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Diagonals Separating the Square of a Continuum

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Abstract

A metric continuum X is indecomposable if it cannot be put as the union of two of its proper subcontinua. A subset R of X is said to be continuumwise connected provided that for each pair of points $p, q \in R$, there exists a subcontinuum M of X such that $\{p, q\} \subset M \subset R$. Let X^2 denote the Cartesian square of X and Δ the diagonal of X^2. Recently, H. Katsuura asked if for a continuum X, distinct from the arc, $X^2 \setminus \Delta$ is continuumwise connected if and only if X is decomposable. In this paper, we show that no implication in this question holds. For the proof of the non-necessity, we use the dynamical properties of a suitable homeomorphism of the Cantor set onto itself to construct an appropriate indecomposable continuum X.

Keywords Continuum · Continuumwise connected · Diagonal · Product · Solenoid · Weakly mixing

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1 Introduction

A *continuum* is a compact connected non-degenerate metric space. A subcontinuum of a continuum X is a nonempty connected closed subset of X, so singletons are subcontinua. An *arc* is a continuum homeomorphic to the interval $[0, 1]$. Let X^2 denote the Cartesian square of X and Δ the diagonal of X^2.

The authors studied in [6] conditions under which Δ satisfies some of the properties described in [2, Table 1] and [8, Definition 1.1], for a number of examples and families of continua.

Recently, H. Katsuura [7] proved that the arc is the only continuum for which Δ does not satisfy that $X^2\setminus\Delta$ is connected, and he included the following question [7, p. 4] (Katsuura mentioned that this question was suggested by Wayne Lewis of Texas Tech University in a private conversation):

Question 1.1 *If X is a continuum other than the arc, is $X^2 \setminus \Delta$ continuumwise connected if and only if X is decomposable?*

As an application of dynamical systems theory to continuum theory, we use the dynamical properties of a particular homeomorphism from the Cantor set onto itself to prove that the necessity in Question 1.1 is not satisfied. In fact, we show that no implication in Question 1.1 is satisfied.

2 No Sufficiency

A *map* is a continuous function. Given continua X and Y, and a number $\varepsilon > 0$, an ε-*map* is an onto map $f : X \to Y$ such that for each $y \in Y$, diameter$(f^{-1}(y)) < \varepsilon$. A continuum X is *arc-like* provided that for each $\varepsilon > 0$, there exists an ε-map $f : X \to [0, 1]$.

Lemma 2.1 *Let X be an arc-like continuum and $p, q \in X$ such that $p \neq q$. Then, for every continuum $K \subset X^2$ containing the points (p, q) and (q, p), $K \cap \Delta \neq \emptyset$.*

Proof Let d be a metric for X. Let K be a continuum in X^2 such that $(p, q), (q, p) \in K$, for some points $p \neq q \in X$.

Suppose that $K \cap \Delta = \emptyset$. Hence, there exists $\varepsilon > 0$ such that for every point $(x, y) \in K$, $d(x, y) > \varepsilon$. Since X is an arc-like continuum, there exists an ε-map $\lambda : X \to [0, 1]$. Then, for each $(x, y) \in K$, $\lambda(x) \neq \lambda(y)$.

If there were points $(x_1, y_1), (x_2, y_2) \in K$ such that $\lambda(x_1) < \lambda(y_1)$ and $\lambda(y_2) < \lambda(x_2)$. Define $g : K \to [-1, 1]$ by $g(x, y) = \lambda(x) - \lambda(y)$. We have that g is continuous, $g(x_1, y_1) < 0$ and $g(x_2, y_2) > 0$. Since K is connected, by the Intermediate Value Theorem, there is $(x_0, y_0) \in K$ such that $g(x_0, y_0) = 0$. So, $\lambda(x_0) = \lambda(y_0)$, a contradiction.

Thus, we obtain that either $\lambda(x) < \lambda(y)$ for each $(x, y) \in K$ or $\lambda(y) < \lambda(x)$ for each $(x, y) \in K$. This contradicts the fact that $(p, q), (q, p) \in K$. □

Corollary 2.2 *Let X be an arc-like continuum. Then, $X^2 \setminus \Delta$ is not continuumwise connected.*

Since there are decomposable arc-like continua (for example, the $\sin(\frac{1}{x})$-curve), Corollary 2.2 shows that the sufficiency in Question 1.1 is not satisfied. We also proved in [6, Sect. 7] that, in fact the $\sin(\frac{1}{x})$-curve belongs to a family of curves for which Δ satisfies a stronger property in X^2. S. B. Nadler, Jr. [10, p. 329] named the compactifications of the ray $[0, \infty)$ whose remainder is an arc as *Elsa continua* and he proved that they are arc-like [9, Lemma 6] (in fact, it is easy to show that a compactification of $[0, \infty)$ with non-degenerate remainder, is arc-like if and only if the remainder is an arc-like continuum). Then, for each Elsa continuum, X is decomposable and $X^2 \setminus \Delta$ is not continuumwise connected.

3 No Necessity

In this section, we present an indecomposable continuum X such that $X^2 \setminus \Delta$ is continuumwise connected.

Definition 3.1 Let X be a continuum and A a subcontinuum of X with $\mathrm{int}_X(A) = \emptyset$. We say that A is a continuum of colocal connectedness in X if for each open subset U of X with $A \subset U$, there exists an open subset V of X such that $A \subset V \subset U$ and $X \setminus V$ is connected.

By [2, Table 1] and [8, Sect. 1], it is known that if X^2 is a continuum and Δ is a subcontinuum of X^2 which has properties from Definition 3.1, then $X^2 \setminus \Delta$ is continuumwise connected.

The construction and proof of the properties of X strongly depends on the dynamical properties of a particular homeomorphism of the Cantor set onto itself.

Let C denote the Cantor ternary set. In this section, we will use a homeomorphism $f : C \to C$ such that f is minimal (C does not contain any proper nonempty closed subset A such that $f(A) = A$, or equivalently, every orbit of f is dense) and f is weakly mixing (for any two nonempty subsets U, V of X^2, there exists $n \in \mathbb{N}$ such that $(f \times f)^n(U) \cap V \neq \emptyset$). A recent reference for the existence of such homeomorphisms is [3, Theorem 4.1]. It is known that the inverse of f^{-1} is also minimal [1, Theorem 6.2 (e)]. By [4, Proposition 2], $f \times f$ has a dense orbit.

We consider the space X obtained by identifying in $C \times [0, 1]$, for each $p \in C$, the points $(p, 1)$ and $(f(p), 0)$. Let $\varphi : C \times [0, 1] \to X$ be the quotient map and let $\sigma = \varphi \times \varphi : (C \times [0, 1])^2 \to X^2$. Let ρ be the metric on $C \times [0, 1]$ given by $\rho((p, s), (q, t)) = |p - q| + |s - t|$. Choose and fix any metric D for the space X.

In the hypothesis of the following theorem, we write the specific properties that we use of the homeomorphism f.

Theorem 3.2 *Suppose that $f : C \to C$ is a homeomorphism such that the orbits of f and f^{-1} are dense and $f \times f : C^2 \to C^2$ has a dense orbit. Then, X is an indecomposable continuum such that the diagonal Δ in X^2 is colocally connected.*

Proof The indecomposability of X is proved in [5, Corollary, p. 552]. So we only need to show that Δ is colocally connected in X^2. In order to do this, take an open subset U in X^2 such that $\Delta \subset U$. Then, there exists $\varepsilon > 0$ such that for any two points

$x, y \in X$, if $D(x, y) < \varepsilon$, then $(x, y) \in U$. Choose and fix $\delta > 0$ such that $\delta < \frac{1}{10}$ and if $(p, s), (q, t) \in C \times [0, 1]$ and $\rho((p, s), (q, t)) < 2\delta$, from the continuity of $\varphi \times \varphi$ it follows that $D(\varphi(p, s), \varphi(q, t)) < \frac{\varepsilon}{3}$ (hence $\sigma((p, s), (q, t)) \in U$).

Define

$$\begin{aligned} V_1 &= \{((p, s), (q, t)) \in (C \times [0, 1])^2 : |p - q| < \delta \text{ and } |t - s| < \delta\},\\ V_2 &= \{((p, s), (q, t)) \in (C \times [0, 1])^2 : |p - f(q)| < \delta, s < \delta \text{ and } 1 - \delta < t\},\\ V_3 &= \{((p, s), (q, t)) \in (C \times [0, 1])^2 : |f(p) - q| < \delta, t < \delta \text{ and } 1 - \delta < s\},\\ V_0 &= V_1 \cup V_2 \cup V_3 \end{aligned}$$

and

$$K = (C \times [0, 1])^2 \setminus V_0.$$

We check that K has the following properties.

(a) $\sigma(K)$ is a continuum, and

(b) $\Delta \subset X^2 \setminus \sigma(K) \subset U$.

In order to prove (a), observe that V_0 is open in $(C \times [0, 1])^2$, so K and $\sigma(K)$ are compact, so we only need to show that $\sigma(K)$ is connected. Let

$$M = (C \times \{0\}) \times (C \times \{\frac{1}{2}\}).$$

Notice that $M \subset K$.

Claim 1. Let $z = ((p, s), (q, t)) \in K$ be such that $s \le t$. Then, there exists a subcontinuum A of K such that $z, ((p, 0), (q, \frac{1}{2})) \in A$.

We prove Claim 1. We consider two cases.

Case 1. $2\delta \le s$ or $t \le 1 - \delta$.

For each $r \in [0, 1]$, let $u(r) = rs$ and $v(r) = rt + (1 - r)(t - s)$. Define $\lambda(r) = ((p, u(r)), (q, v(r)))$ and $A_1 = \{\lambda(r) : r \in [0, 1]\}$. Then, $z, ((p, 0), (q, t - s)) \in A_1$. Take $r \in [0, 1]$, observe that $u(r) \le v(r) \le t$ and $v(r) - u(r) = t - s$. Since $u(r) \le v(r)$ and $\delta < 1 - \delta$, we have that $\lambda(r) \notin V_3$. Since $v(r) - u(r) = t - s$ and $z \notin V_1$, we have that $\lambda(r) \notin V_1$.

In the case that $t \le 1 - \delta$, we have that $v(r) \le 1 - \delta$, Hence, $\lambda(r) \notin V_2$.

Now, we consider the case that $2\delta \le s$. If $r \le \frac{1}{2}$, then $\delta \le (1 - r)2\delta \le (1 - r)s$, so $t \le 1 \le 1 - \delta + (1 - r)s$. This implies that $v(r) = rt + (1 - r)(t - s) \le 1 - \delta$. Thus, $\lambda(r) \notin V_2$. If $\frac{1}{2} \le r$, then $\delta \le \frac{s}{2} \le rs = u(r)$. Hence, $\lambda(r) \notin V_2$.

This completes the proof that for each $r \in [0, 1]$, $\lambda(r) \in K$. Therefore, $A_1 \subset K$.

Given $r \in [0, 1]$, let $w(r) = r(t - s) + (1 - r)\frac{1}{2}$ and $\eta(r) = ((p, 0), (q, w(r)))$. Let $A_2 = \{\eta(r) : r \in [0, 1]\}$. Observe that $((p, 0), (q, t - s)), ((p, 0), (q, \frac{1}{2})) \in A_2$ and $\eta(r) \notin V_3$.

Take $r \in [0, 1]$. Since $\lambda(0) \in K$, we have that $((p, 0), (q, t - s)) \notin V_1 \cup V_2$. Since $((p, 0), (q, t - s)) \notin V_1$, we have that either $\delta \le |p - q|$ or $\delta \le t - s$. If $\delta \le |p - q|$, it is clear that $\eta(r) \notin V_1$. If $\delta \le t - s$, since $\delta \le \frac{1}{2}$, we have that $\delta \le w(r)$. Thus,

$\eta(r) \notin V_1$. Since $((p, 0), (q, t-s)) \notin V_2$, we have that either $\delta \leq |p - f(q)|$ or $t \leq 1-\delta$. If $\delta \leq |p - f(q)|$, it is clear that $\eta(r) \notin V_2$. If $t \leq 1-\delta$, since $\frac{1}{2} \leq 1-\delta$, we have that $w(r) \leq 1-\delta$. Thus, $\eta(r) \notin V_2$. We have shown that for each $r \in [0, 1]$, $\eta(r) \notin V_1 \cup V_2 \cup V_3$. Hence, $A_2 \subset K$.

Define $A = A_1 \cup A_2$. We obtained that $A_1 = \{\lambda(r) : r \in [0, 1]\} \subset K$ and $z, ((p, 0), (q, t-s)) \in A_1$. Also, $A_2 = \{\eta(r) : r \in [0, 1]\} \subset K$ and $((p, 0), (q, t-s)), ((p, 0), (q, \frac{1}{2})) \in A_2$. Hence, $A = A_1 \cup A_2$ is a closed and connected subset of K such that $z, ((p, 0), (q, \frac{1}{2})) \in A$.

Case 2. $s < 2\delta$ and $1-\delta < t$.
Given $r \in [0, 1]$, let $y(r) = r(1-\delta) + (1-r)t$ and $\gamma(r) = ((p, s), (q, y(r)))$. Let $A_3 = \{\gamma(r) : r \in [0, 1]\}$. Notice that $z, ((p, s), (q, 1-\delta)) \in A_3$ and for each $r \in [0, 1]$, $1-\delta \leq y(r)$, so $\gamma(r) \notin V_3$ and $\delta < \frac{7}{10} \leq y(r) - s$. Thus, $\gamma(r) \notin V_1$. Since $z \notin V_2$ and $1-\delta < t$, we have that either $\delta \leq |p - f(q)|$ or $\delta \leq s$, in both cases it is clear that $\gamma(r) \notin V_2$. This completes the proof that for each $r \in [0, 1]$, $\gamma(r) \in K$. Therefore, $A_3 \subset K$.

We apply Case 1 to the point $z_0 = ((p, s), (q, 1-\delta))$, so there exists a subcontinuum A_4 of K such that $z_0, ((p, 0), (q, \frac{1}{2})) \in A_4$. Define $A = A_3 \cup A_4$. Then, A has the required properties.

This finishes the proof of Claim 1.

By the symmetry of the roles of both coordinates in the definition of V_0, we obtain that the following claim also holds.

Claim 2. Let $z = ((p, s), (q, t)) \in K$ be such that $t \leq s$. Then, there exists a subcontinuum A of K such that $z, ((p, \frac{1}{2}), (q, 0)) \in A$.

Claim 3. Let $z = ((p, s), (q, t)) \in K$. Then, there exists a subcontinuum B of $\sigma(K)$ such that $\sigma(z) \in B$ and $B \cap \sigma(M) \neq \emptyset$.

The proof for the case $s \leq t$ follows from Claim 1. So we may suppose that $t \leq s$. By Claim 2, there exists a subcontinuum A_5 of K such that $z, ((p, \frac{1}{2}), (q, 0)) \in A_5$. Let $A_6 = \{((p, \frac{1}{2} + r), (q, r)) : r \in [0, \frac{1}{2}]\}$. Clearly, $((p, \frac{1}{2}), (q, 0)), ((p, 1), (q, \frac{1}{2})) \in A_6$ and $A_6 \subset K$. Let $A = A_5 \cup A_6$. Then, A is a subcontinuum of K, $z \in A$ and $\sigma((p, 1), (q, \frac{1}{2})) = \sigma((f(p), 0), (q, \frac{1}{2})) \in \sigma(A) \cap \sigma(M)$. Hence, $B = \sigma(A)$ satisfies the required properties.

Claim 4. Let $z = ((p, 0), (q, \frac{1}{2})) \in M$. Then, there exists a subcontinuum E of $\sigma(K)$ such that $\sigma(z), \sigma((f(p), 0), (f(q), \frac{1}{2})) \in E$.

In order to prove the existence of E, let $A = \{((p, r), (q, \frac{1}{2} + r)) : r \in [0, \frac{1}{2}]\} \cup \{((p, \frac{1}{2} + r), (f(q), r)) : r \in [0, \frac{1}{2}]\}$. Since $\varphi(q, 1) = \varphi(f(q), 0)$ and $\varphi(p, 1) = \varphi(f(p), 0)$, we obtain that $E = \sigma(A)$ satisfies the required properties. Hence, Claim 4 is proved.

Now, choose and fix a point $(p_0, q_0) \in C^2$ such that (p_0, q_0) has a dense orbit under $f \times f$. By Claim 4, there exists a sequence $E_1, E_2, E_3 \ldots$ of subcontinua of $\sigma(K)$ such that for each $n \in \mathbb{N}$, $\sigma((f^{n-1}(p_0), 0), (f^{n-1}(q_0), \frac{1}{2})), \sigma((f^n(p_0), 0), (f^n(q_0), \frac{1}{2})) \in E_n$. Then, the set $E_0 = E_1 \cup E_2 \cup E_3 \cup \cdots$ is a connected subset of $\sigma(K)$ and $\sigma((p_0, 0), (q_0, \frac{1}{2})) \in E_0$.

Choose any point $((p,0),(q,\frac{1}{2})) \in M$. Since (p_0,q_0) has a dense orbit under $f \times f$ and σ is continuous, we have that $\sigma((p,0),(q,\frac{1}{2})) \in \sigma(\mathrm{cl}_{K^2}(\{((f^n(p_0),0),(f^n(q_0),\frac{1}{2})) : n \in \mathbb{N}\})) \subset \mathrm{cl}_{X^2}(E_0)$. Since E_0 is connected and $E_0 \subset E_0 \cup \sigma(M) \subset \mathrm{cl}_{X^2}(E_0)$, we have that $W = E_0 \cup \sigma(M)$ is a connected subset of $\sigma(K)$.

Given any point $z \in K$, by Claim 3, there is a subcontinuum B_z of $\sigma(K)$ such that $\sigma(z) \in B_z$ and $B_z \cup \sigma(M) \neq \emptyset$. So, $\sigma(K) = W \cup (\cup\{B_z : z \in K\})$ is connected. This ends the proof of (a).

In order to prove (b), take $\sigma(z) \in \Delta$, where $z = ((p,s),(q,t))$. Then, $\varphi(p,s) = \varphi(q,t)$. Thus, either $(p,s) = (q,t)$ or, $t = 1$ and $(p,s) = (f(q),0)$, or, $s = 1$ and $(q,t) = (f(p),0)$. In the first case, $z \in V_1$, in the second $z \in V_2$ and in the third one, $z \in V_3$. In any case, $z \notin K$. We have shown that $\Delta \subset X^2 \setminus \sigma(K)$.

Now, we prove that $X^2 \setminus \sigma(K) \subset U$. Take an element $w = \sigma(z)$, where $z = ((p,s),(q,t)) \notin K$. Then, $z \in V_1 \cup V_2 \cup V_3$. We consider three cases.

Case 1. $z \in V_1$.
In this case, $|p-q| < \delta$ and $|t-s| < \delta$, and by the definition of δ, $\sigma(z) \in U$.

Case 2. $z \in V_2$.
In this case, $|p - f(q)| < \delta, s < \delta$ and $1-\delta < t$. Then, $\rho((p,s),(p,0)) < \delta$, $\rho((p,0),(f(q),0)) < \delta$ and $\rho((q,1),(q,t)) < \delta$. So, $D(\varphi(p,s),\varphi(p,0)) < \frac{\varepsilon}{3}$, $D(\varphi(p,0),\varphi(f(q),0)) < \frac{\varepsilon}{3}$ and $D(\varphi(q,1),\varphi(q,t)) < \frac{\varepsilon}{3}$. Since $\varphi(f(q),0) = \varphi(q,1)$, we have that $D(\varphi(p,s),\varphi(q,t)) < \varepsilon$. By the choice of ε, we conclude that $\sigma(z) = (\varphi(p,s),\varphi(q,t)) \in U$.

Case 3. $z \in V_3$.
This case is similar to Case 2. This completes the proof that $X^2 \setminus \sigma(K) \subset U$. Therefore, (b) holds.

Finally, define $V = X^2 \setminus \sigma(K)$. Then, V is an open subset of X^2 such that $\Delta \subset V \subset U$, and $X^2 \setminus V$ is connected. This finishes the proof that Δ is colocally connected in X^2. □

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