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# Dispersive Estimates and Asymptotic Behavior for a Generalized Boussinesq-Type Equation

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**Abstract**

In this paper, we study the Cauchy problem for a generalized Boussinesq-type equation in $\mathbb{R}^n$. We establish a dispersive estimate for the linear group associated with the generalized Boussinesq-type equation. As applications, the global existence, decay and scattering of solutions are established for small initial data.

**Keywords** Boussinesq equation · Dispersive estimate · Existence · Decay · Scattering

**Mathematics Subject Classification** Primary 35Q35; Secondary 47J35 · 35B40

## 1 Introduction

In this paper, we study the following Cauchy problem of the sixth-order generalized Boussinesq-type equation in $\mathbb{R}^n$, describing the surface waves in shallow waters [1, 2]

$$u_{tt} - \Delta u + \Delta^2 u - \Delta u_{tt} - \Delta^3 u = \Delta f(u), \tag{1.1}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \tag{1.2}$$

where the nonlinear term has the form $f(u) = O(|u|^p)$, $p > 1$.

Boussinesq's theory was the first to give a satisfactory, scientific explanation of the phenomenon of solitary waves discovered by Scott Russell [23]. The classical

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Boussinesq equation can be written

$$u_{tt} - u_{xx} + \alpha u_{xxxx} = (u^2)_{xx}, \tag{1.3}$$

where $\alpha \in \mathbb{R}$ depends on the depth of fluid and the characteristic speed of long waves. Actually, the classical Boussinesq equation is a dispersive equation for $\alpha > 0$. The dispersion comes from the term $u_{xxxx}$. By taking advantage of the dispersion, the well-posedness and scattering of solutions to the Cauchy problem of (1.3) and its generalized versions were established in [5, 7, 11, 13]. For other results on local existence, finite time blowup, stability and instability of solitary waves and so on, see [3, 4, 6, 12, 24, 32] and references therein. Also, the equation (1.3) with the damped term $-\partial_{txx} u$ was studied by many researchers, see [14, 25] and so on.

Following the work of the Boussinesq equation (1.3), various of Boussinesq-type equations have been carried out to describe different physical process. For example, Makhankov [16] modified (1.3) to describe ion-sound waves in plasma as follows

$$u_{tt} - u_{xx} - u_{xxtt} = (u^2)_{xx}. \tag{1.4}$$

Samsonov, Sokurinskaya [21] modified (1.3) and (1.4) to describe the nonlinear waves propagation in waveguide with the possibility of energy exchange through lateral surfaces of the waveguide as follows

$$u_{tt} - u_{xx} + u_{xxxx} - u_{xxtt} = (u^2)_{xx}. \tag{1.5}$$

Furthermore, Schneider and Wayne [22] modified (1.5) to model the water wave problem with surface tension as below

$$u_{tt} - u_{xx} + u_{xxxx} - u_{xxtt} + u_{xxxxtt} = (u^2)_{xx}. \tag{1.6}$$

For the Boussinesq-type equations (1.4)–(1.6) and their generalized versions, all are dispersive equations. The dispersions were regarded as the basic tool for the existence and scattering, see [15, 27, 30]. The local existence and finite time blowup were studied by [9, 31, 33]. For the equations (1.4)–(1.6) with the damped term $-u_{txx}$, there are also many results, see [10, 18, 19, 29] and so on.

For the equation (1.1), it is also a Boussinesq-type equation and dispersive equation. But as far as we know, there are few results. Up to now, there are only some results about the equation (1.1) with the damped term $-\Delta u_t$. For example, the initial boundary value problem was investigated in [34], and they obtained the existence of strong solutions and the long time asymptotic. Later, [26, 28] considered the Cauchy problem, and they established the global existence and asymptotic behavior for small initial data. These results all depended on deeply the important role of the dissipation term $-\Delta u_t$. Inspired by the studies of Boussinesq-type equations (1.3)–(1.6), it is nature to ask whether we can use the dispersion in (1.1) to obtain some fundamental mathematical results without the dissipation term $-\Delta u_t$.

Let's observe the dispersion in (1.1). By the method of the Green function, we can transform the Cauchy problem (1.1)–(1.2) into an integral equation. Considering the

Cauchy problem

$$\begin{cases} \partial_{tt} G - \Delta G + \Delta^2 G - \Delta G_{tt} - \Delta^3 G = 0, \\ G(x,0) = 0,\ \partial_t G(x,0) = \delta. \end{cases} \tag{1.7}$$

By the Fourier transform in (1.7), one has

$$\begin{cases} \partial_{tt} \hat{G} + |\xi|^2 \hat{G} + |\xi|^4 \hat{G} + |\xi|^2 \hat{G}_{tt} + |\xi|^6 \hat{G} = 0, \\ \hat{G}(\xi,0) = 0,\ \partial_t \hat{G}(\xi,0) = 1. \end{cases} \tag{1.8}$$

The characteristic equation of (1.8) is

$$\tau^2 + |\xi|^2 + |\xi|^4 + |\xi|^2\tau^2 + |\xi|^6 = 0,$$

which implies

$$\tau = \pm i p(|\xi|),$$

where

$$p(|\xi|) = |\xi| \sqrt{\frac{1 + |\xi|^2 + |\xi|^4}{1 + |\xi|^2}}.$$

Thus, one can solve the Cauchy problem (1.8)

$$\hat{G}(\xi,t) = \frac{\sin(tp(|\xi|))}{p(|\xi|)}, \quad \partial_t \hat{G}(\xi,t) = cos(tp(|\xi|)).$$

The Duhamel principle implies that the solution of (1.1)–(1.2) is represented by

$$u(t) = \partial_t G(t) * u_0 + G(t) * u_1 + \int_0^t \frac{\Delta}{1-\Delta} G(t-\tau) * f(u)(\tau) \mathrm{d}\tau, \tag{1.9}$$

where $\partial_t G(t)$ and $G(t)$ are defined as

$$\partial_t G(t) = \mathcal{F}^{-1} \cos(tp(|\xi|)), \quad G(t) = \mathcal{F}^{-1} \frac{\sin(tp(|\xi|))}{p(|\xi|)},$$

and $\mathcal{F}^{-1}$ is the inverse Fourier transform. From the expression of the Green function $G$, the equation (1.1) exhibits a dispersion phenomenon which is due to the presence of terms $\Delta u$, $\Delta^2 u$, $\Delta^3 u$. This is closely related to the dispersive estimate for the operator $e^{itp(|\nabla|)}$ defined by the Fourier integral

$$e^{itp(|\nabla|)} f = \mathcal{F}^{-1} e^{itp(|\xi|)} \hat{f} = \int_{\mathbb{R}^n} e^{i(x\xi + tp(|\xi|))} \hat{f} \mathrm{d}\xi. \tag{1.10}$$

In order to describe the main results in this paper, we introduce some notations and spaces. The dual number of $r$ $(1 \leqslant r \leqslant \infty)$ is denoted by $r'$, i.e., $\frac{1}{r} + \frac{1}{r'} = 1$. The notation $f \in g(|\nabla|)X$ means $g^{-1}(|\nabla|)f \in X$ for a function space $X$, where $|\nabla|$ is defined by $(\widehat{|\nabla|f})(\xi) = |\xi|\hat{f}(\xi)$. $L^q = L^q(\mathbb{R}^n)$ and $W^{s,q}(\mathbb{R}^n) = (1-\Delta)^{-\frac{s}{2}}L^q(\mathbb{R}^n)(1 \leqslant q \leqslant \infty, s \in \mathbb{R})$ denote Lebesgue spaces and inhomogeneous Sobolev spaces, respectively. In particular, $H^s = W^{s,2}$. $\dot{B}^s_{r,q}$ and $B^s_{r,q}$ $(1 \leqslant r, q \leqslant \infty, s \in \mathbb{R})$ represent the homogeneous and inhomogeneous Besov spaces, respectively.

The first result in this paper is to obtain the dispersive estimate (1.10). The strategy is described. We can use the stationary phase estimate to get the desired decay estimate in $\mathbb{R}$. Because the symbol $p(|\xi|)$ of the operator is a radial function, we can use the Fourier transform of a radial function to reduce the problem to one-dimensional case in $\mathbb{R}^n(n \geqslant 2)$. This way to deal with dispersive estimates has been applied by many mathematicians [8, 15, 30] an so on.

**Theorem 1.1** *If* $2 \leqslant r \leqslant \infty$, *then we have for* $f \in \Theta^{-(1-\frac{2}{r})}\dot{B}^{\frac{n}{r}}_{r',1} \cap \dot{B}^{\frac{n}{r'}}_{r',1}$ *that*

$$\|e^{itp(|\nabla|)}f\|_{L^\infty} \lesssim (1+|t|)^{-\frac{n}{2}(1-\frac{2}{r})}\|f\|_{\Theta^{-(1-\frac{2}{r})}\dot{B}^{\frac{n}{r}}_{r',1}\cap\dot{B}^{\frac{n}{r'}}_{r',1}},$$

*where* $\Theta$ *is a operator defined by*

$$\Theta g = \mathcal{F}^{-1}\left(\frac{p'(|\xi|)}{|\xi|}\right)^{-\frac{n-1}{2}}(p''|\xi|))^{-\frac{1}{2}}\hat{g}.$$

By making use of the above dispersive estimate, we obtain the estimates in $L^\infty$ space of linear part and nonlinear part associated with the equation (1.1), respectively, which we apply to study the existence and decay of global small amplitude solutions to the Cauchy problem (1.1)–(1.2) by the method of the contractive mapping principle.

**Theorem 1.2** *Suppose when* $n = 1$ *and* $2 < r < 4$ *or when* $n \geqslant 2$ *and* $2 < r < \infty$, $s > \frac{n}{r'}$ *and*

$$p \geqslant s, \;\; p > \frac{2}{r'} + \max\left\{1, \frac{1}{\frac{n}{2}(1-\frac{2}{r})}\right\},$$

*there exists small* $\delta > 0$ *such that*

$$\|u_0\|_{\Theta^{-(1-\frac{2}{r})}\dot{B}^{\frac{n}{r}}_{r',1}\cap\dot{B}^{\frac{n}{r'}}_{r',1}\cap H^s} + \|u_1\|_{p(|\nabla|)\left(\Theta^{-(1-\frac{2}{r})}\dot{B}^{\frac{n}{r}}_{r',1}\cap\dot{B}^{\frac{n}{r'}}_{r',1}\cap H^s\right)} \leqslant \delta.$$

*Then, the Cauchy problem (1.1)–(1.2) possesses a unique solution* $u(x,t) \in \mathcal{C}(\mathbb{R}; H^s)$ *with a positive number* $\rho$ *depending on* $p, \delta, r$ *such that*

$$\sup_{t\in\mathbb{R}}(1+|t|)^{\frac{n}{2}(1-\frac{2}{r})}\|u\|_{L^\infty} + \sup_{t\in\mathbb{R}}\|u\|_{H^s} \leqslant \rho.$$

With the help of the representation of solutions (1.9) and the decay of solutions in Theorem 1.2, we can construct the scattering of solutions.

**Theorem 1.3** *Let $u(x,t)$ be the solution to the Cauchy problem (1.1)–(1.2) in Theorem 1.2. Then, there exists the unique solution $u^{\pm}$ of the linear equation corresponding to (1.1), i.e., $f = 0$, with initial data*

$$\hat{u}_0^{\pm} = \hat{u}_0 + \int_0^{\pm\infty} \sin(\tau p(\xi)) \frac{|\xi|^2}{p(|\xi|)(1+|\xi|^2)} \hat{f}(\xi,\tau)\mathrm{d}\tau,$$
$$\hat{u}_1^{\pm} = \hat{u}_1 - \int_0^{\pm\infty} \cos(\tau p(\xi)) \frac{|\xi|^2}{1+|\xi|^2} \hat{f}(\xi,\tau)\mathrm{d}\tau,$$

*such that*

$$\|u(t) - u^{\pm}(t)\|_{H^s} = O(|t|^{-\theta(p-1)+1}),\ t \to \pm\infty,$$

*where $s, \theta, p$ are the same in Theorem 1.2.*

The paper is organized as follows. We obtain the dispersive estimate in Sect. 2 and establish the existence and decay of global solutions in Sect. 3. Section 4 is to construct the scattering of solutions obtained in Sect. 3.

Throughout this paper, we denote by $\mathbb{R}, \mathbb{Z}$ the set of real numbers and integer numbers, respectively. Positive constants $C$ vary from line to line. $A \lesssim B$ denote $A \leqslant CB$, and $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$ hold at the same time.

## 2 The Dispersive Estimate

In this section, we aim to prove the dispersive estimate. Firstly, let us recall the classical lemmas about the stationary phase estimate and Bessel function.

**Lemma 2.1** (Stationary phase estimate, see [17, 20])

(i) *Suppose $\phi$ is a real-valued function and smooth in $(a,b)$, satisfying $|\phi^{(k)}(x)| \geqslant 1$ for all $x \in (a,b)$. Then,*

$$\left| \int_a^b e^{i\lambda\phi(x)}\mathrm{d}x \right| \leqslant C_k \lambda^{-\frac{1}{k}}$$

*holds when $k \geqslant 2$ or $k = 1$ and $\phi'(x)$ is monotonic.*

(ii) *Let $h(x)$ be a smooth function in $(a,b)$, then under the assumptions on $\phi$ in (i), we have*

$$\left| \int_a^b e^{i\lambda\phi(x)} h(x)\mathrm{d}x \right| \leqslant C_k \lambda^{-\frac{1}{k}} (\|h\|_{L^\infty} + \|h'\|_{L^1}).$$

**Lemma 2.2** (Properties of the Bessel function, see [17, 20])

*The Bessel function* $B_m(r)(0 < r < \infty, m > -\frac{1}{2})$ *is*

$$B_m(r) = \frac{r^m}{2^m \Gamma(m+\frac{1}{2})\pi^{\frac{1}{2}}} \int_{-1}^{1} e^{irt}(1-t^2)^{m-\frac{1}{2}} dt,$$

*which has the properties*

(i) $B_m(r) \leqslant Cr^m$ *and* $\frac{\mathrm{d}}{\mathrm{d}r}(r^{-m}B_m(r)) = -r^{-m}B_{m+1}(r)$.
(ii) $r^{-\frac{n-2}{2}} B_{\frac{n-2}{2}}(r) = C_n \mathrm{Re}(e^{ir}h(r))$, *where* $h(r)$ *is a smooth function satisfying*

$$|\partial_r^k h(r)| \leqslant C_k(1+r)^{-\frac{n-1}{2}-k}, \quad k \geqslant 0.$$

Then, we recall the Littlewood–Paley decomposition. Suppose $\psi:\mathbb{R}^n \to [0, 1]$ be a smooth radial cutoff function

$$\psi(\xi) = \begin{cases} 1, & |\xi| \leqslant 1, \\ smooth, & 1 < |\xi| < 2, \\ 0, & |\xi| \geqslant 2. \end{cases}$$

Set

$$\eta(N^{-1}\xi) = \psi(N^{-1}\xi) - \psi(2N^{-1}\xi), \quad (N \in 2^{\mathbb{Z}}),$$

then the Littlewood–Paley operator $P_N$ can be defined by

$$P_N g = \mathcal{F}^{-1}\left(\eta\left(\frac{\xi}{N}\right)\hat{g}\right).$$

Furthermore, we define the operator $\tilde{P}_N$ by

$$\tilde{P}_N g = \mathcal{F}^{-1}\left\{\left(\eta(\frac{2\xi}{N}) + \eta(\frac{\xi}{N}) + \eta(\frac{\xi}{2N})\right)\hat{g}\right\},$$

and then,

$$\tilde{P}_N P_N = P_N \tilde{P}_N = P_N.$$

From now on, we always set

$$\Theta(|\xi|) = \left(\frac{p'(|\xi|)}{|\xi|}\right)^{-\frac{n-1}{2}} (p''|\xi|))^{-\frac{1}{2}}.$$

In order to prove Theorem 1.1, the embedding $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$ implies that it is enough to prove

$$\|e^{itp(|\nabla|)} f\|_{\dot{B}^0_{\infty,1}} \lesssim (1+|t|)^{-\frac{n}{2}(1-\frac{2}{r})} \|f\|_{\Theta^{-(1-\frac{2}{r})}\dot{B}^{\frac{n}{r}}_{r',1} \cap \dot{B}^{\frac{n}{r'}}_{r',1}}, \tag{2.1}$$

Equivalently,

$$\begin{aligned}&\|e^{itp(|\nabla|)} P_N f\|_{L^\infty}\\ &\quad\lesssim (1+|t|)^{-\frac{n}{2}(1-\frac{2}{r})} \left(\Theta^{1-\frac{2}{r}}(N) N^{\frac{n}{r}} \|\widetilde{P}_N f\|_{L^{r'}} + N^{\frac{n}{r'}} \|\widetilde{P}_N f\|_{L^{r'}}\right).\end{aligned} \tag{2.2}$$

Since

$$e^{itp(|\nabla|)} P_N f = e^{itp(|\nabla|)} P_N \widetilde{P}_N f = \int_{\mathbb{R}^n} e^{i(x\xi + tw(\xi))} \eta(\frac{\xi}{N}) \widehat{\widetilde{P}_N f} \mathrm{d}\xi, \tag{2.3}$$

by the Hölder and Hausdorff–Young inequalities, we have for any $2 \leqslant r \leqslant \infty$ that

$$\|e^{itp(|\nabla|)} P_N f\|_{L^\infty} \lesssim \|\eta(\frac{\xi}{N})\|_{L^{r'}} \|\widehat{\widetilde{P}_N f}(\xi)\|_{L^r} \lesssim N^{\frac{n}{r'}} \|\widetilde{P}_N f\|_{L^{r'}}. \tag{2.4}$$

Thus, it follows from (2.2) and (2.4) that we only need to prove that when $|t| \geqslant 1$,

$$\|e^{itp(|\nabla|)} P_N f\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}(1-\frac{2}{r})} \Theta^{1-\frac{2}{r}}(N) N^{\frac{n}{r}} \|\widetilde{P}_N f\|_{L^{r'}}. \tag{2.5}$$

In order to prove the inequality (2.5), because the proof of the case of $n = 1$ is rather easier than that of the case of $n \geqslant 2$, we divided our proof into the following two lemmas.

**Lemma 2.3** *When $n = 1$ and $2 \leqslant r \leqslant \infty$ and $|t| \geqslant 1$, then*

$$\|e^{itp(|\nabla|)} P_N f\|_{L^\infty} \lesssim |t|^{-\frac{1}{2}(1-\frac{2}{r})} \Theta^{1-\frac{2}{r}}(N) N^{\frac{1}{r}} \|\widetilde{P}_N f\|_{L^{r'}}.$$

***Proof*** By (2.3), the Hölder and Hausdorff–Young inequalities, we have

$$\begin{aligned}\|e^{itp(|\nabla|)} P_N f\|_{L^\infty} &= \left\| \int_{\mathbb{R}} e^{i(x\xi + tp(\xi))} \eta(\frac{\xi}{N}) \widehat{\widetilde{P}_N f} \mathrm{d}\xi \right\|_{L^\infty}\\ &\leqslant \left\| \int_{\mathbb{R}} e^{i(x\xi + tp(\xi))} \eta(\frac{\xi}{N}) \mathrm{d}\xi \right\|_{L^\infty} \|\widehat{\widetilde{P}_N f}\|_{L^\infty}\\ &\lesssim \left\| \int_{\mathbb{R}} e^{i(x\xi + tp(\xi))} \eta(\frac{\xi}{N}) \xi \right\|_{L^\infty} \|\widetilde{P}_N f\|_{L^1}.\end{aligned} \tag{2.6}$$

Next, we need to deal with the estimate of one-dimensional oscillatory integral

$$\left\| \int_{\mathbb{R}} e^{i(x\xi + tp(\xi))} \eta(\frac{\xi}{N}) \mathrm{d}\xi \right\|_{L^\infty}.$$

Let

$$\Psi(\xi) = x\xi + tp(|\xi|),$$

then

$$\Psi''(\xi) = tp''(|\xi|) > 0.$$

We have by Lemma 2.1 (i) that

$$\sup_{x\in\mathbb{R}} \left| \int_{\mathbb{R}} e^{itp(|\xi|)} \eta(\frac{\xi}{N}) \mathrm{d}\xi \right| \lesssim |t|^{-\frac{1}{2}} |p''(N)|^{-\frac{1}{2}} \lesssim |t|^{-\frac{1}{2}} \Theta(N), \tag{2.7}$$

where we have used the fact $|p''(|\xi|)| \geqslant Cp''(N)$ for any $|\xi| \in (\frac{N}{2}, 2N)$. By (2.6) and (2.7), we have

$$\|e^{itp(|\nabla|)} P_N f\|_{L^\infty} \lesssim |t|^{-\frac{1}{2}} \Theta(N) \|\widetilde{P}_N f\|_{L^1}. \tag{2.8}$$

Setting $r' = 2$ in (2.4), we have

$$\|e^{itp(|\nabla|)} P_N f\|_{L^\infty} \lesssim N^{\frac{1}{2}} \|\widetilde{P}_N f\|_{L^2}. \tag{2.9}$$

Interpolating (2.8) with (2.9) implies

$$\|e^{itp(|\nabla|)} P_N f\|_{L^\infty} \lesssim |t|^{-\frac{1}{2}(1-\frac{2}{r})} \Theta^{1-\frac{2}{r}}(N) N^{\frac{1}{r}} \|\widetilde{P}_N f\|_{L^{r'}}.$$

Thus, we complete the proof of Lemma 2.3. □

**Lemma 2.4** *When $n \geqslant 2$ and $2 \leqslant r \leqslant \infty$ and $|t| \geqslant 1$, then*

$$\|e^{itp(|\nabla|)} P_N f\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}(1-\frac{2}{r})} \Theta^{1-\frac{2}{r}}(N) N^{\frac{n}{r}} \|\widetilde{P}_N f\|_{L^{r'}}.$$

***Proof*** A similar estimate with (2.6) shows that

$$\|e^{itp(|\nabla|)} P_N f\|_{L^\infty} \lesssim \left\| \int_{\mathbb{R}^n} e^{i(x\xi+tp(|\xi|))} \eta(\frac{\xi}{N}) \xi \right\|_{L^\infty} \|\widetilde{P}_N f\|_{L^1}. \tag{2.10}$$

Thus, it is necessary to obtain the estimate of the multidimensional oscillatory integral

$$\left\| \int_{\mathbb{R}^n} e^{i(x\xi+tp(|\xi|))} \eta\left(\frac{\xi}{N}\right) \mathrm{d}\xi \right\|_{L^\infty}.$$

By changing the variable $\xi \mapsto N\xi$ and the scaling invariance of $\|\cdot\|_{L^\infty}$, we get

$$\begin{aligned}\left\|\int_{\mathbb{R}^n} e^{i(x\xi+tp(|\xi|))}\eta(\frac{\xi}{N})\mathrm{d}\xi\right\|_{L^\infty} &= N^n\left\|\int_{\mathbb{R}^n} e^{i(Nx\xi+tp(|N\xi|))}\eta(|\xi|)\mathrm{d}\xi\right\|_{L^\infty}\\ &= N^n\left\|\int_{\mathbb{R}^n} e^{i(x\xi+tp(|N\xi|))}\eta(|\xi|)\mathrm{d}\xi\right\|_{L^\infty}.\end{aligned}$$

where supp $\eta(\xi) \subset \{\xi : \frac{1}{2} \leqslant |\xi| \leqslant 2\}$. Furthermore, the Fourier transform of a radial function (see [20]) gives

$$N^n\int_{\mathbb{R}^n} e^{i(x\xi+tp(|N\xi|))}\eta(|\xi|)\mathrm{d}\xi = N^n\int_0^\infty e^{itp(Nr)}\eta(r)r^{n-1}(r|x|)^{-\frac{n-2}{2}}B_{\frac{n-2}{2}}(r|x|)\mathrm{d}r.$$

Thus, we have

$$\begin{aligned}&\left\|\int_{\mathbb{R}^n} e^{i(x\xi+tp(|\xi|))}\eta(\frac{\xi}{N})\mathrm{d}\xi\right\|_{L^\infty}\\ &\quad= N^n\left\|\int_0^\infty e^{itp(Nr)}\eta(r)r^{n-1}(r|x|)^{-\frac{n-2}{2}}B_{\frac{n-2}{2}}(r|x|)\mathrm{d}r\right\|_{L^\infty}.\end{aligned} \tag{2.11}$$

Setting

$$J_N(t,x) = N^n\int_0^\infty e^{itp(Nr)}\eta(r)r^{n-1}(r|x|)^{-\frac{n-2}{2}}B_{\frac{n-2}{2}}(r|x|)\mathrm{d}r,$$

we go to estimate the term $\|J_N(t,x)\|_{L^\infty}$. Some simple calculations give

$$\begin{aligned}p(r) &= r\sqrt{\frac{r^4+r^2+1}{1+r^2}},\\ p'(r) &= \frac{2r^6+4r^4+2r^2+1}{(1+r^2)^{\frac{3}{2}}(r^4+r^2+1)^{\frac{1}{2}}},\\ p''(r) &= \frac{r^3(2r^8+8r^6+18r^4+19r^2+10)}{(1+r^2)^{\frac{5}{2}}(r^4+r^2+1)^{\frac{3}{2}}}.\end{aligned}$$

If $|x| \leqslant 2$, let

$$\mathcal{D}_r g := \frac{1}{itNp'(Nr)}\frac{\mathrm{d}}{\mathrm{d}r}g, \quad (\mathcal{D}_r^*)g := -\frac{1}{itN}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{p'(Nr)}g\right),$$

then

$$\mathcal{D}_r(e^{itp(Nr)}) = e^{itp(Nr)}.$$

Integrating by parts for any $q \in \mathbb{Z}^+$ implies

$$\begin{aligned} J_N(t,x) &= N^n \int_0^\infty e^{itp(Nr)} \eta(r) r^{n-1} (r|x|)^{-\frac{n-2}{2}} B_{\frac{n-2}{2}}(r|x|) \mathrm{d}r \\ &= N^n \int_0^\infty \mathcal{D}_r^q (e^{itp(Nr)}) \eta(r) r^{n-1} (r|x|)^{-\frac{n-2}{2}} B_{\frac{n-2}{2}}(r|x|) \mathrm{d}r \\ &= N^n \int_0^\infty e^{itp(Nr)} (\mathcal{D}_r^*)^q (\eta(r) r^{n-1} (r|x|)^{-\frac{n-2}{2}} B_{\frac{n-2}{2}}(r|x|)) \mathrm{d}r. \end{aligned} \tag{2.12}$$

By the chain rule of derivative, one has

$$\begin{aligned} &(\mathcal{D}_r^*)^q (\eta(r) r^{n-1} (r|x|)^{-\frac{n-2}{2}} B_{\frac{n-2}{2}}(r|x|)) \\ &\quad = \frac{1}{(-itN)^q} \sum_{k=0}^{q} C_{k,q} F_q \partial_r^{q-k} (\eta(r) r^{n-1} (r|x|)^{-\frac{n-2}{2}} B_{\frac{n-2}{2}}(r|x|)), \end{aligned}$$

where

$$F_q = \sum_{q_1,\ldots,q_k \in \Xi_k^q} \prod_{j=1}^{q} \partial_r^{m_j} (\frac{1}{p'(Nr)}),$$

and

$$\Xi_k^q = \{m_1, ..., m_q \in \mathbb{Z}^+ : 0 \leqslant m_1 \leqslant m_2 \leqslant ... \leqslant m_q, m_1 + m_2 + ... + m_q = k\}.$$

For any $m \geqslant 0$, $r \in [\frac{1}{2}, 2]$, we have

$$|\partial_r^m (\frac{1}{p'(Nr)})| \lesssim \begin{cases} 1, & N < 1, \\ N^{-1}, & N \geqslant 1. \end{cases} \tag{2.13}$$

By (i) in Lemma 2.2, we have for $|x| \leqslant 2$ and $m \geqslant 0$,

$$|\partial_r^m (\eta(r) r^{n-1} (r|x|)^{-\frac{n-2}{2}} B_{\frac{n-2}{2}}(r|x|))| \lesssim 1. \tag{2.14}$$

It follows from (2.12)–(2.14) that

$$|J_N(t,x)| \lesssim \begin{cases} |t|^{-q} N^{n-q}, & N < 1, \\ |t|^{-q} N^{n-2q}, & N \geqslant 1. \end{cases} \tag{2.15}$$

If $|x| > 2$, (iii) in Lemma 2.2 implies that

$$\begin{aligned} J_N(t,x) &= N^n \int_0^\infty e^{itp(Nr)}\eta(r)r^{n-1}(r|x|)^{-\frac{n-2}{2}} B_{\frac{n-2}{2}}(r|x|)\mathrm{d}r \\ &= N^n \int_0^\infty e^{itp(Nr)}\eta(r)r^{n-1}(e^{ir|x|}h(r|x|) + e^{-ir|x|}\overline{h}(r|x|))\mathrm{d}r \\ &= J_{N1}(t,x) + J_{N2}(t,x), \end{aligned} \tag{2.16}$$

where

$$J_{N1}(t,x) = N^n \int_0^\infty e^{it(p(Nr)+\frac{r|x|}{t})}\eta(r)r^{n-1}h(r|x|)\mathrm{d}r,$$
$$J_{N2}(t,x) = N^n \int_0^\infty e^{it(p(Nr)-\frac{r|x|}{t})}\eta(r)r^{n-1}\overline{h}(r|x|)\mathrm{d}r.$$

We focus on the case of $t > 0$. For $J_{N1}(t,x)$, we set

$$\Psi_1(r) = p(Nr) + \frac{r|x|}{t}, \quad \Omega_1'(r) = Np'(Nr) + \frac{|x|}{t} > 0.$$

From (iii) in Lemma 2.2, we obtain for $|x| \geqslant 2$ and $m \geqslant 0$,

$$|\partial_r^m(\eta(r)r^{n-1}h(r|x|))| \lesssim |x|^{-\frac{n-1}{2}} \lesssim 1. \tag{2.17}$$

With the help of stationary phase estimate as the case of $|x| < 2$, it follows from (2.13) and (2.17) that for any $q \geqslant 0$,

$$|J_{N1}(t,x)| \lesssim \begin{cases} |t|^{-q}N^{n-q}, & N < 1, \\ |t|^{-q}N^{n-2q}, & N \geqslant 1. \end{cases} \tag{2.18}$$

For $J_{N2}(t,x)$, we set

$$\Psi_2(r) = p(Nr) - \frac{r|x|}{t}, \quad \Psi_2'(r) = Np'(Nr) - \frac{|x|}{t}, \quad \Psi_2''(r) = N^2p''(Nr),$$

which imply that there exists one critical point

$$\frac{|x|}{t} = Np'(Nr).$$

When

$$\frac{|x|}{t} > 100 \sup_{r\in[\frac{1}{2},2]} Np'(Nr) \quad or \quad \frac{|x|}{t} < \frac{1}{100} \inf_{r\in[\frac{1}{2},2]} Np'(Nr),$$

then

$$\Psi_2'(r) \neq 0, \quad \forall r \in \left[\frac{1}{2}, 2\right].$$

Similar to the estimate of $J_{N1}(t,x)$, we have

$$|J_{N2}(t,x)| \lesssim \begin{cases} |t|^{-q} N^{n-q}, & N < 1, \\ |t|^{-q} N^{n-2q}, & N \geqslant 1. \end{cases} \tag{2.19}$$

When

$$\frac{1}{100} \inf_{r\in[\frac{1}{2},2]} Np'(Nr) \leqslant \frac{|x|}{t} \leqslant 100 \sup_{r\in[\frac{1}{2},2]} Np'(Nr),$$

then

$$|x| \sim tNp'(Nr). \tag{2.20}$$

By (ii) in Lemma 2.1, we have that

$$\begin{aligned} J_{N2}(t,x) &= N^n \int_0^\infty e^{it\Psi_2(r)} \eta(r) r^{n-1} \overline{h}(r|x|) \mathrm{d}r \\ &\lesssim N^n (|tN^2 p''(Nr)|)^{-\frac{1}{2}} F(x), \end{aligned} \tag{2.21}$$

where

$$F(x) = \sup_{r\in[\frac{1}{2},2]} |\eta(r) r^{n-1} \overline{h}(r|x|)| + \int_0^\infty |\partial_r(\eta(r) r^{n-1} \overline{h}(r|x|)| \mathrm{d}r.$$

Let us estimate the function $F(x)$. By (iii) in Lemma 2.2, we have

$$|F(x)| \lesssim |x|^{-\frac{n-1}{2}}.$$

Inserting the above estimate into (2.21) and then using (2.20), we have

$$\begin{aligned} J_{N2}(t,x) &\lesssim N^n (|tN^2 p''(Nr)|)^{-\frac{1}{2}} |x|^{-\frac{n-1}{2}} \\ &\lesssim |t|^{-\frac{1}{2}} N^n (N^2 p''(Nr))^{-\frac{1}{2}} (N|t|p'(Nr))^{-\frac{n-1}{2}} \\ &\lesssim |t|^{-\frac{n}{2}} \left(\frac{p'(N)}{N}\right)^{-\frac{n-1}{2}} (p''(N))^{-\frac{1}{2}} \\ &= |t|^{-\frac{n}{2}} \Theta(N). \end{aligned}$$

It follows from

$$\Theta(N) = \left(\frac{p'(N)}{N}\right)^{-\frac{n-1}{2}} (p''(N))^{-\frac{1}{2}} \sim \begin{cases} N^{\frac{n}{2}-2}, & N < 1, \\ 1, & N \geqslant 1, \end{cases} \tag{2.22}$$

and (2.11), (2.15), (2.18), (2.19) with $q = \frac{n}{2}$ that

$$\sup_{x\in\mathbb{R}^n} |J_{N2}(t,x)| \lesssim |t|^{-\frac{n}{2}}\Theta(N). \tag{2.23}$$

It follows from (2.10), (2.11) and (2.23) that

$$\|e^{itp(|\nabla|)}P_N f\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}}\Theta(N)\|\widetilde{P}_N f\|_{L^1}. \tag{2.24}$$

Setting $r' = 2$ in (2.4), we have

$$\|e^{itp(|\nabla|)}P_N f\|_{L^\infty} \lesssim N^{\frac{n}{2}}\|\widetilde{P}_N f\|_{L^2}. \tag{2.25}$$

Interpolating (2.24) with (2.25) implies

$$\|e^{itp(|\nabla|)}P_N f\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}(1-\frac{2}{r})}\Theta^{1-\frac{2}{r}}(N)N^{\frac{n}{r}}\|\widetilde{P}_N f\|_{L^{r'}}.$$

Thus, we complete the proof of Lemma 2.4. □

***The proof of Theorem 1.1:*** It follows from Lemmas 2.3 and 2.4 that the inequality (2.5) actually holds. By (2.4) and (2.5), we deduce that the inequality (2.2) is valid, which results in the inequality (2.1) holds. Thanks to the embedding $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$, the result of Theorem 1.1 is proved.

In fact, the dispersive estimate in Theorem 1.1 is very useful to estimate the linear part $\|(\partial_t G * u_0, G(t) * u_1)\|_{L^\infty}$, but it is not enough to estimate the nonlinear part $\left\|\int_0^t \frac{\Delta}{1-\Delta}G(t-\tau) * f(u)\mathrm{d}\tau\right\|_{L^\infty}$, because we do not have the embedding $L^{r'} \hookrightarrow \dot{B}^0_{r',1}$. In order to overcome the difficulty, we go to refine the dispersive estimate in Theorem 1.1 by using the Besov space $\dot{B}^0_{r',2}$ instead of the Besov space $\dot{B}^0_{r',1}$. Let us introduce the operators

$$\begin{cases} \Lambda_{\alpha,\beta} = \Lambda^\alpha(1+\Lambda^2)^{\frac{\beta-\alpha}{2}}, \\ \hat{\Lambda} = |\xi|. \end{cases}$$

It was known in [5] and [15] for any $\epsilon > 0$ that

$$\Lambda^{-1}_{-\epsilon,\epsilon}\dot{B}^0_{\infty,2} \hookrightarrow L^\infty. \tag{2.26}$$

**Corollary 2.5** *If* $2 \leqslant r \leqslant \infty$ *and suppose* $w(|\nabla|)$ *is a* $L^p (1 \leqslant p \leqslant \infty)$ *bounded operator, then we have for* $f \in (w(|\nabla|)\Theta)^{-(1-\frac{2}{r})} \dot{B}^{\frac{n}{r}}_{r',2} \cap \dot{B}^{\frac{n}{r'}}_{r',2}$ *that*

$$\|e^{itp(|\nabla|)} w(\nabla) f\|_{L^\infty} \lesssim \begin{cases} \|\Lambda_{-\epsilon,\epsilon} f\|_{\dot{B}^{\frac{n}{r'}}_{r',2}}, & t \in \mathbb{R}, \\ |t|^{-\frac{n}{2}(1-\frac{2}{r})} \|\Lambda_{-\epsilon,\epsilon} f\|_{(w(|\nabla|)\Theta)^{-(1-\frac{2}{r})} \dot{B}^{\frac{n}{r}}_{r',2}}, & |t| \geqslant 1. \end{cases}$$

***Proof*** Since $w(|\nabla|)$ is a $L^\infty$ bounded operator, we have

$$\|e^{itp(|\nabla|)} w(\nabla) f\|_{L^\infty} \lesssim \|e^{itp(|\nabla|)} f\|_{L^\infty}.$$

By (2.4), we have for any $\epsilon > 0$

$$\|e^{itp(|\nabla|)} P_N f\|_{L^\infty} \lesssim N^{\frac{n}{r'}} \Lambda_{\epsilon,-\epsilon}(N) \|\widetilde{P}_N \Lambda_{-\epsilon,\epsilon}(N) f\|_{L^{r'}},$$

which implies that

$$\|e^{itp(|\nabla|)} P_N \Lambda_{-\epsilon,\epsilon}(N) f\|_{L^\infty} \lesssim N^{\frac{n}{r'}} \|\widetilde{P}_N \Lambda_{-\epsilon,\epsilon}(N) f\|_{L^{r'}}. \tag{2.27}$$

Taking the $l^2$ norm in (2.27) and using the embedding (2.26) give that

$$\|e^{itp(|\nabla|)} f\|_{L^\infty} \lesssim \|e^{itp(|\nabla|)} \Lambda_{-\epsilon,\epsilon} f\|_{\dot{B}^0_{\infty,2}} \lesssim \|\Lambda_{-\epsilon,\epsilon} f\|_{\dot{B}^{\frac{n}{r'}}_{r',2}}. \tag{2.28}$$

When $|t| \geqslant 1$, by (2.8)–(2.9) and (2.24)–(2.25), we have

$$\|e^{itp(|\nabla|)} P_N w(|\nabla|) f\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}} \Theta(N) w(N) \|\widetilde{P}_N f\|_{L^1},$$

and

$$\|e^{itp(|\nabla|)} P_N w(|\nabla|) f\|_{L^\infty} \lesssim \|e^{itp(|\nabla|)} P_N f\|_{L^\infty} \lesssim N^{\frac{n}{2}} \|\widetilde{P}_N f\|_{L^2},$$

which deduce that

$$\begin{aligned} &\|e^{itp(|\nabla|)} P_N w(|\nabla|) f\|_{L^\infty} \\ &\quad \lesssim |t|^{-\frac{n}{2}(1-\frac{2}{r})} (\Theta w)^{1-\frac{2}{r}}(N) N^{\frac{n}{r}} \Lambda_{\epsilon,-\epsilon}(N) \|\widetilde{P}_N \Lambda_{-\epsilon,\epsilon}(N) f\|_{L^{r'}}, \end{aligned}$$

that is equivalent to

$$\begin{aligned} &\|e^{itp(|\nabla|)} P_N \Lambda_{-\epsilon,\epsilon}(N) w(|\nabla|) f\|_{L^\infty} \\ &\quad \lesssim |t|^{-\frac{n}{2}(1-\frac{2}{r})} (\Theta w)^{1-\frac{2}{r}}(N) N^{\frac{n}{r}} \|\widetilde{P}_N \Lambda_{-\epsilon,\epsilon}(N) f\|_{L^{r'}}. \end{aligned} \tag{2.29}$$

Taking the $l^2$ norm in (2.29) and using the embedding (2.26) give that

$$\|e^{itp(|\nabla|)}w(|\nabla|)f\|_{L^\infty} \lesssim \|e^{itp(|\nabla|)}\Lambda_{-\epsilon,\epsilon}w(|\nabla|)f\|_{\dot{B}^0_{\infty,2}}$$
$$\lesssim |t|^{-\frac{n}{2}(1-\frac{2}{r})}\|\Lambda_{-\epsilon,\epsilon}f\|_{(w(|\nabla|)\Theta)^{-(1-\frac{2}{r})}\dot{B}^{\frac{n}{r}}_{r',2}}. \tag{2.30}$$

It follows from (2.28) and (2.30) that the result of Corollary 2.5 holds. □

## 3 Existence and Decay of Solutions

In this section, we go to establish the global existence and decay of solutions to the Cauchy problem (1.1)–(1.2). In the sequel, we always set

$$\gamma = \frac{n}{2}\left(1 - \frac{2}{r}\right).$$

### 3.1 The Estimate of Linear Part

In this subsection, we aim to establish the $L^\infty$ and $L^2$ estimates of linear part associated with the Cauchy problem (1.1)–(1.2).

**Lemma 3.1** *If* $2 \leqslant r \leqslant \infty$ *and*

$$u_0 \in \Theta^{-(1-\frac{2}{r})}\dot{B}^{\frac{n}{r}}_{r',1} \cap \dot{B}^{\frac{n}{r'}}_{r',1},$$
$$u_1 \in p(|\nabla|)\left(\Theta^{-(1-\frac{2}{r})}\dot{B}^{\frac{n}{r}}_{r',1} \cap \dot{B}^{\frac{n}{r'}}_{r',1}\right),$$

*then*

$$\|(\partial_t G * u_0, G(t) * u_1)\|_{L^\infty}$$
$$\lesssim (1+|t|)^{-\gamma}\left(\|u_0\|_{\Theta^{-(1-\frac{2}{r})}\dot{B}^{\frac{n}{r}}_{r',1}\cap\dot{B}^{\frac{n}{r'}}_{r',1}} + \|u_1\|_{p(|\nabla|)\left(\Theta^{-(1-\frac{2}{r})}\dot{B}^{\frac{n}{r}}_{r',1}\cap\dot{B}^{\frac{n}{r'}}_{r',1}\right)}\right).$$

***Proof*** We first focus on the estimate of $\|\partial_t G * u_0\|_{L^\infty}$.

$$\|\partial_t G * u_0\|_{L^\infty} = \left\|\int_{\mathbb{R}^n} e^{ix\xi} cos(p(\xi)t)\hat{u}_0 \mathrm{d}\xi\right\|_{L^\infty}$$
$$= \left\|\int_{\mathbb{R}^n} e^{ix\xi}\frac{e^{itp(\xi)}+e^{-itp(\xi)}}{2}\hat{u}_0\mathrm{d}\xi\right\|_{L^\infty} \sim \left\|\int_{\mathbb{R}^n} e^{i(x\xi+tp(\xi))}\hat{u}_0\mathrm{d}\xi\right\|_{L^\infty}$$
$$= \left\|e^{itp(|\nabla|)}u_0\right\|_{L^\infty}. \tag{3.1}$$

Theorem 1.1 and (3.1) deduce that

$$\|\partial_t G * u_0\|_{L^\infty} \lesssim (1+|t|)^{-\gamma} \|u_0\|_{\Theta^{-(1-\frac{2}{r})}\dot{B}^{\frac{n}{r}}_{r',1}\cap \dot{B}^{\frac{n}{r'}}_{r',1}}. \tag{3.2}$$

Then, we go to estimate $\|G(t) * u_1\|_{L^\infty}$.

$$\begin{aligned}\|G(t) * u_1\|_{L^\infty} &= \left\| \int_{\mathbb{R}^n} e^{ix\xi} \frac{sin(p(\xi)t)}{p(\xi)} \hat{u}_1 \mathrm{d}\xi \right\|_{L^\infty} \\ &= \left\| \int_{\mathbb{R}^n} e^{ix\xi} \frac{e^{itp(\xi)} - e^{-itp(\xi)}}{2ip(\xi)} \hat{u}_1 \mathrm{d}\xi \right\|_{L^\infty} \\ &\sim \left\| \int_{\mathbb{R}^n} e^{i(x\xi + tp(\xi))} \frac{1}{p(\xi)} \hat{u}_1 \mathrm{d}\xi \right\|_{L^\infty} \\ &= \|e^{itp(|\nabla|)} \frac{1}{p(|\nabla|)} u_1\|_{L^\infty}. \end{aligned} \tag{3.3}$$

It follows from Theorem 1.1 and (3.3) that

$$\|G(t) * u_1\|_{L^\infty} \lesssim (1+|t|)^{-\gamma} \|u_1\|_{p(|\nabla|)\left(\Theta^{-(1-\frac{2}{r})}\dot{B}^{\frac{n}{r}}_{r',1}\cap \dot{B}^{\frac{n}{r'}}_{r',1}\right)}. \tag{3.4}$$

Concluding (3.2) and (3.4) implies Lemma 3.1 holds. □

**Lemma 3.2** *If $s \in \mathbb{R}$ and $u_0 \in H^s$, $u_1 \in p(|\nabla|)H^s$, then*

$$\|(\partial_t G * u_0, G(t) * u_1)\|_{H^s} \lesssim \|u_0\|_{H^s} + \|u_1\|_{p(|\nabla|)H^s}.$$

***Proof*** By the Plancherel theorem, we have

$$\begin{aligned}\|\partial_t G * u_0\|_{H^s} &= \|(1-\Delta)^{\frac{s}{2}} \partial_t G * u_0\|_{L^2} = \|(1+|\xi|^2)^{\frac{s}{2}} \cos(itp(|\xi|))\hat{u}_0\|_{L^2} \\ &= \|(1+|\xi|^2)^{\frac{s}{2}} \hat{u}_0\|_{L^2} = \|u_0\|_{H^s}.\end{aligned}$$

Similarly, we also obtain

$$\|G(t) * u_1\|_{H^s} = \|u_1\|_{p(|\nabla|)H^s}.$$

Concluding the above two equations, we complete the proof of Lemma 3.2. □

### 3.2 The Estimate of Nonlinear Part

In this subsection, we aim to establish the $L^\infty$ and $L^2$ estimates of nonlinear part associated with the Cauchy problem (1.1)–(1.2). Firstly, we recall the chain of fractional derivation.

**Lemma 3.3** *([5, 10, 27]) Suppose s with* $0 \leqslant s \leqslant p$, *then*

$$\|\nabla^s f(u)\|_{L^r} \leqslant \|u\|^{p-1}_{L^{(p-1)r_1}} \|\nabla^s u\|_{L^{r_2}},$$

*for* $r_1 \in (1, \infty]$, $r_2 \in (1, \infty)$, $1/r_1 + 1/r_2 = 1$. *Furthermore,*

$$\|f(u)\|_{H^s} \leqslant \|u\|^{p-1}_{L^\infty} \|u\|_{H^s}.$$
$$\|f(u) - f(v)\|_{L^2} \lesssim (\|u\|^{p-1}_{L^\infty} + \|v\|^{p-1}_{L^\infty})\|u - v\|_{L^2}.$$

Then with the help of Lemma 3.3, we have

**Lemma 3.4** *Suppose when* $n = 1$ *and* $2 \leqslant r < 4$ *or when* $n \geqslant 2$ *and* $2 \leqslant r < \infty$, *then we have for* $s > \frac{n}{r'}$ *that*

$$\left\| \int_0^t \frac{\Delta}{1-\Delta} G(t-\tau) * f \mathrm{d}\tau \right\|_{L^\infty} \lesssim \int_0^t (1 + |t-\tau|)^{-\gamma} \|u\|^{p-\frac{2}{r'}}_{L^\infty} \|u\|^{\frac{2}{r'}}_{H^s} \mathrm{d}\tau.$$

***Proof*** Due to (3.3), we have

$$\begin{aligned}\left\| \int_0^t \frac{\Delta}{1-\Delta} G(t-\tau) * f \mathrm{d}\tau \right\|_{L^\infty} &= \left\| \int_0^t e^{i(t-\tau)p(|\nabla|)} \frac{\Delta}{p(|\nabla|)(1-\Delta)} f \mathrm{d}\tau \right\|_{L^\infty} \\ &\leqslant \int_0^t \left\| e^{i(t-\tau)p(|\nabla|)} \frac{\Delta}{p(|\nabla|)(1-\Delta)} f \right\|_{L^\infty} \mathrm{d}\tau.\end{aligned}$$

Let us compute the pseudo-differential operator

$$\begin{aligned}\frac{\Delta}{p(|\nabla|)(1-\Delta)} &= \frac{-|\nabla|^2}{1+|\nabla|^2} \cdot \frac{\sqrt{1+|\nabla|^2}}{|\nabla|\sqrt{1+|\nabla|^2+|\nabla|^4}} \\ &= \frac{-|\nabla|}{\sqrt{1+|\nabla|^2}\sqrt{1+|\nabla|^2+|\nabla|^4}}.\end{aligned}$$

Denote $w(|\nabla|)$ by

$$\omega(|\nabla|) = \frac{|\nabla|}{\sqrt{1+|\nabla|^2}\sqrt{1+|\nabla|^2+|\nabla|^4}}.$$

Thus, we have

$$\left\| \int_0^t \frac{\Delta}{1-\Delta} G(t-\tau) * f \mathrm{d}\tau \right\|_{L^\infty} \leqslant \int_0^t \left\| e^{i(t-\tau)p(|\nabla|)} w(|\nabla|) f \right\|_{L^\infty} \mathrm{d}\tau. \tag{3.5}$$

Since $w(|\nabla|)$ is a $-2$-order pseudo-differential operator, it is a $L^p(1 \leqslant p \leqslant \infty)$ bounded operator. By Corollary 2.5, we have

$$\left\| e^{i(t-\tau)p(|\nabla|)} w(|\nabla|) f \right\|_{L^\infty} \lesssim \|\Lambda_{-\epsilon,\epsilon} f\|_{\dot{B}^{\frac{n}{r'}}_{r',2}}. \tag{3.6}$$

When $|t| \geqslant 1$, by Corollary 2.5, we have

$$\left\| e^{i(t-\tau)p(|\nabla|)} w(|\nabla|) f \right\|_{L^\infty} \lesssim |t-\tau|^{-\gamma} \|\Lambda_{-\epsilon,\epsilon} f\|_{(w(|\nabla|)\Theta)^{-\left(1-\frac{2}{r}\right)} \dot{B}^{\frac{n}{r}}_{r',2}}. \tag{3.7}$$

Now, we analyze the norm $\|\Lambda_{-\epsilon,\epsilon} f\|_{(w(|\nabla|)\Theta)^{-(1-\frac{2}{r})} \dot{B}^{\frac{n}{r}}_{r',2}}$. Due to

$$w(N) = \frac{N}{(1+N^2)^{\frac{1}{2}}} \cdot \frac{1}{(N^4+N^2+1)^{\frac{1}{2}}} \sim \begin{cases} N, & N<1, \\ N^{-2}, & N \geqslant 1, \end{cases} \tag{3.8}$$

and

$$\Lambda_{-\epsilon,\epsilon}(N) = N^{-\epsilon}(1+N^{2\epsilon}) \sim \begin{cases} N^{-\epsilon}, & N<1, \\ N^{\epsilon}, & N \geqslant 1, \end{cases} \tag{3.9}$$

it follows from (2.22) and (3.8)–(3.9) that

$$\Theta^{1-\frac{2}{r}}(N) w^{1-\frac{2}{r}}(N) \Lambda_{-\epsilon,\epsilon}(N) \sim \begin{cases} N^{\frac{n}{2}-\frac{n}{r}-(1-\frac{2}{r})-\varepsilon}, & N<1, \\ N^{-2(1-\frac{2}{r})+\varepsilon} \leqslant N^{\varepsilon}, & N \geqslant 1. \end{cases} \tag{3.10}$$

By (3.9), we can get for $s > \frac{n}{r'}$ and $\epsilon > 0$ small enough that

$$\|\Lambda_{-\epsilon,\epsilon} f\|_{\dot{B}^{\frac{n}{r'}}_{r',2}} \lesssim \|f\|_{\dot{B}^{\frac{n}{r'}-\epsilon}_{r',2}} + \|f\|_{\dot{B}^{\frac{n}{r'}+\epsilon}_{r',2}} \lesssim \|f\|_{B^s_{r',2}}. \tag{3.11}$$

By (3.10), we have

$$\left\| \Theta^{1-\frac{2}{r}} w^{(1-\frac{2}{r})}(|\nabla|) \Lambda_{-\epsilon,\epsilon} f \right\|_{\dot{B}^{\frac{n}{r}}_{r',2}} \lesssim \|f\|_{\dot{B}^{\frac{n}{2}-(1-\frac{2}{r})-\varepsilon}_{r',2}} + \|f\|_{\dot{B}^{\frac{n}{r}+\varepsilon}_{r',2}}. \tag{3.12}$$

By some computations, we have

$$\begin{cases} s > \max\left\{\frac{2}{r} - \frac{1}{2}, \frac{1}{r'}\right\} = \frac{1}{r'}, & n=1, \quad 2 \leqslant r < 4, \\ s > \max\left\{\frac{n}{2} - (1-\frac{2}{r}), \frac{n}{r'}\right\} = \frac{n}{r'}, & n \geqslant 2, \quad 2 \leqslant r < \infty, \end{cases}$$

which combining with (3.12) shows that

$$\left\| \Theta^{1-\frac{2}{r}} w^{(1-\frac{2}{r})}(|\nabla|) \Lambda_{-\epsilon,\epsilon} f \right\|_{\dot{B}^{\frac{n}{r}}_{r',2}} \lesssim \|f\|_{B^s_{r',2}}. \tag{3.13}$$

By the embedding $W^{s,r'} \hookrightarrow B^s_{r',2}(1 < r \leqslant 2)$ and Lemma 3.3, we obtain

$$\|f\|_{B^s_{r',2}} \lesssim \|f\|_{W^{s,r'}} \lesssim \|u\|^{p-1}_{L^{\frac{2(p-1)r}{r-2}}} \|u\|_{H^s}.$$

The interpolation of Lebesgue spaces implies that

$$\|u\|^{p-1}_{L^{\frac{2(p-1)r}{r-2}}} \lesssim \|u\|^{p-\frac{2}{r'}}_{L^\infty} \|u\|^{\frac{2}{r'}-1}_{H^s}.$$

By the above inequalities, one has

$$\|f\|_{B^s_{r',2}} \lesssim \|u\|^{p-\frac{2}{r'}}_{L^\infty} \|u\|^{\frac{2}{r'}}_{H^s}. \tag{3.14}$$

Thus, it follows from (3.5)–(3.7) and (3.11)–(3.14) that

$$\left\| \int_0^t \frac{\Delta}{1-\Delta} G(t-\tau) * f(\tau) \mathrm{d}\tau \right\|_{L^\infty} \lesssim \int_0^t (1+|t-\tau|)^{-\gamma} \|u\|^{p-\frac{2}{r'}}_{L^\infty} \|u\|^{\frac{2}{r'}}_{H^s} \mathrm{d}\tau.$$

We complete the proof of Lemma 3.4. □

**Lemma 3.5** *It holds that for $s \in \mathbb{R}$,*

$$\left\| \int_0^t \frac{\Delta}{1-\Delta} G(t-\tau) * f(\tau) \mathrm{d}\tau \right\|_{H^s} \leqslant \int_0^t \|u\|^{p-1}_{L^\infty} \|u\|_{H^s} \mathrm{d}\tau.$$

***Proof*** By (3.5), we know that

$$\left\| \int_0^t \frac{\Delta}{1-\Delta} G(t-\tau) * f(\tau) \mathrm{d}\tau \right\|_{H^s} \leqslant \int_0^t \left\| e^{i(t-\tau)p(|\nabla|)} (1-\Delta)^{\frac{s}{2}} w(|\nabla|) f \right\|_{L^2} \mathrm{d}\tau.$$

By the fact $w(|\nabla|)$ is a $L^p(1 \leqslant p \leqslant \infty)$ bounded operator, we have

$$\left\| e^{i(t-\tau)p(|\nabla|)} (1-\Delta)^{\frac{s}{2}} w(|\nabla|) f \right\|_{L^2} \lesssim \left\| (1-\Delta)^{\frac{s}{2}} f \right\|_{L^2}.$$

By Lemma 3.3, we obtain

$$\left\| (1-\Delta)^{\frac{s}{2}} f(u) \right\|_{L^2} \lesssim \|u\|^{p-1}_{L^\infty} \|u\|_{H^s}.$$

Concluding the above inequalities completes the proof of Lemma 3.5 □

### 3.3 Existence and Decay of Global Small Amplitude Solutions

In this subsection, we establish the existence and decay of global small amplitude solutions. Let us introduce a metric space

$$\chi_\rho^{s,\theta} = \{u \in L^\infty(\mathbb{R}; L^\infty) \cap L^\infty(\mathbb{R}; H^s) | \sup_{t\in\mathbb{R}}(1+|t|)^\gamma \|u\|_{L^\infty} + \sup_{t\in\mathbb{R}} \|u\|_{H^s} \leqslant \rho\}$$

with the metric defined by

$$d(u, v) = \|u - v\|_{L^\infty(\mathbb{R}; L^2)}.$$

By the standard way, the metric space $(\chi_\rho^{s,\theta}, d)$ is a complete metric space, see [5].

Then in order to prove Theorem 1.2, we recall a primary lemma.

**Lemma 3.6** *([5, 10, 27]) For any* $a, b > 0$ *and* $\max\{a, b\} > 1$*, it holds*

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b}\mathrm{d}s \leqslant C(1+t)^{-\min\{a,b\}}.$$

***The proof of Theorem 1.2:*** Consider the mapping $M$,

$$M(u) = \partial_t G(t) * u_0 + G(t) * u_1 + \int_0^t \frac{\Delta}{1-\Delta} G(t-\tau) * f(u)(\tau)\mathrm{d}\tau. \tag{3.15}$$

Let $u \in \chi_\rho^{s,\theta}$. By using Lemmas 3.1 and 3.4, we have

$$\begin{aligned}
\|M(u)\|_{L^\infty} &\leqslant \|\partial_t G(t) * u_0 + G(t) * u_1\|_{L^\infty} + \left\| \int_0^t \frac{\Delta}{1-\Delta} G(t-\tau) * f(u)(\tau)\mathrm{d}\tau \right\|_{L^\infty} \\
&\lesssim (1+|t|)^{-\gamma} \left( \|u_0\|_{\Theta^{-(1-\frac{2}{r})}\dot{B}^{\frac{n}{r}}_{r',1} \cap \dot{B}^{\frac{n}{r'}}_{r',1}} + \|u_1\|_{p(|\nabla|)\left(\Theta^{-(1-\frac{2}{r})}\dot{B}^{\frac{n}{r}}_{r',1} \cap \dot{B}^{\frac{n}{r'}}_{r',1}\right)} \right) \\
&\quad + \int_0^t (1+|t-\tau|)^{-\gamma} \|u\|_{L^\infty}^{p-\frac{2}{r'}} \|u\|_{H^s}^{\frac{2}{r'}} \mathrm{d}\tau.
\end{aligned} \tag{3.16}$$

According to the information of space $\chi_\rho^{s,\theta}$, we have from (3.16) that

$$\|N(u)\|_{L^\infty} \lesssim (1+|t|)^{-\gamma}\delta + \rho^p \int_0^t (1+|t-\tau|)^{-\gamma}(1+|\tau|)^{-(p-\frac{2}{r'})\gamma}\mathrm{d}\tau. \tag{3.17}$$

The condition $p > \frac{2}{r'} + \max\{1, \frac{1}{\gamma}\}$ implies that

$$(p - \frac{2}{r'})\gamma > \max\{\gamma, 1\}. \tag{3.18}$$

Combining (3.17) and (3.18) with Lemma 3.6 deduces that for small enough $\delta$ and $\rho$, it holds

$$\sup_{t\in\mathbb{R}}(1+|t|)^{\gamma}\|M(u)\|_{L^\infty}\lesssim\delta+\rho^{p}\leqslant\frac{\rho}{2}. \tag{3.19}$$

Using Lemmas 3.2 and 3.5 in (3.15) deduces that for small enough $\delta$ and $\rho$,

$$\begin{aligned}\|M(u)\|_{H^s}&\leqslant\|\partial_t G(t)*u_0+G(t)*u_1\|_{H^s}+\left\|\int_0^t\frac{\Delta}{1-\Delta}G(t-\tau)*f(u)(\tau)\mathrm{d}\tau\right\|_{H^s}\\&\lesssim\delta+\int_0^t\|u\|_{L^\infty}^{p-1}\|u\|_{H^s}\mathrm{d}\tau\\&\lesssim\delta+\rho^p\int_0^t(1+|\tau|)^{-(p-1)\gamma}\mathrm{d}\tau.\end{aligned} \tag{3.20}$$

The fact $1<r'<2$ and inequality (3.18) imply that

$$(p-1)\gamma>\left(p-\frac{2}{r'}\right)\gamma>\max\{\gamma,1\}. \tag{3.21}$$

By (3.20)–(3.21), we have

$$\|M(u)\|_{H^s}\lesssim\delta+\rho^p\leqslant\frac{\rho}{2}. \tag{3.22}$$

Therefore, the inequalities (3.19) and (3.22) mean that

$$M:\chi_\rho^{s,\theta}\mapsto\chi_\rho^{s,\theta}.$$

For any $u,v\in\chi_\rho^{s,\theta}$, by Lemma 3.3, we have

$$\|f(u)-f(v)\|_{L^2}\lesssim(\|u\|_{L^\infty}^{p-1}+\|v\|_{L^\infty}^{p-1})\|u-v\|_{L^2}.$$

Then,

$$\begin{aligned}\|M(u)-M(v)\|_{L^2}&\lesssim\int_0^t\|f(u)-f(v)\|_{L^2}\mathrm{d}\tau\\&\lesssim\rho^{p-1}d(u,v)\int_0^t(1+|\tau|)^{-(p-1)\theta}\mathrm{d}\tau\lesssim\rho^{p-1}d(u,v),\end{aligned} \tag{3.23}$$

which implies that for small enough $\rho$, $M$ is a contractive mapping in space $\chi_\rho^{s,\theta}$.

Therefore, the existence and uniqueness of solution $u\in\chi_\rho^{s,\theta}$ to (1.1)–(1.2) have been established by the contraction mapping principle. From the standard argument, we can extend $u(t)\in L^\infty(\mathbb{R};H^s)$ to $u(t)\in\mathcal{C}(\mathbb{R};H^s)$. Thus, we complete the proof of Theorem 1.2. □

## 4 Scattering

In this section, we go to establish the scattering of solutions obtained in Sect. 3.

***The proof of Theorem 1.3:*** Let $u^{\pm}$ solve the Cauchy problem

$$\begin{aligned} &u_{tt} - \Delta u + \Delta^2 u - \Delta u_{tt} - \Delta^3 u = 0, \\ &u(x,0) = u_0^{\pm}(x), \quad u_t(x,0) = u_1^{\pm}(x). \end{aligned}$$

Then, $u^{\pm}$ can be expressed by

$$u^{\pm} = \partial_t G(t) * u_0^{\pm} + G(t) * u_1^{\pm}.$$

Equivalently,

$$\hat{u}^{\pm} = \cos(tp(|\xi|))\hat{u}_0^{\pm} + \frac{\sin(tp(|\xi|))}{p(|\xi|)}\hat{u}_1^{\pm}.$$

By the definition of initial data $(u_0^{\pm}, u_1^{\pm})$ in Theorem 1.3, we have

$$\begin{aligned} &\hat{u}^{\pm} = \cos(tp(|\xi|))\hat{u}_0 + \frac{\sin(tp(|\xi|))}{p(|\xi|)}\hat{u}_1 \\ &+ \int_0^{\pm\infty} (\cos(tp(|\xi|))\sin(\tau p(|\xi|)) - \sin(tp(|\xi|))\cos(\tau p(|\xi|)))\frac{|\xi|^2}{p(|\xi|)(1+|\xi|^2)}\hat{f}\mathrm{d}\tau \\ &= \cos(tp(|\xi|))\hat{u}_0 + \frac{\sin(tp(|\xi|))}{p(|\xi|)}\hat{u}_1 + \int_0^{\pm\infty} \sin((t-\tau)p(|\xi|))\frac{-|\xi|^2}{p(|\xi|)(1+|\xi|^2)}\hat{f}\mathrm{d}\tau, \end{aligned}$$

which implies that

$$u^{\pm} = \partial_t G(t) * u_0 + G(t) * u_1 + \int_0^{\pm\infty} \frac{\Delta}{1-\Delta} G(t-\tau) * f(u)(\tau)\mathrm{d}\tau. \tag{4.1}$$

By Lemma 3.3, we have

$$\|f(u)\|_{H^s} \lesssim \|u\|_{L^\infty}^{p-1}\|u\|_{H^s}.$$

By (1.7) and (4.1), one has

$$\begin{aligned} \|u(t) - u^{\pm}(t)\|_{H^s} &\lesssim \left|\int_t^{\pm\infty} \|f(u)\|_{H^s}\mathrm{d}\tau\right| \\ &\lesssim \rho^p \left|\int_t^{\pm\infty} (1+|\tau|)^{-(p-1)\gamma} d\tau\right| \\ &\lesssim \rho^p |t|^{-(p-1)\gamma+1}. \end{aligned}$$

which implies the result of Theorem 1.3. □

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