Bull. Malays. Math. Sci. Soc. (2023) 46:164
https://doi.org/10.1007/s40840-023-01560-9

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

H-eigenvalue Inclusion Sets for Sparse Tensors

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Received: 19 January 2023 / Revised: 14 May 2023 / Accepted: 4 July 2023 /
Published online: 19 July 2023

Abstract

Sparse tensors play fundamental roles in hypergraph data, sensor node network data and remote sensing data. In this paper, we establish new H-eigenvalue inclusion sets for sparse tensors by their majorization matrix's digraph and representation matrix's digraph. Numerical examples are proposed to verify that our conclusions are more accurate and less computations than existing results. As applications, we provide some checkable sufficient conditions for the positive definiteness of even-order sparse tensors, and propose lower and upper bounds of H-spectral radius of nonnegative sparse tensors.

Keywords Sparse tensors · H-eigenvalue inclusion sets · Positive definiteness · H-spectral radius

Mathematics Subject Classification 15A18 · 15A42 · 15A69

1 Introduction

Let $\mathbb{C}(\mathbb{R})$ be the set of complex (real) numbers and $\mathbb{C}^n$ $(\mathbb{R}^n)$ be the set of n-dimensional complex (real) vectors. An m-order n-dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is a multi-way array with entries

$$a_{i_1 i_2 \cdots i_m} \in \mathbb{C}, \quad i_k \in N = \{1, 2, \ldots, n\}, \quad k = 1, 2, \ldots, m.$$

Communicated by Fuad Kittaneh.

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Tensor $\mathcal{A}$ is called nonnegative (positive) if $a_{i_1 i_2 \ldots i_m} \geq 0$ ($a_{i_1 i_2 \ldots i_m} > 0$).

Tensor is a higher-order extension of matrix, and hence many definitions and associated properties for matrix, such as eigenvalue theory, can be extended to higher order tensor by investigating its multilinear algebra analysis [12, 18].

Definition 1.1 Let $\mathcal{A}$ be an m-order n-dimensional tensor. Then (λ, x) is called an eigenpair of tensor $\mathcal{A}$ if

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \tag{1}$$

where $(\mathcal{A}x^{m-1})_i = \sum\limits_{i_2,\ldots,i_m \in N} a_{i i_2 \ldots i_m} x_{i_2} \ldots x_{i_m}$ and $x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})^\top$. Further, (λ, x) is called an H-eigenpair if they are both real.

Due to their numerous applications in fields such as higher-order Markov chains [14], medical resonance imaging [1, 18, 19], Hypergraph [2, 16] and positive definiteness of multivariate forms in automatical control [15, 18, 20, 23], tensor eigenvalue problems have attracted a great deal of critical attention. For example, we can use the smallest H-eigenvalue to ascertain whether a multivariate form is positive definite [15]. However, it is NP-hard to locate all H-eigenvalues or the smallest H-eigenvalue [6, 18]. To check the positive definiteness, researchers created a set that contained all H-eigenvalues [4, 8, 9, 24, 25, 28, 29]. Recently, sparse tensor eigenvalue problems, which the number of non-zero elements is far less than the number of zero elements, have recently been crucial in data problems, such as hypergraph data, sensor node network data and remote sensing data [2, 21, 26, 27]. Unfortunately, the computing effort is large if we use the aforementioned methods to build the H-eigenvalue inclusion set for sparse tensor eigenvalue problems with high-dimensional variables. Therefore, the sparsity of tensors encourages us to develop new H-eigenvalue inclusion sets. Very recently, Liu et al. [13] established bounds for the spectral radius of a nonnegative sparse tensor by its majorization matrix's digraph. There are two intriguing issues that come up: can we use the aforementioned method for generic sparse tensors? can new matrix's digraph be introduced to characterize H-eigenvalues of both generic and nonnegative sparse tensors?

Motivated and inspired by the above works, we explore the relations between general sparse tensors and their majorization matrix's digraph and representation matrix's digraph introduced by [5, 7, 16]. By drawing on the information of $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(i)$ and $\Gamma_{|\mathring{\mathcal{A}}|}(i)$ of majorization matrix's digraph and representation matrix's digraph, we establish tight H-eigenvalue inclusion sets with reduced calculations, which enhances the results [8, 9]. Based on new H-eigenvalue inclusion sets, we propose several sufficient conditions to identify positive definiteness of even-order real supersymmetric sparse tensors. Finally, we estimate sharp lower and upper bounds for H-spectral radius of nonnegative sparse tensors with simple computations.

The remainder of the paper is organized as follows. In Sect. 2, important definitions and preliminary results are recalled. In Sect. 3, we establish the improved H-eigenvalue inclusion sets and show that they have their own advantages by Examples 3.1 and 3.2. In Sect. 4, we check the positive definiteness of even-order real supersymmetric sparse tensors and estimate the bounds for H-spectral radius of nonnegative sparse tensors using these H-eigenvalue inclusion sets.

2 Preliminaries

In this section, we introduce some definitions and related properties of the tensor analysis.

Definition 2.1 [3, 18] Let $\mathcal{A}$ be an m-order n-dimensional tensor.

(i) Tensor $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $I \subset \{1, 2, \ldots, n\}$ such that

$$a_{i_1 i_2 \ldots i_m} = 0, \quad \forall i_1 \in I, i_2, \ldots, i_m \notin I.$$

If $\mathcal{A}$ is not reducible, then it is called irreducible.

(ii) Tensor $\mathcal{A}$ is called supersymmetric if its entries are invariant under any permutation of their indices.

(iii) Let $\sigma(\mathcal{A})$ be the set of all H-eigenvalues of $\mathcal{A}$. Then, H-spectral radius $\rho(\mathcal{A})$ is denoted by

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

As we know, H-spectral radius $\rho(\mathcal{A})$ coincides with the maximum eigenvalue of nonnegative tensors.

In what follows, we introduce the relations between directed graph and matrices (tensors).

The directed graph of a nonnegative matrix $A = (a_{ij})$ has as vertices the indices $\{1, \cdots, n\}$, and there is an arc from vertex i to vertex j if $a_{ij} \neq 0$. Matrix A is irreducible, if and only if one can get from any vertex to any other vertex (perhaps in several steps) and is called a strongly connected graph [16].

Definition 2.2 [5, 7, 16] Let $\mathcal{A}$ be an m-order n-dimensional nonnegative tensor.

(i) A nonnegative matrix $\mathring{\mathcal{A}} = (a_{ij})_{n\times n}$ is called the majorization associated to tensor $\mathcal{A}$, if the (i, j)-th element of $\mathring{\mathcal{A}}$ is defined to be $a_{ij\ldots j}$ for any $i, j \in N$.

(ii) A nonnegative matrix $\mathcal{G}(\mathcal{A}) = (a_{ij})_{n\times n}$ is called the representation associated to the tensor $\mathcal{A}$, if the (i, j)-th element of $\mathcal{G}(\mathcal{A})$ is defined to be $\sum\limits_{j\in\{i_2,\cdots,i_m\}} a_{ii_2\cdots i_m}$.

(iii) We associate with $\mathring{\mathcal{A}}$ digraphs as $\Gamma_{\mathring{\mathcal{A}}} = \left(V(\mathring{\mathcal{A}}), E(\mathring{\mathcal{A}})\right)$, where $V(\mathring{\mathcal{A}}) = \{1, \cdots, n\}$ is the vertex set of $\Gamma_{\mathring{\mathcal{A}}}$, and $E(\mathring{\mathcal{A}}) = \{e_{ij} : e_{ij} = a_{ij\ldots j} \neq 0, i \neq j\}$ is the arc set of $\Gamma_{\mathring{\mathcal{A}}}$, i.e., e_{ij} is the directed edge of $\Gamma_{\mathring{\mathcal{A}}}$.

(iv) We associate with $\mathcal{G}(\mathcal{A})$ digraphs as $\Gamma_{\mathcal{G}(\mathcal{A})} = (V(\mathcal{G}(\mathcal{A})), E(\mathcal{G}(\mathcal{A})))$, where $V(\mathcal{G}(\mathcal{A})) = \{1, \cdots, n\}$ is the vertex set of $\Gamma_{\mathcal{G}(\mathcal{A})}$, and $E(\mathcal{G}(\mathcal{A})) = \{g_{ij} : g_{ij} = \sum\limits_{j\in\{i_2,\cdots,i_m\}} a_{ii_2\cdots i_m} \neq 0, i \neq j\}$ the arc set of $\Gamma_{\mathcal{G}(\mathcal{A}))}$, i.e., g_{ij} is the directed edge of $\Gamma_{\mathcal{G}(\mathcal{A})}$.

(v) Tensor $\mathcal{A}$ is called weakly irreducible if $\mathcal{G}(\mathcal{A})$ is irreducible.

From Theorem 2.3 of [16], if $\mathring{\mathcal{A}}$ is irreducible, then $\mathcal{A}$ is irreducible. Further, if $\mathcal{A}$ is irreducible, then $\mathcal{A}$ is weakly irreducible in [7]. When $\mathcal{A}$ is a general tensor, we use $|\mathcal{A}|$ to denote the nonnegative tensor composed of $\mathcal{A}$. In this paper, $|\mathring{\mathcal{A}}|$ and $\mathcal{G}(|\mathcal{A}|)$ denote the majorization matrix and the representation matrix of general tensors, respectively.

We end this section with important results of [8, 9, 25]. Given an m-order n-dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$, denote

$$\begin{aligned}
\Delta_i &= \{(i_2, \cdots, i_m) : i_j = i \ for\ some\ j \in \{2, \cdots, m\},\ where\ i, i_2, \cdots, i_m \in N\}, \\
\overline{\Delta}_i &= \{(i_2, \cdots, i_m) : i_j \neq i \ for\ any\ \{2, \cdots, m\},\ where\ i, i_2, \cdots, i_m \in N\}, \\
r_i(\mathcal{A}) &= \sum_{\substack{i_2, \ldots, i_m \in N \\ \delta_{i i_2 \ldots i_m} = 0}} |a_{i i_2 \ldots i_m}|, \quad r_i^{\Delta_i}(\mathcal{A}) = \sum_{\substack{(i_2, \cdots, i_m) \in \Delta_i \\ \delta_{i i_2 \ldots i_m} = 0}} |a_{i i_2 \cdots i_m}|, \\
r_i^{\overline{\Delta}_i}(\mathcal{A}) &= \sum_{(i_2, \cdots, i_m) \in \overline{\Delta}_i} |a_{i i_2 \cdots i_m}|, \quad r_i(\mathcal{A}) = r_i^{\Delta_i}(\mathcal{A}) + r_i^{\overline{\Delta}_i}(\mathcal{A}),
\end{aligned}$$

where

$$\delta_{i_1 i_2 \ldots i_m} = \begin{cases} 1, & if\ i_1 = i_2 = \cdots = i_m, \\ 0, & otherwise. \end{cases}$$

Lemma 2.1 *Let $\mathcal{A}$ be an m-order n-dimensional tensor. Then,*

(I) *(Theorem 6 of [18])*

$$\sigma(\mathcal{A}) \subseteq \Upsilon(\mathcal{A}) = \bigcup_{i \in N} \Upsilon_i(\mathcal{A}),$$

where $\Upsilon_i(\mathcal{A}) = \{z \in \mathbb{R} : |z - a_{i \ldots i}| \leq r_i(\mathcal{A})\}$.

(II) *(Theorem 2.1 of [8])*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) = \bigcup_{i, j \in N, i \neq j} \mathcal{K}_{i,j}(\mathcal{A}),$$

where $\mathcal{K}_{i,j}(\mathcal{A}) = \{z \in \mathbb{R} : (|z - a_{i \ldots i}| - r_i^j(\mathcal{A}))|z - a_{j \ldots j}| \leq |a_{ij \ldots j}| r_j(\mathcal{A})\}$ and $r_i^j(\mathcal{A}) = r_i(\mathcal{A}) - |a_{ij \cdots j}|$.

(III) *(Theorem 2.1 of [9])*

$$\sigma(\mathcal{A}) \subseteq \Theta(\mathcal{A}) = \bigcup_{i, j \in N, i \neq j} \Theta_{i,j}(\mathcal{A}),$$

where $\Theta_{i,j}(\mathcal{A}) = \{z \in \mathbb{R} : (|z - a_{i \cdots i}| - r_i^{\Delta_i}(\mathcal{A}))|z - a_{j \cdots j}| \leq r_i^{\overline{\Delta}_i}(\mathcal{A}) r_j(\mathcal{A})\}$.

3 *H*-eigenvalue Inclusion Sets of Sparse Tensors

In this section, we establish two tight H-eigenvalue inclusion sets of a sparse tensor by its majorization matrix's digraph and representation matrix's digraph, which can reduce calculations and improve H-eigenvalue inclusion sets in [8, 11].

Theorem 3.1 *Let $\mathcal{A}$ be an m-order n-dimensional tensor with $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(i) = \{i : \exists\, j \in N$ such that $g_{ij} \in E(\mathcal{G}(|\mathcal{A}|))\} \neq \emptyset$. Then,*

$$\sigma(\mathcal{A}) \subseteq \widetilde{\Theta}(\mathcal{A}) = \bigcup_{g_{ij} \in E(\mathcal{G}(|\mathcal{A}|))} \Theta_{i,j}(\mathcal{A}),$$

where $\Theta_{i,j}(\mathcal{A}) = \left\{z \in \mathbb{R} : \left(|z - a_{i\cdots i}| - r_i^{\Delta_i}(\mathcal{A})\right) |z - a_{j\cdots j}| \leq r_i^{\overline{\Delta}_i}(\mathcal{A}) r_j(\mathcal{A})\right\}$.

Proof Let (λ, x) be an H-eigenpair of $\mathcal{A}$, i.e.

$$(\lambda - a_{i\cdots i}) x_i^{m-1} = \sum_{\delta_{ii_2\cdots i_m = 0}} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}. \tag{2}$$

Without loss of generality, we assume

$$|x_{t_1}| \geq |x_{t_2}| \geq \cdots \geq |x_{t_n}| \geq 0.$$

Since $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(t_1) \neq \emptyset$, we set $|x_{t_s}| = \max\{|x_{t_i}| : \sum_{(i_2,\cdots,i_m) \in \overline{\Delta}_{t_1}} | a_{t_1 i_2 \cdots i_m} | \neq 0, i \in N\}$, which means $g_{t_1 t_s} \in E(\mathcal{G}(|\mathcal{A}|))$. In view of the t_1-th equation of (2), we deduce

$$\begin{aligned}
|(\lambda - a_{t_1\cdots t_1}) x_{t_1}^{m-1}| &= |\sum_{\substack{(i_2,\cdots,i_m) \in \Delta_{t_1} \\ \delta_{t_1 i_2\cdots i_m = 0}}} a_{t_1 i_2\cdots i_m} x_{i_2} \cdots x_{i_m} + \sum_{(i_2,\cdots,i_m) \in \overline{\Delta}_{t_1}} a_{t_1 i_2\cdots i_m} x_{i_2} \cdots x_{i_m}| \\
&\leq \sum_{\substack{(i_2,\cdots,i_m) \in \Delta_{t_1} \\ \delta_{t_1 i_2\cdots i_m = 0}}} |a_{t_1 i_2\cdots i_m}| |x_{i_2}| \cdots |x_{i_m}| + \sum_{(i_2,\cdots,i_m) \in \overline{\Delta}_{t_1}} |a_{t_1 i_2\cdots i_m}| |x_{i_2}| \cdots |x_{i_m}| \\
&\leq r_{t_1}^{\Delta_{t_1}}(\mathcal{A}) |x_{t_1}|^{m-1} + r_{t_1}^{\overline{\Delta}_{t_1}}(\mathcal{A}) |x_{t_s}|^{m-1},
\end{aligned}$$

equivalently,

$$\left(|\lambda - a_{t_l\cdots t_l}| - r_{t_1}^{\Delta_{t_1}}(\mathcal{A})\right) |x_{t_1}|^{m-1} \leq r_{t_1}^{\overline{\Delta}_{t_1}}(\mathcal{A}) |x_{t_s}|^{m-1}. \tag{3}$$

We now break up the argument into two cases.

Case 1: $|x_{t_s}| = 0$. Then, $|\lambda - a_{t_l\cdots t_l}| - r_{t_1}^{\Delta_{t_1}}(\mathcal{A}) \leq 0$ and it is obvious that $\lambda \in \Theta_{t_1,t_s}(\mathcal{A}) \subseteq \widetilde{\Theta}(\mathcal{A})$.

Case 2: $|x_{t_s}| > 0$. It follows from (2) and $i = t_s$ that

$$\begin{aligned}|(\lambda - a_{t_s\cdots t_s})x_{t_s}^{m-1}| &= |\sum_{\delta_{t_s i_2\cdots i_m}=0} a_{t_s i_2\cdots i_m} x_{i_2}\cdots x_{i_m}| \\ &\leq \sum_{\delta_{t_s i_2\cdots i_m}=0} |a_{t_s i_2\cdots i_m}||x_{i_2}|\cdots|x_{i_m}| \\ &\leq r_{t_s}(\mathcal{A})|x_{t_1}|^{m-1}.\end{aligned} \tag{4}$$

Multiplying inequalities (3) and (4) gives

$$\left(|\lambda - a_{t_l\cdots t_l}| - r_{t_1}^{\Delta_{t_1}}(\mathcal{A})\right)|\lambda - a_{t_s\cdots t_s}||x_{t_1}|^{m-1}|x_{t_s}|^{m-1} \leq r_{t_1}^{\overline{\Delta}_{t_1}}(\mathcal{A})r_{t_s}(\mathcal{A})|x_{t_1}|^{m-1}|x_{t_s}|^{m-1}.$$

From $|x_{t_1}|^{m-1}|x_{t_s}|^{m-1} > 0$, it holds that

$$\left(|\lambda - a_{t_l\cdots t_l}| - r_{t_1}^{\Delta_{t_1}}(\mathcal{A})\right)|\lambda - a_{t_s\cdots t_s}| \leq r_{t_1}^{\overline{\Delta}_{t_1}}(\mathcal{A})r_{t_s}(\mathcal{A}),$$

which implies $\lambda \in \Theta_{t_1,t_s}(\mathcal{A}) \subseteq \widetilde{\Theta}(\mathcal{A})$. □

Corollary 3.1 *Let $\mathcal{A}$ be an m-order n-dimensional tensor. Then,*

$$\sigma(\mathcal{A}) \subseteq \widetilde{\widetilde{\Theta}}(\mathcal{A}) = \bigcup_{g_{ij}\in E(\mathcal{G}(|\mathcal{A}|))\bigcup j-i=1,1-n} \Theta_{i,j}(\mathcal{A}),$$

where $\Theta_{i,j}(\mathcal{A})$ is defined in Theorem 3.1.

Proof When $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(i) \neq \emptyset$, by Theorem 3.1, the results hold. We only prove $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(i) = \emptyset$. For any $\epsilon > 0$, set

$$|\mathcal{A}(\epsilon)| = |\mathcal{A}| + \Phi(\epsilon) \quad and \quad \Phi(\epsilon) = (\theta_{i_1\cdots i_m}),$$

where

$$\theta_{i_1\cdots i_m} = \begin{cases} \theta_{ij\cdots j} = \epsilon, & if j - i = 1, 1-n, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\Gamma_{\mathcal{G}(|\mathcal{A}(\epsilon)|)}(i) \neq \emptyset$. Following the similar arguments to the proof of Theorem 3.1, we have

$$\sigma(\mathcal{A}(\epsilon)) \subseteq \widetilde{\Theta}(\mathcal{A}(\epsilon)),$$

Letting $\epsilon \to 0$, we obtain

$$\sigma(\mathcal{A}) \subseteq \widetilde{\widetilde{\Theta}}(\mathcal{A}).$$

□

Remark 3.1 Compared with Theorem 2.1 of [9], the result of Corollary 3.1 has minor computations and tight H-eigenvalue inclusion sets, i.e., $\widetilde{\widetilde{\Theta}}(\mathcal{A}) \subseteq \Theta(\mathcal{A})$. Indeed, $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(i) \neq \emptyset$ is a condition easy to verify and meet.

Now, we arrive at the following H-eigenvalue inclusion sets for sparse tensors by their majorization matrix's digraph.

Theorem 3.2 *Let $\mathcal{A}$ be an m-order n-dimensional tensor with $\Gamma_{|\mathring{A}|}(i) = \{i : \exists\ j \in N$ such that $e_{ij} \in E(|\mathring{A}|)\} \neq \emptyset$. Then,*

$$\sigma(\mathcal{A}) \subseteq \varpi(\mathcal{A}) = \bigcup_{e_{ij} \in E(|\mathring{A}|)} \varpi_{i,j}(\mathcal{A}),$$

where $\varpi_{i,j}(\mathcal{A}) = \left\{z \in \mathbb{R} : \left(|z - a_{i\cdots i}| - r_i^{'}(\mathcal{A})\right) |z - a_{j\cdots j}| \leq \widetilde{r}_i(\mathcal{A}) r_j(\mathcal{A})\right\}$, $\widetilde{r}_i(\mathcal{A}) = \sum_{\substack{\delta_{i_2\cdots i_m}=1 \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|$ *and* $r_i^{'}(\mathcal{A}) = r_i(\mathcal{A}) - \widetilde{r}_i(\mathcal{A})$.

Proof Let (λ, x) be an H-eigenpair of $\mathcal{A}$. Without loss of generality, we assume that $|x_{t_1}| \geq |x_{t_2}| \geq \cdots \geq |x_{t_n}|$. Since $\Gamma_{|\mathring{A}|}(i) \neq \emptyset$, there exists $j \neq t_1$ with $a_{t_1 j \ldots j} \neq 0$. Assume

$$a_{t_1 t_l \cdots t_l} = 0,\ l = 2, 3, \cdots, s-1,\ a_{t_1 t_s \cdots t_s} \neq 0\ (2 \leq s \leq n),$$

which implies $e_{t_1 t_s} \in E(|\mathring{A}|)$. Recalling the t_1-th equation of (2), we have

$$\begin{aligned} |(\lambda - a_{t_1\cdots t_1}) x_{t_1}^{m-1}| &= |\sum_{\delta_{t_1 i_2\cdots i_m}=0} a_{t_1 i_2\cdots i_m} x_{i_2} \cdots x_{i_m}| \\ &\leq \sum_{\delta_{i_2\cdots i_m}=0} |a_{t_1 i_2\cdots i_m}||x_{i_2}| \cdots |x_{i_m}| + \sum_{\substack{\delta_{i_2\cdots i_m}=1 \\ \delta_{t_1 i_2\cdots i_m}=0}} |a_{t_1 i_2\cdots i_m}||x_{i_2}| \cdots |x_{i_m}| \\ &\leq r_{t_1}^{'}(\mathcal{A})|x_{t_1}|^{m-1} + \widetilde{r}_{t_1}(\mathcal{A})|x_{t_s}|^{m-1}, \end{aligned}$$

equivalently,

$$\left(|\lambda - a_{t_1\cdots t_1}| - r_{t_1}^{'}(\mathcal{A})\right) |x_{t_1}|^{m-1} \leq \widetilde{r}_{t_1}(\mathcal{A})|x_{t_s}|^{m-1}. \tag{5}$$

Next, we break up the argument into two cases.

Case 1: $x_{t_s} = 0$. Then, $|(\lambda - a_{t_1\cdots t_1})| - r_{t_1}^{'}(\mathcal{A}) \leq 0$. Clearly, $\lambda \in \varpi_{t_1,t_s}(\mathcal{A})$.

Case 2: $x_{t_s} \neq 0$. It follows from (2) and $i = t_s$ that

$$\begin{aligned} |(\lambda - a_{t_s\cdots t_s}) x_{t_s}^{m-1}| &= |\sum_{\delta_{t_s i_2\cdots i_m}=0} a_{t_s i_2\cdots i_m} x_{i_2} \cdots x_{i_m}| \\ &\leq \sum_{\delta_{t_s i_2\cdots i_m}=0} |a_{t_s i_2\cdots i_m}||x_{i_2}| \cdots |x_{i_m}| \\ &\leq r_{t_s}(\mathcal{A})|x_{t_1}|^{m-1}. \end{aligned} \tag{6}$$

Multiplying inequalities (5) and (6) gives

$$\left(|\lambda - a_{t_1\cdots t_1}| - r'_{t_1}(\mathcal{A})\right)|\lambda - a_{t_s\cdots t_s}||x_{t_1}|^{m-1}|x_{t_s}|^{m-1} \leq \widetilde{r}_{t_1}(\mathcal{A})r_{t_s}(\mathcal{A})|x_{t_1}|^{m-1}|x_{t_s}|^{m-1}.$$

From $|x_{t_1}|^{m-1}|x_{t_s}|^{m-1} > 0$, it holds that

$$\left(|\lambda - a_{t_1\cdots t_1}| - r'_{t_1}(\mathcal{A})\right)|\lambda - a_{t_s\cdots t_s}| \leq \widetilde{r}_{t_1}(\mathcal{A})r_{t_s}(\mathcal{A}),$$

which implies $\lambda \in \varpi_{t_1,t_s}(\mathcal{A}) \subseteq \varpi(\mathcal{A})$. □

Following the similar arguments to the proof of Corollary 3.1, we obtain the desired conclusions.

Corollary 3.2 *Let $\mathcal{A}$ be an m-order n-dimensional tensor. Then,*

$$\sigma(\mathcal{A}) \subseteq \widehat{\varpi}(\mathcal{A}) = \bigcup_{e_{ij}\in E(|\mathring{\mathcal{A}}|)\bigcup j-i=1,1-n} \varpi_{i,j}(\mathcal{A}).$$

Compared with Theorem 2.1 of [8], the result of Corollary 3.2 requires minor calculations but has accurate results. Detailed investigation is given in Corollary 3.3.

Lemma 3.1 (Lemma 2.2 of [9])

(i) *Let $a, b, c \geq 0$ and $d > 0$. If $\frac{a}{b+c+d} \leq 1$, then*

$$\frac{a-(b+c)}{d} \leq \frac{a-b}{c+d} \leq \frac{a}{b+c+d}.$$

(ii) *Let $a, b, c \geq 0$ and $d > 0$. If $\frac{a}{b+c+d} \geq 1$, then*

$$\frac{a-(b+c)}{d} \geq \frac{a-b}{c+d} \geq \frac{a}{b+c+d}.$$

Corollary 3.3 *Let $\mathcal{A}$ be an m-order n-dimensional tensor with $n \geq 2$. Then,*

$$\widehat{\varpi}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}).$$

Proof Let $z \in \widehat{\varpi}(\mathcal{A})$. Then there exist $p, q \in N$ with $p \neq q$ such that $z \in \varpi_{p,q}(\mathcal{A})$, i.e.

$$\left(|z - a_{q\cdots q}| - r'_q(\mathcal{A})\right)|z - a_{p\cdots p}| \leq \widetilde{r}_q(\mathcal{A})r_p(\mathcal{A}). \tag{7}$$

We now break up the argument into two cases.

Case 1: $\widetilde{r}_q(\mathcal{A})r_p(\mathcal{A}) = 0$, it holds that $\widetilde{r}_q(\mathcal{A}) = 0$ or $r_p(\mathcal{A}) = 0$.

When $\widetilde{r}_q(\mathcal{A}) = 0$, we have $|a_{qp\cdots p}| = 0$ and $r_q^p(\mathcal{A}) = r_q^{'}(\mathcal{A})$,

$$\begin{aligned}\left(|z - a_{q\cdots q}| - r_q^p(\mathcal{A})\right)|z - a_{p\cdots p}| &= (|z - a_{q\cdots q}| - r_q^{'}(\mathcal{A}))|z - a_{p\cdots p}| \\ &\leq \widetilde{r}_q(\mathcal{A}) r_p(\mathcal{A}) = |a_{qp\cdots p}| r_p(\mathcal{A}),\end{aligned}$$

which implies that $z \in \mathcal{K}_{q,p}(\mathcal{A})$. Consequently, $\widehat{\varpi}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$.

When $r_p(\mathcal{A}) = 0$, one has $r_q^p(\mathcal{A}) \geq r_q^{'}(\mathcal{A})$ and

$$\begin{aligned}\left(|z - a_{q\cdots q}| - r_q^p(\mathcal{A})\right)|z - a_{p\cdots p}| &\leq (|z - a_{q\cdots q}| - r_q^{'}(\mathcal{A}))|z - a_{p\cdots p}| \\ &\leq \widetilde{r}_q(\mathcal{A}) r_p(\mathcal{A}) = 0 = |a_{qp\cdots p}| r_p(\mathcal{A}),\end{aligned}$$

which leads to $z \in \mathcal{K}_{q,p}(\mathcal{A})$. Certainly, $\widehat{\varpi}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$.

Case 2: $\widetilde{r}_q(\mathcal{A}) r_p(\mathcal{A}) > 0$, dividing both sides by $\widetilde{r}_q(\mathcal{A}) r_p(\mathcal{A})$ on (7), one has

$$\frac{(|z - a_{q\cdots q}| - r_q^{'}(\mathcal{A}))}{\widetilde{r}_q(\mathcal{A})} \frac{|z - a_{p\cdots p}|}{r_p(\mathcal{A})} \leq 1, \tag{8}$$

which implies

$$\frac{(|z - a_{q\cdots q}| - r_q^{'}(\mathcal{A}))}{\widetilde{r}_q(\mathcal{A})} \leq 1 \tag{9}$$

or

$$\frac{|z - a_{p\cdots p}|}{r_p(\mathcal{A})} \leq 1. \tag{10}$$

Let $a = |z - a_{q\cdots q}|$, $b = r_q^{'}(\mathcal{A})$, $c = \sum\limits_{\substack{i_2\cdots i_m=1 \\ \delta_{qi_2\cdots i_m}=0}}^{n} a_{qi_2\cdots i_m} - |a_{qp\cdots p}|$ and $d = |a_{qp\cdots p}|$.

If (9) holds with $d = |a_{qp\cdots p}| > 0$, it follows from Lemma 3.1 and (8) that

$$\frac{|z - a_{q\cdots q}| - r_q^p(\mathcal{A})}{|a_{qp\cdots p}|} \frac{|z - a_{p\cdots p}|}{r_p(\mathcal{A})} \leq \frac{|z - a_{q\cdots q}| - r_q^{'}(\mathcal{A})}{\widetilde{r}_q(\mathcal{A})} \frac{|z - a_{p\cdots p}|}{r_p(\mathcal{A})} \leq 1,$$

equivalently,

$$(|z - a_{q\cdots q}| - r_q^p(\mathcal{A}))|z - a_{p\cdots p}| \leq |a_{qp\cdots p}| r_p(\mathcal{A}).$$

This implies $\widehat{\varpi}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$.

If (9) holds with $d = |a_{qp\cdots p}| = 0$, we obtain

$$|z - a_{q\cdots q}| - r_q^p(\mathcal{A}) \leq 0 = |a_{qp\cdots p}|.$$

Hence,

$$(|z - a_{q\cdots q}| - r_q^p(\mathcal{A}))|z - a_{p\cdots p}| \leq 0 = |a_{qp\cdots p}|r_p(\mathcal{A}),$$

which shows $\widehat{\varpi}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$.

Otherwise, (10) holds. We only prove $\widehat{\varpi}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$ under the case that

$$\frac{(|z - a_{q\cdots q}| - r_q^{'}(\mathcal{A}))}{\widetilde{r}_q(\mathcal{A})} > 1. \tag{11}$$

Owing to $\widetilde{r}_i(\mathcal{A}) = r_i(\mathcal{A}) - r_i^{'}(\mathcal{A})$, from (11), we deduce

$$\frac{|z - a_{q\cdots q}|}{r_q(\mathcal{A})} > 1.$$

If $d = |a_{pq\cdots q}| > 0$, from Lemma 3.1 and (8), we have

$$\frac{(|z - a_{p\cdots p}| - r_p^q(\mathcal{A}))}{|a_{pq\cdots q}|}\frac{|z - a_{q\cdots q}|}{r_q(\mathcal{A})} \leq \frac{(|z - a_{q\cdots q}| - r_q^{'}(\mathcal{A}))}{\widetilde{r}_q(\mathcal{A})}\frac{|z - a_{p\cdots p}|}{r_p(\mathcal{A})} \leq 1,$$

equivalently,

$$(|z - a_{p\cdots p}| - r_p^q(\mathcal{A}))|z - a_{q\cdots q}| \leq |a_{pq\cdots q}|r_q(\mathcal{A}).$$

This implies $\widehat{\varpi}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$.

If $d = |a_{pq\cdots q}| = 0$, from Lemma 3.1 and (8), it holds that

$$|z - a_{p\cdots p}| - r_p^q(\mathcal{A}) \leq 0 = |a_{pq\cdots q}|.$$

Hence,

$$(|z - a_{p\cdots p}| - r_p^q(\mathcal{A}))|z - a_{q\cdots q}| \leq 0 = |a_{pq\cdots q}|r_q(\mathcal{A}).$$

Consequently, $\widehat{\varpi}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$.

Based on the above two cases, we obtain the desired results. □

To illustrate the validity of Theorems 3.1 and 3.2, we employ a running example.

Example 3.1 Let $\mathcal{A}$ be a 3-order 4-dimensional tensor defined as follows:

$$a_{ijk} = \begin{cases} a_{111} = 1; a_{112} = 3; a_{121} = -1; a_{122} = 2; \\ a_{222} = 2; a_{232} = -1; a_{233} = 1; \\ a_{333} = 2; a_{334} = -3; a_{343} = 1; a_{344} = -1; \\ a_{411} = 1; a_{414} = -1; a_{421} = 2; a_{424} = 1; a_{444} = 3. \end{cases}$$

Table 1 Inclusion sets for Theorem 2.1 of [8]

$\mathcal{K}_{1,2}(\mathcal{A}) = [-3.702, 6.000]$	$\mathcal{K}_{1,3}(\mathcal{A}) = [-5.000, 7.000]$	$\mathcal{K}_{1,4}(\mathcal{A}) = [-5.000, 7.000]$
$\mathcal{K}_{2,1}(\mathcal{A}) = [0.000, 4.000]$	$\mathcal{K}_{2,3}(\mathcal{A}) = [-0.791, 4.791]$	$\mathcal{K}_{2,4}(\mathcal{A}) = [0.000, 4.000]$
$\mathcal{K}_{3,1}(\mathcal{A}) = [-3.000, 7.000]$	$\mathcal{K}_{3,2}(\mathcal{A}) = [-3.000, 7.000]$	$\mathcal{K}_{3,4}(\mathcal{A}) = [-2.854, 7.193]$
$\mathcal{K}_{4,1}(\mathcal{A}) = [-2.646, 7.873]$	$\mathcal{K}_{4,2}(\mathcal{A}) = [-2.000, 8.000]$	$\mathcal{K}_{4,3}(\mathcal{A}) = [-2.000, 8.000]$

Table 2 Inclusion sets for Theorem 3.1

$\Theta_{1,2}(\mathcal{A}) = [-3.702, 6.000]$	$\Theta_{2,3}(\mathcal{A}) = [-0.791, 4.791]$	$\Theta_{3,4}(\mathcal{A}) = [-2.854, 7.193]$
$\Theta_{4,1}(\mathcal{A}) = [-3.243, 7.690]$	$\Theta_{4,2}(\mathcal{A}) = [-1.000, 6.372]$	

Recalling Definition 2.2, we obtain

$$\mathcal{G}(|\mathcal{A}|) = \begin{pmatrix} 5 & 6 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 6 & 5 \\ 4 & 3 & 0 & 5 \end{pmatrix} \quad and \quad |\mathring{\mathcal{A}}| = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 3 \end{pmatrix}.$$

By virtue of Theorem 6 of [18], one has

$$-5.000 \le \lambda \le 8.000.$$

From Theorem 2.1 of [8], we obtain Table 1 and

$$-5.000 \le \lambda \le 8.000.$$

From Theorem 2.1 of [9], following the similar computations of $\mathcal{K}_{i,j}(\mathcal{A})$, we calculate 12 times $\Theta_{i,j}(\mathcal{A})$ with $i \neq j \in \{1, 2, 3, 4\}$ and obtain

$$-4.531 \le \lambda \le 7.690.$$

From representation matrix $\mathcal{G}(|\mathcal{A}|)$, for any $i \in \{1, 2, 3, 4\}$, we verify $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(i) \neq \emptyset$. In view of Theorem 3.1, we only compute Table 2 and

$$-3.702 \le \lambda \le 7.690.$$

By majorization matrix $|\mathring{\mathcal{A}}|$, for any $i \in \{1, 2, 3, 4\}$, we observe $\Gamma_{|\mathring{\mathcal{A}}|}(i) \neq \emptyset$. It follows from Theorem 3.2 that
and

$$-3.702 \le \lambda \le 7.873.$$

Table 3 Inclusion sets for Theorem 3.2

$\varpi_{1,2}(\mathcal{A}) = [-3.702, 6.000]$	$\varpi_{2,3}(\mathcal{A}) = [-0.791, 4.791]$
$\varpi_{3,4}(\mathcal{A}) = [-2.854, 7.193]$	$\varpi_{4,1}(\mathcal{A}) = [-2.646, 7.873]$

Table 4 Inclusion sets for Theorem 3.1

$\Theta_{1,2}(\mathcal{A}) = [-2.275, 6.000]$	$\Theta_{1,3}(\mathcal{A}) = [-2.000, 4.000]$	$\Theta_{1,4}(\mathcal{A}) = [-2.606, 5.464]$
$\Theta_{2,1}(\mathcal{A}) = [-2.000, 6.275]$	$\Theta_{2,3}(\mathcal{A}) = [-1.000, 5.372]$	$\Theta_{3,1}(\mathcal{A}) = [-1.562, 3.562]$
$\Theta_{3,4}(\mathcal{A}) = [-1.236, 4.000]$	$\Theta_{4,1}(\mathcal{A}) = [-2.372, 5.702]$	$\Theta_{4,2}(\mathcal{A}) = [-1.702, 6.372]$

Table 5 Inclusion sets for Theorem 3.2

$\varpi_{1,2}(\mathcal{A}) = [-2.464, 5.828]$	$\varpi_{2,3}(\mathcal{A}) = [-1.000, 5.828]$
$\varpi_{3,4}(\mathcal{A}) = [-1.236, 4.000]$	$\varpi_{4,1}(\mathcal{A}) = [-2.236, 5.828]$

Tight bounds and simple computations are advantages of the H-eigenvalue inclusion sets given by Theorems 3.1 and 3.2 over Theorems 2.1 of [8] and Theorem 2.1 of [9]. The conclusions of Theorems 3.1 and 3.2 generally have their own benefits. The conclusion of Theorem 3.1 in Example 3.1 is more precise than Theorem 3.2. The following example implies the converse results.

Example 3.2 Let $\mathcal{A}$ be an 3-order 4-dimensional tensor defined as follows:

$$a_{ijk} = \begin{cases} a_{111} = 1;\, a_{121} = -1;\, a_{122} = 2;\, a_{134} = -1; \\ a_{212} = -1;\, a_{213} = 1;\, a_{222} = 3;\, a_{233} = 1; \\ a_{331} = -1;\, a_{333} = 1;\, a_{344} = 1; \\ a_{411} = 1;\, a_{412} = -1;\, a_{414} = 2;\, a_{444} = 2. \end{cases}$$

We can obtain

$$\mathcal{G}(|\mathcal{A}|) = \begin{pmatrix} 2 & 3 & 1 & 1 \\ 2 & 4 & 3 & 0 \\ 1 & 0 & 2 & 1 \\ 4 & 1 & 0 & 4 \end{pmatrix} \quad and \quad |\mathring{\mathcal{A}}| = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{pmatrix}.$$

From representation matrix $\mathcal{G}(|\mathcal{A}|)$, for any $i \in \{1, 2, 3, 4\}$, we know $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(i) \neq \emptyset$. In view of Theorem 3.1, we compute Table 4 and

$$-2.606 \leq \lambda \leq 6.372.$$

By majorization matrix $|\mathring{\mathcal{A}}|$, for any $i \in \{1, 2, 3, 4\}$, we observe $\Gamma_{|\mathring{\mathcal{A}}|}(i) \neq \emptyset$. It follows from Theorem 3.2 that and

$$-2.464 \le \lambda \le 5.828.$$

4 Applications

4.1 Testing Positive Definiteness of Even-Order Real Supersymmetric Sparse Tensors

This subsection focuses on proving that an even-order real supersymmetric sparse tensor is positive definite based on the principle that $\mathcal{A}$ is positive definite if and only if all of its H-eigenvalues are positive [19]. In order to achieve this, we provide the following adequate conditions for the positive definiteness of sparse tensors via H-eigenvalue inclusion sets in Sect. 3.

Theorem 4.1 *Let $\mathcal{A}$ be an even m-order n-dimensional supersymmetric tensor with $\Gamma_{\mathcal{G}(|\mathcal{A}|)}(i) \neq \emptyset$. If all $(i, j) \in \{(k, l) : g_{kl} \in E(\mathcal{G}(|\mathcal{A}|)), l \neq k\}$ and $a_{i\cdots i} > 0, i \in N$ such that*

$$\left(a_{i\cdots i} - r_i^{\Delta_i}(\mathcal{A})\right) a_{j\cdots j} > r_i^{\overline{\Delta}_i}(\mathcal{A}) r_j(\mathcal{A}),$$

then $\mathcal{A}$ is positive definite.

Proof Let λ be an H-eigenvalue of $\mathcal{A}$. Suppose on the contrary that $\lambda \le 0$. It follows from Theorem 3.1 that there is a $g_{i_0 j_0} \in E(\mathcal{G}(|\mathcal{A}|))$ such that $\lambda \in \Phi_{i_0, j_0}(\mathcal{A})$, that is,

$$\left(|\lambda - a_{i_0\cdots i_0}| - r_{i_0}^{\Delta_{i_0}}(\mathcal{A})\right) |\lambda - a_{j_0\cdots j_0}| \le r_{i_0}^{\overline{\Delta}_{i_0}}(\mathcal{A}) r_{j_0}(\mathcal{A}).$$

From $a_{i\cdots i} > 0$ and $a_{j\cdots j} > 0$, we have

$$\begin{aligned}&\left(|\lambda - a_{i_0\cdots i_0}| - r_{i_0}^{\Delta_{i_0}}(\mathcal{A})\right) |\lambda - a_{j_0\cdots j_0}| \\ &\quad \ge \left(a_{i_0\cdots i_0} - r_{i_0}^{\Delta_{i_0}}(\mathcal{A})\right) a_{j_0\cdots j_0} > r_{i_0}^{\overline{\Delta}_{i_0}}(\mathcal{A}) r_{j_0}(\mathcal{A}) \ge 0.\end{aligned}$$

This is a contradiction. Hence, $\lambda > 0$ and $\mathcal{A}$ is positive definite. □

Corollary 4.1 *Let $\mathcal{A}$ be an even m-order n-dimensional supersymmetric tensor. If all $(i, j) \in \{(k, l) : g_{kl} \in E(\mathcal{G}(|\mathcal{A}|)) \bigcup l - k = 1$ or $1 - n, l \neq k\}$ and $a_{i\cdots i} > 0, i \in N$ such that*

$$\left(a_{i\cdots i} - r_i^{\Delta_i}(\mathcal{A})\right) a_{j\cdots j} > r_i^{\overline{\Delta}_i}(\mathcal{A}) r_j(\mathcal{A}),$$

then $\mathcal{A}$ is positive definite.

Proof Following the proof of Theorem 4.1, we obtain the desired conclusions. □

Theorem 4.2 *Let $\mathcal{A}$ be an even m-order n-dimensional supersymmetric tensor with $\Gamma_{|\mathring{\mathcal{A}}|}(i) \neq \emptyset$. If all $(i, j) \in \{(k, l) : e_{kl} \in E(|\mathring{\mathcal{A}}|)\}, l \neq k\}$ and $a_{i\cdots i} > 0, i \in N$ such that*

$$\left(a_{i\cdots i} - r_i^{'}(\mathcal{A})\right) a_{j\cdots j} > \widetilde{r}_i(\mathcal{A}) r_j(\mathcal{A}),$$

then $\mathcal{A}$ is positive definite.

Proof Let λ be an H-eigenvalue of $\mathcal{A}$. Suppose on the contrary that $\lambda \leq 0$. It follows from Theorem 3.2 that there is a $e_{i_0 j_0} \in E(|\mathring{\mathcal{A}}|)$ such that $\lambda \in \varpi_{i_0, j_0}(\mathcal{A})$, that is

$$\left(|\lambda - a_{i_0\cdots i_0}| - r_{i_0}^{'}(\mathcal{A})\right) |\lambda - a_{j_0\cdots j_0}| \leq \widetilde{r}_{i_0}(\mathcal{A}) r_{j_0}(\mathcal{A}).$$

Since $a_{i\cdots i} > 0$ and $a_{j\cdots j} > 0$, it holds that

$$\left(|\lambda - a_{i_0\cdots i_0}| - r_{i_0}^{'}(\mathcal{A})\right) |\lambda - a_{j_0\cdots j_0}| \geq \left(a_{i_0\cdots i_0} - r_{i_0}^{'}(\mathcal{A})\right) a_{j_0\cdots j_0} > \widetilde{r}_{i_0}(\mathcal{A}) r_{j_0}(\mathcal{A}) \geq 0.$$

which a contradiction arises. Therefore, $\lambda > 0$ and $\mathcal{A}$ is positive definite. □

In virtue of Theorem 4.2 and Corollary 3.2, we can get the following conclusions.

Corollary 4.2 *Let $\mathcal{A}$ be an even m-order n-dimensional supersymmetric tensor. If all $(i, j) \in \{(k, l) : e_{kl} \in E(|\mathring{\mathcal{A}}|) \bigcup l - k = 1$ or $1 - n,\ l \neq k\}$ and $a_{i\cdots i} > 0, i \in N$ such that*

$$\left(a_{i\cdots i} - r_i^{'}(\mathcal{A})\right) a_{j\cdots j} > \widetilde{r}_i(\mathcal{A}) r_j(\mathcal{A}),$$

then $\mathcal{A}$ is positive definite.

The following example shows that our results can exactly judge the positive definiteness of an even-order real supersymmetric sparse tensor.

Example 4.1 Let $\mathcal{A}$ be a 4-order 4-dimensional symmetric tensor defined as follows:

$$a_{ijkl} = \begin{cases} a_{1111} = 3; a_{2222} = 3; a_{3333} = 1; a_{4444} = 7; \\ a_{1222} = a_{2221} = a_{2212} = a_{2122} = -\frac{1}{2}; a_{3444} = a_{4443} = a_{4434} = a_{4344} = 1; \\ a_{1331} = a_{3311} = a_{3113} = a_{1133} = \frac{1}{4}; \\ a_{ijkl} = 0, \quad otherwise, \end{cases}$$

We get the minimum H-eigenvalues is 1.000. Hence, $\mathcal{A}$ is positive definite. We verify

$$\mathcal{G}(|\mathcal{A}|) = \begin{pmatrix} 4 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{2} & \frac{9}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} & 1 \\ 0 & 0 & 3 & 10 \end{pmatrix} \quad and \quad |\mathring{\mathcal{A}}| = \begin{pmatrix} 3 & \frac{1}{2} & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 7 \end{pmatrix}.$$

Table 6 Numerical results of Theorem 4.1

	$\left(a_{i\cdots i} - r_i^{\Delta_i}(\mathcal{A})\right) a_{j\cdots j}$	$r_i^{\overline{\Delta}_i}(\mathcal{A}) r_j(\mathcal{A})$
i=1,j=2	7.500	0.750
i=1,j=3	2.500	0.750
i=2,j=1	4.500	0.000
i=3,j=1	1.500	1.000
i=3,j=4	3.500	3.000
i=4,j=3	4.000	0.000

Table 7 Numerical results of Corollary 4.2

	$\left(a_{i\cdots i} - r_i^{'}(\mathcal{A})\right) a_{j\cdots j}$	$\widetilde{r}_i(\mathcal{A}) r_j(\mathcal{A})$
i=1,j=2	7.500	0.750
i=2,j=3	1.500	0.000
i=3,j=4	3.500	3.000
i=4,j=1	4.000	0.000

According to the Theorem 4.1, we compute Table 6 and

$$\left(a_{i\cdots i} - r_i^{\Delta_i}(\mathcal{A})\right) a_{j\cdots j} > r_i^{\overline{\Delta}_i}(\mathcal{A}) r_j(\mathcal{A}), \ \forall (i, j) \in \{(k, l) : g_{kl} \in E(\mathcal{G}(|\mathcal{A}|)), l \neq k\},$$

which shows that $\mathcal{A}$ is positive definite.

In view of Corollary 4.2, we compute Table 7 and

$$\left(a_{i\cdots i} - r_i^{'}(\mathcal{A})\right) a_{j\cdots j} > \widetilde{r}_i(\mathcal{A}) r_j(\mathcal{A}), \forall (i, j) \in \left\{(k, l) : e_{kl} \in E(|\mathring{\mathcal{A}}|) \bigcup l - k = 1 \ or \ 1 - n, \ l \neq k\right\}.$$

Therefore, $\mathcal{A}$ is positive definite.

However, from Theorem 4.1 of [9], we obtain

$$(a_{3333} - r_3^{\Delta_3}(\mathcal{A}))a_{2222} = 1.500 = 1.500 = r_3^{\overline{\Delta}_3}(\mathcal{A}) r_2(\mathcal{A}),$$

which implies that Theorem 4.1 of [9] cannot identify the positiveness of $\mathcal{A}$.

Unfortunately, from Theorem 4.2 of [8], we obtain

$$(a_{3333} - r_3^1(\mathcal{A})) = -1.500 < 0.000 = |a_{3111}| r_1(\mathcal{A}),$$
$$(a_{3333} - r_3^2(\mathcal{A})) = -1.500 < 0.000 = |a_{3222}| r_2(\mathcal{A}),$$

which implies that Theorem 4.2 of [8] are not suitable to test the positiveness of $\mathcal{A}$.

4.2 Bounds for *H*-spectral radius of nonnegative sparse tensors

We give bounds of the H-spectral radius a nonnegative sparse tensor via H-eigenvalue inclusion theorems in Sect. 3. We start this subsection with some fundamental results of nonnegative tensors.

Lemma 4.1 (Lemma 3.2 of [8]) *Let $\mathcal{A}$ be a nonnegative tensor with order m and dimension $n \geq 2$. Then,*

$$\rho(\mathcal{A}) \geq \max_{i \in N} a_{i \dots i}.$$

Lemma 4.2 (Theorem 4.1 of [5]) *Let $\mathcal{A}$ be an m-order n-dimensional weakly irreducible nonnegative tensor. Then there exists a unique x such that $(\rho(\mathcal{A}), x)$ is a positive eigenpair.*

Theorem 4.3 *Let $\mathcal{A}$ be an m-order n-dimensional nonnegative tensor with $\Gamma_{\mathcal{G}(\mathcal{A})}(i) \neq \emptyset$ and $n \geq 2$. Then*

$$\min_{g_{ij} \in E(\mathcal{G}(\mathcal{A})) \bigcup j-i=1,1-n} \Psi_{i,j}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \max_{g_{ij} \in E(\mathcal{G}(\mathcal{A}))} \Psi_{i,j}(\mathcal{A}),$$

where

$$\Psi_{i,j}(\mathcal{A}) = \frac{1}{2}\left\{a_{i \cdots i} + a_{j \cdots j} + r_i^{\Delta_i}(\mathcal{A}) + \sqrt{\left(a_{i \cdots i} - a_{j \cdots j} + r_i^{\Delta_i}(\mathcal{A})\right)^2 + 4 r_i^{\overline{\Delta}_i}(\mathcal{A}) r_j(\mathcal{A})}\right\}.$$

Proof Let $\rho(\mathcal{A})$ be H-spectral radius of $\mathcal{A}$. Since $\mathcal{A}$ is nonnegative, then $\rho(\mathcal{A})$ is an H-eigenvalue of $\mathcal{A}$ with $\rho(\mathcal{A}) \in \widetilde{\Theta}(\mathcal{A})$ from Theorem 3.1, that is,

$$\rho(\mathcal{A}) \in \bigcup_{g_{ij} \in E(\mathcal{G}(\mathcal{A}))} \Theta_{i,j}(\mathcal{A}). \tag{12}$$

(i) We show $\rho(\mathcal{A}) \leq \max_{g_{ij} \in E(\mathcal{G}(\mathcal{A}))} \Psi_{i,j}(\mathcal{A})$.
Referring to $i = t_1,\ j = t_s$ of (12), one has

$$\left(|\rho(\mathcal{A}) - a_{t_1 \cdots t_1}| - r_{t_1}^{\Delta_{t_1}}(\mathcal{A})\right) |\rho(\mathcal{A}) - a_{t_s \cdots t_s}| \leq r_{t_1}^{\overline{\Delta}_{t_1}}(\mathcal{A}) r_{t_s}(\mathcal{A}).$$

This together with $\rho(\mathcal{A}) \geq a_{i \cdots i}$ yields

$$\left(\rho(\mathcal{A}) - a_{t_1 \cdots t_1} - r_{t_1}^{\Delta_{t_1}}(\mathcal{A})\right)(\rho(\mathcal{A}) - a_{t_s \cdots t_s}) \leq r_{t_1}^{\overline{\Delta}_{t_1}}(\mathcal{A}) r_{t_s}(\mathcal{A}),$$

equivalently,

$$\rho(\mathcal{A})^2 - \left(a_{t_1 \cdots t_1} + a_{t_s \cdots t_s} + r_{t_1}^{\Delta_{t_1}}(\mathcal{A})\right)\rho(\mathcal{A}) + a_{t_s \cdots t_s}\left(a_{t_1 \cdots t_1} + r_{t_1}^{\Delta_{t_1}}(\mathcal{A})\right)$$

$$- r_{t_1}^{\overline{\Delta}_{t_1}}(\mathcal{A})r_{t_s}(\mathcal{A}) \le 0.$$

Solving for $\rho(\mathcal{A})$ gives

$$\rho(\mathcal{A}) \le \frac{1}{2}\Bigg\{a_{t_1\cdots t_1} + a_{t_s\cdots t_s} + r_{t_1}^{\Delta_{t_1}}(\mathcal{A}) + \sqrt{\left(a_{t_1\cdots t_1} - a_{t_s\cdots t_s} + r_{t_1}^{\Delta_{t_1}}(\mathcal{A})\right)^2 + 4r_{t_1}^{\overline{\Delta}_{t_1}}(\mathcal{A})r_{t_s}(\mathcal{A})}\Bigg\},$$

which implies

$$\rho(\mathcal{A}) \le \Psi_{t_1,t_s}(\mathcal{A}) \le \max_{g_{ij}\in E(\mathcal{G}(\mathcal{A}))} \Psi_{i,j}(\mathcal{A}).$$

(ii) We prove $\min\limits_{g_{ij}\in E(\mathcal{G}(\mathcal{A}))\bigcup j-i=1,1-n} \Psi_{i,j}(\mathcal{A}) \le \rho(\mathcal{A})$.

Let $x = (x_1, x_2, \cdots, x_n)^\top$ be an H-eigenvector of $\mathcal{A}$ corresponding to $\rho(\mathcal{A})$. We break up the argument into two cases.

Case 1: $\mathcal{G}(\mathcal{A})$ is irreducible. Then, $\mathcal{A}$ is weakly irreducible. Therefore, $x = (x_1, x_2, \cdots, x_n)^\top$ is a positive vector from Lemma 4.2. Suppose

$$0 < x_{t_n} \le x_{t_r} = \min\{x_{t_j} : \sum_{(i_2,\cdots,i_m)\in\overline{\Delta}_{t_n}} a_{t_n i_2\cdots i_m} \ne 0,\ j \in N\}.$$

In view of the t_n-th equation of (2), we deduce

$$\begin{aligned}(\rho(\mathcal{A}) - a_{t_n\cdots t_n})x_{t_n}^{m-1} &= \sum_{\substack{(i_2,\cdots,i_m)\in\Delta_{t_n}\\ \delta_{t_n i_2\cdots i_m}=0}} a_{t_n i_2\cdots i_m}x_{i_2}\cdots x_{i_m} + \sum_{(i_2,\cdots,i_m)\in\overline{\Delta}_{t_n}} a_{t_n i_2\cdots i_m}x_{i_2}\cdots x_{i_m}\\ &\ge \sum_{\substack{(i_2,\cdots,i_m)\in\Delta_{t_n}\\ \delta_{t_n i_2\cdots i_m}=0}} a_{t_n i_2\cdots i_m}x_{t_n}^{m-1} + \sum_{(i_2,\cdots,i_m)\in\overline{\Delta}_{t_n}} a_{t_n i_2\cdots i_m}x_{t_r}^{m-1}\\ &= r_{t_n}^{\Delta_{t_n}}(\mathcal{A})x_{t_n}^{m-1} + r_{t_n}^{\overline{\Delta}_{t_n}}(\mathcal{A})x_{t_r}^{m-1},\end{aligned}$$

equivalently,

$$\left(\rho(\mathcal{A}) - a_{t_n\cdots t_n} - r_{t_n}^{\Delta_{t_n}}(\mathcal{A})\right)x_{t_n}^{m-1} \ge r_{t_n}^{\overline{\Delta}_{t_n}}(\mathcal{A})x_{t_r}^{m-1} \ge 0. \tag{13}$$

Referring to t_r-th equation of (2), we have

$$(\rho(\mathcal{A}) - a_{t_r\cdots t_r})x_{t_r}^{m-1} = \sum_{\delta_{t_r i_2\cdots i_m}=0} a_{t_r i_2\cdots i_m}x_{i_2}\cdots x_{i_m} \ge r_{t_r}(\mathcal{A})x_{t_n}^{m-1} \ge 0. \tag{14}$$

Multiplying (13) and (14) yields

$$\left(\rho(\mathcal{A}) - a_{t_n\cdots t_n} - r_{t_n}^{\Delta_{t_n}}(\mathcal{A})\right)(\rho(\mathcal{A}) - a_{t_r\cdots t_r})x_{t_n}^{m-1}x_{t_r}^{m-1} \ge r_{t_n}^{\overline{\Delta}_{t_n}}(\mathcal{A})r_{t_r}(\mathcal{A})x_{t_n}^{m-1}x_{t_r}^{m-1}.$$

By virtue of $x_{t_r} \geq x_{t_n} > 0$, we obtain

$$\left(\rho(\mathcal{A}) - a_{t_n\cdots t_n} - r_{t_n}^{\Delta_{t_n}}(\mathcal{A})\right)(\rho(\mathcal{A}) - a_{t_r\cdots t_r}) \geq r_{t_n}^{\overline{\Delta}_{t_n}}(\mathcal{A}) r_{t_r}(\mathcal{A}).$$

Solving for $\rho(\mathcal{A})$ gives

$$\rho(\mathcal{A}) \geq \frac{1}{2}\left\{a_{t_n\cdots t_n} + a_{t_r\cdots t_r} + r_{t_n}^{\Delta_{t_n}}(\mathcal{A}) + \sqrt{\left(a_{t_n\cdots t_n} - a_{t_r\cdots t_r} + r_{t_n}^{\Delta_{t_n}}(\mathcal{A})\right)^2 + 4 r_{t_n}^{\overline{\Delta}_{t_n}}(\mathcal{A}) r_{t_r}(\mathcal{A})}\right\},$$

which shows

$$\rho(\mathcal{A}) \geq \Psi_{t_n,t_r}(\mathcal{A}) \geq \min_{g_{ij}\in E(\mathcal{G}(\mathcal{A}))} \Psi_{i,j}(\mathcal{A}).$$

Case 2: $\mathcal{G}(\mathcal{A})$ is reducible. For any $\epsilon > 0$, set

$$\mathcal{A}(\epsilon) = \mathcal{A} + \Phi(\epsilon) \quad and \quad \Phi(\epsilon) = (\theta_{i_1\cdots i_m}),$$

where

$$\theta_{i_1\cdots i_m} = \begin{cases} \theta_{ij\cdots j} = \epsilon, & if j - i = 1, 1-n, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\mathcal{A}(\epsilon)$ is irreducible. Following the similar proof of Case 1 in Theorem 4.3, we have

$$\min_{g_{ij}\in E(\mathcal{G}(\mathcal{A}))} \Psi_{i,j}(\mathcal{A}(\epsilon)) \leq \rho(\mathcal{A}(\epsilon)).$$

Letting $\epsilon \to 0$, we obtain

$$\min_{g_{ij}\in E(\mathcal{G}(\mathcal{A}))\bigcup j-i=1,1-n} \Psi_{i,j}(\mathcal{A}) \leq \rho(\mathcal{A}).$$

Combining Cases 1 and 2, we obtain the desired results.

□

When the condition $\Gamma_{\mathcal{G}(\mathcal{A})}(i) \neq \emptyset$ is replaced by weak irreducibility of $\mathcal{A}$, we obtain accurate conclusion.

Corollary 4.3 *Let $\mathcal{A}$ be an m-order n-dimensional weakly irreducible nonnegative tensor. Then,*

$$\min_{g_{ij}\in E(\mathcal{G}(\mathcal{A}))} \Psi_{i,j}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \max_{g_{ij}\in E(\mathcal{G}(\mathcal{A}))} \Psi_{i,j}(\mathcal{A}).$$

When the condition $\Gamma_{\mathcal{G}(\mathcal{A})}(i) \neq \emptyset$ is omitted, we obtain general results.

Corollary 4.4 *Let $\mathcal{A}$ be an m-order n-dimensional nonnegative tensor. Then,*

$$\min_{g_{ij}\in E(\mathcal{G}(\mathcal{A}))\bigcup j-i=1,1-n} \Psi_{i,j}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \max_{g_{ij}\in E(\mathcal{G}(\mathcal{A}))\bigcup j-i=1,1-n} \Psi_{i,j}(\mathcal{A}).$$

Based on Theorem 3.2, we can establish the conclusions.

Theorem 4.4 *Let $\mathcal{A}$ be an m-order n-dimensional nonnegative tensor with $\Gamma_{\mathring{\mathcal{A}}}(i) \neq \emptyset$ and $n \geq 2$. Then,*

$$\min_{e_{ij}\in E(\mathring{\mathcal{A}})\bigcup j-i=1,1-n} \kappa_{i,j}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \max_{e_{ij}\in E(\mathring{\mathcal{A}})} \kappa_{i,j}(\mathcal{A}),$$

where

$$\kappa_{i,j}(\mathcal{A}) = \frac{1}{2}\left\{a_{i\cdots i} + a_{j\cdots j} + r_i^{'}(\mathcal{A}) + \sqrt{(a_{i\cdots i} - a_{j\cdots j} + r_i^{'}(\mathcal{A}))^2 + 4\widetilde{r}_i(\mathcal{A})r_j(\mathcal{A})}\right\}.$$

Remark 4.1 Compared with Theorem 2.1 of [13], the results of Theorem 4.4 is sharp under the condition $\Gamma_{\mathring{\mathcal{A}}}(i) \neq \emptyset$. Compared with Corollary 4 of [13] under irreducibility of $\mathcal{A}$, we deduce the following resluts with weak irreducibility of $\mathcal{A}$

$$\min_{e_{ij}\in E(\mathring{\mathcal{A}})} \kappa_{i,j}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \max_{e_{ij}\in E(\mathring{\mathcal{A}})} \kappa_{i,j}(\mathcal{A}).$$

Corollary 4.5 *Let $\mathcal{A}$ be an m-order n-dimensional nonnegative tensor. Then,*

$$\min_{e_{ij}\in E(\mathring{\mathcal{A}})\bigcup j-i=1,1-n} \kappa_{i,j}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \max_{e_{ij}\in E(\mathring{\mathcal{A}})\bigcup j-i=1,1-n} \kappa_{i,j}(\mathcal{A}).$$

In what follows, we test the efficiency of the obtained results.

Example 4.2 Let $\mathcal{A}$ be a 3-order 4-dimensional nonnegative tensor defined as follows:

$$a_{ijk} = \begin{cases} a_{111} = 1; a_{112} = 1; a_{122} = 2; a_{123} = 1; a_{133} = 2; \\ a_{222} = 3; a_{232} = 1; a_{233} = 1; a_{243} = 1; \\ a_{333} = 2; a_{334} = 1; a_{343} = 1; a_{344} = 4; \\ a_{421} = 6; a_{422} = 1; a_{424} = 1; a_{444} = 1. \end{cases}$$

We compute $\rho(\mathcal{A}) = 7.112$ and identify

$$\mathcal{G}(\mathcal{A}) = \begin{pmatrix} 2&4&3&0 \\ 0&4&3&1 \\ 0&0&4&6 \\ 6&8&0&2 \end{pmatrix} \quad and \quad \mathring{\mathcal{A}} = \begin{pmatrix} 1&2&2&0 \\ 0&3&1&0 \\ 0&0&2&4 \\ 0&1&0&1 \end{pmatrix}.$$

Table 8 Bounds for Theorem 3.3 of [10]

$\Omega_{1,2}(\mathcal{A}) = 6.646$	$\Omega_{1,3}(\mathcal{A}) = 7.275$	$\Omega_{1,4}(\mathcal{A}) = 7.000$
$\Omega_{2,1}(\mathcal{A}) = 6.000$	$\Omega_{2,3}(\mathcal{A}) = 6.372$	$\Omega_{2,4}(\mathcal{A}) = 6.000$
$\Omega_{3,1}(\mathcal{A}) = 8.000$	$\Omega_{3,2}(\mathcal{A}) = 8.000$	$\Omega_{3,4}(\mathcal{A}) = 8.352$
$\Omega_{4,1}(\mathcal{A}) = 9.000$	$\Omega_{4,2}(\mathcal{A}) = 8.541$	$\Omega_{4,3}(\mathcal{A}) = 9.000$

Table 9 Bounds for Theorem 2.1 of [13]

$\kappa_{1,2}(\mathcal{A}) = 6.464$	$\kappa_{1,3}(\mathcal{A}) = 7.424$	$\kappa_{2,3}(\mathcal{A}) = 6.372$
$\kappa_{3,4}(\mathcal{A}) = 8.352$	$\kappa_{4,1}(\mathcal{A}) = 8.772$	$\kappa_{4,2}(\mathcal{A}) = 8.541$

Table 10 Bounds of Corollary 4.3

$\Psi_{1,2}(\mathcal{A}) = 6.405$	$\Psi_{1,3}(\mathcal{A}) = 7.477$	$\Psi_{2,3}(\mathcal{A}) = 6.606$	$\Psi_{2,4}(\mathcal{A}) = 6.772$
$\Psi_{3,4}(\mathcal{A}) = 8.352$	$\Psi_{4,1}(\mathcal{A}) = 8.000$	$\Psi_{4,2}(\mathcal{A}) = 7.109$	

By Lemma 5.2 of [25], one has

$$6 \leq \rho(\mathcal{A}) \leq 9.$$

From Theorem 3.3 of [10], Table 8 holds
and

$$\min_{i,j \in \{1,2,3,4\}, i \neq j} \Omega_{i,j}(\mathcal{A}) = 6 \leq \rho(\mathcal{A}) \leq 9 = \max_{i,j \in \{1,2,3,4\}, i \neq j} \Omega_{i,j}(\mathcal{A}).$$

From Theorem 5 of [11], following the similar computations of $\Omega_{i,j}(\mathcal{A})$, we obtain

$$\min_{i,j \in \{1,2,3,4\}, i \neq j} \Delta_{i,j}(\mathcal{A}) = 6.275 \leq \rho(\mathcal{A}) \leq 8.481 = \max_{i,j \in \{1,2,3,4\}, i \neq j} \Delta_{i,j}(\mathcal{A}).$$

We observe $\mathring{\mathcal{A}}$ is reducible. It follows from Theorem 2.1 of [13] that
i.e.,

$$\min_{e_{ij} \in E(\mathring{\mathcal{A}}) \bigcup j-i=1,1-n} \kappa_{i,j}(\mathcal{A}) = 6.372 \leq \rho(\mathcal{A}) \leq 8.772 = \max_{e_{ij} \in E(\mathring{\mathcal{A}}) \bigcup j-i=1,1-n} \kappa_{i,j}(\mathcal{A}).$$

From representation matrix $\mathcal{G}(\mathcal{A})$, we know that $\mathcal{A}$ is weakly irreducible. Based on Corollary 4.3, we obtain Table 10 and

$$\min_{g_{ij} \in E(\mathcal{G}(\mathcal{A}))} \Psi_{i,j}(\mathcal{A}) = 6.405 \leq \rho(\mathcal{A}) \leq 8.352 = \max_{g_{ij} \in E(\mathcal{G}(\mathcal{A}))} \Psi_{i,j}(\mathcal{A}).$$

By majorization matrix $\mathring{\mathcal{A}}$, for any $i \in \{1, 2, 3, 4\}$, we observe $\Gamma_{\mathring{\mathcal{A}}}(i) \neq \emptyset$. Recalling Theorem 4.4, we calculate Table 11 and

Table 11 Bounds of Theorem 4.4

$\kappa_{1,2}(\mathcal{A}) = 6.464$	$\kappa_{1,3}(\mathcal{A}) = 7.424$	$\kappa_{2,3}(\mathcal{A}) = 6.372$
$\kappa_{3,4}(\mathcal{A}) = 8.352$	$\kappa_{4,1}(\mathcal{A}) = 8.772$	$\kappa_{4,2}(\mathcal{A}) = 8.541$

$$\min_{e_{ij} \in E(\mathring{\mathcal{A}}) \bigcup j-i=1,1-n} \kappa_{i,j}(\mathcal{A}) = 6.372 \le \rho(\mathcal{A}) \le 8.541 = \max_{e_{ij} \in E(\mathring{\mathcal{A}})} \kappa_{i,j}(\mathcal{A}).$$

It is easy to see that the bounds in Corollary 4.3 and Theorem 4.4 are sharper than those of Lemma 5.2 in [25], Theorem 3.3 in [10], Theorem 5 in [11] and Theorem 2.1 of [13].

5 Conclusion

In this paper, we established the improved H-eigenvalue inclusion sets of a sparse tensor by its majorization matrix's digraph and representation matrix's digraph, which have advantages of tight bounds and minor computations. Meanwhile, two sufficient conditions were proposed to check positive definiteness of an even-order real supersymmetric sparse tensor. Further studies can be considered to develop certain algorithms for solving image restoration from sparse tensor data based on improved H-eigenvalue inclusion sets.

Acknowledgements This work was supported by the Natural Science Foundation of China (12071250) and the Natural Science Foundation of Shandong Province (ZR2020MA025).

Data Availability All relevant data are within the paper.

Declarations

Conflict of interest The authors declare that they have no competing interests.

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