G then A is called a *permutable* [1] or *quasinormal* [2] *subgroup* of G.

Ore proved that each quasinormal subgroup of a finite group is subnormal [2]. Since then his result has been generalized in different ways (see [3–5]).

Communicated by Peyman Niroomand.

The author is supported by RAS Fundamental Research Program (project FWNF-2022-0002).

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Given two subgroups A and B of a group G, a common situation is that $AB \neq BA$ but there exists an element $x \in G$ such that $AB^x = B^x A$. A minimal example of such case is the symmetric group S_3 with two different subgroups of order 2 in S_3. Following [6, 7], we mention the following cases.

(a) Let $G = AB$ be a finite group. Let A_p and B_p be Sylow p-subgroups of A and B respectively. We have $A_p B_p \neq B_p A_p$ in general, but $A_p B_p^x = B_p^x A_p$ for some element $x \in G$.
(b) Let A and B be Hall subgroups of a solvable group G. Then there exists an element $x \in G$ such that $AB^x = B^x A$.
(c) Let A and B are normally embedded subgroups of a finite solvable group G. Then there exists an element $x \in G$ such that $AB^x = B^x A$ (see [1, I]).
(d) Let A be a maximal subgroup of G and $|G : A|$ is a prime power. Then every conjugacy class of Sylow subgroups contains a subgroup that permutable with A.

In order to study the situations of such kind the following natural definitions were introduced in [8].

Definition Let A, B be subgroups of a group G and $\emptyset \neq X \subseteq G$. Then

(1) A is called *X-permutable* with B if there exists an element $x \in X$ such that $AB^x = B^x A$;
(2) A is called *hereditarily X-permutable* with B if $AB^x = B^x A$ for some $x \in X \cap \langle A, B \rangle$;
(3) A is called *(hereditarily) X-permutable in G* if A is (hereditarily) X-permutable with all subgroups of G.

The concept of a X-permutable subgroup has been intensively studied in recent years in the papers of various authors. We recommend the monograph [9] for background and results in this direction. However, the further application of this concept in solving various problems was restrained by the lack of information about G-permutable (hereditarily G-permutable) subgroups that are in composition factors of groups. Therefore, in the Kourovka Notebook the following question was posed.

Problem 1 [10, 17.37] *Is there an integer n such that for all $m > n$ the alternating group A_m has no non-trivial A_m-permutable subgroups?*

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It is clear that by a non-trivial subgroup in Problem 1 we mean a subgroup distinct from the identity group and the group itself, i.e., a proper subgroup.

Problem 1 is part of a more general question, also posed in the Kourovka Notebook.

Problem 2 [10, 17.112] *Which finite non-abelian simple groups G possess*

(a) *a non-trivial G-permutable subgroup?*
(b) *a non-trivial hereditarily G-permutable subgroup?*

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Problem 2(b) was answered in the negative for the alternating groups and for the sporadic groups in [11, 12], respectively. The authors are not aware of examples of hereditarily G-permutable subgroups in finite simple non-abelian groups.

The situation with G-permutable subgroups is different. If $G \simeq \mathrm{A}_5$ or $G \simeq \mathrm{A}_6$ then subgroups of order 2 are G-permutable subgroups in G. Indeed, since there is only one conjugacy class of involutions in G, the group $A = \langle (1,2)(3,4) \rangle$ is contained up to conjugation in all subgroups of even order and A normalizes all odd-order subgroups.

Example 1 Consider the group $G = \mathrm{PSL}_2(7)$ of order $2^3 \cdot 3 \cdot 7$. There are 3 classes of maximal subgroups of G: S_4, S_4 and $7:3$. Subgroups of orders 2, 4, 6, 8, 12 are not G-permutable with subgroup of order 7. Subgroups of order 3 are not G-permutable with cyclic subgroups of order 4. Maximal subgroups of order 21 are not G-permutable with subgroups of order 2. Finally, two classes of maximal subgroups of order 24 are not G-permutable with each other (their orders are not divisible by 7). Thus, G has no proper G-permutable subgroups.

Problem 2(a) was solved for sporadic groups and the Tits group ${}^2\mathrm{F}_4(2)'$ in [13]. Among sporadic groups and the Tits group, the only group G with proper G-permutable subgroups is the Janko group J_1.

Example 2 The simple group $G = \mathrm{PSL}_2(2^k)$ has a non-trivial G-permutable subgroup of order 2. Indeed, let A be a subgroup of order 2. Since there is only one conjugacy class of involutions in G, for every subgroup L of even order we have $A \leqslant L^g$ for some $g \in G$. Each subgroup M of odd order is contained in a cyclic subgroup of order $q \pm 1$, where $q = 2^k$. Therefore, M and A^x are contained in dihedral subgroup $D_{2(q\pm1)}$ for some $x \in G$ and MA^x is a subgroup.

Example 2 provides an infinite series of finite simple linear groups G containing G-permutable subgroups. Problem 1 asks a similar question about alternating groups. Direct computer calculations show that A_n does not contain proper A_n-permutable subgroups for $n \in \{7, 8, 9, 10\}$. It would be interesting to have some algorithm to check this for higher n and for other groups as well.

Given $A \leqslant H \leqslant G$, Examples 1 and 2 of [13] show that

$$A \text{ is } G\text{-permutable in } G \nRightarrow A \text{ is } H\text{-permutable in } H;$$
$$A \text{ is } H\text{-permutable in } H \nRightarrow A \text{ is } G\text{-permutable in } G.$$

It means that the G-permutability property is not inherited by subgroups and overgroups. It is clear (see [6, Lemma 2.1(3)]) that if A is G-permutable in G then A/K is G/K-permutable in G/K for every $K \trianglelefteq G$. In other words, the G-permutability property is inherited by factor groups.

Example 3 Consider the group $G = G_1 \times G_2 \leqslant \mathrm{A}_{10}$, where $G_1 = \langle (1,2)(3,4), (2,3,5) \rangle \simeq \mathrm{A}_5$ and $G_2 = \langle (6,7)(8,9), (6,8,10) \rangle \simeq \mathrm{A}_5$. The group $H_1 = \langle (1,2)(3,4) \rangle$ is G_1-permutable in G_1 and $H_2 = \langle (6,7)(8,9) \rangle$ is G_2-permutable in G_2.

However, $H = H_1 \times H_2$ is not G-permutable with the subgroup $B = \langle(1, 2, 3)(6, 7, 8)\rangle$ and some other subgroups of G. The group $C = \langle(1, 2)(3, 4)(6, 7)(8, 9)\rangle$ of order 2 is not G-permutable up to conjugation with exactly two subgroups $D_1 = \langle(1, 2, 3)(6, 7, 8), (1, 3)(2, 5)\rangle \simeq \mathrm{A}_4$ and $D_2 = \langle(1, 2, 3)(6, 7, 8), (6, 7)(8, 10)\rangle \simeq \mathrm{A}_4$. Moreover, G does not contain proper G-permutable subgroups except two normal subgroups G_1 and G_2.

Example 3 shows that the G-permutability property is not closed under extensions. It would be useful to find some additional conditions to deduce permutable subgroups of a finite group from permutable subgroups of its composition factors.

In order to formulate our main result we define the following property. Given $\theta > 11/20$, we say that odd integer n satisfies $(*)$ if it can be written as a sum of three primes $n = p_1 + p_2 + p_3$ with $|p_i - n/3| \leqslant n^\theta$ for $i \in \{1, 2, 3\}$.

Theorem 1 *Let $n \geqslant 710$. Suppose that n satisfies $(*)$ if n is odd, and $n - 3$ satisfies $(*)$ if n is even. Then the alternating group A_n has no proper A_n-permutable subgroups.*

As a corollary of Theorem 1 and Proposition 3 below we obtain a positive answer to Problem 1.

Corollary *There exists an integer N such that for all $n > N$ the alternating group A_n has no proper A_n-permutable subgroups.*

In the case of prime degree we obtain the following result.

Theorem 2 *Let p be a prime. Then A_p has proper A_p-permutable subgroups if and only if $p = 5$.*

The paper is organized as follows. In Sect. 2, we recall the notation and prove some auxiliary results. In Sect. 3, we prove Theorem 2. Finally, in Sect. 4, we prove our main result which is Theorem 1.

2 Notation and Preliminary Results

A cyclic group of order k is denoted by $\mathbb{Z}_k$. Given a finite set Ω, we denote the symmetric group of Ω by $\mathrm{S}(\Omega)$. In the case $\Omega = \{1, 2, \ldots, k\}$, we also write S_k. Similarly, we denote the alternating group of Ω by $\mathrm{A}(\Omega)$. In the case $\Omega = \{1, 2, \ldots, k\}$, we also write A_k. We denote the number of moved points of an element $g \in \mathrm{S}(\Omega)$ by $|\operatorname{supp}(g)|$. The set of all Sylow p-subgroups of a group G is denoted by $\mathrm{Syl}_p(G)$. A semidirect product of a normal subgroup A and a group B is denoted by $A : B$.

We use a short notation $H = \mathrm{S}_{k_1} \times \mathrm{S}_{k_2} \ldots \times \mathrm{S}_{k_s}$ assuming that $\mathrm{S}_{k_i} = \mathrm{S}(\Omega_i)$, where $\Omega = \Omega_1 \sqcup \Omega_2 \sqcup \ldots \sqcup \Omega_s$ is a partition of Ω and $\Omega_1 = \{1, \ldots, k_1\}$, $\Omega_2 = \{k_1 + 1, \ldots, k_1 + k_2\}, \ldots, \Omega_s = \{k_1 + \ldots + k_{s-1} + 1, \ldots, k_1 + \ldots + k_s\}$.

A maximal subgroup M of a group H is denoted by $M \lessdot H$. Given a prime p, by $\mathrm{AGL}_1(p)$ we denote a one-dimensional affine group over a field of p elements (see [14, §2.8]).

Proposition 1 [15] *If $p > 23$ is a prime and $X \in \{\mathrm{A}_p, \mathrm{S}_p\}$ then $\mathrm{AGL}_1(p) \cap X \lessdot X$.*

In what follows, we will use the following well-known fact. If $X \in \{\mathrm{A}_n, \mathrm{S}_n\}$ and A is a proper subgroup of X with $A \neq \mathrm{A}_n$, then A is contained in a maximal subgroup M of X and one of the following holds:

(a) $M = (\mathrm{S}_m \times \mathrm{S}_k) \cap X$, where $n = m + k$ and $m \neq k$ (intransitive case);
(b) $M = (\mathrm{S}_m \wr \mathrm{S}_k) \cap X$, where $n = mk$, $m > 1$, $k > 1$ (imprimitive case);
(c) M is primitive on Ω.

We need the following facts from number theory.

Proposition 2 [16, Lemma 1.9] *If $n > 53$ then the interval $(8n/9, n)$ contains a prime number.*

Proposition 3 [17, Theorem 1.1] *Let $\theta > 11/20$. Every sufficiently large odd integer n can be written as a sum of three primes $n = p_1 + p_2 + p_3$ with $|p_i - n/3| \leqslant n^\theta$ for $i \in \{1, 2, 3\}$.*

Recall that if n satisfies the conclusions of Proposition 3 we say that n satisfies $(*)$. The following two lemmas will be used in the proofs of the theorems.

Lemma 1 *Let $G = \mathrm{A}_n$ and $1 < A < G$. Let $M = (\mathrm{S}_k \times \mathrm{S}_{n-k}) \cap G$ be a maximal subgroup of G, where $1 \leqslant k < [\frac{n}{2}]$. If $A^x \nleqslant M$ for every $x \in G$ then A is not G-permutable subgroup of G.*

Proof Assume that A is a G-permutable subgroup of G. Then there exists $g \in G$ such that $A^g M = M A^g$, that is $A^g M \leqslant G$. Since $A^g \nleqslant M$ we have $A^g M = G$. According to [18, Theorem D] we obtain that A acts transitively on Ω.

Suppose that A acts m-transitively on Ω. Let A_{n-m-1} be a stabilizer of $(m+1)$ points in G. Then there exists $y \in G$ such that $A\,\mathrm{A}_{n-m-1}^y \leqslant G$. The group A_{n-m-1}^y is also a stabilizer of some $(m + 1)$ points, say $\{\alpha_1, \ldots, \alpha_{m+1}\}$. Since A acts m-transitively on Ω we have

$$\begin{aligned}|A| &= n|A \cap \mathrm{A}_{n-1}| = n(n-1)|A \cap \mathrm{A}_{n-2}| \\ &= \ldots = n(n-1)\ldots(n-m+1)|A \cap \mathrm{A}_{n-m}|,\end{aligned}$$

where A_{n-i} is the stabilizer in G of i points $\alpha_1, \ldots, \alpha_i$. Since $\mathrm{A}_{n-m-1} \leqslant \mathrm{A}_{n-m}$ we obtain that

$$\frac{|A|}{|A \cap \mathrm{A}_{n-m-1}|} \geqslant \frac{|A|}{|A \cap \mathrm{A}_{n-m}|} = n(n-1)\ldots(n-m+1)$$

and

$$\begin{aligned}|A\,\mathrm{A}_{n-m-1}^y| &= \frac{|\mathrm{A}_{n-m-1}||A|}{|A \cap \mathrm{A}_{n-m-1}^y|} \geqslant \frac{(n-m-1)!}{2} n(n-1)\ldots(n-m+1) \\ &> \frac{(n-1)!}{2} = |\mathrm{A}_{n-1}|\end{aligned}$$

Hence, $A\,\mathrm{A}_{n-m-1}^y = G$ and A is $(m+1)$-transitive. It follows that $A = G$; a contradiction. □

Lemma 2 *Let $G = \mathrm{A}_n$ and $1 < A < G$. Suppose that A contains a cycle of length $s > n/2 + 1$. Then A is not G-permutable subgroup of G.*

Proof Define $k = [\frac{n}{2}]$ and $M = (\mathrm{S}_{k+1} \times \mathrm{S}_{n-k-1}) \cap G$. Then $M \lessdot G$ and it follows from the conditions of the lemma that $A^x \nleqslant M$ for any $x \in G$. The conclusion of the lemma follows from Lemma 1. □

3 Proof of Theorem 2

In this section G denotes the group A_p acting on the set $\Omega = \{1, 2, \dots, p\}$. Assume that there exists a non-trivial A_p-permutable subgroup A in A_p. If A is transitive, then A is not contained in any stabilizer of a point. This is a contradiction with Lemma 1 for $k = 1$.

Consider the case $p > 23$. According to [15, Table I], G contains an affine subgroup $H = \mathrm{AGL}_1(p) \cap G$, which is maximal in G.

(a) Suppose that $A \nleqslant H^g$ for every $g \in G$. Then there exists $x \in G$ such that $AH^x = G$, and hence $AH = G$. According to [18, Theorem D], we have $\mathrm{A}_{p-k} \leqslant A$ for some k with $1 \leqslant k \leqslant 5$, and H is k-homogeneous on Ω. Let $b = (1, 2, \dots, p)$ be a cycle of length p and $B = \langle b \rangle$. Then there exists $x \in G$ such that $AB^x \leqslant G$. Since AB^x contains b and a cycle of length three, we have $AB^x = G$ by the theorem of Jordan. This implies that

$$|AB^x| = |\,\mathrm{A}_p\,| = \frac{p!}{2} \text{ and } |A| \geqslant \frac{(p-1)!}{2}.$$

It follows that $A = \mathrm{A}_{p-1}$; a contradiction with Lemma 2.

(b) Suppose that $A \leqslant H$. Then $A \leqslant \mathbb{Z}_p : \mathbb{Z}_{p-1}$. Since A is not transitive, then by Lemma 2 there are no elements of order p in A and every element of A fixes exactly one point of Ω. It follows from Lemma 1 that A is contained in the maximal subgroup $(\mathrm{S}_{p-2} \times \mathrm{S}_2) \cap G$. Since every element of H consists of a product of cycles of the same length, all non-identity elements of A have the following cyclic structure:

$$(\alpha_1, \alpha_2)(\alpha_3, \alpha_4) \dots (\alpha_{p-2}, \alpha_{p-1}), \alpha_i \in \Omega.$$

Hence, $A = \langle a \rangle \simeq \mathbb{Z}_2$. Define $B = (\mathrm{S}_3 \times \mathrm{S}_5 \times \mathrm{S}_{p-8}) \cap G$. Then there exists $x \in G$ such that $A^x B = K \leqslant G$. It is clear that $a^z \notin B$ for every $z \in G$, therefore $|K| = 2|B|$.

Since K contains a 3-cycle, K cannot be primitive. Since K contains a $(p-8)$-cycle, K cannot be transitive. Thus, K is intransitive and $K \leqslant M$, where M is one of the following groups

$$(\mathrm{S}_8 \times \mathrm{S}_{p-8}) \cap G, \quad (\mathrm{S}_3 \times \mathrm{S}_{p-3}) \cap G, \quad (\mathrm{S}_5 \times \mathrm{S}_{p-5}) \cap G.$$

Since B is maximal in each of these groups and $B < K$, we have $K = M$; a contradiction with $|M| > 2|B|$.

Now consider the cases with $p \leqslant 23$.

The case $p = 7$ is mentioned in [13].

Let $p = 11$ and $B \in \mathrm{Syl}_{11}(G)$. Since A is not transitive, we have $\gcd(|A|, 11) = 1$. There exists $x \in G$ such that $H = AB^x \leqslant G$. If $H = G$, then $A = \mathrm{A}_{10}$; a contradiction with Lemma 2. The only maximal subgroups of G with order divisible by 11 are Mathieu groups M_{11}. Hence, $H \leqslant M_{11}$ and the order of H is either 55, 660 or 7920. It follows that A contains an element a of order 5. The group H is a primitive and proper subgroup of G, so it cannot contain a cycle of length 5. Therefore, a is a product of two 5-cycles. Let $M = (\mathrm{S}_9 \times \mathrm{S}_2) \cap G$ be a maximal subgroup of G. Then $a^z \notin M$ for every $z \in G$. This is a contradiction with Lemma 1.

Let $p = 13$. According to [15, Table I] an affine subgroup $H = \mathrm{AGL}_1(13) \cap G$ is maximal in G. The same arguments as in (a) imply that $A \leqslant H$. Since $H \simeq \mathbb{Z}_{13}:\mathbb{Z}_6$, we have $A \leqslant \mathbb{Z}_6$, where $\mathbb{Z}_6 = \langle x \rangle$ and $|\operatorname{supp}(x)| = 12$. If there exists $a \in A$ with $|a| = 3$, then $a^z \notin M = (\mathrm{S}_{11} \times \mathrm{S}_2) \cap G$ for every $z \in G$. This is a contradiction with Lemma 1. Hence, $|A| = 2$. Define

$$b = (1, 2, 3)(4, 5, 6, 7, 8, 9, 10, 11, 12), \quad B = \langle b \rangle.$$

Then there exists $y \in G$ such that $A^y B \leqslant G$. In particular, $A^y \leqslant N_G(B)$. This is a contradiction with $|N_G(B)| = 81$ is odd.

Let $p = 17$ and $B \in \mathrm{Syl}_{17}(G)$. Since A is not transitive, we have $\gcd(|A|, 17) = 1$. There exists $x \in G$ such that $H = AB^x \leqslant G$. If $H = G$, then $A = \mathrm{A}_{16}$; a contradiction with Lemma 2. According to [15, Table I] the only maximal subgroups of G with order divisible by 17 are $\mathrm{PSL}_2(16):4$. It follows from [19, p. 12] that either $\mathrm{PSL}_2(16) \leqslant H \leqslant \mathrm{PSL}_2(16):4$ or $H \leqslant 17:8$.

In former case, A contains a group $2^4:15$. In particular, A contains a cycle of length 15 and therefore $A^z \nleqslant M = (\mathrm{S}_{14} \times \mathrm{S}_3) \cap G$ for every $z \in G$. This is a contradiction with Lemma 1.

In latter case $H \simeq \mathbb{Z}_{17}:\mathbb{Z}_8$ and $A \leqslant \mathbb{Z}_8$, where $\mathbb{Z}_8 = \langle x \rangle$ and $|\operatorname{supp}(x)| = 16$. If there exists $a \in A$ with $|a| > 2$, then $a^z \notin M = (\mathrm{S}_{15} \times \mathrm{S}_2) \cap G$ for every $z \in G$, that contradicts Lemma 1. Therefore, $A = \langle a \rangle$, where $|a| = 2$ and $|\operatorname{supp}(a)| = 16$. Define

$$b = (1, 2, 3)(4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16), \quad B = \langle b \rangle.$$

Then there exists $y \in G$ such that $A^y B \leqslant G$. In particular, $A^y \leqslant N_G(B)$. On the other hand, it is easy to check that an involution with such cyclic structure does not normalize B.

Let $p = 19$. According to [15, Table I], an affine subgroup $H = \mathrm{AGL}_1(19) \cap G$ is maximal in G. The same arguments as in (a) imply that $A \leqslant H$. Since $H \simeq \mathbb{Z}_{19}:\mathbb{Z}_9$, we have $A \leqslant \mathbb{Z}_9$, where $\mathbb{Z}_9 = \langle x \rangle$ and $|\operatorname{supp}(x)| = 18$. Let $a \in A$ be an element of order 3. Then $a^z \notin M = (\mathrm{S}_{17} \times \mathrm{S}_2) \cap G$ for every $z \in G$, a contradiction with Lemma 1.

Let $p = 23$ and $B \in \mathrm{Syl}_{23}(G)$. There exists $x \in G$ such that $H = AB^x \leqslant G$. If $H = G$, then $A = \mathrm{A}_{22}$; a contradiction with Lemma 2. The only maximal subgroups of G with order divisible by 23 are Mathieu groups M_{23}. It follows from [19, p.71] that

$H \simeq M_{23}$ or $H \simeq 23{:}11$, therefore $A \simeq M_{22}$ or $A \simeq \mathbb{Z}_{11}$. In both cases A contains an element a of order 11 which is a product of two 11-cycles. Let $M = (\mathrm{S}_{21} \times \mathrm{S}_2) \cap G$ be a maximal subgroup of G. Then $a^z \notin M$ for every $z \in G$, a contradiction with Lemma 1.

4 Proof of Theorem 1

The proof of Theorem 1 consists of a series of lemmas given below. Recall that G always denotes the group A_n acting on the set $\Omega = \{1, 2, \dots, n\}$ and $1 < A < G$.

Lemma 3 *Let A be a G-permutable subgroup of G and p be the largest prime number not exceeding $(n-3)$. If $n > 56$, then there exists $x \in G$ such that*

$$A^x \leqslant (\langle(1, 2, \dots, p-1)\rangle \times \mathrm{S}_{n-p}) \cap G.$$

Proof Define $H = (\mathrm{AGL}_1(p) \times \mathrm{S}_{n-p}) \cap G$. There exists $g \in G$ such that $A^g H \leqslant G$ and we can assume that $g = 1$.

Suppose that $AH = G$. Then we have

$$|G| = \frac{n!}{2} = \frac{|A| \cdot |H|}{|A \cap H|} = \frac{p(p-1)(n-p)!}{2} \frac{|A|}{|A \cap H|},$$
$$|A| = |A \cap H| \cdot n(n-1) \dots (p+1) \cdot (p-2)(p-3) \dots (n-p+1).$$

There exists a prime number p_1 in the interval $(\frac{n}{2}+1, p-2)$. Hence, p_1 divides $|A|$ and A contains a cycle of length p_1, contradicting Lemma 2.

Thus, the group AH is contained in a maximal subgroup M of G. If AH is transitive and imprimitive, then there exist positive integers m, k such that $n = mk$ and $M \simeq (\mathrm{S}_m \wr \mathrm{S}_k) \cap G$. However, the group $\mathrm{S}_m \wr \mathrm{S}_k$ does not contain elements of order p.

Assume that AH is primitive. Since $n - p \geqslant 3$ we have that H contains a cycle of length 3. By the theorem of Jordan, we get $AH = G$; a contradiction.

Therefore, AH is intransitive and

$$AH \leqslant M = (\mathrm{S}_p \times \mathrm{S}_{n-p}) \cap G.$$

If $AH = M$ we have

$$|M| = \frac{p!(n-p)!}{2} = \frac{p(p-1) \cdot (n-p)!}{2} \frac{|A|}{|A \cap H|}.$$

Then $(p-2)!$ divides $|A|$ and A contains a cycle of length p_1, where p_1 is a prime in the interval $(\frac{n}{2}+1, p-2)$. This is a contradiction with Lemma 2.

According to Proposition 1 a subgroup $\mathrm{AGL}_1(p) \simeq \mathbb{Z}_p : \mathbb{Z}_{p-1}$ is maximal in S_p for $p > 23$. Hence, $H \lessdot M$ and $H \leqslant AH < M$. Thus, $AH = H$ and $A \leqslant H$. By Lemma 2 it follows that A does not contain elements of order p. Therefore,

$$A \leqslant (\mathbb{Z}_{p-1} \times \mathrm{S}_{n-p}) \cap G.$$

There exists $x \in G$ such that $\mathbb{Z}_{p-1}^x = \langle(1, 2, \ldots, p-1)\rangle$ and $S_{n-p}^x = S_{n-p}$. Hence, $A^x \leqslant (\langle(1, 2, \ldots, p-1)\rangle \times S_{n-p}) \cap G$. □

Lemma 4 *Let $n = p_1 + p_2 + p_3$ satisfies $(*)$ and $p_1 \leqslant p_2 \leqslant p_3$. Let p be the largest prime number not exceeding $(n-3)$. If $n > 56$ and $|A| \geqslant (p_1 - 2)!$, then A is not G-permutable subgroup of G.*

Proof Assume that A is G-permutable subgroup of G. According to Lemma 3 and Proposition 2 we have

$$|A| \leqslant (n-p)!(p-1) \leqslant \left[\frac{n}{9}\right]!p.$$

By Proposition 3, we have $|p_1 - n/3| \leqslant n^\theta$, therefore

$$p_1 \geqslant \frac{n}{3} - n^\theta \geqslant \frac{n}{6}.$$

Thus, $|A| \geqslant (p_1-2)! \geqslant \left(\left[\frac{n}{6}\right] - 2\right)! > \left[\frac{n}{9}\right]!p \geqslant |A|$, a contradiction. □

Notice that a similar conclusion of Lemma 4 is true if we replace n by $n-3$.

Lemma 5 *Let A be a G-permutable subgroup of G. Let $n = p_1 + p_2 + p_3$ satisfies $(*)$ and $p_i > 23$. Let $H = \mathrm{AGL}_1(p_1) \times \mathrm{AGL}_1(p_2) \times \mathrm{AGL}_1(p_3) \cap G$. If p_1, p_2, p_3 are pairwise distinct, then there exists $x \in G$ such that $A^x \leqslant H$.*

Proof There exists $x \in G$ such that $A^x H \leqslant G$. We can assume that $x = 1$.

If $AH = G$, then A acts transitively on Ω, and we can repeat the arguments from the proof of Lemma 1 to get a contradiction.

Hence, AH is contained in a maximal subgroup M of G. If AH is transitive and imprimitive, then there exist $m \geqslant 3$, $k \geqslant 3$ such that $n = mk$ and $M \simeq (S_m \wr S_k) \cap G$. However, the group $S_m \wr S_k$ does not contain elements of prime order s, where $s = \max\{p_1, p_2, p_3\}$.

Assume that AH is primitive. Since H contains a cycle of length p_1 then according to [5, Theorem 8.23] we have $AH = G$; a contradiction.

Thus, AH is intransitive and

$$AH \leqslant M = (S_k \times S_{n-k}) \cap G, \text{ where } k \in \{p_1, p_2, p_3\}.$$

We can assume that $k = p_1$ and $AH \leqslant (S_{p_1} \times S_{p_2+p_3}) \cap G$.

Let $AH_{2,3}$ be the projection of the group AH on $S_{p_2+p_3}$. It is clear that

$$\mathrm{AGL}_1(p_2) \times \mathrm{AGL}_1(p_3) \leqslant AH_{2,3} \leqslant S_{p_2+p_3}.$$

Assume that $AH_{2,3} = S_{p_2+p_3}$. Let $A_{2,3}$ and $H_{2,3}$ be the projections of the groups A and H on $S_{p_2+p_3}$, respectively. Then we have

$$|S_{p_2+p_3}| = (p_2+p_3)! = |AH_{2,3}| \leqslant |A_{2,3}| \cdot |H_{2,3}| = |A_{2,3}| \cdot p_2(p_2-1)p_3(p_3-1).$$

Hence, $|A| \geqslant |A_{2,3}| > (p_2 - 2)!$, which is a contradiction with Lemma 4.

Therefore, $AH_{2,3}$ is contained in a maximal subgroup $M_{2,3}$ of $S_{p_2+p_3}$. If $AH_{2,3}$ is transitive and imprimitive, then there exist m', k' such that $p_2 + p_3 = m'k'$ and $M_{2,3} \simeq (S'_m \wr S'_k) \cap G$. However, the group $S'_m \wr S'_k$ does not contain elements of order p', where $p' = \max\{p_2, p_3\}$.

Assume that $AH_{2,3}$ is primitive. Since H contains a cycle of length p_2 then according to [5, Theorem 8.23] we have $AH_{2,3} = S_{p_2+p_3}$; a contradiction.

Thus, $AH_{2,3}$ is intransitive and

$$AH_{2,3} \leqslant S_{p_2} \times S_{p_3}, \quad AH \leqslant (S_{p_1} \times S_{p_2} \times S_{p_3}) \cap G.$$

Let AH_i be the projection of AH on S_{p_i}, where $i \in \{1, 2, 3\}$. Then $\mathrm{AGL}_1(p_i) \leqslant AH_i \leqslant S_{p_i}$. Since $p_i > 23$ we have $\mathrm{AGL}_1(p_i) \lessdot S_{p_i}$.

Assume that $AH_j = S_{p_j}$ for some $j \in \{1, 2, 3\}$. Let B_j and H_j be projections of groups A and H on S_{p_j}, respectively. Then we have

$$|S_{p_j}| = p_j! = |AH_j| \leqslant |B_j| \cdot |H_j| = p_j(p_j - 1)|B_j|, \quad |A| \geqslant |B_j| \geqslant (p_j - 2)!$$

which is a contradiction with Lemma 4.

Hence, $AH_i = \mathrm{AGL}_1(p_i)$ and $B_i \leqslant \mathrm{AGL}_1(p_i)$ for $i \in \{1, 2, 3\}$. Therefore,

$$A \leqslant (B_1 \times B_2 \times B_3) \cap G \leqslant \mathrm{AGL}_1(p_1) \times \mathrm{AGL}_1(p_2) \times \mathrm{AGL}_1(p_3) \cap G.$$

□

Lemma 6 *Let A be a G-permutable subgroup of G. Let $n = p_1 + p_1 + p_1$ satisfies* (∗) *and $p_1 > 23$. Let $H = (\mathrm{AGL}_1(p_1) \times \mathrm{AGL}_1(p_1) \times \mathrm{AGL}_1(p_1)) : S_3 \cap G$. Then there exists $x \in G$ such that $A^x \leqslant H$.*

Proof There exists $x \in G$ such that $A^x H \leqslant G$ and we can assume that $x = 1$.

If $AH = G$, then $|A| \geqslant |G|/|H| \geqslant (p_1 - 2)!$, which is a contradiction with Lemma 4.

Hence, AH is contained in a maximal subgroup M of G. Assume that AH is primitive. Since H contains a cycle of length p_1 then according to [5, Theorem 8.23] we have $AH = G$; a contradiction.

Since H is transitive and AH is imprimitive then $AH \leqslant (S_{p_1} \wr S_3) \cap G$ and

$$(\mathrm{AGL}_1(p_1) \times \mathrm{AGL}_1(p_1) \times \mathrm{AGL}_1(p_1)) : S_3 \cap G \leqslant AH \leqslant (S_{p_1} \times S_{p_1} \times S_{p_1}) : S_3 \cap G.$$

Let AH_i be the projections of AH on the corresponding groups $S_{p_1} = S(\Omega_i)$, where $i \in \{1, 2, 3\}$. Then $\mathrm{AGL}_1(p_1) \leqslant AH_i \leqslant S_{p_1}$. Since $p_1 > 23$ it follows that $\mathrm{AGL}_1(p_1) \lessdot S_{p_1}$. Applying the same arguments as in the proof of Lemma 5 we have $AH_i \neq S(\Omega_i)$.

Hence, $AH_i = \mathrm{AGL}_1(p_1)$ and $B_i \leqslant \mathrm{AGL}_1(p_1)$ for $i \in \{1, 2, 3\}$. Thus,

$$A \leqslant (B_1 \times B_2 \times B_3) : S_3 \cap G \leqslant (\mathrm{AGL}_1(p_1) \times \mathrm{AGL}_1(p_1) \times \mathrm{AGL}_1(p_1)) : S_3 \cap G = H.$$

□

Lemma 7 *Let A be a G-permutable subgroup of G. Let $n = p_1 + p_1 + p_2$ satisfies $(*)$, $p_i > 23$, and $p_1 \neq p_2$. Let $H = ((\mathrm{AGL}_1(p_1) \times \mathrm{AGL}_1(p_1)) : \mathrm{S}_2) \times \mathrm{AGL}_1(p_2) \cap G$. Then there exists $x \in G$ such that $A^x \leqslant H$.*

Proof It follows from the proofs of Lemmas 5 and 6. □

Lemma 8 *Let A be a G-permutable subgroup of G. Let n be an even number, $n - 3 = p_1 + p_2 + p_3$ satisfies $(*)$, and $p_i > 23$. Let $H = \mathrm{AGL}_1(p_1) \times \mathrm{AGL}_1(p_2) \times \mathrm{AGL}_1(p_3) \times \mathrm{S}_3 \cap G$. If p_1, p_2, p_3 are pairwise distinct, then there exists $x \in G$ such that $A^x \leqslant H$.*

Proof The proof is similar to the proof of Lemma 5. □

Lemma 9 *Let A be a G-permutable subgroup of G. Let n be an even number, $n - 3 = p_1 + p_1 + p_1$ satisfies $(*)$, and $p_1 > 23$. Let $H = (\mathrm{AGL}_1(p_1) \wr \mathrm{S}_3) \times \mathrm{S}_3 \cap G$. Then there exists $x \in G$ such that $A^x \leqslant H$.*

Proof There exists $x \in G$ such that $A^x H \leqslant G$ and we can assume that $x = 1$.

If $AH = G$, then $|A| \geqslant |G|/|H| \geqslant (p_1 - 2)!$, contradicting Lemma 4.

Hence, AH is contained in a maximal subgroup M of G. Assume that AH is primitive. Since H contains a cycle of length p_1 then according to [5, Theorem 8.23] we have $AH = G$; a contradiction.

If AH is transitive and imprimitive, then $M \simeq (\mathrm{S}_m \wr \mathrm{S}_k) \cap G$ for some $m, k \geqslant 2$. However, the group $\mathrm{S}_m \wr \mathrm{S}_k$ does not contain elements of the following cyclic structure

$$(\alpha_1, \ldots, \alpha_{p_1})(\beta_1, \ldots, \beta_{p_1})(\gamma_1, \ldots, \gamma_{p_1})(\alpha, \beta, \gamma),$$

which are contained in H.

Thus, AH is intransitive and $AH \leqslant \mathrm{S}_{3p_1} \times \mathrm{S}_3$.

Let AH_1 be the projection of AH on S_{3p_1}. Applying the same arguments as in the proof of Lemma 6, we have $AH_1 \leqslant \mathrm{AGL}_1(p_1) \wr \mathrm{S}_3$. Hence, $A \leqslant AH_1 \times \mathrm{S}_3 \leqslant (\mathrm{AGL}_1(p_1) \wr \mathrm{S}_3) \times \mathrm{S}_3$ and $A \leqslant G$, that is $A \leqslant H$. □

Lemma 10 *Let A be a G-permutable subgroup of G. Let n be an even number and $n - 3 = p_1 + p_1 + p_2$ satisfies $(*)$, where $p_i > 23$ and $p_1 \neq p_2$. Let $H = ((\mathrm{AGL}_1(p_1) \times \mathrm{AGL}_1(p_1)) : \mathrm{S}_2) \times \mathrm{AGL}_1(p_2) \times \mathrm{S}_3 \cap G$. Then there exists $x \in G$ such that $A^x \leqslant H$.*

Proof It follows from the proofs of Lemmas 5 and 9. □

Lemma 11 *Let $n \geqslant 710$, $|A| \leqslant 2n^3$ and $|a| \leqslant 7$ for every element $a \in A$ of prime order. Suppose that there exists $x \in A$ of prime order such that $|\operatorname{supp}(x)| \geqslant n - 6$. Then A is not a G-permutable subgroup of G.*

Proof Define

$$B = \mathrm{S}_7 \times \mathrm{S}_{11} \times \mathrm{S}_{19} \times \mathrm{S}_{41} \times \mathrm{S}_{79} \times \mathrm{S}_{163} \times \mathrm{S}_{331} \times \mathrm{S}_{n-651} \cap G.$$

It follows from the cyclic structure of x that $x \notin B$. Assume that A is G-permutable with B. Then there exists $y \in G$ such that $H = A^y B \leqslant G$. Since $x \notin B$ we have $H > B$.

The group H cannot be primitive because it contains a 3-cycle and cannot be transitive because it contains a cycle of prime order $q > n/2$. Hence, H is intransitive and $H \leqslant \mathrm{S}_k \times \mathrm{S}_{n-k}$ for some $k < n$.

Define $\Delta = \{7, 11, 19, 41, 79, 163, 331, n - 651\}$. If $k \notin \Delta$, then we define H_k as a projection of H on S_k. Assume that H_k is primitive. Since H_k contains a 3-cycle, then $H_k = \mathrm{S}_k$. It is clear that k is the sum of numbers from Δ. For $k > n - 651$ we have $k \geqslant n - 651 + 7$ and

$$|A^y| \cdot |B| \geqslant |A^y B| = |H| \geqslant \frac{|B| \cdot \prod_{i=1}^{7}(n - 651 + i)}{|\,\mathrm{S}_7\,|} > |B| \cdot 2n^3,$$

where the last inequality is true for all $n \geqslant 710$. This is a contradiction with $|A| \leqslant 2n^3$.

For $k < n - 651$ we obtain that the number $\frac{|H_k|}{|B|} = \frac{|\,\mathrm{S}_k\,|}{|B|}$ contains a prime divisor q greater than 7. Since $\frac{|H_k|}{|B|}$ divides $\frac{|H|}{|B|}$ and $\frac{|H|}{|B|}$ divides $|A|$, it follows that A contains an element of order q, which contradicts the conditions of the lemma.

Therefore, H_k is imprimitive. Each successive number from Δ (except maybe the last one) is a prime number greater than the sum of all the previous ones, so the group H_k contains a cycle of prime order $s > k/2$, which implies that H_k is intransitive. Hence, $H_k \leqslant \mathrm{S}_{k_1} \times \mathrm{S}_{k_2}$ for some $1 < k_1, k_2 < k$. Continuing in this way, we obtain that $H \leqslant H_{k_1} \times H_{k_2} \times \ldots \times H_{k_8} \cap G = B$. This is a contradiction with $H > B$. □

Lemma 12 *Let $n \geqslant 710$ and $n = p_1 + p_1 + p_1$ ($n = p_1 + p_1 + p_2$) satisfies $(*)$. Then G does not contain proper G-permutable subgroups.*

Proof Assume that A is a proper G-permutable subgroup of G and $n = p_1 + p_1 + p_1$. Then by Lemmas 3 and 6 there exist elements $x, y \in G$ such that

$$A^x \leqslant \langle(1, 2, \ldots, p - 1)\rangle \times \mathrm{S}_{n-p} \cap G, \quad A^y \leqslant (\mathrm{AGL}_1(p_1) \\ \times \mathrm{AGL}_1(p_1) \times \mathrm{AGL}_1(p_1)) : \mathrm{S}_3 \cap G.$$

Define

$$K = \langle(1, 2, \ldots, p - 1)\rangle \times \mathrm{S}_{n-p} \cap G, L = (\mathrm{AGL}_1(p_1) \times \mathrm{AGL}_1(p_1) \\ \times \mathrm{AGL}_1(p_1)) : \mathrm{S}_3 \cap G.$$

By Propositions 2 and 3 we get that $p > 2p_1 + 1$. Then for every element $g \in K$ we have either $|\,\mathrm{supp}(g)| \geqslant p - 1 > 2p_1$ or $|\,\mathrm{supp}(g)| \leqslant n - p < p_1 - 1$. On the other hand, for every element $g \in L$ we have

$$|\,\mathrm{supp}(g)| \in \{p_1 - 1, p_1, 2p_1 - 2, 2p_1 - 1, 2p_1, 3p_1 - 3, 3p_1 - 2, 3p_1 - 1, 3p_1\}.$$

Thus, if $g \in A$, then $|\,\mathrm{supp}(g)| \in \{3p_1 - 3, 3p_1 - 2, 3p_1 - 1, 3p_1\}$. It follows from Lemma 1 that $A^z \leqslant \mathrm{S}_{n-1}$ for some $z \in G$. In particular, for every $g \in A$ we have

$|\operatorname{supp}(g)| < 3p_1$ and A does not contain elements of order p_1. Therefore,

$$|A| \leqslant (p_1-1)^3 \cdot |\,\mathrm{S}_3\,| \leqslant 6\left(\frac{n}{3}\right)^3 = \frac{2}{9}n^3.$$

Let $a \in A$ be an element of prime order l. Then

$$a = (\alpha_1, \ldots, \alpha_l)\ldots(\beta_1, \ldots, \beta_l), \quad |\operatorname{supp}(a)| \geqslant n-3.$$

If $l > 3$, then for every element $g \in G$ we have $a^g \notin \mathrm{S}_{n-3} \times \mathrm{S}_3$, which contradicts Lemma 1. Hence, prime orders of the elements of A do not exceed 3. By Lemma 11 we obtain that A is not a G-permutable subgroup of G.

The case $n = p_1 + p_1 + p_2$ is proved similarly using Lemmas 3 and 7. □

Lemma 13 *Let $n \geqslant 710$ and $n = p_1 + p_2 + p_3$ satisfies $(*)$. If p_1, p_2, p_3 are pairwise distinct then G does not contain proper G-permutable subgroups.*

Proof Assume that A is a proper G-permutable subgroup of G. Then by Lemmas 3 and 5 there exist elements $x, y \in G$ such that

$$A^x \leqslant \langle(1, 2, \ldots, p-1)\rangle \times \mathrm{S}_{n-p} \cap G, \quad A^y \leqslant \mathrm{AGL}_1(p_1) \times \mathrm{AGL}_1(p_2) \times \mathrm{AGL}_1(p_3) \cap G.$$

Define $K = \langle(1, 2, \ldots, p-1)\rangle \times \mathrm{S}_{n-p} \cap G$. By Propositions 2 and 3 we get that $p > p_i + p_j + 1$, where $i, j \in \{1, 2, 3\}$. Then for every element $g \in K$ we have either $|\operatorname{supp}(g)| \geqslant p-1 > p_i + p_j$ or $|\operatorname{supp}(g)| \leqslant n-p < p_i$. Hence, A does not contain elements of prime orders p_i, where $i \in \{1, 2, 3\}$. Thus,

$$|A| \leqslant (p_1-1)(p_2-1)(p_3-1) \leqslant n^3.$$

Moreover, for every element $g \in A$ we have $|\operatorname{supp}(g)| = p_1 + p_2 + p_3 - 3 = n-3$.

Similarly to the proof of Lemma 12 we obtain that prime orders of elements of A do not exceed 3. This is a contradiction with Lemma 11. □

Lemma 14 *Let $n \geqslant 710$ be an even number and $n-3 = p_1 + p_1 + p_1$ ($n-3 = p_1 + p_1 + p_2$) satisfies $(*)$. Then G does not contain proper G-permutable subgroups.*

Proof Assume that A is a proper G-permutable subgroup of G and $n-3 = p_1 + p_1 + p_1$. By Lemmas 3 and 9 there exist elements $x, y \in G$ such that

$$A^x \leqslant \langle(1, 2, \ldots, p-1)\rangle \times \mathrm{S}_{n-p} \cap G, \quad A^y \leqslant (\mathrm{AGL}_1(p_1) \wr \mathrm{S}_3) \times \mathrm{S}_3 \cap G.$$

Define $K = \langle(1, 2, \ldots, p-1)\rangle \times \mathrm{S}_{n-p} \cap G$ and $L = (\mathrm{AGL}_1(p_1) \wr \mathrm{S}_3) \times \mathrm{S}_3 \cap G$. Then for every element $g \in K$ we have either $|\operatorname{supp}(g)| \geqslant p-1 > 2p_1 + 3$ or $|\operatorname{supp}(g)| \leqslant n-p < p_1 - 1$. On the other hand, for every element $g \in L$ we have

$$|\operatorname{supp}(g)| \in \{\delta, p_1-1+\delta, p_1+\delta, 2p_1-2+\delta, 2p_1-1+\delta, 2p_1+\delta, 3p_1-3+\delta, 3p_1-2+\delta, 3p_1-1+\delta, 3p_1+\delta\},$$

where $\delta \in \{0, 2, 3\}$. Thus, if $g \in A$ then

$$|\operatorname{supp}(g)| \in \{\delta, 3p_1 - 3 + \delta, 3p_1 - 2 + \delta, 3p_1 - 1 + \delta, 3p_1 + \delta\}.$$

If A contains an element v of order p_1, then it has a cyclic form

$$v = (\alpha_1, \ldots, \alpha_{p_1})(\beta_1, \ldots, \beta_{p_1})(\gamma_1, \ldots, \gamma_{p_1}).$$

For every element $g \in G$ we get that $v^g \notin \mathrm{S}_{n-7} \times \mathrm{S}_7$ and $A^g \not\leqslant \mathrm{S}_{n-7} \times \mathrm{S}_7$, which contradicts Lemma 1. Thus, A does not contain elements of prime order p_1 and

$$|A| \leqslant (p_1 - 1)^3 \cdot |\,\mathrm{S}_3\,| \cdot |\,\mathrm{S}_3\,| \leqslant 36 \left(\frac{n}{3}\right)^3 = \frac{4}{3} n^3.$$

If $A \leqslant \mathrm{S}_3 = \mathrm{S}(\alpha_1, \alpha_2, \alpha_3)$, then A is not G-permutable with B, where $B = \langle (1, 2, 3, \ldots, n) \rangle$.

Thus, A contains an element w such that $|\operatorname{supp}(w)| \geqslant 3p_1 - 3 = n - 6$. The element w has a form $w = ab$, where $a \in \mathrm{AGL}_1(p_1) \wr \mathrm{S}_3$, $b \in \mathrm{S}_3$. Then either $|w|$ is a prime number or $w' = w^{|b|} \neq 1$ and $w' \in \mathrm{AGL}_1(p_1) \wr \mathrm{S}_3$. Taking a suitable power of the element w', we obtain an element $u \in A$ of prime order. Since for every nonidentity element $g \in \mathrm{AGL}_1(p_1) \wr \mathrm{S}_3$ we have $|\operatorname{supp}(g)| \geqslant p_1 - 1$, then $|\operatorname{supp}(u)| \geqslant 3p_1 - 3 = n - 6$.

Let $z \in A$ be an element of prime order l. Then either $z \in \mathrm{S}_3$ and $|z| \in \{2, 3\}$ or

$$z = (\alpha_1, \ldots, \alpha_l) \ldots (\beta_1, \ldots, \beta_l), \quad |\operatorname{supp}(z)| \geqslant n - 6.$$

If $l > 7$, then for every element $g \in G$ we have $z^g \notin \mathrm{S}_{n-7} \times \mathrm{S}_7$, which contradicts Lemma 1. Hence, prime orders of the elements of A do not exceed 7. By Lemma 11 we obtain that A is not a G-permutable subgroup of G.

The case $n = p_1 + p_1 + p_2$ is proved similarly using Lemmas 3 and 10. □

Lemma 15 *Let $n \geqslant 710$ be an even number and $n - 3 = p_1 + p_2 + p_3$ satisfies* (∗). *If p_1, p_2, p_3 are pairwise distinct, then G does not contain proper G-permutable subgroups.*

Proof Assume that A is a proper G-permutable subgroup of G. By Lemmas 3 and 9 there exist elements $x, y \in G$ such that

$$A^x \leqslant \langle (1, 2, \ldots, p - 1) \rangle \times \mathrm{S}_{n-p} \cap G, \quad A^y \leqslant \mathrm{AGL}_1(p_1) \times \mathrm{AGL}_1(p_2) \times \mathrm{AGL}_1(p_3) \times \mathrm{S}_3 \cap G$$

Define $K = \langle (1, 2, \ldots, p - 1) \rangle \times \mathrm{S}_{n-p} \cap G$, $L = \mathrm{AGL}_1(p_1) \times \mathrm{AGL}_1(p_2) \times \mathrm{AGL}_1(p_3) \times \mathrm{S}_3 \cap G$.

By Propositions 2 and 3 it follows that $p > p_i + p_j + 4$, where $i, j \in \{1, 2, 3\}$. Then for every element $g \in K$ we have either $|\operatorname{supp}(g)| \geqslant p - 1 > p_i + p_j + 3$ or $|\operatorname{supp}(g)| \leqslant n - p < p_i - 1$.

Since $A \leqslant K^{x^{-1}} \cap L^{y^{-1}}$ then for every element $g \in A$ we have

$$|\operatorname{supp}(g)| \in \{\delta, n-6+\delta, n-5+\delta, n-4+\delta, n-3+\delta\},$$

where $\delta \in \{0, 2, 3\}$. In particular, since p_1, p_2, p_3 are pairwise distinct, we have that A does not contain elements of orders p_i, where $i \in \{1, 2, 3\}$. Thus,

$$|A| \leqslant (p_1-1)(p_2-1)(p_3-1) \cdot |\,S_3\,| \leqslant 6\left(\frac{n}{2}\right)^3 \leqslant n^3.$$

Further, similarly to the proof of Lemma 14 we obtain that prime orders of elements of A do not exceed 7 and there exists an element $a \in A$ of prime order such that $|\operatorname{supp}(a)| \geqslant n-6$. This is a contradiction with Lemma 11. □

Theorem 2 now follows from Lemmas 12–15.

Acknowledgements The authors are deeply grateful to the referee for carefully reading the paper and helpful suggestions.

Data availability All relevant data are within the paper.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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