Bull. Malays. Math. Sci. Soc. (2023) 46:158
https://doi.org/10.1007/s40840-023-01553-8

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

# On the Second Eigenvalue of Certain Cayley Graphs on the Symmetric Group

**Roghayeh Maleki**[1,2] · **Andriaherimanana Sarobidy Razafimahatratra**[1,2]

Received: 22 February 2023 / Revised: 15 June 2023 / Accepted: 26 June 2023 /
Published online: 11 July 2023


**Abstract**

In 2020, Siemons and Zalesski determined the second eigenvalue of the Cayley graph $\Gamma_{n,k} = \mathrm{Cay}(\mathrm{Sym}(n), C(n,k))$ for $k = 0$ and $k = 1$, where $C(n,k)$ is the conjugacy class of $(n-k)$-cycles. In this paper, it is proved that for any $n \geq 3$ and $k \in \mathbb{N}$ relatively small compared to $n$, the second eigenvalue of $\Gamma_{n,k}$ is the eigenvalue afforded by the irreducible character of $\mathrm{Sym}(n)$ that corresponds to the partition $[n-1,1]$. As a byproduct of our method, the result of Siemons and Zalesski when $k \in \{0,1\}$ is retrieved. Moreover, we prove that the second eigenvalue of $\Gamma_{n,n-5}$ is also equal to the eigenvalue afforded by the irreducible character of the partition $[n-1,1]$.

**Keywords** Second eigenvalue · Cayley graphs · Symmetric group · Representation theory

**Mathematics Subject Classification** Primary 05C50 · Secondary 20C30

## 1 Introduction

This paper is concerned with the second largest eigenvalue of certain Cayley graphs on the symmetric group. Let $G$ be a finite group with identity $e$, and let $C$ be a subset of $G \setminus \{e\}$ with the property that if $x \in C$ then $x^{-1} \in C$. The *Cayley graph* $\mathrm{Cay}(G,C)$ is the undirected graph with vertex set $G$ and two group elements $g$ and $h$ are adjacent if and only if $hg^{-1} \in C$. Since the right-regular representation

Communicated by Wen Chean Teh.

✉ Andriaherimanana Sarobidy Razafimahatratra
sarobidy@phystech.edu

[1] UP FAMNIT, University of Primorska, Glagoljaška 8, 6000 Koper, Slovenia

[2] UP IAM, University of Primorska, Muzejski trg 2, 6000 Koper, Slovenia

$\mathrm{R}(G) = \left\{\rho_g : G \to G,\ \rho_g(x) = xg \mid g \in G\right\}$ of $G$ acts as a subgroup of automorphisms of $\mathrm{Cay}(G, C)$, the latter is vertex transitive. We say that the graph $\mathrm{Cay}(G, C)$ is a *normal Cayley graph* if $gCg^{-1} = C$, for any $g \in G$. In other words, $\mathrm{Cay}(G, C)$ is a normal Cayley graph if and only if $C$ is a union of conjugacy classes of $G$. The graph $\mathrm{Cay}(G, C)$ is regular with valency equal to $|C|$.

Given a graph $X = (V, E)$, the eigenvalues of $X$ (i.e., the eigenvalues of its adjacency matrix) are real and its distinct eigenvalues can thus be arranged in a decreasing order $\lambda_1(X) > \lambda_2(X) > \ldots > \lambda_s(X)$, for some positive integer $s$. The eigenvalues $\lambda_1(X)$ and $\lambda_2(X)$ are called *first* and *second eigenvalues* of $X$, respectively. When $X$ is a regular graph, the first eigenvalue $\lambda_1(X)$ is equal to the valency of $X$.

In [1], Babai proved that the eigenvalues of normal Cayley graphs can be determined by the irreducible characters of the underlying group. For any group $G$, we let $\mathrm{Irr}(G)$ be the set of all non-equivalent irreducible characters of $G$.

**Theorem 1.1** (Babai [1]) *Let $X = \mathrm{Cay}(G, C)$ be a normal Cayley graph. The eigenvalues of $X$ are determined by the irreducible characters of $G$. In particular, if $\chi \in \mathrm{Irr}(G)$ then the eigenvalue of $X$ afforded by $\chi$ is*

$$\xi_\chi = \frac{1}{\chi(e)} \sum_{g \in C} \chi(g).$$

Consequently, the eigenvalues of a normal Cayley graph of $\mathrm{Sym}(n)$ can be determined using its irreducible characters. Recall that a partition of the integer $n$ is a non-increasing sequence of positive integers summing to $n$. If $\lambda = [n_1, n_2, \ldots, n_t]$ is a partition of $n$, then we write $\lambda \vdash n$. It is well known that there is a bijective correspondence between the irreducible representations of $\mathrm{Sym}(n)$ and the set of all partitions of $n$. We refer the readers to [10] for more details on the representation theory of the symmetric group. Each $\lambda \vdash n$ corresponds to an irreducible $\mathbb{C}\,\mathrm{Sym}(n)$-module $S^\lambda$ (i.e., a complex irreducible representation of $\mathrm{Sym}(n)$) called the $\lambda$-*Specht* module. Let $\chi^\lambda$ be the character afforded by $S^\lambda$. For any $\lambda \vdash n$, we denote the degree of $\chi^\lambda$ by $f^\lambda$ (i.e., $f^\lambda = \chi^\lambda(id)$). If $\lambda \vdash n$ and $\sigma \in \mathrm{Sym}(n)$ has cycle type $\tau$, then we define $\chi^\lambda_\tau := \chi^\lambda(\sigma)$.

One of the most important characters of $\mathrm{Sym}(n)$ is the (natural) permutation character which is the induced character $\mathbf{1}_{\mathrm{Sym}(n-1)}^{\mathrm{Sym}(n)}$ of the trivial character $\mathbf{1}_{\mathrm{Sym}(n-1)}$ of the stabilizer of a point to the symmetric group $\mathrm{Sym}(n)$. Since the natural action of $\mathrm{Sym}(n)$ is 2-transitive, there are two constituents in the permutation character of $\mathrm{Sym}(n)$, one of which is the trivial character of $\mathrm{Sym}(n)$, i.e., $\chi^{[n]}$. Using standard results in the representation theory of the symmetric group, it is not hard to see that the other character is $\chi^{[n-1,1]}$. In other words, $\mathbf{1}_{\mathrm{Sym}(n-1)}^{\mathrm{Sym}(n)} = \chi^{[n]} + \chi^{[n-1,1]}$. In [8], it is proved that for $n$ large enough, the second eigenvalue of any Cayley graph of $\mathrm{Sym}(n)$ with connection set equal to a conjugacy class with at least 2 fixed points is afforded by $\chi^{[n-1,1]}$. In the case where the connection set of a normal Cayley graph on the symmetric group is a conjugacy class of derangements (i.e., permutations without a fixed point), the second eigenvalue need not be afforded by $\chi^{[n-1,1]}$. For instance, the eigenvalue afforded by

$\chi^{[n-1,1]}$ in the Cayley graph of the symmetric group with connection set equal to the conjugacy class of $n$-cycles, is the smallest eigenvalue.

The permutation character $\mathbf{1}_{\mathrm{Sym}(n-1)}^{\mathrm{Sym}(n)}$ corresponds to the character of the matrix representation of $\mathrm{Sym}(n)$ which maps a permutation to its corresponding permutation matrix. Therefore, $\mathbf{1}_{\mathrm{Sym}(n-1)}^{\mathrm{Sym}(n)}$ coincides with the map **fix**, which for any $\sigma \in \mathrm{Sym}(n)$ gives

$$\mathbf{fix}(\sigma) = |\{i \in [n] \mid \sigma(i) = i\}|.$$

Consequently, we have $\chi^{[n-1,1]}(\sigma) = \mathbf{fix}(\sigma) - 1$, for any $\sigma \in \mathrm{Sym}(n)$. We observe the following.

$$\chi^{[n-1,1]}(\sigma) = \begin{cases} -1 & \text{if } \sigma \text{ is a derangement} \\ 0 & \text{if } \sigma \text{ has one fixed point} \\ c \geq 1 & \text{if } \sigma \text{ has at least two fixed points.} \end{cases}$$

The irreducible character $\chi^{[n-1,1]}$ plays an important role in many spectral results for normal Cayley graphs on the symmetric group. For instance, it was shown in [9] that the smallest eigenvalue of the Cayley graph of the symmetric group with connection set equal to the set of all derangements on $n$ is afforded by $\chi^{[n-1,1]}$. Hence, it does not afford the second eigenvalue. Here, we note that since the connection set contains only derangements, the value of the $\chi^{[n-1,1]}$ on every element of the connection set is equal to $-1$. However, if the connection set of $\mathrm{Cay}(\mathrm{Sym}(n), S)$ contains an element on which $\chi^{[n-1,1]}$ is non-negative, then the irreducible character affording the second eigenvalue may completely change. As shown in [8, Proposition 2.3 and Proposition 2.4], when the connection set is a conjugacy class of an element $\sigma \in \mathrm{Sym}(n)$, the character affording the second eigenvalue depend on whether $\chi^{[n-1,1]}(\sigma)$ is positive or not; see [6] and [7, Theorem1.3 and Theorem 1.4] for more examples.

Hence, there are grounds to believe that the second eigenvalue of a normal Cayley graph of the symmetric group depends on the value of $\chi^{[n-1,1]}$. Some further evidence for this will be given in the main results of this paper, which leads us to give an interesting conjecture.

The objective of this paper is to study the second eigenvalue of a particular normal Cayley graph of $\mathrm{Sym}(n)$. Let $n$ and $k$ be two integers such that $0 \leq k \leq n-1$. We denote the conjugacy class of the $(n-k)$-cycles of $\mathrm{Sym}(n)$ by $C(n,k)$. That is,

$$C(n,k) = \left\{\sigma(1,2,3,\ldots,n-k)\sigma^{-1} \mid \sigma \in \mathrm{Sym}(n)\right\}.$$

Let $\Gamma_{n,k}$ be the Cayley graph of $\mathrm{Sym}(n)$ with connection set equal to $C(n,k)$. In other words, $\Gamma_{n,k} := \mathrm{Cay}(\mathrm{Sym}(n), C(n,k))$ is the graph whose vertex set is $\mathrm{Sym}(n)$ and there is an edge between two permutations $\sigma$ and $\tau$ of $\mathrm{Sym}(n)$ if and only if $\sigma\tau^{-1} \in C(n,k)$. The graph $\Gamma_{n,k}$ is vertex transitive and regular of valency $c_{n,k} := |C(n,k)| = \binom{n}{k}(n-k-1)!$. Since $C(n,k)$ is a conjugacy class, $\Gamma_{n,k}$ is also a *normal Cayley graph*. We note that it follows from a result in [8] that the second eigenvalue of $\Gamma_{n,k}$ is equal to the eigenvalue afforded by $\chi^{[n-1,1]}$, when $n$ is large enough. However, it is not known if this result holds for any $n$. We believe that this result holds in general, not just when $n$ is large enough. In this paper, we obtain the second eigenvalue of $\Gamma_{n,k}$, when $k$ is relatively small, for any value of $n$.

The following theorem was recently proved by Siemons and Zalesski [11].

**Theorem 1.2** *For $n \geq 5$, the second eigenvalue of $\Gamma_{n,k}$, for $k \in \{0, 1\}$, is as follows.*

- *If $k = 0$, then $\lambda_2(\Gamma_{n,k}) = (n-2)!$ whenn is even and $\lambda_2(\Gamma_{n,k}) = 2(n-3)!$ when $n$ is odd.*
- *If $k = 1$, then $\lambda_2(\Gamma_{n,k}) = 3(n-3)(n-5)!$ when $n$ is even and $\lambda_2(\Gamma_{n,k}) = 2(n-2)(n-4)!$ when $n$ is odd.*

Finding the second eigenvalue of $\Gamma_{n,k}$, for arbitrary $k \leq n$, is an open problem posed by Siemons and Zalesski in [11] (although we note that this eigenvalue is predicted in [8]). Our main result is the following.

**Theorem 1.3** *For any $n \geq 3$ and $k \in \mathbb{N}$ such that $2 \leq k \leq \min\left(n, 2\log_{\frac{k}{e}}\left(\frac{n(n-2)}{2e}\right) - 1\right)$, the second eigenvalue of $\Gamma_{n,k}$ is*

$$\lambda_2\left(\Gamma_{n,k}\right) = \frac{k-1}{n-1}\binom{n}{k}(n-k-1)!.$$

*Moreover, $\lambda_2\left(\Gamma_{n,k}\right)$ is afforded by the irreducible character of $\mathrm{Sym}(n)$ that corresponds to the partition $[n-1, 1]$.*

It is well known that a Cayley graph $\mathrm{Cay}(G, C)$ is connected if and only if $\langle C\rangle = G$. Note that for $n \geq 5$, if the graph $\Gamma_{n,k}$ is connected, then $C(n, k)$ has to contain an odd permutation. Indeed, if $C(n, k)$ only contains even permutations, then we have $\langle C(n, k)\rangle \leq \mathrm{Alt}(n)$, where $\mathrm{Alt}(n)$ is the alternating group on $\{1, 2, \ldots, n\}$. In general, if $X = \mathrm{Cay}(\mathrm{Sym}(n), T)$ is a normal Cayley graph, then it is straightforward that $\langle T\rangle \trianglelefteq \mathrm{Sym}(n)$. Since $\mathrm{Alt}(n)$ is the only minimal normal subgroup of $\mathrm{Sym}(n)$, for $n \geq 5$, we have $\langle T\rangle \in \{\mathrm{Alt}(n), \mathrm{Sym}(n)\}$.

Consequently, if $n \geq 5$ and $n - k$ is odd, then the graph $\Gamma_{n,k}$ is disconnected and has two components. Hence, the second eigenvalue of $\Gamma_{n,k}$ is equal to the second eigenvalue of $\mathrm{Cay}(\mathrm{Alt}(n), C(n, k))$, when $n - k$ is odd. The following result was proved by Huang and Huang [5].

**Theorem 1.4** *For any $n \geq 5$, the second eigenvalue of $\mathrm{Cay}(\mathrm{Alt}(n), C(n, n-3))$ is $\frac{n(n-2)(n-4)}{3}$.*

Therefore, the second eigenvalue of $\Gamma_{n,n-3}$ is $\frac{n(n-2)(n-4)}{3}$. It can be verified that this eigenvalue is also the one afforded by irreducible character corresponding to the partition $[n-1, 1]$.

Other results on the second eigenvalue of $\Gamma_{n,k}$, when $k$ is large are also known. For instance, Diaconis and Shahshahani [3] proved that $\lambda_2(\Gamma_{n,n-2}) = \binom{n}{2} - n$, which coincides with our result in Theorem 1.3 when $k = n-2$. This eigenvalue corresponds to the irreducible character of the partition $[n-1, 1]$. Moreover, Huang et al. [6] proved that $\lambda_2(\Gamma_{n,n-4}) = \frac{n(n-2)(n-3)(n-5)}{4}$ is the eigenvalue corresponding to the partition $[n-1, 1]$.

We believe that our main result, Theorem 1.3, also holds in the general case when $2 \leq k \leq n-2$. For $n \in \{3, 4, 5, 6, \ldots, 21\}$ and $2 < k < n-2$, an exhaustive search on `Sagemath` [12] for the second eigenvalue of $\Gamma_{n,k}$, showed that $\lambda_2(\Gamma_{n,k}) = \frac{k-1}{n-1}\binom{n}{k}(n-k-1)!$. We make the following conjecture.

**Conjecture 1.5** *For any $n \in \mathbb{N}$ and $2 \leq k \leq n-2$, the second eigenvalue of $\Gamma_{n,k}$ is given by the irreducible character corresponding to $[n-1, 1]$, and its value is*

$$\lambda_2(\Gamma_{n,k}) = \frac{k-1}{n-1}\binom{n}{k}(n-k-1)!.$$

Our next result is about Conjecture 1.5, for short cycles.

**Theorem 1.6** *For any $n \geq 7$, $\lambda_2(\Gamma_{n,n-5}) = \frac{n(n-2)(n-3)(n-4)(n-6)}{5}$. Moreover, this eigenvalue is afforded by the irreducible character corresponding to the partition $[n-1, 1]$.*

This paper is organized as follows. In Sect. 3.1, we prove Theorem 1.3. In our proof of Theorem 1.3, we use the representation theory of the symmetric group. In particular, our proof relies on the recursive Murnaghan-Nakayama rule (see [10]). In Sect. 3.2, we show that our method can also be applied to retrieve the result of Siemons and Zalesski in Theorem 1.2. We prove Theorem 1.6 in Sect. 4.

## 2 Background Results

It is assumed that the reader is familiar with the notion of Specht modules [10, Section 2.3], the Hook Length Formula [10, Section 3.10] and the Murnaghan-Nakayama Rule [10, Section 4.10].

Given two Young diagrams $\lambda$ and $\mu$ such that $\mu \subset \lambda$, the *skew diagram* $\lambda/\mu$ is the set of cells of $\lambda$ that are not in $\mu$. A *rim hook* $\rho$ of a Young diagram $\lambda \vdash n$ is a skew diagram whose cells are on a path with only upward and rightward steps (see [10] for details on this). The length of $\rho$ denoted by $|\rho|$ is the number of its cells, and the *leg-length* $\ell\ell(\rho)$ is the number of rows that $\rho$ spans minus 1. For any $\lambda \vdash n$, we let $\mathsf{RH}_{n-k}(\lambda)$ be the set of all rim hooks of length $n-k$ of the partition $\lambda$. As a rim hook $\rho \in \mathsf{RH}_{n-k}(\lambda)$ is a skew diagram, removal of the cells of $\lambda$ which are in $\rho$ results in a Young diagram that corresponds to a partition of $k$. We denote the Young diagram obtained from such removal by $\lambda \setminus \rho$.

We recall the following lemma which is proved in [2].

**Lemma 2.1** *Let $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_q] \vdash n$. Assume that $\lambda$ has a rim hook $\rho$ of length $n-k$ and $\mu = [\mu_1, \mu_2, \ldots, \mu_t] \vdash k$ is the Young diagram obtained from $\lambda$ by removing $\rho$. That is, $\mu = \lambda \setminus \rho$.*

*If $\ell$ is a positive integer such that $2\ell - n \geq k+1$, then $\lambda$ has at most one rim hook of length $\ell$. In particular, if $3k+1 < n$, then there is at most one rim hook of length $n-k$ in any Young diagram $\lambda \vdash n$.*

By Lemma 2.1, there is at most one rim hook of length $k$ in any partition $\lambda \vdash n$, whenever $2 \leq k \leq \left\lfloor \frac{n-1}{3} \right\rfloor$.

Next, we present a lemma on the low degree irreducible characters of $\mathrm{Sym}(n)$.

**Lemma 2.2** *Let $n \geq 19$. If $\phi$ is a character of $\mathrm{Sym}(n)$ of degree less than $3\binom{n}{3}$ and $\chi^\lambda$ a constituent of $\phi$, then $\lambda$ is one of $[n], [1^n], [n-1,1], [2,1^{n-2}], [n-2,2], [2^2,1^{n-4}], [n-2,1^2], [3,1^{n-3}], [n-3,3], [2^3,1^{n-6}], [n-3,1^3], [4,1^{n-4}]$ , $[n-3,2,1]$, or $[3,2,1^{n-5}]$.*

A proof of Lemma 2.2 can be found in [2, Lemma 3.4 and Remark 3.5].

## 3 Proof of the Main Results

### 3.1 Proof of Theorem 1.3

Throughout this section, we let $n, k \in \mathbb{N}$ such that $2 \leq k \leq \min\left(n, 2\log_{\frac{k}{e}}\left(\frac{n(n-2)}{2e}\right) - 1\right)$. Using Theorem 1.1, the eigenvalues of $\Gamma_{n,k}$ are given in the following lemma.

**Lemma 3.1** *For any $n, k \in \mathbb{N}$ such that $1 \leq k \leq n$, the eigenvalues of $\Gamma_{n,k}$ are the numbers*

$$\xi_{\chi^\lambda} = \frac{\chi^\lambda_{[n-k,1^k]}}{f^\lambda}\binom{n}{k}(n-k-1)!,$$

*for all $\lambda \vdash n$.*

Using the recursive Murnaghan-Nakayama rule (see [10]), we obtain the following lemma.

**Lemma 3.2** *For any $\lambda \vdash n$, the character value of the irreducible character $\chi^\lambda$ is*

$$\chi^\lambda_{[n-k,1^k]} = \sum_{\rho \in \mathsf{RH}_{n-k}(\lambda)} (-1)^{\ell\ell(\rho)} f^{\lambda\setminus\rho}.$$

***Proof*** Let $\rho_1, \rho_2, \ldots, \rho_\ell$ be all the distinct rim hooks of length $k$ in $\lambda$. By the Murnaghan-Nakayama rule, we have

$$\chi^\lambda_{[n-k,1^k]} = \sum_{i=1}^{\ell}(-1)^{\ell\ell(\rho_i)}\chi^{\lambda\setminus\rho_i}_{[1^k]} = \sum_{i=1}^{\ell}(-1)^{\ell\ell(\rho_i)} f^{\lambda\setminus\rho_i}.$$

□

Our strategy is to use the fact that $k$ is relatively small so that there is at most one rim hook of length $n-k$ in any Young diagram $\lambda$. Since $2 \le k \le \min\left(n, 2\log_{\frac{k}{e}}\left(\frac{n(n-2)}{2e}\right)-1\right) \le \frac{n-1}{3}$, the non-zero eigenvalues of $\Gamma_{n,k}$ are of the form

$$\xi_{\chi^\lambda} = (-1)^{\ell\ell(\rho)} \frac{f^{\lambda\setminus\rho}}{f^\lambda}\binom{n}{k}(n-k-1)!, \tag{1}$$

where $\lambda \vdash n$ and $\rho$ is the unique rim hook of length $n-k$ of $\lambda$. We prove that the maximum of $\left\{\xi_{\chi^\lambda}\right\}_{\lambda\vdash n, \lambda\neq[n],[1^n]}$ is attained by the partition $[n-1, 1]$. The transpose $[2, 1^{n-2}]$ of $[n-1, 1]$ also gives the second eigenvalue depending on the parity of $n$ and $k$.

Now, we divide the proof of Theorem 1.3 into two cases, depending on whether $k=2$ or $k \ge 3$.

### 3.1.1 When $k=2$

By Lemma 2.1, $\lambda \vdash n$ has at most one rim hook of length $n-2$. If $\lambda \vdash n$ does not have a rim hook of length $n-2$, then the corresponding eigenvalue is equal to 0. If $\rho$ is the unique rim hook of length $n-2$ of $\lambda \vdash n$, then $\lambda\setminus\rho \in \left\{\square\square, \begin{smallmatrix}\square\\ \square\end{smallmatrix}\right\}$ and thus $f^{\lambda\setminus\rho}=1$. Consequently, the non-zero eigenvalues of $\Gamma_{n,2}$ are

$$\xi_{\chi^\lambda} = (-1)^{\ell\ell(\rho)} \frac{1}{f^\lambda}\binom{n}{k}(n-k-1)!,$$

where $\lambda \vdash n$ has a rim hook of length $n-2$. It is easy to see that the second eigenvalue of $\Gamma_{n,2}$ is obtained from the irreducible characters of smallest degree which is not equal to 1.

- When $n \ge 7$, it is well known that the only irreducible characters of $\mathrm{Sym}(n)$ of degree less than $n$ are those corresponding to the partitions $[n]$, $[1^n]$, $[n-1, 1]$ and $[2, 1^{n-2}]$. It is obvious that the partition giving the second eigenvalue (independently of $k$) is $[n-1, 1]$ since the leg-length of a rim hook of length $n-2$ on $[n-1, 1]$ is equal to 0.
- When $3 \le n \le 6$, we use `Sagemath` [12] to verify that the second eigenvalue is given by $[n-1, 1]$. Therefore, $\lambda_2(\Gamma_{n,2}) = \frac{1}{n-1}\binom{n}{2}(n-3)!$.

### 3.1.2 When $k \ge 3$

First, we recall a basic result about the largest degree of irreducible characters of finite groups. If $G$ is a finite group, then we let $b(G)$ be the largest degree of an irreducible character of $G$. We recall the following proposition.

**Proposition 3.3** ([4]) *If $G$ is a finite group, then $b(G) < \sqrt{|G|}$.*

For the case where $3 \leq n \leq 18$ and $3 \leq k \leq n-2$, we use Sagemath [12] to verify that the second eigenvalue is given by the partition $[n-1, 1]$ and it is equal to $\lambda_2(\Gamma_{n,k}) = \frac{k-1}{n-1}c_{n,k}$.

Suppose that $n \geq 19$. It is easy to see that (1) depends only on $(-1)^{\ell\ell(\rho)}\frac{f^{\lambda\setminus\rho}}{f^\lambda}$. Using Proposition 3.3, if $f^\lambda > 3\binom{n}{3}$, then

$$\frac{f^{\lambda\setminus\rho}}{f^\lambda} \leq \frac{b(\mathrm{Sym}(k))}{3\binom{n}{3}} \leq \frac{\sqrt{k!}}{3\binom{n}{3}}.$$

Using the classical bound $\frac{k^k}{e^{k-1}} \leq k! \leq \frac{k^{k+1}}{e^{k-1}}$, it is easy to verify that when $3 \leq k \leq \min\left(n, 2\log_{\frac{k}{e}}\left(\frac{n(n-2)}{2e}\right) - 1\right)$, we have $\sqrt{k!} \leq \frac{3}{n-1}\binom{n}{3}$. Consequently, if $f^\lambda \geq 3\binom{n}{3}$, then

$$0 < \frac{f^{\lambda\setminus\rho}}{f^\lambda} \leq \frac{1}{n-1}. \tag{2}$$

This implies that the eigenvalue afforded by $\lambda \vdash n$, with $f^\lambda > 3\binom{n}{3}$, is less than $\frac{c_{n,k}}{n-1}$, in absolute value.

Now, we compute the eigenvalues that correspond to the irreducible characters of $\mathrm{Sym}(n)$ with degree less than $3\binom{n}{3}$. The trivial character always affords the valency of $\Gamma_{n,k}$, which is $c_{n,k} := \binom{n}{k}(n-k-1)!$. We note that if $\lambda \vdash n$ and $\lambda'$ is the transpose of $\lambda$, then $\chi^{\lambda'}(\sigma) = \chi^{[1^n]}(\sigma)\chi^\lambda(\sigma)$, for any $\sigma \in \mathrm{Sym}(n)$. Therefore, the eigenvalue afforded by the partition $[1^n]$ is either equal to $c_{n,k}$ or $-c_{n,k}$, in which case $\Gamma_{n,k}$ is bipartite. The eigenvalue corresponding to $[1^n]$ is precisely $(-1)^{n-k-1}c_{n,k}$. The eigenvalues of $\Gamma_{n,k}$ corresponding to the other irreducible characters of degree less than $3\binom{n}{3}$ are given in Table 1. These values were computed using the Murnaghan-Nakayama rule and the hook length formula.

An analysis of the eigenvalues in Table 1 shows that the largest is $\frac{k-1}{n-1}c_{n,k} = \frac{k-1}{n-1}\binom{n}{k}(n-k-1)!$, when $n \geq 19$ and $2 \leq k \leq n-2$. Moreover, this eigenvalue is afforded by the irreducible character of $\mathrm{Sym}(n)$ corresponding to the partition $[n-1, 1]$.

We conclude that $\lambda_2(\Gamma_{n,k}) = \frac{k-1}{n-1}c_{n,k}$, when $n \geq 19$ and $3 \leq k \leq \min\left(n, 2\log_{\frac{k}{e}}\left(\frac{n(n-2)}{2e}\right) - 1\right)$. This completes the proof of Theorem 1.3.

### 3.2 Proof of Theorem 1.2

In this section, we use Table 1 to retrieve the result of Siemons and Zalesski in Theorem 1.2. First, we note that the calculation leading to (2) is independent of $k$, so (2) holds for $k \in \{0, 1\}$.

**Table 1** Eigenvalues $\Gamma_{n,k}$ corresponding to irreducible characters of degree less than $3\binom{n}{3}$

| Partition | Eigenvalue |
|---|---|
| $[n-1, 1]$ | $\frac{k-1}{n-1} c_{n,k}$ |
| $[2, 1^{n-2}]$ | $(-1)^{n-k+1} \frac{k-1}{n-1} c_{n,k}$ |
| $[n-2, 2]$ | $\frac{\binom{k}{2}-k}{\binom{n}{2}-n} c_{n,k}$ |
| $[2^2, 1^{n-4}]$ | $(-1)^{n-k+1} \frac{\binom{k}{2}-k}{\binom{n}{2}-n} c_{n,k}$ |
| $[n-2, 1^2]$ | $\frac{\binom{k-1}{2}}{\binom{n-1}{2}} c_{n,k}$ |
| $[3, 1^{n-3}]$ | $(-1)^{n-k+1} \frac{\binom{k-1}{2}}{\binom{n-1}{2}} c_{n,k}$ |
| $[n-3, 3]$ | $\frac{\binom{k}{3}-\binom{k}{2}}{\binom{n}{3}-\binom{n}{2}} c_{n,k}$ |
| $[2^3, 1^{n-6}]$ | $(-1)^{n-k+1} \frac{\binom{k}{3}-\binom{k}{2}}{\binom{n}{3}-\binom{n}{2}} c_{n,k}$ |
| $[n-3, 1^3]$ | $\frac{\binom{k-1}{3}}{\binom{n-1}{3}} c_{n,k}$ |
| $[4, 1^{n-4}]$ | $(-1)^{n-k+1} \frac{\binom{k-1}{3}}{\binom{n-1}{3}} c_{n,k}$ |
| $[n-3, 2, 1]$ | $\frac{k(k-2)(k-4)}{n(n-2)(n-4)} c_{n,k}$ |
| $[3, 2, 1^{n-5}]$ | $(-1)^{n-k+1} \frac{k(k-2)(k-4)}{n(n-2)(n-4)} c_{n,k}$ |

### 3.2.1 The Case $k = 0$

The eigenvalue afforded by the irreducible character corresponding to $[n-1, 1]$ is $\frac{-c_{n,0}}{n-1} < 0$. When $n$ is odd, it is easy to see that the largest eigenvalue in Table 1 is $\binom{n-1}{2}^{-1} c_{n,0} = 2(n-3)!$, which is afforded by $[n-2, 1^2]$. When $n$ is even, the largest eigenvalue is $\frac{c_{n,0}}{n-1} = (n-2)!$, which is afforded by $[2, 1^{n-2}]$.

### 3.2.2 The Case $k = 1$

The eigenvalues afforded by $[n-1, 1]$ and $[2, 1^{n-2}]$ are both equal to 0 in this case. When $n$ is odd, the second eigenvalue is afforded by $[2^2, 1^{n-4}]$ and is equal to $(-1)^n \frac{-1}{\binom{n}{2}-n} c_{n,1} = 2(n-2)(n-4)!$. When $n$ is even, the eigenvalue afforded by $[2^2, 1^{n-4}]$ is negative and the second eigenvalue of $\Gamma_{n,k}$ is equal to $\frac{(-1)(-3)}{n(n-2)(n-4)} c_{n,1} = 3(n-3)(n-5)!$, which is afforded by $[n-3, 2, 1]$.

## 4 Second Eigenvalue of $\Gamma_{n,n-5}$

In this section, we prove Theorem 1.6. The first two subsections give some background material that are needed for the proof.

### 4.1 Equitable Partitions

In this subsection, we show that the eigenvalue afforded by the irreducible character corresponding to the partition $[n-1, 1]$ appears as an eigenvalue of an equitable partition of $\Gamma_{n,k}$.

Given a graph $X = (V, E)$ and a partition $\Pi = \{V_1, V_2, \ldots, V_t\}$ of the vertices of $X$, we say that $\pi$ is an *equitable partition* of $X$ if for any $i, j \in \{1, 2, \ldots, t\}$, there exists a number $a_{ij}$ with the property that the number of vertices in $V_j$ adjacent to a fix vertex in $V_i$ is equal to $a_{ij}$. If $\Pi = \{V_1, V_2, \ldots, V_t\}$ is an equitable partition of $X$, then the *quotient matrix* of $\Pi$ is the $t \times t$ matrix indexed by the set $\{1, 2, \ldots, t\}$ in both rows and columns, and whose $(i, j)$-entry is equal to $a_{ij}$.

The following lemma is straightforward and is given without a proof (see [6, Lemma 5]).

**Lemma 4.1** *Let $G$ be a finite group and $H \leq G$. If $X = \mathrm{Cay}(G, C)$ is a Cayley graph of $G$, then the partition of $X$ into left cosets of $H$ is an equitable partition of $X$.*

Let $G = \mathrm{Sym}(n)$ and for any $i \in \{1, 2, \ldots, n\}$, let $G_i$ be the stabilizer of $i$ in the natural action of $G$ on the set $\{1, 2, \ldots, n\}$. For any $s, t \in \{1, 2, \ldots, n\}$, we define $G_{t,s} := \{\sigma \in G \mid \sigma(t) = s\}$. By Lemma 4.1, the partition $G/G_i$ is equitable for any $i \in \{1, 2, \ldots, n\}$. If $B_{\Pi_i} = (b_{s,t})$ is the quotient matrix corresponding to the equitable partition $\Pi_i$ of $\Gamma_{n,k}$ given by $G/G_i$, then by [6]

$$b_{s,t} = |C(n,k) \cap G_{t,s}| = \begin{cases} \binom{n-1}{n-k}(n-k-1)! & \text{if } t = s \\ \binom{n-2}{n-k-2}(n-k-2)! & \text{if } t \neq s. \end{cases}$$

It is not hard to see that

$$B_{\Pi_i} = \binom{n-2}{n-k-2}(n-k-2)!(J-I) + \binom{n-1}{n-k}(n-k-1)!I,$$

where $J$ is the $n \times n$ all ones matrix and $I$ is the $n \times n$ identity matrix. The eigenvalues of $B_{\Pi_i}$ are

$$\binom{n}{k}(n-k-1)! \text{ and } \frac{k-1}{n-1}\binom{n}{k}(n-k-1)!,$$

which are the eigenvalues afforded by the irreducible characters corresponding to $[n]$ and $[n-1, 1]$, respectively.

### 4.2 A Recursive Method

In [6], Huang et al. gave a recursive method to compute the second eigenvalue of highly transitive groups.

Throughout this section, we let $G \leq \mathrm{Sym}(\Omega)$ be a finite transitive group acting on $\Omega = \{1, 2, \ldots, n\}$. Let $G^{(0)} = G$ and for any $k \geq 1$, let

$$G^{(k)} = G_n \cap G_{n-1} \cap \ldots \cap G_{n-k+1}.$$

Let $T$ be a union of conjugacy classes of $G$. Define

$$\begin{cases} T_k = T_{k-1} \setminus (T_{k-1} \cap G_k), & \text{for any } k \geq 1, \\ T_0 = T. \end{cases}$$

In other words, if $\mathrm{Supp}(\sigma) = \{i \in \Omega \mid \sigma(i) \neq i\}$, then $T_k = \{\sigma \in T \mid \{1, 2, \ldots, k\} \subset \mathrm{Supp}(\sigma)\}$. For any $i \geq 0$ and $k \geq 0$, let

$$X_{k,i} = \mathrm{Cay}(G^{(i)}, T_k \cap G^{(i)}). \tag{3}$$

Let $\Pi$ be the partition of $G$ into left cosets of any stabilizer of a point of $G$. From the results in the previous subsection, we know that this partition is equitable. Let $B_\Pi$ be the quotient matrix corresponding to this equitable partition. By Lemma 4.1, the partition of $G^{(i)}$ into left cosets of any of its point-stabilizers is an equitable partition of $X_{k,i}$, for any $k \geq 0$. Let $B_\Pi^{(k,i)}$ be the quotient matrix of this equitable partition. The second eigenvalue $\lambda_2(B_\Pi^{(k,i)})$ of $B_\Pi^{(k,i)}$ was computed in [6] and it is equal to

$$\lambda_2\left(B_\Pi^{(k,i)}\right) = |T_k \cap G^{(i)} \cap G_{k+1}| - |T_k \cap G^{(i)} \cap G_{k+2,k+1}|.$$

When $G$ is highly transitive, the second eigenvalue of the normal Cayley graph $\mathrm{Cay}(G, T)$ can be computed via a recursive method on the graphs defined in (3).

**Lemma 4.2** [6, Theorem 14] *Let* $m = \max_{\sigma \in T} |\mathrm{supp}(\sigma)|$. *If* $G$ *is* $(m+a)$*-transitive for some* $a \geq 1$ *and* $\lambda_2(X_{k,a-1}) = \lambda_2\left(B_\Pi^{(k,a-1)}\right)$ *for any* $0 \leq k \leq m-1$, *then*

$$\lambda_2(\mathrm{Cay}(G, T)) = \lambda_2(X_{0,0}) = \lambda_2(B_\Pi).$$

In the next subsection, we find the second eigenvalue of $\Gamma_{n,n-5}$ using this recursive method.

### 4.3 Proof of Theorem 1.6

Let $T = C(n, n-5)$ be the conjugacy class of 5-cycles of $\mathrm{Sym}(n)$. Our main tool to prove Theorem 1.6 is Lemma 4.2. Since $\langle T \rangle = \mathrm{Alt}(n)$, the graph $\Gamma_{n,n-5}$ is disconnected and is the disjoint union of two copies of $\mathrm{Cay}(\mathrm{Alt}(n), T)$. Therefore, $\lambda_2(\Gamma_{n,n-5}) = \lambda_2(\mathrm{Cay}(\mathrm{Alt}(n), T))$. Let us apply Lemma 4.2 on $X = \mathrm{Cay}(\mathrm{Alt}(n), T)$.

As the elements of $T$ are 5-cycles, $m = \max_{\sigma \in T} |\operatorname{supp}(\sigma)| = 5$. Since $\operatorname{Alt}(n)$ is $(n-2)$-transitive, it is easy to see that $\operatorname{Alt}(n)$ is $(m+a)$-transitive, whenever $a \leq n-7$. Let $a = n-7$ and $G = \operatorname{Alt}(n)$. Now, it is enough to verify that

$$\lambda_2(X_{k,n-8}) = |T_k \cap G^{(n-8)} \cap G_{k+1}| - |T_k \cap G^{(n-8)} \cap G_{k+2,k+1}|, \qquad (4)$$

for any $0 \leq k \leq 4$.

Note that $G^{(n-8)} = \operatorname{Alt}(8)$ and for any $0 \leq k \leq 4$, $X_{k,n-8} = \operatorname{Cay}(\operatorname{Alt}(8), T_k \cap \operatorname{Alt}(8))$. Using `Sagemath` [12] and `SciPy` [13], we were able to verify (4) for any $0 \leq k \leq 4$. We have compiled in the following table the values of $\lambda_2(X_{k,n-8})$.

| $k$ | $\lambda_1(\operatorname{Cay}(\operatorname{Alt}(8), T_k \cap \operatorname{Alt}(8)))$ | $\lambda_2(\operatorname{Cay}(\operatorname{Alt}(8), T_k \cap \operatorname{Alt}(8)))$ |
|---|---|---|
| 0 | 1344 | 384 |
| 1 | 840 | 300 |
| 2 | 480 | 216 |
| 3 | 240 | 138 |
| 4 | 96 | 72 |

By Lemma 4.2, we conclude that $\lambda_2(\Gamma_{n,n-5}) = \frac{n-6}{n-1}\binom{n}{5}4! = \frac{n(n-2)(n-3)(n-4)(n-6)}{5}$. This completes the proof of Theorem 1.6.

**Acknowledgements** We are grateful to the anonymous reviewers for their insightful comments. The research of both authors is supported in part by the Ministry of Education, Science and Sport of Republic of Slovenia (University of Primorska Developmental funding pillar).

**Data availability** No data used. All computations are available in the table in Section 4.3

## Declarations

**Conflict of interest** No conflict of interest to declare.

## References

1. Babai, L.: Spectra of Cayley graphs. J. Combin. Theory Ser. B **27**(2), 180–189 (1979)
2. Behajaina, A., Maleki, R., Rasoamanana, A.T., Razafimahatratra, A.S.: 3-setwise intersecting families of the symmetric group. Discrete Math. **344**(8), 112467 (2021)
3. Diaconis, P., Shahshahani, M.: Generating a random permutation with random transpositions. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete **57**(2), 159–179 (1981)
4. Halasi, Z., Hannusch, C., Nguyen, H.: The largest character degrees of the symmetric and alternating groups. Proc. Am. Math. Soc. **144**(5), 1947–1960 (2016)
5. Huang, X., Huang, Q.: The second largest eigenvalues of some Cayley graphs on alternating groups. J. Algebraic Combin. **50**(1), 99–111 (2019)
6. Huang, X., Huang, Q., Cioabă, S.M.: The second eigenvalue of some normal Cayley graphs of highly transitive groups. Electron. J. Combin., **26**(2), P2.44 (2019)
7. Ku, C.Y., Lau, T., Wong, K.B.: The spectrum of eigenvalues for certain subgraphs of the $k$-point fixing graph. Linear Algebra Appl. **543**, 72–91 (2018)

8. Parzanchevski, O., Puder, D.: Aldous's spectral gap conjecture for normal sets. Trans. Am. Math. Soc. **373**(10), 7067–7086 (2020)
9. Renteln, P.: On the spectrum of the derangement graph. Electron. J. Combin. **14**(1), R82 (2007)
10. Sagan, B.E.: The symmetric group: representations, combinatorial algorithms, and symmetric functions (graduate texts in mathematics). Springer, New York (2001)
11. Siemons, J., Zalesski, A.: On the second largest eigenvalue of some Cayley graphs of the symmetric group. J. Algebraic Combin. **55**(3), 989–1005 (2020)
12. The Sage Developers. SageMath, the Sage Mathematics Software System (Version 9.7), (2022). https://www.sagemath.org
13. Virtanen, P., et al.: SciPy 1.0: fundamental algorithms for scientific computing in Python. Nat. Methods **17**(3), 261–272 (2020)