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# Some Applications of $L^1$-Estimates of Fractional Integral Operators in Lorentz Spaces

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**Abstract**

In this paper, we establish some application of $L^1$-estimates for the Riesz potentials of order $\alpha$ in some Lorentz spaces. We use this estimate to improve certain Olsen-type inequalities in Lorentz spaces. In addition, some endpoint vector-valued inequalities for the Riesz potentials in Lebesgue spaces are obtained.

**Keywords** Riesz potentials · Olsen inequality · Lorentz spaces · Vector-valued inequality

**Mathematics Subject Classification** Primary 47G40 · Secondary 46E30 · 42B30

## 1 Introduction

Let $f$ measurable function on $\mathbb{R}^n, n \geq 1$, and $0 < \alpha < n$. The fractional integral operator or Riesz potential of $f$ of order $\alpha$ is defined as

$$I_\alpha f(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \mathrm{d}y, \, x \in \mathbb{R}^n$$

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with $\gamma(\alpha) := \dfrac{\pi^{n/2}2^\alpha\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}$. Especially for $n > 2$ and $\alpha = 2$, solution of Poisson equation $\Delta u = f$ for $f \in C_c^\infty$ is $u = -I_2 f$. One of the most important results related to $I_\alpha$ is the Hardy–Littlewood–Sobolev theorem; namely, the fractional integral operator $I_\alpha$ is bounded form $L^p(\mathbb{R}^n)$ $(1 < p < \frac{n}{\alpha})$ to $L^q(\mathbb{R}^n)$ if and only if $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{\alpha}{n}$. However, the operator $I_\alpha$ is unbounded form $L^1(\mathbb{R}^n)$ to $L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$. One of the direct consequences of the Hardy–Littlewood–Sobolev theorem is the Sobolev embedding theorem, namely the inclusion of the Sobolev spaces $W^{1,p}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 < p < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. This inclusion was later improved so that it applies for $p = 1$ and $q = \dfrac{n}{n-1}$. See [1–4] for future studies about fractional integral operators in Lebesgue spaces.

In 1960, Stein and Weiss [5] constructed a subset of the Lebesgue space which was later named the Hardy space $H^p(\mathbb{R}^n)$ and succeeded in showing the boundedness of $I_\alpha$ of $H^p(\mathbb{R}^n)$ to $L^{\frac{np}{n-\alpha p}}(\mathbb{R}^n)$ for $1 \le p < n/\alpha$. There are some equivalent definitions for $H^p$. For this paper, we use the definition

$$H^p(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : Rf = \nabla I_1 f \in L^p(\mathbb{R}^n, \mathbb{R}^n)\}.$$

Note that $Rf = \nabla I_1 f$ can be written as $(1-n)(R_1 f, \dots, R_n f)$ with

$$R_j f(x) = \frac{1}{\gamma(\alpha)} \text{p.v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\mathrm{d}y,\ j = 1, \dots, n.$$

For $1 < p < \infty$, $H^p = L^p$, but for $p = 1$, $H^1$ is strictly contained in $L^1$. Here is Stein and Weiss result about the boundedness of fractional integral operators in $H^p(\mathbb{R}^n)$.

**Theorem 1.1** *Let $0 < \alpha < n$ and $1 \le p < n/\alpha$. There exists $C = C(\alpha, n, p) > 0$ such that*

$$\|I_\alpha f\|_{L^{np/(n-\alpha p)}(\mathbb{R}^n)} \le C(\|f\|_{L^p(\mathbb{R}^n)} + \|Rf\|_{L^p(\mathbb{R}^n,\mathbb{R}^n)})$$

*for all $f \in H^p(\mathbb{R}^n)$.*

From now on, we abbreviate $A \le CB$ by $A \lesssim B$.
In 2017, Schikora et al. [6] prove the special case for Theorem 1.1; namely,

$$\|I_\alpha f\|_{L^{n/(n-\alpha)}(\mathbb{R}^n)} \lesssim \|Rf\|_{L^1(\mathbb{R}^n,\mathbb{R}^n)} \tag{1.1}$$

hold for all $f \in C_c^\infty(\mathbb{R}^n)$ such that $Rf \in L^1(\mathbb{R}^n, \mathbb{R}^n)$. Two years later, Spector [7] improve inequalities (1.1) to Lorentz spaces. To state this result, we need to recall the definition of Lorentz spaces. The Lorentz space $L^{p,q}(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ such that

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} := p^{\frac{1}{q}} \left( \int_0^\infty \left[ t |\{x \in \mathbb{R}^n : |f(x)| > t\}|^{\frac{1}{p}} \right]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} < \infty$$

with $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. Now, we ready to state an improvement of (1.1). In [7], Spector proves the following theorem.

**Theorem 1.2** [7] *Let $n \geq 2$ and $0 < \alpha < n$. Then*

$$\|I_\alpha f\|_{L^{n/(n-\alpha),1}(\mathbb{R}^n)} \lesssim \|Rf\|_{L^1(\mathbb{R}^n,\mathbb{R}^n)}$$

*for all $f \in C_c^\infty(\mathbb{R}^n)$ such that $Rf \in L^1(\mathbb{R}^n, \mathbb{R}^n)$.*

For Lorentz spaces, we have inclusion $L^p \subset L^{p,1}$. In the other words, Theorem 1.2 is the improvement of the inequality (1.1).

As an application of Theorem 1.2, we investigate some endpoint case of Olsen-type inequalities. To formulate these inequalities, let us recall the definition of Morrey spaces. For $1 \leq p < \infty$ and $0 \leq \lambda \leq n$, the (classical) Morrey space $L_\lambda^p = L_\lambda^p(\mathbb{R}^n)$ is defined to be the space of all functions $f \in L_{loc}^p(\mathbb{R}^n)$ for which

$$\|f\|_{L_\lambda^p(\mathbb{R}^n)} = \sup_{B(a,r)} \left( \frac{1}{r^\lambda} \int_{B(a,r)} |f(y)|^p \mathrm{d}y \right)^{1/p} < \infty,$$

where $B(a, r)$ denotes the (open) ball centered at $a \in R$ with radius $r > 0$. Suppose $1 < p < \frac{n}{\alpha}, 0 \leq \lambda < n - \alpha p$, Olsen proved the following results (see [8, Theorem 2])

$$\|g \cdot I_\alpha f\|_{L_\lambda^p(\mathbb{R}^n)} \lesssim \|g\|_{L_\lambda^{(n-\lambda)/\alpha}(\mathbb{R}^n)} \|f\|_{L_\lambda^p(\mathbb{R}^n)}.$$

In particular, for $\lambda = 0$, we have

$$\|g \cdot I_\alpha f\|_{L^p(\mathbb{R}^n)} \lesssim \|g\|_{L^{\frac{n}{\alpha}}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}. \tag{1.2}$$

This inequality and Theorem 1.2 motivate us to find out what happen if $p = 1$. Before we state the main result of this paper, we will recall the definition of fractional maximal operators. Suppose $0 < \alpha < n$ and $f$ is locally integrable, define

$$M_\alpha(f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|<r} |f(x-y)| \mathrm{d}y.$$

Note that if $\alpha = 0$, the definition above becomes Hardy–Littlewood maximal operators. Here is the main result of this paper.

**Theorem 1.3**

1. *Suppose that $g \in L^{\frac{n}{\alpha},\infty}(\mathbb{R}^n)$ and $f \in C_c^\infty(\mathbb{R}^n)$ such that $Rf \in L^1(\mathbb{R}^n, \mathbb{R}^n)$. Then*

$$\|g \cdot I_\alpha f\|_{L^1(\mathbb{R}^n)} \lesssim \|g\|_{L^{\frac{n}{\alpha},\infty}(\mathbb{R}^n)} . \|Rf\|_{L^1(\mathbb{R}^n,\mathbb{R}^n)}. \tag{1.3}$$

2. *Suppose that* $g \in L^{\frac{n}{\alpha},\infty}(\mathbb{R}^n)$ *and* $f \in C_c^\infty(\mathbb{R}^n)$ *such that* $R(|f|) \in L^1(\mathbb{R}^n, \mathbb{R}^n)$. *Then*

$$\|g \cdot M_\alpha f\|_{L^1(\mathbb{R}^n)} \lesssim \|g\|_{L^{\frac{n}{\alpha},\infty}(\mathbb{R}^n)} . \|R(|f|)\|_{L^1(\mathbb{R}^n,\mathbb{R}^n)}. \tag{1.4}$$

***Remark 1.4***

1. We can replace $L^1$ in the left-hand side of inequality (1.3) and (1.4) by $L^{1,\frac{n}{n-\alpha}}$. This statement can be proved using Theorem 1.2 and inclusion in Lorentz spaces.
2. The term in left-hand side can be seen as integral $I_\alpha f$ with weight $g$. In particular, if $g(x) = |x|^{-\alpha}$, then $g \in L^{\frac{n}{\alpha},\infty}(\mathbb{R}^n)$ and $g \notin L^{\frac{n}{\alpha}}(\mathbb{R}^n)$. Therefore, (1.3) can be viewed as an extension of (1.2). We will give the proof of Theorem 1.3 in Sect. 2. (see [9] for the classical results on the boundedness of fractional integral operators on weighted Lebesgue spaces).
3. The characterization of $f \in C_c^\infty(\mathbb{R}^n)$ such that $R(|f|) \in L^1(\mathbb{R}^n, \mathbb{R}^n)$ is given as follows. From our observation, $f$ should not be a nonnegative function and $|f| \notin C_c^\infty(\mathbb{R}^n)$. To prove this, let $u = |f| \in C_c^\infty(\mathbb{R}^n)$ and assume that $Ru \in L^1(\mathbb{R}^n, \mathbb{R}^n)$. Then, $u$ satisfies the assumption of Theorem 1.2. However, there is a function $u_0 \in C_c^\infty(\mathbb{R}^n)$ and that $u_0$ is nonnegative, such that $Ru_0 \notin L^1(\mathbb{R}^n, \mathbb{R}^n)$.

Now, we will give vector-valued version for Theorems 1.2 and 1.3. In this case, we assume that $\vec{f} \in \ell^r$ such that $\|\vec{f}\|_{\ell^r} \in C_c^\infty(\mathbb{R}^n)$ with $R(\|\vec{f}\|_{\ell^r}) \in L^1(\mathbb{R}^n, \mathbb{R}^n)$. Naturally we can just change $f$ in Theorem 1.2 and 1.3 by $\|\vec{f}\|_{\ell^r}$. But, we have some better results related to inequalities in Theorem 1.2 and 1.3 for vector-valued functions. Here is the vector version for Theorem 1.2 and 1.3.

**Theorem 1.5** *Let* $r \geq 1$ *and* $0 < \alpha < n$ *then*

$$\left\| \|I_\alpha \vec{f}\|_{\ell^r} \right\|_{L^{n/(n-\alpha),1}(\mathbb{R}^n)} \lesssim \left\| R(\|\vec{f}\|_{\ell^r}) \right\|_{L^1(\mathbb{R}^n,\mathbb{R}^n)}$$

*for every* $\vec{f}$ *with* $\|\vec{f}\|_{\ell^r} \in C_c^\infty(\mathbb{R}^n)$ *such that* $R(\|\vec{f}\|_{\ell^r}) \in L^1(\mathbb{R}^n, \mathbb{R}^n)$.

**Theorem 1.6** *Suppose that* $0 < \alpha < n$ *and that* $r \geq 1$ *satisfy* $\frac{1}{r} + \frac{1}{r'} = 1$. *Then*

$$\left\| \vec{g} \cdot I_\alpha \vec{f} \right\|_{L^1(\mathbb{R}^n)} \lesssim \left\| \|\vec{g}\|_{\ell^{r'}} \right\|_{L^{\frac{n}{\alpha},\infty}(\mathbb{R}^n)} \left\| R(\|\vec{f}\|_{\ell^r}) \right\|_{L^1(\mathbb{R}^n,\mathbb{R}^n)}$$

*for every* $\vec{f}$ *and* $\vec{g}$ *with* $\|\vec{g}\|_{\ell^{r'}} \in L^{\frac{n}{\alpha},\infty}(\mathbb{R}^n)$, $\|\vec{f}\|_{\ell^r} \in C_c^\infty(\mathbb{R}^n)$ *such that* $R(\|\vec{f}\|_{\ell^r}) \in L^1(\mathbb{R}^n, \mathbb{R}^n)$.

Theorems 1.5 and 1.6 will be proved in Sect. 3. This theorem can be proved using duality and Theorem 1.3.

## 2 Olsen-Type Inequalities in Lorentz Spaces

In this section, we provide some theorems that will be very helpful in proving the main result. First, let us recall the Hölder inequality in Lebesgue spaces.

**Theorem 2.1** *Suppose* $p, p_1, p_2 \in [1, \infty]$ *satisfy* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *Then*

$$\|f \cdot g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^{p_1}(\mathbb{R}^n)} . \|g\|_{L^{p_2}(\mathbb{R}^n)}$$

*for all* $f \in L^{p_1}(\mathbb{R}^n)$ *and* $g \in L^{p_2}(\mathbb{R}^n)$.

For the Lorentz space, we have the following generalization of Hölder inequality in Lorentz spaces due to O'Neil [10].

**Theorem 2.2** *Suppose* $0 < p_1, p_2, p < \infty$ *and* $0 < q_1, q_2, q \leq \infty$ *satisfy* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. *Then*

$$\|fg\|_{L^{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p_1,q_1}(\mathbb{R}^n)} \|g\|_{L^{p_2,q_2}(\mathbb{R}^n)}$$

*for all* $f \in L^{p_1,q_1}(\mathbb{R}^n)$ *and* $g \in L^{p_2,q_2}(\mathbb{R}^n)$.

We are now ready to prove Theorem 1.3 as follows.

***Proof of Theorem 1.3*** Suppose that $f \in C_c^\infty(\mathbb{R}^n)$, with $Rf \in L^1(\mathbb{R}^n, \mathbb{R}^n)$ and $g \in L^{\frac{n}{\alpha},\infty}(\mathbb{R}^n)$. By Hölder's inequality in Lorentz spaces, we have

$$\|g \cdot I_\alpha f\|_{L^1(\mathbb{R}^n)} \lesssim \|g\|_{L^{\frac{n}{\alpha},\infty}(\mathbb{R}^n)} \|I_\alpha f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)}.$$

Apply Theorem 1.2 to the right-hand side, we get

$$\|g\|_{L^{\frac{n}{\alpha},\infty}(\mathbb{R}^n)} \|I_\alpha f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} \lesssim \|g\|_{L^{\frac{n}{\alpha},\infty}(\mathbb{R}^n)} \|Rf\|_{L^1(\mathbb{R}^n,\mathbb{R}^n)}.$$

For the second part, suppose that $f \in C_c^\infty(\mathbb{R}^n)$, with $R(|f|) \in L^1(\mathbb{R}^n, \mathbb{R}^n)$ and $g \in L^{\frac{n}{\alpha},\infty}$. We already know that

$$M_\alpha f(x) \lesssim I_\alpha(|f|)(x).$$

Combining this inequality with Theorems 1.2 and 2.2, we have

$$\begin{aligned} F\|g \cdot M_\alpha(f)\|_{L^1(\mathbb{R}^n)} &\lesssim \|g\|_{L^{\frac{n}{\alpha},\infty}(\mathbb{R}^n)} \|I_\alpha(|f|)\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} \\ &\lesssim \|g\|_{L^{\frac{n}{\alpha},\infty}(\mathbb{R}^n)} \|R(|f|)\|_{L^1(\mathbb{R}^n,\mathbb{R}^n)}. \end{aligned}$$

This completes the proof. □

## 3 Vector-Valued Olsen-Type Inequalities

In this section, we will give the proof of Theorems 1.5 and 1.6.

### 3.1 Proof of Theorem 1.5

Let $r \geq 1$ and $0 < \alpha < n$. First, we will proof the inequality

$$\|I_\alpha \vec{f}(x)\|_{\ell^r} \lesssim I_\alpha(\|\vec{f}\|_{\ell^r})(x).$$

Actually, this proof is inspired by [11]. By using duality on the left-hand side, we get

$$\begin{aligned}
\|I_\alpha \vec{f}(x)\|_{\ell^r} &= \sup_{a=\{a_j(x)\}_{j=1}^\infty \in \ell^{r'}:\|a\|_{\ell^{r'}}=1} \left|\sum_{j=1}^\infty a_j(x) I_\alpha f_j(x)\right| \\
&\leq \sup_{a=\{a_j(x)\}_{j=1}^\infty \in \ell^{r'}:\|a\|_{\ell^{r'}}=1} \sum_{j=1}^\infty |a_j(x) I_\alpha f_j(x)| \\
&\lesssim \sup_{a=\{a_j(x)\}_{j=1}^\infty \in \ell^{r'}:\|a\|_{\ell^{r'}}=1} \sum_{j=1}^\infty \int_{\mathbb{R}^n} \frac{|a_j(x) f_j(y)|}{|x-y|^{n-\alpha}} \mathrm{d}y.
\end{aligned}$$

Observe that

$$\sum_{j=1}^k \int_{\mathbb{R}^n} \frac{|a_j(x) f_j(y)|}{|x-y|^{n-\alpha}} \mathrm{d}y = \int_{\mathbb{R}^n} \sum_{j=1}^k \frac{|a_j(x) f_j(y)|}{|x-y|^{n-\alpha}} \mathrm{d}y \leq \int_{\mathbb{R}^n} \sum_{j=1}^\infty \frac{|a_j(x) f_j(y)|}{|x-y|^{n-\alpha}} \mathrm{d}y.$$

To simplify the notation, we write

$$\sup_{a(x)=\{a_j(x)\}_{j=1}^\infty \in \ell^{r'}:\|a(x)\|_{\ell^{r'}}=1}$$

as $\sup_{\|a\|_{\ell^{r'}}=1}$. From the above observation, we have

$$\begin{aligned}
\sup_{\|a\|_{\ell^{r'}}=1} \sum_{j=1}^\infty \int_{\mathbb{R}^n} \frac{|a_j(x) f_j(y)|}{|x-y|^{n-\alpha}} \mathrm{d}y &\leq \sup_{\|a\|_{\ell^{r'}}=1} \int_{\mathbb{R}^n} \sum_{j=1}^\infty \frac{|a_j(x) f_j(y)|}{|x-y|^{n-\alpha}} \mathrm{d}y \\
&\leq \int_{\mathbb{R}^n} \sup_{\|a\|_{\ell^{r'}}=1} \sum_{j=1}^\infty \frac{|a_j(x) f_j(y)|}{|x-y|^{n-\alpha}} \mathrm{d}y.
\end{aligned}$$

Again, by duality

$$\int_{\mathbb{R}^n} \sup_{\|a\|_{\ell^{r'}}=1} \sum_{j=1}^\infty \frac{|a_j(x) f_j(y)|}{|x-y|^{n-\alpha}} \mathrm{d}y \lesssim \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{\|\vec{f}(y)\|_{\ell^r}}{|x-y|^{n-\alpha}} \mathrm{d}y = I_\alpha(\|\vec{f}\|_{\ell^r})(x).$$

Finally, we have

$$\|I_\alpha \vec{f}(x)\|_{\ell^r} \leq I_\alpha(\|\vec{f}\|_{\ell^r})(x). \tag{3.1}$$

By Theorem 1.2 and the inequality (3.1), we conclude that

$$\Big\| \|I_\alpha \vec{f}(x)\|_{\ell^r} \Big\|_{L^{n/(n-\alpha),1}(\mathbb{R}^n)} \lesssim \Big\| R(\|\vec{f}\|_{\ell^r}) \Big\|_{L^1(\mathbb{R}^n,\mathbb{R}^n)}$$

as desired.

### 3.2 Proof of Theorem 1.6

We know from Hölder inequality

$$|\vec{g}(x) \cdot I_\alpha \vec{f}(x)| \le \|\vec{g}(x)\|_{\ell^{r'}} \|I_\alpha \vec{f}(x)\|_{\ell^r}.$$

From inequality (3.1), we get

$$|\vec{g}(x) \cdot I_\alpha \vec{f}(x)| \le \|\vec{g}(x)\|_{\ell^{r'}} I_\alpha(\|\vec{f}\|_{\ell^r})(x).$$

Applying Theorem 1.3, we obtain

$$\begin{aligned} \Big\| |\vec{g}(\cdot) \cdot I_\alpha \vec{f}(\cdot)| \Big\|_{L^1(\mathbb{R}^n)} &\le \Big\| \|\vec{g}(\cdot)\|_{\ell^{r'}} . I_\alpha(\|\vec{f}\|_{\ell^r})(\cdot) \Big\|_{L^1(\mathbb{R}^n)} \\ &\lesssim \| \|\vec{g}(\cdot)\|_{\ell^{r'}} \|_{L^{\frac{n}{\alpha},\infty}(\mathbb{R}^n)} . \|R(\|\vec{f}\|_{\ell^r})\|_{L^1(\mathbb{R}^n,\mathbb{R}^n)}. \end{aligned}$$

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## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

## References

1. Grafakos, L.: Modern Fourier Analysis. Graduate Texts in Mathematics, 2nd edn. Springer, New York (2008)
2. Lu, S., Ding, Y., Yan, D.: Singular Integrals and Related Topics. World Scientific Publishing, Singapore (2006)
3. Sobolev, S.L.: On a theorem of functional analysis, Mat. Sb. 4(46), pp. 471–497 (in Russian); English transl. Amer. Math. Soc. **34**, 39–68 (1938)
4. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton (1970)
5. Stein, E.M., Weiss, G.: On the theory of harmonic functions of several variables, I. The theory of $H^p$-spaces. Acta Math. **103**, 25–62 (1960)
6. Schikorra, A., Spector, D., Van Schaftingen, J.: An $L^1$-type estimate for Riesz potentials. Rev. Mat. Iberoam. **33**(1), 291–303 (2017)
7. Spector, D.: An optimal Sobolev embedding for $L^1$. J. Funct. Anal. **279**, 108559 (2020)
8. Olsen, P.A.: Fractional integration, Morrey spaces and a Schrödinger equation. Commun. Partial Differ. Equ. **20**, 2005–2055 (1995)
9. Muckenhoupt, B., Wheeden, R.L.: Weighted norm inequalities for singular and fractional integrals. Trans. Am. Math. Soc. **161**, 249–258 (1971)

10. O'Neil, R.: Convolution operators and L(p, q) spaces. Duke Math. J. **30**(1), 129–142 (1963)
11. Sawano, Y.: $\ell^q$ valued extension of the fractional maximal operator for non-doubling measures via potential operators. Int. J. Pure Appl. Math. **26**(4), 505–522 (2006)
12. Morrey, C.B.: Functions of several variables and absolute continuity. Duke Math. J. **6**(1), 187–215 (1940)
13. Adams, D.R.: A note on Riesz potentials. Duke Math. J. **42**, 765–778 (1975)