



# Unstable Manifolds for Rough Evolution Equations

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## Abstract

In this paper, we consider a class of evolution equations driven by finite-dimensional  $\gamma$ -Hölder rough paths, where  $\gamma \in (1/3, 1/2]$ . We prove the global-in-time solutions of rough evolution equations (REEs) in a suitable space, also obtain that the solutions generate random dynamical systems. Meanwhile, we derive the existence of local unstable manifolds for such equations by a properly discretized Lyapunov–Perron method.

**Keywords** Rough evolution equations · Random dynamical system · Unstable manifolds · Lyapunov–Perron method

**Mathematics Subject Classification** 60H15 · 60H05 · 37H10 · 37D10

## 1 Introduction

Invariant manifolds are one of the cornerstones of nonlinear dynamical systems and have been widely studied in deterministic systems. However, in practical applications, nonlinear dynamical systems are always affected by noises. Invariant manifolds have

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been widely studied for stochastic ordinary differential equations (SDEs) (see [1, 4, 5]) and stochastic partial differential equations (SPDEs) (see Chen et al. [6, 8, 9, 14]). One of the key difficulties in studying invariant manifolds of a stochastic partial differential equation is to prove that it generates a random dynamical system. As we all know that a large class of partial differential equations with stationary random coefficients and Itô stochastic ordinary differential equations generate random dynamical systems (see Arnold [1]). Nevertheless, for stochastic partial differential equations driven by the standard Brownian motion, it is unknown that how to obtain random dynamical systems. The reasons are: (i) the stochastic integral is only defined almost surely where the exceptional set may depend on the initial state; (ii) Kolmogorov's theorem is only true for finite dimensional random fields. However, there are some results for additive and linear multiplicative noise (see [8–10]).

A way to obtain a random dynamical system for a stochastic differential equation is that this equation is driven by  $\gamma$ -Hölder continuous paths. In this sense, there are two techniques of defining the stochastic integral that are in pathwise sense. For  $\gamma > 1/2$ , these integrals are consistent with the well-known Young integral (see Young [26] and Zähle [27]). One of the techniques is to define the integral based on fractional derivatives. There are already some investigations which have proven that the (pathwise) solutions driven by fractional Brownian motion with  $\gamma > 1/2$  generate random dynamic systems, obtained that the existence of random attractors and invariant manifolds that describe the longtime behaviors of the solutions (see Chen et al. [7], Gao et al. [12], Garrido-Atienza et al. [13, 14]). For  $1/3 < \gamma < 1/2$ , Garrido-Atienza et al. [15] have obtained random dynamical systems for stochastic evolution equations driven by multiplicative fractional Brownian motion; Another one is to interpret integral in the rough path sense. Rough path theory (see [11, 16, 17, 23]) is close to deterministic analytical methods, Bailleul [2] analyzed flows driven by rough paths and Bailleul et al. [3] studied random dynamical systems for rough differential equations. Kuehn and Neamțu [24] have proven the existence and regularity of local center manifolds for rough differential equations by means of a suitably discretized Lyapunov–Perron-type method. Gubinelli and Tindel [18] generalised theory of rough paths to solve not only SDEs but also SPDEs: evolution equations driven by the infinite dimensional Gaussian process. Gerasimovičs and Hairer [16] have developed a pathwise local solution theory for a class semilinear SPDEs with multiplicative noise driven by a finite dimensional Wiener process. Hesse and Neamțu [19, 20] have investigated local, global mild solutions and random dynamical systems for rough partial differential equations. Recently, Hesse and Neamțu [21] have obtained global-in-time solutions and random dynamical systems for semilinear parabolic rough partial differential equations. Furthermore, based on the structure of solution in [21], Neamțu and Kuehn [22] have derived the center manifolds for rough partial differential equations.

However, so far, there are little works relate to unstable manifolds of rough evolution equations. Therefore, in this paper, based on [6, 14, 16, 21, 22] and [24], we are going to study random dynamical systems and local unstable manifolds for (2.2). In order to overcome the obstacle that how to obtain random dynamical systems of SPDEs with nonlinear multiplicative noises, similar to [21] and [24], we choose a proper space that is different from [16] and [22], give a simpler proof of the local solutions for rough evolution equations than [16] and obtain the global solutions, also, we obtain

random dynamical systems by using the rough integral developed in [16] and rough path cocycle of [3]. Meanwhile, we obtain the contraction properties of the Lyapunov–Perron operator by using rough path estimates. Moreover, by using properly discretized Lyapunov–Perron method, we derive the existence of local unstable manifolds for rough evolution equations.

This paper is structured as follows. In Sect. 2 we provide background on mildly controlled rough paths and study the global solutions of rough evolution equations. Section 3 is devoted to dynamics of rough evolution equations. In Sect. 4 we derive the existence of local unstable manifolds which are based on a discrete-time Lyapunov–Perron method. Since we work with pathwise integral, so, at each step, it is necessary to control the norms of the random input on a fixed time-interval. By deriving suitable estimates of the mildly controlled rough integrals, the unstable manifolds is obtained by employing a random dynamical systems approach. The results obtained for the discrete Lyapunov–Perron map can then be extended to the time-continuous one (further details please refer to [14, 24]).

## 2 Rough evolution equations

Throughout this paper, let  $T > 0$ , we consider a separable Hilbert space  $\mathcal{H}$  and  $A$  is a generator of analytic  $C_0$ -semigroup  $\{S_t : t \geq 0\}$  on the interpolation space  $(\mathcal{H}_\alpha = \text{Dom}(-A)^\alpha; \alpha \in \mathbb{R})$ . We will use the following fact that for all  $\alpha \geq \beta$ ,  $\gamma \in [0, 1]$  and  $u \in \mathcal{H}_\beta$ , one has

$$\|S_t u\|_{\mathcal{H}_\alpha} \leq C_\beta t^{\beta-\alpha} \|u\|_{\mathcal{H}_\beta}, \quad \|S_t u - u\|_{\mathcal{H}_{\beta-\gamma}} \leq C_\gamma t^\gamma \|u\|_{\mathcal{H}_\beta} \tag{2.1}$$

uniformly over  $t \in (0, T]$ . For an introduction to semigroup theory, one can refer to [25].

**Notation:** We denote  $\mathcal{H}_\alpha^d := \mathcal{L}(\mathbb{R}^d, \mathcal{H}_\alpha)$  ( $\mathcal{H}_\alpha^{d \times d} := \mathcal{L}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathcal{H}_\alpha)$ ) as the space of continuously linear operators from  $\mathbb{R}^d(\mathbb{R}^d \otimes \mathbb{R}^d)$  to  $\mathcal{H}_\alpha$ . For some fixed  $\alpha, \beta \in \mathbb{R}$  and  $k \in \mathbb{N}$ , we denote  $\mathcal{C}_{\alpha,\beta}^k(\mathcal{H}, \mathcal{H}^n)$  as the space of  $k$ -order continuously Fréchet-differentiable functions  $g : \mathcal{H}_\theta \rightarrow \mathcal{H}_{\theta+\beta}^n$  for any  $\theta \geq \alpha$ ,  $n \in \mathbb{N}$  with bounded derivatives  $D^i g$ , for all  $i = 1, \dots, k$ . Furthermore, we denote  $\mathcal{C}_{\alpha,\beta}(\mathcal{H}, \mathcal{H})$  as the space of continuous functions  $f : \mathcal{H}_\theta \rightarrow \mathcal{H}_{\theta+\beta}$  for any  $\theta \geq \alpha$ .  $\mathcal{C}_n([0, T]; V)$  as the space of continuous functions from  $\Delta_n$  to  $V$  where  $\Delta_n := \{(t_1, \dots, t_n) : T \geq t_1 \geq \dots \geq t_n \geq 0\}$  for  $n \geq 1$  and, for notational simplicity, denote  $\mathcal{C}([0, T]; V) = \mathcal{C}_1([0, T]; V)$ .  $C$  stands for a universal constant which may vary from line to line, the dependence of this constant  $C = C_{\dots}$  on certain parameters will be explicitly stated in subscripts.

In this article, we will consider rough evolution equations

$$\begin{cases} dy_u = (Ay_u + f(y_u)) du + g(y_u) d\mathbf{w}_u, u \in [0, T], \\ y_0 = \xi \in \mathcal{H}, \end{cases} \tag{2.2}$$

where we assume:

- $f \in \mathcal{C}_{-2\gamma,0}(\mathcal{H}, \mathcal{H})$  is global Lipschitz continuous,

- $g \in \mathcal{C}_{-2\gamma,0}^3(\mathcal{H}, \mathcal{H}^d)$  and such that  $\|g(0)\|_{\mathcal{H}_\theta^d} = C_0$  for  $\theta \geq -2\gamma$ ,
- $\mathbf{w}$  is a  $\gamma$ -Hölder rough path with  $\gamma \in (\frac{1}{3}, \frac{1}{2}]$  that will be defined as below.

The mild solution of (2.2) can be given by

$$y_t = S_t \xi + \int_0^t S_{tu} f(y_u) du + \int_0^t S_{tu} g(y_u) d\mathbf{w}_u, \tag{2.3}$$

where the last integral is rough integral, which is pathwise, will be defined below. From now on, for notational simplicity, we denote  $S_{ts} := S_{t-s}$  for  $0 \leq s < t \leq T$ . In this section, we will prove the global in time solution of (2.2) and its truncated equation, this is essential for one to consider the invariant manifolds for rough evolution equation.

First of all, we review some concepts and results on rough path theory, for more details, please refer to [11] and [16]. Given a Banach space  $V$  endowed with the norm  $\|\cdot\|_V$ , for  $h \in \mathcal{C}([0, T]; V)$ ,  $p \in \mathcal{C}_2([0, T]; V)$ , let

$$\begin{aligned} \delta h_{t,s} &= h_t - h_s, & \delta p_{t,u,s} &= p_{t,s} - p_{t,u} - p_{u,s}. \\ \hat{\delta} h_{t,s} &= h_t - S_{ts} h_s, & \hat{\delta} p_{t,u,s} &= p_{t,s} - p_{t,u} - S_{tu} p_{u,s}. \end{aligned}$$

Notice that  $V$  is one of the spaces in which the action of the semigroup  $S$  makes sense. Then, for  $0 < \gamma < 1$  we set

$$\begin{aligned} |h|_{\gamma,V} &= \sup_{s,t \in [0,T]} \frac{\|\delta h_{t,s}\|_V}{|t-s|^\gamma}, & \|h\|_{\gamma,V} &= \sup_{s,t \in [0,T]} \frac{\|\hat{\delta} h_{t,s}\|_V}{|t-s|^\gamma}, \\ |p|_{\gamma,V} &= \sup_{s,t \in [0,T]} \frac{\|p_{t,s}\|_V}{|t-s|^\gamma}. \end{aligned}$$

Consequently, one can define the spaces as below:

$$\begin{aligned} \mathcal{C}^\gamma([0, T]; V) &= \{h \in \mathcal{C}([0, T]; V) : |h|_{\gamma,V} < \infty\}, \\ \mathcal{C}_2^\gamma([0, T]; V) &= \{p \in \mathcal{C}_2([0, T]; V) : |p|_{\gamma,V} < \infty\}, \\ \hat{\mathcal{C}}^\gamma([0, T]; V) &= \{h \in \mathcal{C}([0, T]; V) : \|h\|_{\gamma,V} < \infty\}. \end{aligned}$$

**Remark 2.1** Since the semigroup  $S$  is not Hölder continuous at  $t = 0$ , hence, from now on, we will choose  $\hat{\delta}$  operator and  $\hat{\mathcal{C}}^\gamma$  type Hölder spaces for our evolution setting to overcome this obstacle.

In addition, we endow  $\mathcal{C}([0, T]; V)$  with the supremum norm  $\|h\|_{\infty,V} = \sup_{0 \leq t \leq T} \|h_t\|_V$ . For notational simplicity, in the cases of  $V = \mathcal{H}_\alpha, \mathcal{H}_\alpha^d$  or  $\mathcal{H}_\alpha^{d \times d}$ , we will denote  $|h|_V = |h|_{\gamma,\alpha}$ ,  $\|h\|_{\gamma,V} = \|h\|_{\gamma,\alpha}$ ,  $\|h\|_{\infty,V} = \|h\|_{\infty,\alpha}$ .

**Definition 2.1** For  $\gamma \in (\frac{1}{3}, \frac{1}{2}]$ , we define the space of  $\gamma$ -Hölder rough paths(over  $\mathbb{R}^d$ ) as those pairs  $\mathbf{w} = (w, w^2) \in \mathcal{C}^\gamma([0, T]; \mathbb{R}^d) \times \mathcal{C}_2^{2\gamma}([0, T]; \mathbb{R}^d \otimes \mathbb{R}^d)$  satisfying the

Chen’s relation, i.e. for  $s \leq u \leq t \in [0, T]$

$$w_{t,s}^2 - w_{t,u}^2 - w_{u,s}^2 = \delta w_{u,s} \otimes \delta w_{t,u}.$$

This space is denoted as  $\mathcal{C}^\gamma([0, T]; \mathbb{R}^d)$ . For two rough paths  $\mathbf{w} = (w, w^2)$ ,  $\tilde{\mathbf{w}} = (\tilde{w}, \tilde{w}^2) \in \mathcal{C}^\gamma([0, T]; \mathbb{R}^d)$ , we define the rough metric  $\varrho_\gamma$  as:

$$\varrho_\gamma(\mathbf{w}, \tilde{\mathbf{w}}) = |w - \tilde{w}|_\gamma + |w^2 - \tilde{w}^2|_{2\gamma}.$$

**Definition 2.2** Let  $\mathbf{w} \in \mathcal{C}^\gamma([0, T]; \mathbb{R}^d)$ , for some  $\gamma \in (\frac{1}{3}, \frac{1}{2}]$ , we call  $(y, y') \in \hat{\mathcal{C}}^\gamma([0, T]; \mathcal{H}_\alpha) \times \hat{\mathcal{C}}^\gamma([0, T]; \mathcal{H}_\alpha^d)$  a mildly controlled rough path, if the remainder term  $R^y$  is defined by

$$R_{t,s}^y = \hat{\delta}y_{t,s} - S_{t,s}y'_s \delta w_{t,s}, \quad \text{for } s \leq t \in [0, T], \tag{2.4}$$

which belongs to  $\mathcal{C}_2^{2\gamma}([0, T]; \mathcal{H}_\alpha)$ , then we call  $y'$  mildly Gubinelli derivative of  $y$  and denote  $(y, y') \in \mathcal{D}_{S,w}^{2\gamma}([0, T]; \mathcal{H}_\alpha)$ .

Notice that, when one replaces  $\mathcal{H}_\alpha$  by  $\mathcal{H}_\alpha^d$ , the above definition is also true. Meanwhile, a seminorm on this space is defined as

$$\|y, y'\|_{w,2\gamma,\alpha} = \|y'\|_{\gamma,\alpha} + |R^y|_{2\gamma,\alpha}.$$

The norm of  $\mathcal{D}_{S,w}^{2\gamma}([0, T]; \mathcal{H}_\alpha)$  is defined as

$$\|y, y'\|_{\mathcal{D}_{S,w}^{2\gamma}} = \|y_0\|_{\mathcal{H}_\alpha} + \|y'_0\|_{\mathcal{H}_\alpha^d} + \|y, y'\|_{w,2\gamma,\alpha}.$$

**Remark 2.2** [22] have used controlled rough path given in [21] which is different from the one we use. Here we incorporate semigroup into the definition of controlled rough path as in [16].

According to (2.4), one can easily derive that

$$\|y\|_{\gamma,\alpha} \leq |R^y|_{2\gamma,\alpha} T^\gamma + \|y'\|_{\infty,\alpha} |w|_\gamma \leq (1 + |w|_\gamma)(\|y'_0\|_{\mathcal{H}_\alpha^d} + \|y, y'\|_{w,2\gamma,\alpha} T^\gamma). \tag{2.5}$$

Furthermore, given a mildly controlled rough path, one can define the rough integral as below:

**Theorem 2.1** Let  $T > 0$  and  $\mathbf{w} \in \mathcal{C}^\gamma([0, T]; \mathbb{R}^d)$  for some  $\gamma \in (\frac{1}{3}, \frac{1}{2}]$ . Let  $(y, y') \in \mathcal{D}_{S,w}^{2\gamma}([0, T]; \mathcal{H}_\alpha^d)$ . Furthermore,  $\mathcal{P}$  stands for a partition of  $[0, T]$ . Then the integral defined as

$$\int_s^t S_{tu} y_u d\mathbf{w}_u := \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} S_{tu}(y_u \delta w_{v,u} + y'_u w_{v,u}^2) \tag{2.6}$$

exists as an element of  $\hat{\mathcal{C}}^\gamma([0, T]; \mathcal{H}_\alpha)$  and satisfies that for every  $0 \leq \beta < 3\gamma$  we have

$$\begin{aligned} & \left\| \int_s^t S_{tu} y_u d\mathbf{w}_u - S_{ts} y_s \delta w_{t,s} - S_{ts} y'_s w_{t,s}^2 \right\|_{\mathcal{H}_{\alpha+\beta}} \\ & \lesssim (|R^y|_{2\gamma,\alpha} |w|_\gamma + \|y'\|_{\gamma,\alpha} |w^2|_{2\gamma}) |t - s|^{3\gamma-\beta}. \end{aligned} \tag{2.7}$$

Moreover, the map

$$(y, y') \rightarrow (z, z') := \left( \int_0^\cdot S_{\cdot u} y_u d\mathbf{w}_u, y \right)$$

is continuous from  $\mathcal{D}_{S,w}^{2\gamma}([0, T]; \mathcal{H}_\alpha^d)$  to  $\mathcal{D}_{S,w}^{2\gamma}([0, T]; \mathcal{H}_\alpha)$ . Here the underlying constant depends on  $\gamma, d$  and  $T$  and can be chosen uniformly over  $T \in (0, 1]$ .

In our case, one needs to consider a suitable class of nonlinearities integrands, according to Lemma 3.14 of [16], we consider mildly controlled rough paths compose with regular functions as follows, since the proof is identical to the one of Lemma 3.7 of [16], we omit it here.

**Lemma 2.1** *Let  $g \in \mathcal{C}_{\alpha,0}^{2\gamma}(\mathcal{H}, \mathcal{H}^d)$ ,  $T > 0$  and  $(y, y') \in \mathcal{D}_{S,w}^{2\gamma}([0, T]; \mathcal{H}_\alpha)$ , for some  $\mathbf{w} \in \mathcal{C}^\gamma([0, T]; \mathbb{R}^d)$ ,  $\gamma \in (1/3, 1/2]$ . Moreover, suppose  $y \in \hat{\mathcal{C}}^\eta([0, T]; \mathcal{H}_{\alpha+2\gamma})$ ,  $\eta \in [0, 1]$  and  $y' \in L^\infty([0, T]; \mathcal{H}_{\alpha+2\gamma}^d)$ . Define  $(z_t, z'_t) = (g(y_t), Dg(y_t)y'_t)$ , then,  $(z, z') \in \mathcal{D}_{S,w}^{2\gamma}([0, T]; \mathcal{H}_\alpha^d)$  and satisfies the bound*

$$\begin{aligned} & \|z, z'\|_{w,2\gamma,\alpha} \\ & \leq C_{g,T} (1 + |w|_\gamma)^2 (1 + \|y'_0\|_{\mathcal{H}_\alpha^d} + \|y, y'\|_{w,2\gamma,\alpha}) \\ & \cdot (1 + \|y_0\|_{\mathcal{H}_{\alpha+2\gamma}} + \|y'_0\|_{\mathcal{H}_\alpha^d} + \|y\|_{\eta,\alpha+2\gamma} + \|y'\|_{\infty,\alpha+2\gamma} + \|y, y'\|_{w,2\gamma,\alpha}). \end{aligned} \tag{2.8}$$

The constant  $C_{g,T}$  depends on  $g$  and the bounds of its derivatives, meanwhile, it depends on time  $T$ , but can be chosen uniformly over  $T \in (0, 1]$ .

According to Lemma 2.1, the composition with regular functions requires higher spatial regularity conditions for mildly controlled rough path. Hence, in our evolution setting, in order to obtain the global in time solutions of (2.2) in a suitable space, as in [16], we need the following space:

$$\mathcal{D}_{S,w}^{2\gamma,\beta,\eta}([0, T]; \mathcal{H}_\alpha) = \mathcal{D}_{S,w}^{2\gamma}([0, T]; \mathcal{H}_\alpha) \cap (\hat{\mathcal{C}}^\eta([0, T]; \mathcal{H}_{\alpha+\beta}) \times L^\infty([0, T]; \mathcal{H}_{\alpha+\beta}^d)),$$

where  $\beta \in \mathbb{R}$  and  $\eta \in [0, 1]$ . Let  $(y, y') \in \mathcal{D}_{S,w}^{2\gamma,\beta,\eta}([0, T]; \mathcal{H}_\alpha)$ , the seminorm of this space is defined as:

$$\|y, y'\|_{w,2\gamma,\beta,\eta} = \|y\|_{\eta,\alpha+\beta} + \|y'\|_{\infty,\alpha+\beta} + \|y, y'\|_{w,2\gamma,\alpha}.$$

The norm of this space is defined as below:

$$\|y, y'\|_{\mathcal{D}_{S,w}^{2\gamma,\beta,\eta}} = \|y_0\|_{\mathcal{H}_{\alpha+\beta}} + \|y'_0\|_{\mathcal{H}_\alpha^d} + \|y\|_{\eta,\alpha+\beta} + \|y'\|_{\infty,\alpha+\beta} + \|y, y'\|_{w,2\gamma,\alpha}.$$

Moreover, we will denote  $\hat{\mathcal{C}}^0 = \mathcal{C}$  for  $\eta = 0$ .

Furthermore, from Lemma 2.1, we know that composition with regular functions maps  $\mathcal{D}_{S,w}^{2\gamma,2\gamma,\eta}([0, T]; \mathcal{H}_\alpha)$  to  $\mathcal{D}_{S,w}^{2\gamma,2\gamma,0}([0, T]; \mathcal{H}_\alpha^d)$ , for  $\eta \in [0, 1]$ . For notational simplicity, we denote

$$\mathcal{D}_w^{2\gamma,\eta}([0, T]; \mathcal{H}_\alpha) := \mathcal{D}_{S,w}^{2\gamma,2\gamma,\eta}([0, T]; \mathcal{H}_{\alpha-2\gamma}), \quad 0 \leq \eta < \gamma,$$

the seminorm and norm of  $\mathcal{D}_w^{2\gamma,\eta}([0, T]; \mathcal{H}_\alpha)$  are respectively denoted as  $\|\cdot, \cdot\|_{w,2\gamma,2\gamma,\eta}$  and  $\|\cdot, \cdot\|_{\mathcal{D}_w^{2\gamma,\eta}}$ .

**Remark 2.3** Notice that, in [16], for notational simplicity, the authors have denoted  $\mathcal{D}_w^{2\gamma}([0, T]; \mathcal{H}_\alpha) := \mathcal{D}_{S,w}^{2\gamma,2\gamma,\gamma}([0, T]; \mathcal{H}_{\alpha-2\gamma})$  and considered the solution in  $\mathcal{D}_w^{2\gamma}([0, T]; \mathcal{H})$  which is differ from our case. In our situation, in order to facilitate the study of the global in time solution of (2.2), we will choose to consider (2.2) in the space  $\mathcal{D}_w^{2\gamma,\eta}([0, T]; \mathcal{H})$  which is bigger than the space  $\mathcal{D}_w^{2\gamma}([0, T]; \mathcal{H})$  of [16].

**Lemma 2.2** Let  $T > 0$ ,  $g \in \mathcal{C}_{-2\gamma,0}^3(\mathcal{H}, \mathcal{H}^d)$ ,  $(y, y') \in \mathcal{D}_w^{2\gamma,\eta}([0, T]; \mathcal{H})$ , for some  $w \in \mathcal{C}^\gamma([0, T]; \mathbb{R}^d)$  with  $\gamma \in (\frac{1}{3}, \frac{1}{2}]$ . We have

$$\left( \int_0^\cdot S_{\cdot u} g(y_u) d\mathbf{w}_u, g(y) \right) \in \mathcal{D}_w^{2\gamma,\eta}([0, T]; \mathcal{H})$$

and

$$\left\| \int_0^\cdot S_{\cdot u} g(y_u) d\mathbf{w}_u, g(y) \right\|_{\mathcal{D}_w^{2\gamma,\eta}} \leq C_{\gamma,d,T} (1 + |w|_\gamma + |w^2|_{2\gamma}) \|g(y), (g(y))'\|_{\mathcal{D}_{S,w}^{2\gamma,2\gamma,0}}, \quad (2.9)$$

where the constant  $C_{\gamma,d,T}$  depends on  $\gamma, d$  and  $T$  and can be chosen uniformly over  $T \in (0, 1]$ .

**Proof** According to (2.1) and (2.7) we obtain that

$$\begin{aligned} \left\| R_{t,s}^{\int_0^\cdot S_{\cdot u} g(y_u) d\mathbf{w}_u} \right\|_{\mathcal{H}_{-2\gamma}} &\leq \left\| \int_s^t S_{t u} g(y_u) d\mathbf{w}_u - S_{t s} g(y_s) \delta w_{t,s} - S_{t s} (g(y_s))' w_{t,s}^2 \right\|_{\mathcal{H}_{-2\gamma}} \\ &\quad + \|S_{t s} (g(y_s))' w_{t,s}^2\|_{\mathcal{H}_{-2\gamma}} \\ &\lesssim \left( |R^{g(y)}|_{2\gamma,-2\gamma} |w|_\gamma + \|(g(y))'\|_{\gamma,-2\gamma} |w^2|_{2\gamma} \right) |t - s|^{3\gamma} \\ &\quad + \|(g(y_s))'\|_{\mathcal{H}_{-2\gamma}^{d \times d}} |w^2|_{2\gamma} |t - s|^{2\gamma}, \end{aligned}$$

then we have

$$\begin{aligned} \left\| R_{t,s}^{\int_0^\cdot S_{\cdot u} g(y_u) d\mathbf{w}_u} \right\|_{2\gamma,-2\gamma} &\lesssim T^\gamma (|w|_\gamma + |w^2|_{2\gamma}) \|g(y), (g(y))'\|_{w,2\gamma,-2\gamma} \\ &\quad + |w^2|_{2\gamma} \|(g(y))'\|_{\infty,-2\gamma}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & \left\| \int_s^t S_{tu} g(y_u) d\mathbf{w}_u \right\|_{\mathcal{H}} \\ & \leq \left\| \int_s^t S_{tu} g(y_u) d\mathbf{w}_u - S_{ts} g(y_s) \delta w_{t,s} - S_{ts} (g(y_s))' w_{t,s}^2 \right\|_{\mathcal{H}} \\ & \quad + \|S_{ts} g(y_s) \delta w_{t,s}\|_{\mathcal{H}} + \|S_{ts} (g(y_s))' w_{t,s}^2\|_{\mathcal{H}} \\ & \lesssim \left( |R^{g(y)}|_{2\gamma, -2\gamma} |w|_{\gamma} + \|(g(y))'\|_{\gamma, -2\gamma} |w^2|_{2\gamma} \right) |t - s|^{\gamma} \\ & \quad + \|g(y_s)\|_{\mathcal{H}^d} |w|_{\gamma} |t - s|^{\gamma} + \|(g(y_s))'\|_{\mathcal{H}^{d \times d}} |w^2|_{2\gamma} |t - s|^{2\gamma}, \end{aligned}$$

then

$$\begin{aligned} \left\| \int_0^\cdot S_{\cdot u} g(y_u) d\mathbf{w}_u \right\|_{\eta, 0} & \leq T^{\gamma-\eta} (|w|_{\gamma} + |w^2|_{2\gamma}) \|g(y), (g(y))'\|_{w, 2\gamma, -2\gamma} \\ & \quad + T^{\gamma-\eta} \|g(y)\|_{\infty, 0} |w|_{\gamma} + T^{2\gamma-\eta} \|(g(y))'\|_{\infty, 0} |w^2|_{2\gamma}. \end{aligned}$$

From (2.5) one has

$$\|g(y)\|_{\gamma, -2\gamma} \leq (1 + |w|_{\gamma}) (\|(g(y_0))'\|_{\mathcal{H}_{-2\gamma}^{d \times d}} + T^{\gamma} \|g(y), (g(y))'\|_{w, 2\gamma, -2\gamma}).$$

Consequently, from above estimates we have

$$\begin{aligned} & \left\| \int_0^\cdot S_{\cdot u} g(y_u) d\mathbf{w}_u, g(y) \right\|_{\mathcal{D}_w^{2\gamma, \eta}} \\ & \lesssim (1 + |w|_{\gamma} + |w^2|_{2\gamma}) \|(g(y_0))'\|_{\mathcal{H}_{-2\gamma}^{d \times d}} + T^{\gamma} |w^2|_{2\gamma} \|(g(y))'\|_{\gamma, -2\gamma} \\ & \quad + \|g(y)\|_{\infty, 0} + T^{\gamma-\eta} |w|_{\gamma} \|g(y)\|_{\infty, 0} + T^{2\gamma-\eta} |w^2|_{2\gamma} \|(g(y))'\|_{\infty, 0} \\ & \quad + \|g(y_0)\|_{\mathcal{H}_{-2\gamma}^d} + (1 + |w|_{\gamma} + |w^2|_{2\gamma}) T^{\gamma} \|g(y), (g(y))'\|_{w, 2\gamma, -2\gamma} \\ & \quad + (|w|_{\gamma} + |w^2|_{2\gamma}) T^{\gamma-\eta} \|g(y), (g(y))'\|_{w, 2\gamma, -2\gamma}. \end{aligned} \tag{2.10}$$

Finally, using (2.10), we easily obtain the desired result. □

**Lemma 2.3** *Let  $T > 0$ ,  $g \in \mathcal{C}_{-2\gamma, 0}^3(\mathcal{H}, \mathcal{H}^d)$ ,  $(y, y')$  and  $(v, v') \in \mathcal{D}_w^{2\gamma, \eta}([0, T]; \mathcal{H})$ , for some  $\mathbf{w} \in \mathcal{C}^{\gamma}([0, T]; \mathbb{R}^d)$  and there exists  $M > 0$  such that  $|w|_{\gamma}$ ,  $|w^2|_{2\gamma}$ ,  $\|y, y'\|_{\mathcal{D}_w^{2\gamma, \eta}}$  and  $\|v, v'\|_{\mathcal{D}_w^{2\gamma, \eta}} \leq M$ , then the following estimate holds true*

$$\|g(y) - g(v), (g(y) - g(v))'\|_{\mathcal{D}_{S, w}^{2\gamma, 2\gamma, 0}} \leq C_{M, g, T} (1 + |w|_{\gamma})^2 \|y - v, (y - v)'\|_{\mathcal{D}_w^{2\gamma, \eta}}. \tag{2.11}$$

The constant  $C_{M, g, T}$  depends on  $M$ ,  $g$  and the bounds of its derivatives. At the same time, it depends on time  $T$ , but can be chosen uniformly over  $T \in (0, 1]$ .

**Proof** Firstly, we give an inequality which will be used throughout the proof: for  $g \in \mathcal{C}_{-2\gamma, 0}^3(\mathcal{H}, \mathcal{H}^d)$ ,  $x_1, x_2, x_3, x_4 \in \mathcal{H}_{\theta}$ ,  $\theta \geq -2\gamma$ , the following bound holds

$$\|g(x_1) - g(x_2) - g(x_3) + g(x_4)\|_{\mathcal{H}_{\theta}^d}$$



$$\leq C_g (\|x_1 - x_2 - x_3 + x_4\|_{\mathcal{H}_\theta} + (\|x_1 - x_3\|_{\mathcal{H}_\theta} + \|x_2 - x_4\|_{\mathcal{H}_\theta}) \|x_3 - x_4\|_{\mathcal{H}_\theta}). \tag{2.12}$$

Due to

$$\begin{aligned} & \|g(y_t) - g(v_t) - S_{ts}(g(y_s) - g(v_s))\|_{\mathcal{H}_{-2\gamma}^d} \\ & \leq \|g(y_t) - g(v_t) - (g(y_s) - g(v_s))\|_{\mathcal{H}_{-2\gamma}^d} + \|(S_{ts} - I)(g(y_s) - g(v_s))\|_{\mathcal{H}_{-2\gamma}^d} \\ & \leq \|g(y_t) - g(v_t) - (g(y_s) - g(v_s))\|_{\mathcal{H}_{-2\gamma}^d} + \|(g(y_s) - g(v_s))\|_{\mathcal{H}^d} |t - s|^{2\gamma}, \end{aligned}$$

so, we have

$$\|g(y) - g(v)\|_{\gamma, -2\gamma} \leq |g(y) - g(v)|_{\gamma, -2\gamma} + T^\gamma \|g(y) - g(v)\|_{\infty, 0}.$$

Similarly, we have

$$|g(y) - g(v)|_{\gamma, -2\gamma} \leq \|g(y) - g(v)\|_{\gamma, -2\gamma} + T^\gamma \|g(y) - g(v)\|_{\infty, 0}.$$

Using (2.5) and (2.12), we derive that

$$\begin{aligned} |g(y) - g(v)|_{\gamma, -2\gamma} & \leq C_g (|y - v|_{\gamma, -2\gamma} + (|y|_{\gamma, -2\gamma} + |v|_{\gamma, -2\gamma}) \|y - v\|_{\infty, -2\gamma}) \\ & \leq C_g (\|y - v\|_{\gamma, -2\gamma} + T^\gamma \|y - v\|_{\infty, 0}) \\ & \quad + C_g (\|y\|_{\gamma, -2\gamma} + T^\gamma \|y\|_{\infty, 0} + \|v\|_{\gamma, -2\gamma} \\ & \quad + T^\gamma \|v\|_{\infty, 0}) \|y - v\|_{\infty, -2\gamma} \\ & \leq C_g (\|y - v\|_{\gamma, -2\gamma} + T^\gamma \|y - v\|_{\infty, 0}) \\ & \quad + C_g (1 + |w|_\gamma) (\|y'_0\|_{\mathcal{H}_{-2\gamma}^d} + T^\gamma \|y_0\|_{\mathcal{H}} \\ & \quad + T^{\gamma+\eta} \|y\|_{\eta, 0} + T^\gamma \|y, y'\|_{w, 2\gamma, -2\gamma} \\ & \quad + \|v'_0\|_{\mathcal{H}_{-2\gamma}^d} + T^\gamma \|v_0\|_{\mathcal{H}} + T^{\gamma+\eta} \|v\|_{\eta, 0} \\ & \quad + T^\gamma \|v, v'\|_{w, 2\gamma, -2\gamma}) \|y - v\|_{\infty, -2\gamma} \\ & \leq C_{g, T, M} (1 + |w|_\gamma) (\|y - v\|_{\gamma, -2\gamma} + \|y - v\|_{\infty, -2\gamma} + T^\gamma \|y - v\|_{\infty, 0}), \end{aligned}$$

therefore

$$\begin{aligned} \|g(y) - g(v)\|_{\gamma, -2\gamma} & \leq C_{g, T, M} (1 + |w|_\gamma) (\|y - v\|_{\gamma, -2\gamma} \\ & \quad + \|y - v\|_{\infty, -2\gamma} + T^\gamma \|y - v\|_{\infty, 0}). \end{aligned}$$

Similarly, we can obtain that

$$\begin{aligned} \|Dg(y)y' - Dy(v)v'\|_{\gamma, -2\gamma} & \leq |Dg(y)y' - Dg(v)v'|_{\gamma, -2\gamma} \\ & \quad + \|Dg(y)y' - Dg(v)v'\|_{\infty, 0} T^\gamma, \\ |Dg(y)y' - Dg(v)v'|_{\gamma, -2\gamma} & \leq |Dg(y)(y' - v')|_{\gamma, -2\gamma} \\ & \quad + |(Dg(y) - Dg(v))v'|_{\gamma, -2\gamma} \\ & := I + II. \end{aligned}$$

$$\begin{aligned}
 I &\leq \|Dg(y)\|_{\infty, \mathcal{L}(\mathcal{H}_{-2\gamma} \otimes \mathbb{R}^d, \mathcal{H}_{-2\gamma})} |y' - v'|_{\gamma, -2\gamma} \\
 &\quad + |Dg(y)|_{\gamma, \mathcal{L}(\mathcal{H}_{-2\gamma} \otimes \mathbb{R}^d, \mathcal{H}_{-2\gamma})} \|y' - v'\|_{\infty, -2\gamma} \\
 &\leq C_g (|y' - v'|_{\gamma, -2\gamma} + |y|_{\gamma, -2\gamma} \|y' - v'\|_{\infty, -2\gamma}) \\
 &\leq C_{g, M} (1 + |w|_{\gamma}) (\|y' - v'\|_{\gamma, -2\gamma} + \|y' - v'\|_{\infty, -2\gamma} \\
 &\quad + T^\gamma \|y' - v'\|_{\infty, 0}),
 \end{aligned}$$

meanwhile, we have

$$\begin{aligned}
 II &\leq C_{g, M, T} (1 + |w|_{\gamma}) (\|y - v\|_{\gamma, -2\gamma} + \|y - v\|_{\infty, -2\gamma} + T^\gamma \|y - v\|_{\infty, 0}), \\
 \|Dg(y)y' - Dg(v)v'\|_{\infty, 0} &\leq \|Dg(y)(y' - v')\|_{\infty, 0} + \|(Dg(y) - Dg(v))v'\|_{\infty, 0} \\
 &\leq C_g (\|y' - v'\|_{\infty, 0} + \|y - v\|_{\infty, 0} \|v'\|_{\infty, 0}) \\
 &\leq C_{g, M} (\|y' - v'\|_{\infty, 0} + \|y - v\|_{\infty, 0})
 \end{aligned}$$

and

$$\|y - v\|_{\infty, 0} \lesssim T^\eta \|y - v\|_{\eta, 0} + \|y_0 - v_0\|_{\mathcal{H}},$$

according to above estimates, we obtain

$$\|Dg(y)y' - Dg(v)v'\|_{\gamma, -2\gamma} \leq C_{g, M, T} (1 + |w|_{\gamma}) \|y - v, (y - v)'\|_{\mathcal{D}_w^{2\gamma, \eta}}.$$

Since

$$\begin{aligned}
 R_{t,s}^{g(y)} &= g(y_t) - g(y_s) - Dg(y_s)S_{ts}y'_s\delta w_{t,s} + Dg(y_s)S_{ts}y'_s\delta w_{t,s} - Dg(y_s)y'_s\delta w_{t,s} \\
 &\quad + Dg(y_s)y'_s\delta w_{t,s} - S_{ts}Dg(y_s)y'_s\delta w_{t,s} + g(y_s) - S_{ts}g(y_s) \\
 &= g(y_t) - g(y_s) - Dg(y_s)\hat{\delta}y_{t,s} + Dg(y_s)R_{t,s}^y + Dg(y_s)(S_{ts} - I)y'_s\delta w_{t,s} \\
 &\quad - (S_{ts} - I)Dg(y_s)y'_s\delta w_{t,s} - (S_{ts} - I)g(y_s) \\
 &= g(y_t) - g(y_s) - Dg(y_s)\delta y_{t,s} + Dg(y_s)R_{t,s}^y + Dg(y_s)(S_{ts} - I)y'_s\delta w_{t,s} \\
 &\quad - (S_{ts} - I)Dg(y_s)y'_s\delta w_{t,s} - (S_{ts} - I)g(y_s) + Dg(y_s)(S_{ts} - I)y_s,
 \end{aligned}$$

hence, we have

$$\begin{aligned}
 R_{t,s}^{g(y)} - R_{t,s}^{g(v)} &= g(y_t) - g(y_s) - Dg(y_s)\delta y_{t,s} - (g(v_t) - g(v_s) - Dg(v_s)\delta v_{t,s}) \\
 &\quad + Dg(y_s)R_{t,s}^y - Dg(v_s)R_{t,s}^v \\
 &\quad - (S_{ts} - I)(g(y_s) - g(v_s)) \\
 &\quad + Dg(y_s)(S_{ts} - I)y'_s\delta w_{t,s} - Dg(v_s)(S_{ts} - I)v'_s\delta w_{t,s} \\
 &\quad - (S_{ts} - I)(Dg(y_s)y'_s\delta w_{t,s} - Dg(v_s)v'_s\delta w_{t,s}) \\
 &\quad + Dg(y_s)(S_{ts} - I)y_s - Dg(v_s)(S_{ts} - I)v_s \\
 &= i + ii + iii + iv + v + vi.
 \end{aligned} \tag{2.13}$$

For  $i$ , applying (44) of [24], we have

$$\begin{aligned} \|i\|_{\mathcal{H}_{-2\gamma}^d} &\leq \left\| \int_0^1 \int_0^1 C_g[\tau r^2(y_t - v_t) + (r - \tau r^2)(y_s - v_s)]d\tau dr \delta y_{t,s} \otimes \delta y_{t,s} \right\|_{\mathcal{H}_{-2\gamma}} \\ &\quad + \left\| \int_0^1 \int_0^1 C_g[\tau r^2 v_t + (r - \tau r^2)v_s]d\tau dr (\delta y_{t,s} \otimes \delta y_{t,s} - \delta v_{t,s} \otimes \delta v_{t,s}) \right\|_{\mathcal{H}_{-2\gamma}} \\ &\leq C_g \|y - v\|_{\infty, -2\gamma} \|\hat{\delta} y_{t,s}\| + (S_{ts} - I)y_s \|\mathcal{H}_{-2\gamma}^2\| \\ &\quad + C_g \|v\|_{\infty, -2\gamma} (\|\hat{\delta} y_{t,s} + (S_{ts} - I)y_s\|_{\mathcal{H}_{-2\gamma}} + \|\hat{\delta} v_{t,s} + (S_{ts} - I)v_s\|_{\mathcal{H}_{-2\gamma}}) \\ &\quad \cdot \|\hat{\delta}(y - v)_{t,s} + (S_{ts} - I)(y - v)_s\|_{\mathcal{H}_{-2\gamma}} \\ &\leq C_g \|y - v\|_{\infty, -2\gamma} (\|y\|_{\gamma, -2\gamma} |t - s|^\gamma + \|y\|_{\infty, 0} |t - s|^{2\gamma})^2 \\ &\quad + C_g \|v\|_{\infty, -2\gamma} (\|y\|_{\gamma, -2\gamma} |t - s|^\gamma + \|y\|_{\infty, 0} |t - s|^{2\gamma} + \|v\|_{\gamma, -2\gamma} |t - s|^\gamma \\ &\quad + \|v\|_{\infty, 0} |t - s|^{2\gamma}) \cdot (\|y - v\|_{\gamma, -2\gamma} |t - s|^\gamma + \|y - v\|_{\infty, 0} |t - s|^{2\gamma}). \end{aligned}$$

For  $ii$ , we have

$$\begin{aligned} \|ii\|_{\mathcal{H}_{-2\gamma}^d} &= \|Dg(y_s)R_{t,s}^y - Dg(y_s)R_{t,s}^v + Dg(y_s)R_{t,s}^v - Dg(v_s)R_{t,s}^v\|_{\mathcal{H}_{-2\gamma}^d} \\ &\leq \|Dg(y_s)(R_{t,s}^y - R_{t,s}^v)\|_{\mathcal{H}_{-2\gamma}^d} + \|(Dg(y_s) - Dg(v_s))R_{t,s}^v\|_{\mathcal{H}_{-2\gamma}^d} \\ &\leq C_g |R^y - R^v|_{2\gamma, -2\gamma} |t - s|^{2\gamma} + C_g \|y - v\|_{\infty, -2\gamma} |R^v|_{2\gamma, -2\gamma} |t - s|^{2\gamma}. \end{aligned}$$

For  $iii$ , we easily have

$$\|iii\|_{\mathcal{H}_{-2\gamma}^d} \leq C_g \|y - v\|_{\infty, 0} |t - s|^{2\gamma}.$$

For  $iv$ , we have

$$\begin{aligned} \|iv\|_{\mathcal{H}_{-2\gamma}^d} &\leq \|(Dg(y_s) - Dg(v_s))(S_{ts} - I)y'_s \delta w_{t,s}\|_{\mathcal{H}_{-2\gamma}^d} \\ &\quad + \|Dg(v_s)(S_{ts} - I)(y'_s - v'_s) \delta w_{t,s}\|_{\mathcal{H}_{-2\gamma}^d} \\ &\leq C_g \|y - v\|_{\infty, -2\gamma} \|y'\|_{\infty, 0} |w|_\gamma |t - s|^{2\gamma} + C_g \|y' - v'\|_{\infty, 0} |w|_\gamma |t - s|^{2\gamma}. \end{aligned}$$

For  $v$  and  $vi$ , similar to  $iv$ , we obtain

$$\begin{aligned} \|v\|_{\mathcal{H}_{-2\gamma}^d} &\leq C_{g,M} |w|_\gamma (\|y - v\|_{\infty, 0} + \|y' - v'\|_{\infty, 0}) |t - s|^{3\gamma}, \\ \|vi\|_{\mathcal{H}_{-2\gamma}^d} &\leq C_{g,M} (\|y - v\|_{\infty, 0} + \|y - v\|_{\infty, -2\gamma}) |t - s|^{2\gamma}. \end{aligned}$$

Consequently, we easily obtain

$$|R^{g(y)} - R^{g(v)}|_{2\gamma, -2\gamma} \leq C_{g,M,T} (1 + |w|_\gamma)^2 \|y - v, (y - v)'\|_{\mathcal{D}_w^{2\gamma, \eta}}.$$

Finally, according to previous estimates and the norm of  $\mathcal{D}_{S,w}^{2\gamma, 2\gamma, 0}([0, T]; \mathcal{H})$ , our result can be easily derived. □

By substituting (2.11) into (2.9), we easily obtain the following result.

**Lemma 2.4** *Let  $T > 0$ ,  $g \in C^3_{-2\gamma,0}(\mathcal{H}, \mathcal{H}^d)$ ,  $(y, y')$  and  $(v, v') \in \mathcal{D}^{2\gamma,\eta}_w([0, T]; \mathcal{H})$ , for some  $w \in \mathcal{C}^\gamma([0, T]; \mathbb{R}^d)$  and there exists  $M > 0$  such that  $|w|, |w^2|, \|y, y'\|_{\mathcal{D}^{2\gamma,\eta}_w}$  and  $\|v, v'\|_{\mathcal{D}^{2\gamma,\eta}_w} \leq M$ , then, there exists a constant  $C$  such that*

$$\begin{aligned} & \left\| \int_0^\cdot S_{\cdot u}(g(y_u) - g(v_u))d\mathbf{w}_u, g(y) - g(v) \right\|_{\mathcal{D}^{2\gamma,\eta}_w} \\ & \leq C_{g,M,T}(1 + |w|_\gamma + |w^2|_{2\gamma})(1 + |w|_\gamma)^2 \|y - v, (y - v)'\|_{\mathcal{D}^{2\gamma,\eta}_w}. \end{aligned} \tag{2.14}$$

The constant  $C_{M,g,T}$  depends on  $M$ ,  $g$  and the bounds of its derivatives, at the same time, it depends on time  $T$ , but is consistent with time  $T \in (0, 1]$ .

However, in  $\mathcal{D}^{2\gamma,\eta}_w([0, T]; \mathcal{H})$ , we also need to estimate the terms containing the initial condition and the drift of rough evolution equation (2.2). Hence, we will focus on this in the following Lemma 2.5.

**Lemma 2.5** *Let  $T > 0$ ,  $\xi \in \mathcal{H}$ ,  $f \in C_{-2\gamma,0}(\mathcal{H}, \mathcal{H})$  be global Lipschitz continuous, and  $(y, y') \in \mathcal{D}^{2\gamma,\eta}_w([0, T]; \mathcal{H})$ , we have that the mildly Gubinelli derivative*

$$\left( S.\xi + \int_0^\cdot S_{\cdot u} f(y_u) du \right)' = 0, \tag{2.15}$$

also have the estimate

$$\left\| S.\xi + \int_0^\cdot S_{\cdot u} f(y_u) du, 0 \right\|_{\mathcal{D}^{2\gamma,\eta}_w} \leq C_{\gamma,T}(\|\xi\| + \|f(y)\|_{\infty,-2\gamma} + \|f(y)\|_{\infty,0}). \tag{2.16}$$

Moreover, for two mildly controlled rough paths  $(y, y')$  and  $(v, v')$  with  $y_0 = \xi$  and  $v_0 = \tilde{\xi}$ , we have

$$\begin{aligned} & \left\| S.(\xi - \tilde{\xi}) + \int_0^\cdot S_{\cdot u}(f(y_u) - f(v_u))du, 0 \right\|_{\mathcal{D}^{2\gamma,\eta}_w} \\ & \leq C_{\gamma,T}(\|\xi - \tilde{\xi}\| + \|f(y) - f(v)\|_{\infty,-2\gamma} + \|f(y) - f(v)\|_{\infty,0}). \end{aligned} \tag{2.17}$$

**Proof** Let  $0 < T \leq 1$ . Since

$$\begin{aligned} & \|S_t \xi - S_{ts} S_s \xi\|_{\mathcal{H}_{-2\gamma}} = 0, \\ & \|S_t \xi - S_{ts} S_s \xi\|_{\mathcal{H}} = 0, \\ & \|S_0 \xi\|_{\mathcal{H}} \lesssim \|\xi\|_{\mathcal{H}}, \end{aligned}$$

hence we have

$$\begin{aligned} & (S.\xi)' = 0, \\ & |R^{S.\xi}|_{2\gamma,-2\gamma} = 0, \\ & \|S.\xi, 0\|_{\mathcal{D}^{2\gamma,\eta}_w} \leq C \|\xi\|. \end{aligned} \tag{2.18}$$

Meanwhile, due to

$$\begin{aligned} & \left\| \int_0^t S_{tu} f(y_u) du - S_{ts} \int_0^s S_{su} f(y_u) du \right\|_{\mathcal{H}_{-2\gamma}} \\ &= \left\| \int_s^t S_{tu} f(y_u) du \right\|_{\mathcal{H}_{-2\gamma}} \\ &\leq \int_s^t \|f(y_u)\|_{\mathcal{H}_{-2\gamma}} du \\ &= \|f(y)\|_{\infty, -2\gamma}(t - s), \\ & \left\| \int_0^0 S_{0u} f(y_u) du \right\|_{\mathcal{H}_{-2\gamma}} = 0, \\ & \left\| \int_s^t S_{tu} f(y_u) du \right\|_{\mathcal{H}} \leq \int_s^t \|f(y_u)\|_{\mathcal{H}} du \leq (t - s) \|f(y)\|_{\infty, 0}, \end{aligned}$$

thus we have

$$\begin{aligned} & \left( \int_0^t S_{tu} f(y_u) du \right)' = 0, \\ & \left\| \int_0^\cdot S_{\cdot u} f(y_u) du \right\|_{\eta, 0} \leq \|f(y)\|_{\infty, 0} |t - s|^{1-\eta}, \\ & |Rf_0^{\int_0^\cdot S_{\cdot u} f(y_u) du}|_{2\gamma, -2\gamma} \leq \|f(y)\|_{\infty, -2\gamma} (t - s)^{1-2\gamma}, \\ & \left\| \int_0^\cdot S_{\cdot u} f(y_u) du, 0 \right\|_{\mathcal{D}_w^{2\gamma, \eta}} \leq C_\gamma (T^{1-2\gamma} \|f(y)\|_{\infty, -2\gamma} + T^{1-\eta} \|f(y)\|_{\infty, 0}). \end{aligned} \tag{2.19}$$

Finally, (2.16) is proved, consequently, (2.17) can be easily obtained. □

In  $\mathcal{D}_w^{2\gamma, \eta}([0, T]; \mathcal{H})$ , because of above preliminary results, similar to Theorem 4.1 of [16] one can then easily derive a local solution for (2.2) by a fixed-point argument, i.e.:

**Theorem 2.2** *Let  $T > 0$ , given  $\xi \in \mathcal{H}$  and  $\mathbf{w} = (w, w^2) \in \mathcal{C}^\gamma([0, T]; \mathbb{R}^d)$  with  $\gamma \in (\frac{1}{3}, \frac{1}{2}]$ . Then there exists  $0 < T_0 \leq T$  such that the rough evolution equation (2.2) has a unique local solution represented by a mildly controlled rough path  $(y, y') \in \mathcal{D}_w^{2\gamma, \eta}([0, T_0]; \mathcal{H})$  with  $y' = g(y)$ , for all  $0 \leq t \leq T_0$*

$$y_t = S_t \xi + \int_0^t S_{tu} f(y_u) du + \int_0^t S_{tu} g(y_u) d\mathbf{w}_u. \tag{2.20}$$

**Proof** Let  $0 < T \leq 1$ ,

$$\mathcal{M}(y, y')_t = \left( S_t \xi + \int_0^t S_{tu} f(y_u) du + \int_0^t S_{tu} g(y_u) d\mathbf{w}_u, g(y_t) \right).$$

It is easy to obtain that if  $(y_0, y'_0) = (\xi, g(\xi))$ , then the same is true for  $\mathcal{M}(y, y')$ . Thus we can regard  $\mathcal{M}_T$  as a mapping on the complete metric space:

$$\{(y, y') \in \mathcal{D}_w^{2\gamma, \eta}([0, T]; \mathcal{H}) : y_0 = \xi, y'_0 = g(\xi)\}.$$

Meanwhile, since

$$\|S.\xi + S.g(\xi)\delta w_{\cdot,0}, S.g(\xi)\|_{w,2\gamma,-2\gamma} = 0,$$

hence we easily have that this is also true for the closed ball  $B_T(w, r)$  centred at  $t \rightarrow (S_t\xi + S_tg(\xi)\delta w_{t,0}, S_tg(\xi)) \in \mathcal{D}_w^{2\gamma, \eta}([0, T]; \mathcal{H})$ , i.e.

$$B_T(w, r) = \{(y, y') \in \mathcal{D}_w^{2\gamma, \eta}([0, T]; \mathcal{H}) : y_0 = \xi, y'_0 = g(\xi), \|y - (S.\xi + S.g(\xi)\delta w_{\cdot,0})\|_{\eta,0} + \|y' - S.g(\xi)\|_{\infty,0} + \|y - (S.\xi + S.g(\xi)\delta w_{\cdot,0}), y' - S.g(\xi)\|_{w,2\gamma,-2\gamma} \leq r\}.$$

Since, by triangle inequality, for  $(y, y') \in B_T(w, r)$  we have

$$\begin{aligned} \|S.\xi + S.g(\xi)\delta w_{\cdot,0}, S.g(\xi)\|_{w,2\gamma,2\gamma,\eta} &\leq \|g(\xi)\|_{\mathcal{H}^d} + T^{\gamma-\eta}\|g(\xi)\|_{\mathcal{H}^d}|w|_{\gamma}, \\ \|y, y'\|_{w,2\gamma,2\gamma,\eta} &\leq r + \|g(\xi)\|_{\mathcal{H}^d} + \|g(\xi)\|_{\mathcal{H}^d}|w|_{\gamma}. \end{aligned}$$

Then, one obtains

$$\begin{aligned} &\|\mathcal{M}(y) - (S.\xi + S.g(\xi)\delta w_{\cdot,0}), g(y) - S.g(\xi)\|_{w,2\gamma,2\gamma,\eta} \\ &\leq \|\mathcal{M}(y), g(y)\|_{w,2\gamma,2\gamma,\eta} + \|S.\xi + S.g(\xi)\delta w_{\cdot,0}, S.g(\xi)\|_{w,2\gamma,2\gamma,\eta} \\ &\leq T^{1-2\gamma}(\|f(y)\|_{\infty,-2\gamma} + \|f(y)\|_{\infty,0}) + T^{\gamma-\eta}\|g(y)\|_{\infty,0}|w|_{\gamma} + \|g(y)\|_{\infty,0} \\ &\quad + T^{\gamma-\eta}(|w|_{\gamma} + |w^2|_{2\gamma})\|g(y), (g(y))'\|_{w,2\gamma,-2\gamma} \\ &\quad + (1 + |w|_{\gamma} + |w^2|_{2\gamma})\|(g(y_0))'\|_{\mathcal{H}^{d \times d}_{-2\gamma}} \\ &\quad + T^{2\gamma-\eta}\|(g(y))'\|_{\infty,0}|w^2|_{2\gamma} + T^{\gamma}(1 + |w|_{\gamma} + |w^2|_{2\gamma})\|g(y), (g(y))'\|_{w,2\gamma,-2\gamma} \\ &\quad + T^{\gamma}|w^2|_{2\gamma}\|(g(y))'\|_{\gamma,-2\gamma} + \|g(\xi)\|_{\mathcal{H}^d} + T^{\gamma-\eta}\|g(\xi)\|_{\mathcal{H}^d}|w|_{\gamma}, \end{aligned}$$

since  $g \in \mathcal{C}^3_{-2\gamma,0}(\mathcal{H}, \mathcal{H}^d)$  and  $\|g(0)\|_{\mathcal{H}^d_\theta}$  for  $\theta \geq -2\gamma$ , by mean value theorem we easily obtain  $\|g(y)\|_{\mathcal{H}^d_\theta} \leq 1 + \|y\|_{\mathcal{H}_\theta}$ , consequently, according to (2.8) and above estimates we easily have

$$\begin{aligned} &\|\mathcal{M}(y) - (S.\xi + S.g(\xi)\delta w_{\cdot,0}), g(y) - S.g(\xi)\|_{w,2\gamma,2\gamma,\eta} \\ &\leq C_{L_f,g,\varrho}(w,0) + T^{\gamma-\eta}C_{g,\varrho}(w,0), \|\xi\|, \|y, y'\|_{w,2\gamma,2\gamma,\eta}. \end{aligned}$$

Let  $r = 2C_{L_f,g,\varrho}(w,0)$ , then for  $\forall (y, y') \in B_T(w, r)$ , we have

$$\|\mathcal{M}(y) - (S.\xi + S.g(\xi)\delta w_{\cdot,0}), g(y) - S.g(\xi)\|_{w,2\gamma,2\gamma,\eta} \leq \frac{r}{2} + T^{\gamma-\eta}C_{g,\varrho}(w,0), \|\xi\|, r.$$

By letting  $T = T_1$  to be sufficient small such that

$$T_1^{\gamma-\eta}C_{g,\varrho}(w,0), \|\xi\|, r < \frac{r}{2},$$

then one obtains

$$\mathcal{M}(B_{T_1}(w, r)) \subseteq B_{T_1}(w, r).$$

For the sake of proving contractivity of  $\mathcal{M}_T$ , one can use steps that are similar to the previous steps to show

$$\|\mathcal{M}(y) - \mathcal{M}(v), g(y) - g(v)\|_{w, 2\gamma, 2\gamma, \eta} \leq T_2^{\eta \wedge (\gamma - \eta)} C_{g, \rho(w, 0), \|\xi\|, r} \|y - v, y' - v'\|_{w, 2\gamma, 2\gamma, \eta}.$$

This ensures contractivity when  $T_2$  is sufficient small. Let  $T_0 = \min\{T_1, T_2\}$ , by the Banach fixed point theorem, one has that there is a unique  $(y, y') \in B_{T_0}(w, r)$  satisfies  $\mathcal{M}(y) = y$ , i.e. a solution of REE (2.2) on the small time interval  $[0, T_0]$ .  $\square$

**Remark 2.4** Our proof of Theorem 2.2 is simpler than the one of Theorem 4.1 in [16]. This is also the key that we choose to study (2.2) in the space  $\mathcal{D}_w^{2\gamma, \eta}([0, T]; \mathcal{H})$ . Here, we directly view  $\mathcal{M}_T$  as mapping from the space  $\mathcal{D}_w^{2\gamma, \eta}([0, T]; \mathcal{H})$  into itself, however, in [16], the technique is to take  $\varepsilon \in (1/3, \gamma)$  and view  $\mathcal{M}_T$  as map from  $\mathcal{D}_{S, w}^{2\gamma, 2\gamma, \varepsilon}([0, T]; \mathcal{H}_{2\varepsilon - 2\gamma})$ , rather than  $\mathcal{D}_{S, w}^{2\gamma, 2\gamma, \gamma}([0, T]; \mathcal{H})$ .

### 2.1 Global in time solution of rough evolution equation

As we all known, the global in time solution is the key that allows one to consider the longtime behaviour of rough evolution equation (2.2), so in this subsection we will focus on this issue. Similar to [20] and [21], we will derive the following result which is fundamental importance for the discussion of global in time solution for (2.2). According to (2.8), (2.10), (2.18) and (2.19), we obtain that:

**Corollary 2.1** *Let  $(y, g(y)) \in \mathcal{D}_w^{2\gamma, \eta}([0, T]; \mathcal{H})$  with  $0 < T \leq 1$  be the solution of (2.2) with the initial condition  $y_0 = \xi \in \mathcal{H}$ . Then one has the following estimate*

$$\|y, g(y)\|_{\mathcal{D}_w^{2\gamma, \eta}} \lesssim 1 + \|\xi\| + T^{\gamma - \eta} \|y, g(y)\|_{\mathcal{D}_w^{2\gamma, \eta}}. \tag{2.21}$$

**Proof** Since  $(y, g(y))$  is the solution of (2.2) we have

$$\begin{aligned} \|y, g(y)\|_{\mathcal{D}_w^{2\gamma, \eta}} &\leq \|S\xi, 0\|_{\mathcal{D}_w^{2\gamma, \eta}} + \left\| \int_0^\cdot S_u f(y_u) du, 0 \right\|_{\mathcal{D}_w^{2\gamma, \eta}} \\ &\quad + \left\| \int_0^\cdot S_u g(y_u) d\mathbf{w}_u, g(y) \right\|_{\mathcal{D}_w^{2\gamma, \eta}}. \end{aligned}$$

According to (2.10), (2.18) and (2.19), we obtain

$$\begin{aligned} &\|y, g(y)\|_{\mathcal{D}_w^{2\gamma, \eta}} \\ &\leq \|\xi\| + T^{1 - 2\gamma} (\|f(y)\|_{\infty, -2\gamma} + \|f(y)\|_{\infty, 0}) + \|g(y_0)\|_{\mathcal{H}_{-2\gamma}^d} + \|(g(y_0))'\|_{\mathcal{H}_{-2\gamma}^{d \times d}} \\ &\quad + \|g(y)\|_{\infty, 0} + T^{2\gamma - \eta} \|(g(y))'\|_{\infty, 0} + T^{\gamma - \eta} \|g(y), (g(y))'\|_{w, 2\gamma, -2\gamma}. \end{aligned}$$

Meanwhile, from (2.8) we have

$$\|g(y), (g(y))'\|_{w,2\gamma,-2\gamma} \lesssim 1 + \|y, g(y)\|_{\mathcal{D}_w^{2\gamma,\eta}}.$$

Combining  $\|g(y)\|_{\mathcal{H}_\theta^d} \leq 1 + \|y\|_{\mathcal{H}_\theta}$  and the bounds of its derivatives with previous estimates, we obtain

$$\begin{aligned} \|y, g(y)\|_{\mathcal{D}_w^{2\gamma,\eta}} &\lesssim 1 + \|\xi\| + T^{1-2\gamma}(1 + \|y\|_{\infty,-2\gamma} + \|y\|_{\infty,0}) \\ &\quad + T^{2\gamma-\eta}\|(g(y))'\|_{\infty,0} + T^{\gamma-\eta}\|g(y), (g(y))'\|_{w,2\gamma,-2\gamma} \\ &\lesssim 1 + \|\xi\| + T^{2\gamma-\eta}(1 + \|y\|_{\infty,-2\gamma} + \|y\|_{\infty,0} + \|g(y)\|_{\infty,0}) \\ &\quad + T^{\gamma-\eta}\|y, g(y)\|_{\mathcal{D}_w^{2\gamma,\eta}} \\ &\lesssim 1 + \|\xi\| + T^{\gamma-\eta}\|y, g(y)\|_{\mathcal{D}_w^{2\gamma,\eta}}. \end{aligned}$$

Finally we obtain the desired result. □

Applying a concatenation discussion of [20] and [21], according to (2.21) we obtain an a-priori bound for the solution of (2.2). The technique of proof is identical to the one of [20] Lemma 5.8, we omit here.

**Lemma 2.6** *Let  $T > 0$ ,  $(y, g(y)) \in \mathcal{D}_w^{2\gamma,\eta}([0, T]; \mathcal{H})$  be the solution of (2.2), where the initial condition  $y_0 = \xi \in \mathcal{H}$  with  $\|\xi\| \leq \rho$ . Let  $\tilde{r} = 1 \vee \rho$ , then there exists constant  $M$  such that*

$$\|y\|_{\infty,0,[0,T]} \leq M\tilde{r}e^{MT}.$$

Lemma 2.6 ensures that the solution of (2.2) does not explode in any finite time, therefore, in  $\mathcal{D}_w^{2\gamma,\eta}([0, T]; \mathcal{H})$ , according to above preliminary results, based on Theorem 2.2, we have the following result that the local solution of (2.2) can be extended to global one by a standard concatenation discussion, the details of proof one can refer to [20] Theorem 5.10 and [21] Theorem 3.9, we omit here.

**Theorem 2.3** *Let  $T > 0$ , given  $\xi \in \mathcal{H}$  and  $\mathbf{w} = (w, w^2) \in \mathcal{C}^\gamma([0, T]; \mathbb{R}^d)$ . The rough evolution equation (2.2) has a unique global solution represented by a mildly controlled rough path  $(y, y') \in \mathcal{D}_w^{2\gamma,\eta}([0, T]; \mathcal{H})$  given by*

$$(y, y') = \left( S_\cdot \xi + \int_0^\cdot S_\cdot f(y_u) du + \int_0^\cdot S_\cdot g(y_u) d\mathbf{w}_\cdot, g(y) \right). \tag{2.22}$$

**Remark 2.5** We emphasis the fact that the solution of (2.2) is global in time. Rough paths and rough drivers are usually defined on compact intervals, according to [3] and [21], we say  $\mathbf{w} = (w, w^2) \in \mathcal{C}^\gamma(\mathbb{R}; \mathbb{R}^d)$  is a  $\gamma$ -Hölder rough path if  $\mathbf{w}|_I \in \mathcal{C}^\gamma(I; \mathbb{R}^d)$  for every compact interval  $I \subseteq \mathbb{R}$  containing 0. Hence, in our setting, we have that  $(y, y') \in \mathcal{D}_w^{2\gamma,\eta}([0, \infty); \mathcal{H})$  if  $(y, y') \in \mathcal{D}_w^{2\gamma,\eta}([0, T]; \mathcal{H})$  for every  $T > 0$ . Therefore, we set  $C_g|_{[0,\infty)} = \max_{I \subseteq [0,\infty)} C_g|_I$ ,  $\varrho(w, 0)|_{[0,\infty)} = \max_{I \subseteq [0,\infty)} \varrho(w, 0)|_I$ , according to



Theorem 2.2, letting  $r = 2C_{L_{f,g,q}(w,0)}|_{[0,\infty)}$ , and previous deliberations we have that  $r$  keeps invariant in concatenation arguments and one can obtain a unique solution of (2.2) in  $\mathcal{D}_w^{2\gamma,\eta}([0, \infty); \mathcal{H})$ .

### 2.2 Truncated rough evolution equation

We will prove a global unstable manifold for a modified equation of (2.2) by using cut-off function over a random neighborhood in the following Sect. 4. Hence we will construct a local unstable manifold depended on the size of perturbations and the spectral gap of the linear part of (2.2). In order to consider the existence of local invariant manifolds by using Lyapunov–Perron method, in this subsection, we modify these nonlinear  $f$  and  $g$  by applying appropriate cut-off technique to make their Lipschitz constants small enough. Since in contrast to the classical cut-off techniques (as in [6, 8, 9] and so on), in our case, similar to [22] and [24], we truncate the norm of mildly controlled rough path  $(y, y')$ . Due to the technical reasons of Lyapunov–Perron method, which we will use in Sect. 4, we fix the time interval as  $[0, 1]$  in this subsection.

Meanwhile, we assume the following restrictions on the drift and diffusion coefficients:

- $f \in \mathcal{C}_{-2\gamma,0}^1(\mathcal{H}, \mathcal{H})$  is global Lipschitz continuous with  $f(0) = Df(0) = 0$ ;
- $g \in \mathcal{C}_{-2\gamma,0}^3(\mathcal{H}, \mathcal{H}^d)$  with  $g(0) = Dg(0) = D^2g(0) = 0$ ,

so one easily obtains that  $(y = 0, y' = 0)$  is a stationary solution of (2.2).

Let  $\chi : \mathcal{D}_w^{2\gamma,\eta}([0, 1]; \mathcal{H}) \rightarrow \mathcal{D}_w^{2\gamma,\eta}([0, 1]; \mathcal{H})$  be a Lipschitz continuous cut-off function:

$$\chi(y) := \begin{cases} y, & \|y, y'\|_{\mathcal{D}_w^{2\gamma,\eta}} \leq \frac{1}{2}, \\ 0, & \|y, y'\|_{\mathcal{D}_w^{2\gamma,\eta}} \geq 1. \end{cases}$$

As examples in subsection 2.1 of [24], we can take  $\varphi : \mathbb{R}^+ \rightarrow [0, 1]$  is a  $C_b^3$  Lipschitz cut-off function, then  $\chi(y)$  can be constructed as

$$\chi(y) = y\varphi(\|y, y'\|_{\mathcal{D}_w^{2\gamma,\eta}}).$$

In the following, we assume that  $\chi$  is constructed by  $\varphi$ . According to Definition 2.2, one has

$$\chi'(y) = y'\varphi(\|y, y'\|_{\mathcal{D}_w^{2\gamma,\eta}}),$$

this construction indicates that

$$(\chi(y), \chi'(y)) := \begin{cases} (y, y'), & \|y, y'\|_{\mathcal{D}_w^{2\gamma,\eta}} \leq \frac{1}{2}, \\ 0, & \|y, y'\|_{\mathcal{D}_w^{2\gamma,\eta}} \geq 1. \end{cases}$$

For a positive number  $R$ , we define

$$\chi_R(y) = R\chi(y/R),$$

this means that

$$\chi_R(y) := \begin{cases} y, & \|y, y'\|_{\mathcal{D}_w^{2\gamma,\eta}} \leq \frac{R}{2}, \\ 0, & \|y, y'\|_{\mathcal{D}_w^{2\gamma,\eta}} \geq R, \end{cases}$$

then

$$(\chi_R(y), \chi'_R(y)) := \begin{cases} (y, y'), & \|y, y'\|_{\mathcal{D}_w^{2\gamma,\eta}} \leq \frac{R}{2}, \\ 0, & \|y, y'\|_{\mathcal{D}_w^{2\gamma,\eta}} \geq R. \end{cases}$$

For a mildly controlled rough path  $(y, y') \in \mathcal{D}_w^{2\gamma,\eta}([0, 1]; \mathcal{H})$ , we introduce the operators

$$f_R(y_t) := f \circ \chi_R(y_t), \quad g_R(y_t) := g \circ \chi_R(y_t).$$

Based on Lemma 2.1, we obtain the mildly Gubinelli derivative of  $g_R(y)$ :

$$(g_R(y))' = Dg(\chi_R(y))\chi'_R(y) = Dg(y\varphi(\|y, y'\|_{\mathcal{D}_w^{2\gamma,\eta}/R}))y'\varphi(\|y, y'\|_{\mathcal{D}_w^{2\gamma,\eta}/R}).$$

It is directly obtained that if  $\|y, y'\|_{\mathcal{D}_w^{2\gamma,\eta}} \leq R/2$ , we have that  $f_R(y) = f(y)$  and  $g_R(y) = g(y)$ .

Next, we will discuss the Lipschitz continuity of  $f_R$  and  $g_R$ , and the Lipschitz constants are supposed to be strictly increasing in  $R$ .

**Lemma 2.7** *Let  $(y, y')$  and  $(v, v') \in \mathcal{D}_w^{2\gamma,\eta}([0, 1]; \mathcal{H})$ , then there exists a constant  $C = C_{f,\chi,|w|_\gamma}$  such that*

$$\|f_R(y) - f_R(v)\|_{\infty,0} + \|f_R(y) - f_R(v)\|_{\infty,-2\gamma} \leq CR\|y - v, y' - v'\|_{\mathcal{D}_w^{2\gamma,\eta}}. \tag{2.23}$$

**Proof** We easily have

$$\sup_{t \in [0,1]} \|f_R(y_t) - f_R(v_t)\|_{\mathcal{H}_{-2\gamma}} = \sup_{t \in [0,1]} \|f(\chi_R(y_t)) - f(\chi_R(v_t))\|_{\mathcal{H}_{-2\gamma}}.$$

Firstly, since  $f \in \mathcal{C}^1_{-2\gamma,0}(\mathcal{H}, \mathcal{H})$  is global Lipschitz continuous and  $Df(0) = 0$ , thus we have

$$\begin{aligned} & \|f(\chi_R(y_t)) - f(\chi_R(v_t))\|_{\mathcal{H}_{-2\gamma}} \\ & \leq \int_0^1 \|Df(r\chi_R(y_t) + (1-r)\chi_R(v_t))\|_{\mathcal{L}(\mathcal{H}_{-2\gamma}, \mathcal{H}_{-2\gamma})} dr \|\chi_R(y_t) - \chi_R(v_t)\|_{\mathcal{H}_{-2\gamma}} \end{aligned}$$

$$\leq C_f \max \{ \|\chi_R(y_t)\|_{\mathcal{H}_{-2\gamma}}, \|\chi_R(v_t)\|_{\mathcal{H}_{-2\gamma}} \} \|\chi_R(y_t) - \chi_R(v_t)\|_{\mathcal{H}_{-2\gamma}}.$$

Secondly, due to

$$\|y\|_{\infty, -2\gamma} \leq \|y_0\|_{\mathcal{H}_{-2\gamma}} + \|y\|_{\gamma, -2\gamma} \leq \|y_0\|_{\mathcal{H}} + \|y\|_{\gamma, -2\gamma}$$

and

$$\begin{aligned} \|y\|_{\gamma, -2\gamma} &\leq (1 + |w|_{\gamma})(\|y'_0\|_{\mathcal{H}_{-2\gamma}^d} + \|y, y'\|_{w, 2\gamma, -2\gamma}) \\ &\leq (1 + |w|_{\gamma})\|y, y'\|_{\mathcal{D}_w^{2\gamma, \eta}}, \end{aligned}$$

hence we obtain

$$\begin{aligned} \|y\|_{\infty, -2\gamma} &\leq \|y_0\|_{\mathcal{H}} + (1 + |w|_{\gamma})(\|y'_0\|_{\mathcal{H}_{-2\gamma}^d} + \|y, y'\|_{w, 2\gamma, -2\gamma}) \\ &\leq (1 + |w|_{\gamma})(\|y_0\|_{\mathcal{H}} + \|y'_0\|_{\mathcal{H}_{-2\gamma}^d} + \|y, y'\|_{w, 2\gamma, -2\gamma}) \\ &\leq (1 + |w|_{\gamma})\|y, y'\|_{\mathcal{D}_w^{2\gamma, \eta}}. \end{aligned}$$

In addition,

$$\begin{aligned} \|\chi_R(y)\|_{\infty, -2\gamma} &= \|y\varphi(\|y, y'\|_{\mathcal{D}_w^{2\gamma, \eta}/R})\|_{\infty, -2\gamma} = \|y\|_{\infty, -2\gamma}\varphi(\|y, y'\|_{\mathcal{D}_w^{2\gamma, \eta}/R}) \\ &\leq C(1 + |w|_{\gamma})\|y, y'\|_{\mathcal{D}_w^{2\gamma, \eta}} \leq C|w|_{\gamma}R, \end{aligned} \tag{2.24}$$

and  $\varphi : \mathbb{R}^+ \rightarrow [0, 1]$  is  $C_b^3$ , then we have

$$\begin{aligned} \|\chi_R(y_t) - \chi_R(v_t)\|_{\mathcal{H}_{-2\gamma}} &= \|y_t\varphi(\|y, y'\|_{\mathcal{D}_w^{2\gamma, \eta}/R}) - v_t\varphi(\|v, v'\|_{\mathcal{D}_w^{2\gamma, \eta}/R})\|_{\mathcal{H}_{-2\gamma}} \\ &\leq \|(y_t - v_t)\varphi(\|y, y'\|_{\mathcal{D}_w^{2\gamma, \eta}/R})\|_{\mathcal{H}_{-2\gamma}} \\ &\quad + \|v_t(\varphi(\|y, y'\|_{\mathcal{D}_w^{2\gamma, \eta}/R}) - \varphi(\|v, v'\|_{\mathcal{D}_w^{2\gamma, \eta}/R}))\|_{\mathcal{H}_{-2\gamma}} \\ &\leq \|y - v\|_{\infty, -2\gamma} \\ &\quad + \|v\|_{\infty, -2\gamma}\|D\varphi\|_{\infty}(\|y, y'\|_{\mathcal{D}_w^{2\gamma, \eta}/R} - \|v, v'\|_{\mathcal{D}_w^{2\gamma, \eta}/R}) \\ &\leq C_{\chi, |w|_{\gamma}}\|y - v, (y - v)'\|_{\mathcal{D}_w^{2\gamma, \eta}}. \end{aligned}$$

Consequently, we have

$$\|f(\chi_R(y)) - f(\chi_R(v))\|_{\infty, -2\gamma} \leq C_{\chi, |w|_{\gamma}, f}R\|y - v, (y - v)'\|_{\mathcal{D}_w^{2\gamma, \eta}}.$$

Similarly, we have

$$\|f(\chi_R(y)) - f(\chi_R(v))\|_{\infty, 0} \leq C_{\chi, f}R\|y - v, (y - v)'\|_{\mathcal{D}_w^{2\gamma, \eta}}.$$

Finally, according to above estimates, we easily obtain the desired result. □

**Lemma 2.8** *Let  $(y, y')$  and  $(v, v') \in \mathcal{D}_w^{2\gamma, \eta}([0, 1]; \mathcal{H})$ , then there exists a constant  $C = C[g, \chi, |w|_\gamma]$  such that*

$$\|g_R(y) - g_R(v), (g_R(y) - g_R(v))'\|_{\mathcal{D}_{S,w}^{2\gamma, 0}} \leq C(R)\|y - v, (y - v)'\|_{\mathcal{D}_w^{2\gamma, \eta}}. \tag{2.25}$$

**Proof** In the beginning, we give the inequality below which will be used in the following process of proof. Let  $g \in \mathcal{C}^3_{-2\gamma, 0}(\mathcal{H}, \mathcal{H}^d)$ ,  $x_1, x_2, x_3, x_4 \in \mathcal{H}_\theta$ ,  $\theta \geq -2\gamma$ , the estimate as below holds true

$$\begin{aligned} & \|g(x_1) - g(x_2) - g(x_3) + g(x_4)\|_{\mathcal{H}_\theta^d} \\ & \leq C_g \max\{\|x_1\|_{\mathcal{H}_\theta}, \|x_2\|_{\mathcal{H}_\theta}\} \|x_1 - x_2 - x_3 + x_4\|_{\mathcal{H}_\theta} \\ & \quad + C_g (\|x_1 - x_3\|_{\mathcal{H}_\theta} + \|x_2 - x_4\|_{\mathcal{H}_\theta}) \|x_3 - x_4\|_{\mathcal{H}_\theta}. \end{aligned} \tag{2.26}$$

The key of this lemma is to estimate terms of  $\|g(\chi_R(y)) - g(\chi_R(v))\|_{\gamma, -2\gamma}$ ,  $\|(g(\chi_R(y)) - g(\chi_R(v)))'\|_{\gamma, -2\gamma}$  and  $|R^{g(\chi_R(y))} - R^{g(\chi_R(v))}|_{2\gamma, -2\gamma}$ . Based on the construction of  $\chi_R(y)$ , we easily have the following estimates

$$\begin{aligned} R_{t,s}^{\chi_R(y)} &= \hat{\delta}\chi_R(y)_{t,s} - S_{ts}\chi'_R(y)_s\delta w_{t,s} \\ &= \hat{\delta}y_{t,s}\varphi(\|y, y'\|_{\mathcal{D}_w^{2\gamma, \eta}/R}) - S_{ts}y'_s\varphi(\|y, y'\|_{\mathcal{D}_w^{2\gamma, \eta}/R})\delta w_{t,s} \\ &= R_{t,s}^y\varphi(\|y, y'\|_{\mathcal{D}_w^{2\gamma, \eta}/R}), \\ & \quad \| \chi_R(y) \|_{\gamma, -2\gamma} \leq \| y \varphi(\| y, y' \|_{\mathcal{D}_w^{2\gamma, \eta} / R}) \|_{\gamma, -2\gamma} \leq \varphi(\| y, y' \|_{\mathcal{D}_w^{2\gamma, \eta} / R}) \| y \|_{\gamma, -2\gamma} \\ & \leq (1 + |w|_\gamma) \| y, y' \|_{\mathcal{D}_w^{2\gamma, \eta}} \leq C_{|w|_\gamma} R, \\ \| \chi_R(y) \|_{\infty, 0} &= \| y \varphi(\| y, y' \|_{\mathcal{D}_w^{2\gamma, \eta} / R}) \|_{\infty, 0} = \varphi(\| y, y' \|_{\mathcal{D}_w^{2\gamma, \eta} / R}) \| y \|_{\infty, 0} \\ & \leq \| y \|_{\infty, 0} \leq \| y, y' \|_{\mathcal{D}_w^{2\gamma, \eta}} \leq R, \\ | \chi_R(y) - \chi_R(v) |_{\gamma, -2\gamma} &\leq \varphi(\| y, y' \|_{\mathcal{D}_w^{2\gamma, \eta} / R}) | y - v |_{\gamma, -2\gamma} \\ & \quad + \| v \|_{\gamma, -2\gamma} \| D\varphi \|_\infty (\| y, y' \|_{\mathcal{D}_w^{2\gamma, \eta} / R} - \| v, v' \|_{\mathcal{D}_w^{2\gamma, \eta} / R}) \\ & \leq C_{|w|_\gamma, \chi} \| y - v, (y - v)' \|_{\mathcal{D}_w^{2\gamma, \eta}}, \\ | \chi'_R(y) - \chi'_R(v) |_{\gamma, -2\gamma} &= | y' \varphi(\| y, y' \|_{\mathcal{D}_w^{2\gamma, \eta} / R}) - v' \varphi(\| v, v' \|_{\mathcal{D}_w^{2\gamma, \eta} / R}) |_{\gamma, -2\gamma} \\ & \leq \varphi(\| y, y' \|_{\mathcal{D}_w^{2\gamma, \eta} / R}) | y' - v' |_{\gamma, -2\gamma} \\ & \quad + \| v' \|_{\gamma, -2\gamma} \| D\varphi \|_\infty (\| y, y' \|_{\mathcal{D}_w^{2\gamma, \eta} / R} - \| v, v' \|_{\mathcal{D}_w^{2\gamma, \eta} / R}) \\ & \leq C_{|w|_\gamma, \chi} \| y - v, (y - v)' \|_{\mathcal{D}_w^{2\gamma, \eta}}, \\ | R^{\chi_R(y)} - R^{\chi_R(v)} |_{2\gamma, -2\gamma} &= | R^y \varphi(\| y, y' \|_{\mathcal{D}_w^{2\gamma, \eta} / R}) - R^v \varphi(\| v, v' \|_{\mathcal{D}_w^{2\gamma, \eta} / R}) |_{2\gamma, -2\gamma} \\ & \leq \varphi(\| y, y' \|_{\mathcal{D}_w^{2\gamma, \eta} / R}) | R^y - R^v |_{2\gamma, -2\gamma} \\ & \quad + | R^v |_{2\gamma, -2\gamma} \| D\varphi \|_\infty (\| y, y' \|_{\mathcal{D}_w^{2\gamma, \eta} / R} - \| v, v' \|_{\mathcal{D}_w^{2\gamma, \eta} / R}) \\ & \leq C_{|w|_\gamma, \chi} \| y - v, (y - v)' \|_{\mathcal{D}_w^{2\gamma, \eta}}. \end{aligned}$$

Firstly, applying above estimates and (2.24), we have

$$\begin{aligned} & \|g(\chi_R(y)) - g(\chi_R(v))\|_{\gamma, -2\gamma} \leq |g(\chi_R(y)) - g(\chi_R(v))|_{\gamma, -2\gamma} + \|g(\chi_R(y)) - g(\chi_R(v))\|_{\infty, 0}, \\ & |g(\chi_R(y)) - g(\chi_R(v))|_{\gamma, -2\gamma} \\ & \leq C_g \max\{\|\chi_R(y)\|_{\infty, -2\gamma}, \|\chi_R(v)\|_{\infty, -2\gamma}\} |\chi_R(y) - \chi_R(v)|_{\gamma, -2\gamma} \\ & \quad + C_g (|\chi_R(y)|_{\gamma, -2\gamma} + |\chi_R(v)|_{\gamma, -2\gamma}) \|\chi_R(y) - \chi_R(v)\|_{\infty, -2\gamma} \\ & \leq C_{g, |w|_{\gamma, \chi} R} (\|\chi_R(y) - \chi_R(v)\|_{\gamma, -2\gamma} + \|\chi_R(y) - \chi_R(v)\|_{\infty, 0}) \\ & \quad + C_g (\|\chi_R(y)\|_{\gamma, -2\gamma} + \|\chi_R(y)\|_{\infty, 0} + \|\chi_R(v)\|_{\gamma, -2\gamma} + \|\chi_R(v)\|_{\infty, 0}) \|\chi_R(y) - \chi_R(v)\|_{\infty, -2\gamma} \\ & \leq C_{g, |w|_{\gamma, \chi} R} \|y - v, (y - v)'\|_{\mathcal{D}_w^{2\gamma, \eta}} \end{aligned}$$

and

$$\begin{aligned} & \|g(\chi_R(y)) - g(\chi_R(v))\|_{\infty, 0} \leq C_g \max\{\|\chi_R(y)\|_{\infty, 0}, \|\chi_R(v)\|_{\infty, 0}\} \|\chi_R(y) - \chi_R(v)\|_{\infty, 0} \\ & \leq C_{g, \chi} R \|\chi_R(y) - \chi_R(v)\|_{\infty, 0}, \end{aligned}$$

hence, based on above estimates we obtain

$$\|g(\chi_R(y)) - g(\chi_R(v))\|_{\gamma, -2\gamma} \leq C_{g, |w|_{\gamma, \chi} R} \|y - v, (y - v)'\|_{\mathcal{D}_w^{2\gamma, \eta}}.$$

Secondly, since

$$\begin{aligned} & \|Dg(\chi_R(y))\chi'_R(y) - Dg(\chi_R(v))\chi'_R(v)\|_{\gamma, -2\gamma} \leq |Dg(\chi_R(y))\chi'_R(y) - Dg(\chi_R(v))\chi'_R(v)|_{\gamma, -2\gamma} \\ & \quad + \|Dg(\chi_R(y))\chi'_R(y) - Dg(\chi_R(v))\chi'_R(v)\|_{\infty, 0}, \\ & |Dg(\chi_R(y))\chi'_R(y) - Dg(\chi_R(v))\chi'_R(v)|_{\gamma, -2\gamma} \\ & \leq |Dg(\chi_R(y))(\chi'_R(y) - \chi'_R(v))|_{\gamma, -2\gamma} + |(Dg(\chi_R(y)) - Dg(\chi_R(v)))\chi'_R(v)|_{\gamma, -2\gamma} \\ & \leq \|Dg(\chi_R(y))\|_{\infty, \mathcal{L}(\mathcal{H}_{-2\gamma} \otimes \mathbb{R}^d, \mathcal{H}_{-2\gamma})} |\chi'_R(y) - \chi'_R(v)|_{\gamma, -2\gamma} \\ & \quad + \|Dg(\chi_R(y))\|_{\gamma, \mathcal{L}(\mathcal{H}_{-2\gamma} \otimes \mathbb{R}^d, \mathcal{H}_{-2\gamma})} |\chi'_R(y) - \chi'_R(v)|_{\infty, -2\gamma} \\ & \quad + |Dg(\chi_R(y)) - Dg(\chi_R(v))|_{\gamma, \mathcal{L}(\mathcal{H}_{-2\gamma} \otimes \mathbb{R}^d, \mathcal{H}_{-2\gamma})} \|\chi'_R(v)\|_{\infty, -2\gamma} \\ & \quad + |Dg(\chi_R(y)) - Dg(\chi_R(v))|_{\infty, \mathcal{L}(\mathcal{H}_{-2\gamma} \otimes \mathbb{R}^d, \mathcal{H}_{-2\gamma})} \|\chi'_R(v)\|_{\gamma, -2\gamma} \\ & \leq C_g (\|\chi_R(y)\|_{\infty, -2\gamma} \|\chi'_R(y) - \chi'_R(v)\|_{\gamma, -2\gamma} + |\chi_R(y)|_{\gamma, -2\gamma} \|\chi'_R(y) - \chi'_R(v)\|_{\infty, -2\gamma}) \\ & \quad + C_g (\|\chi'_R(v)\|_{\infty, -2\gamma} |\chi_R(y) - \chi_R(v)|_{\gamma, -2\gamma} + |\chi'_R(v)|_{\gamma, -2\gamma} \|\chi_R(y) - \chi_R(v)\|_{\infty, -2\gamma}) \\ & \leq C_{g, |w|_{\gamma, \chi} R} (\|\chi'_R(y) - \chi'_R(v)\|_{\gamma, -2\gamma} + \|\chi'_R(y) - \chi'_R(v)\|_{\infty, 0} + \|\chi'_R(y) - \chi'_R(v)\|_{\infty, -2\gamma}) \\ & \quad + C_{g, \chi} R (\|\chi_R(y) - \chi_R(v)\|_{\gamma, -2\gamma} + \|\chi_R(y) - \chi_R(v)\|_{\infty, 0} + \|\chi_R(y) - \chi_R(v)\|_{\infty, -2\gamma}) \end{aligned}$$

and

$$\begin{aligned} & \|Dg(\chi_R(y))\chi'_R(y) - Dg(\chi_R(v))\chi'_R(v)\|_{\infty, 0} \leq \|Dg(\chi_R(y))(\chi'_R(y) - \chi'_R(v))\|_{\infty, 0} \\ & \quad + \|(Dg(\chi_R(y)) - Dg(\chi_R(v)))\chi'_R(v)\|_{\infty, 0} \\ & \leq C_g (\|\chi_R(y)\|_{\infty, 0} \|\chi'_R(y) - \chi'_R(v)\|_{\infty, 0} \\ & \quad + \|\chi_R(y) - \chi_R(v)\|_{\infty, 0} \|\chi'_R(v)\|_{\infty, 0}) \\ & \leq C_{g, |w|_{\gamma, \chi} R} (\|\chi'_R(y) - \chi'_R(v)\|_{\infty, 0} \\ & \quad + \|\chi_R(y) - \chi_R(v)\|_{\infty, 0}). \end{aligned}$$

Using above estimates we easily obtain

$$\|Dg(\chi_R(y))\chi'_R(y) - Dg(\chi_R(v))\chi'_R(v)\|_{\gamma, -2\gamma} \leq C_{g, |w|_{\gamma, \chi} R} \|y - v, (y - v)'\|_{\mathcal{D}_w^{2\gamma, \eta}}.$$

For the remainder term of  $g(\chi_R(y)) - g(\chi_R(v))$ , using (2.13) we have

$$\begin{aligned}
 &R_{t,s}^{g(\chi_R(y))} - R_{t,s}^{g(\chi_R(v))} \\
 &= g(\chi_R(y_t)) - g(\chi_R(y_s)) - Dg(\chi_R(y_s))\delta\chi_R(y)_{t,s} - (g(\chi_R(v_t)) - g(\chi_R(v_s))) \\
 &\quad - Dg(\chi_R(v_s))\delta\chi_R(v)_{t,s} + Dg(\chi_R(y))_s R_{t,s}^{\chi_R(y)} - Dg(\chi_R(v))_s R_{t,s}^{\chi_R(v)} \\
 &\quad - (S_{ts} - I)(g(\chi_R(y_s)) - g(\chi_R(v_s))) \\
 &\quad + Dg(\chi_R(y_s))(S_{ts} - I)\chi'_R(y)_s \delta w_{t,s} - (Dg(\chi_R(v_s))(S_{ts} - I)\chi'_R(v)_s \delta w_{t,s}) \\
 &\quad - (S_{ts} - I)(Dg(\chi_R(y_s))\chi'_R(y)_s \delta w_{t,s} - Dg(\chi_R(v_s))\chi'_R(v)_s \delta w_{t,s}) \\
 &\quad + Dg(\chi_R(y_s))(S_{ts} - I)\chi_R(y)_s - Dg(\chi_R(v_s))(S_{ts} - I)\chi_R(v)_s \\
 &= i + ii + iii + iv + v + vi.
 \end{aligned}$$

For  $i$ , using (44) of [24] twice, we have

$$\begin{aligned}
 &\|i\|_{\mathcal{H}_{-2\gamma}^d} \\
 &\leq \left\| \int_0^1 \int_0^1 C_g[\tau r^2(\chi_R(y_t) - \chi_R(v_t)) + (r - \tau r^2)(\chi_R(y_s) - \chi_R(v_s))] d\tau dr \delta\chi_R(y)_{t,s} \otimes \delta\chi_R(y)_{t,s} \right\|_{\mathcal{H}_{-2\gamma}} \\
 &\quad + \left\| \int_0^1 \int_0^1 C_g[\tau r^2\chi_R(v_t) + (r - \tau r^2)\chi_R(v_s)] d\tau dr (\delta\chi_R(y)_{t,s} \otimes \delta\chi_R(y)_{t,s} - \delta\chi_R(v)_{t,s} \otimes \delta\chi_R(v)_{t,s}) \right\|_{\mathcal{H}_{-2\gamma}} \\
 &\leq C_g \|\chi_R(y) - \chi_R(v)\|_{\infty, -2\gamma} \|\hat{\delta}\chi_R(y)_{t,s} + (S_{ts} - I)\chi_R(y_s)\|_{\mathcal{H}_{-2\gamma}}^2 \\
 &\quad + C_g \|\chi_R(v)\|_{\infty, -2\gamma} (\|\hat{\delta}\chi_R(y)_{t,s} + (S_{ts} - I)\chi_R(y_s)\|_{\mathcal{H}_{-2\gamma}} \\
 &\quad + \|\hat{\delta}\chi_R(v)_{t,s} + (S_{ts} - I)\chi_R(v_s)\|_{\mathcal{H}_{-2\gamma}}) \\
 &\quad \cdot \|\hat{\delta}(\chi_R(y) - \chi_R(v))_{t,s} + (S_{ts} - I)(\chi_R(y) - \chi_R(v))_s\|_{\mathcal{H}_{-2\gamma}} \\
 &\leq C_g \|\chi_R(y) - \chi_R(v)\|_{\infty, -2\gamma} (\|\chi_R(y)\|_{\gamma, -2\gamma} |t - s|^\gamma + \|\chi_R(y)\|_{\infty, 0} |t - s|^{2\gamma})^2 \\
 &\quad + C_g \|\chi_R(v)\|_{\infty, -2\gamma} (\|\chi_R(y)\|_{\gamma, -2\gamma} |t - s|^\gamma + \|\chi_R(y)\|_{\infty, 0} |t - s|^{2\gamma} \\
 &\quad + \|\chi_R(v)\|_{\gamma, -2\gamma} |t - s|^\gamma \\
 &\quad + \|\chi_R(v)\|_{\infty, 0} |t - s|^{2\gamma}) \cdot (\|\chi_R(y) - \chi_R(v)\|_{\gamma, -2\gamma} |t - s|^\gamma \\
 &\quad + \|\chi_R(y) - \chi_R(v)\|_{\infty, 0} |t - s|^{2\gamma}),
 \end{aligned}$$

hence,

$$\|i\|_{2\gamma, -2\gamma} \leq C_{g, |w|_\gamma, \chi} R^2 \|y - v, (y - v)'\|_{\mathcal{D}_w^{2\gamma, \eta}}.$$

For *ii*,

$$\begin{aligned} \|ii\|_{\mathcal{H}_{-2\gamma}^d} &= \|Dg(\chi_R(y_s))R_{t,s}^{\chi_R(y)} - Dg(\chi_R(y_s))R_{t,s}^{\chi_R(v)} + Dg(\chi_R(y_s))R_{t,s}^{\chi_R(v)} \\ &\quad - Dg(\chi_R(v_s))R_{t,s}^{\chi_R(v)}\|_{\mathcal{H}_{-2\gamma}^d} \\ &\leq \|Dg(\chi_R(y_s))(R_{t,s}^{\chi_R(y)} - R_{t,s}^{\chi_R(v)})\|_{\mathcal{H}_{-2\gamma}^d} + \|(Dg(\chi_R(y_s)) - Dg(\chi_R(v_s)))R_{t,s}^{\chi_R(v)}\|_{\mathcal{H}_{-2\gamma}^d} \\ &\leq C_g \|\chi_R(y)\|_{\infty, -2\gamma} |R^{\chi_R(y)} - R^{\chi_R(v)}|_{2\gamma, -2\gamma} |t - s|^{2\gamma} \\ &\quad + C_g \|\chi_R(y) - \chi_R(v)\|_{\infty, -2\gamma} |R^{\chi_R(v)}|_{2\gamma, -2\gamma} |t - s|^{2\gamma}, \end{aligned}$$

hence,

$$\|ii\|_{2\gamma, -2\gamma} \leq C_{g, |w|_\gamma, \chi} R \|y - v, (y - v)'\|_{\mathcal{D}_w^{2\gamma, \eta}}.$$

For *iii*, we easily have

$$\begin{aligned} \|iii\|_{\mathcal{H}_{-2\gamma}^d} &\leq \|Dg(\chi_R(y_s))(\chi_R(y_s) - \chi_R(v_s))\|_{\mathcal{H}^d} |t - s|^{2\gamma} \\ &\leq C_g \|\chi_R(y)\|_{\infty, 0} \|\chi_R(y) - \chi_R(v)\|_{\infty, 0} |t - s|^{2\gamma} \\ &\leq C_{g, |w|_\gamma, \chi} R \|\chi_R(y) - \chi_R(v)\|_{\infty, 0} |t - s|^{2\gamma}, \end{aligned}$$

hence,

$$\|iii\|_{2\gamma, -2\gamma} \leq C_{g, |w|_\gamma, \chi} R \|y - v, (y - v)'\|_{\mathcal{D}_w^{2\gamma, \eta}}.$$

For *iv*, we have

$$\begin{aligned} \|iv\|_{\mathcal{H}_{-2\gamma}^d} &\leq \|(Dg(\chi_R(y_s)) - Dg(\chi_R(v_s)))(S_{ts} - I)\chi_R'(y)_s \delta w_{t,s}\|_{\mathcal{H}_{-2\gamma}^d} \\ &\quad + \|Dg(\chi_R(v_s))(S_{ts} - I)(\chi_R'(y)_s - \chi_R'(v)_s) \delta w_{t,s}\|_{\mathcal{H}_{-2\gamma}^d} \\ &\leq C_g \|\chi_R(y) - \chi_R(v)\|_{\infty, -2\gamma} \|\chi_R'(y)\|_{\infty, 0} |w|_\gamma |t - s|^{3\gamma} \\ &\quad + C_g \|\chi_R(y)\|_{\infty, -2\gamma} \|\chi_R'(y) - \chi_R'(v)\|_{\infty, 0} |w|_\gamma |t - s|^{3\gamma} \\ &\leq C_{g, |w|_\gamma, \chi} R \|\chi_R(y) - \chi_R(v)\|_{\infty, -2\gamma} |w|_\gamma |t - s|^{3\gamma} \\ &\quad + C_{g, |w|_\gamma, \chi} R \|\chi_R'(y) - \chi_R'(v)\|_{\infty, 0} |w|_\gamma |t - s|^{3\gamma}, \end{aligned}$$

hence,

$$\|iv\|_{2\gamma, -2\gamma} \leq C_{g, |w|_\gamma, \chi} R \|y - v, (y - v)'\|_{\mathcal{D}_w^{2\gamma, \eta}}.$$

For *v*, we have

$$\begin{aligned} \|v\|_{\mathcal{H}_{-2\gamma}^d} &\leq \|(S_{ts} - I)(Dg(\chi_R(y_s)) - Dg(\chi_R(v_s)))\chi_R'(y)_s \delta w_{t,s}\|_{\mathcal{H}_{-2\gamma}^d} \\ &\quad + \|(S_{ts} - I)Dg(\chi_R(v_s))(\chi_R'(y)_s - \chi_R'(v)_s) \delta w_{t,s}\|_{\mathcal{H}_{-2\gamma}^d} \\ &\leq C_g \|\chi_R(y) - \chi_R(v)\|_{\infty, 0} \|\chi_R'(y)\|_{\infty, 0} |w|_\gamma |t - s|^{3\gamma} \\ &\quad + C_g \|\chi_R(y)\|_{\infty, 0} \|\chi_R'(y) - \chi_R'(v)\|_{\infty, 0} |w|_\gamma |t - s|^{3\gamma} \\ &\leq C_{g, \chi} R \|\chi_R(y) - \chi_R(v)\|_{\infty, 0} |w|_\gamma |t - s|^{3\gamma} \\ &\quad + C_{g, \chi} R \|\chi_R'(y) - \chi_R'(v)\|_{\infty, 0} |w|_\gamma |t - s|^{3\gamma}, \end{aligned}$$

hence,

$$\|v\|_{2\gamma, -2\gamma} \leq C_{g, |w|_\gamma, \chi} R \|y - v, (y - v)'\|_{\mathcal{D}_w^{2\gamma, \eta}}.$$

For  $vi$ , we have

$$\begin{aligned} \|vi\|_{\mathcal{H}^d_{-2\gamma}} &\leq \| (Dg(\chi_R(y_s)) - Dg(\chi_R(v_s)))(S_{ts} - I)\chi_R(y)_s \|_{\mathcal{H}^d_{-2\gamma}} \\ &\quad + \| Dg(\chi_R(v_s))(S_{ts} - I)(\chi_R(y)_s - \chi_R(v)_s) \|_{\mathcal{H}^d_{-2\gamma}} \\ &\leq C_g \|\chi_R(y) - \chi_R(v)\|_{\infty, -2\gamma} \|\chi_R(y)\|_{\infty, 0} |t - s|^{2\gamma} \\ &\quad + C_g \|\chi_R(y)\|_{\infty, -2\gamma} \|\chi_R(y) - \chi_R(v)\|_{\infty, 0} |t - s|^{2\gamma} \\ &\leq C_{g, \chi} R \|\chi_R(y) - \chi_R(v)\|_{\infty, -2\gamma} |t - s|^{2\gamma} \\ &\quad + C_{g, |w|_\gamma, \chi} R \|\chi'_R(y) - \chi'_R(v)\|_{\infty, 0} |t - s|^{2\gamma}, \end{aligned}$$

hence,

$$\|vi\|_{2\gamma, -2\gamma} \leq C_{g, |w|_\gamma, \chi} R \|y - v, (y - v)'\|_{\mathcal{D}^{2\gamma, \eta}_w}.$$

Consequently, according to above estimates, we obtain

$$\begin{aligned} |R^{g(\chi_R(y))} - R^{g(\chi_R(v))}|_{2\gamma, -2\gamma} &\leq C_{g, |w|_\gamma, \chi} (R + R^2) \|y - v, (y - v)'\|_{\mathcal{D}^{2\gamma, \eta}_w} \\ &\leq C_{g, |w|_\gamma, \chi} (R) \|y - v, (y - v)'\|_{\mathcal{D}^{2\gamma, \eta}_w}. \end{aligned}$$

Finally, one can easily obtain (2.25). □

According to above lemmas, we will derive that the modified equation of (2.2) obtained by replacing  $f$  and  $g$  with  $f_R$  and  $g_R$  has a unique solution. To this end, for  $(y, y') \in \mathcal{D}^{2\gamma, \eta}_w([0, 1]; \mathcal{H})$  and  $t \in [0, 1]$ , we introduce

$$\mathcal{T}_R(w, y, y')[t] := \int_0^t S_{tu} f_R(y_u) du + \int_0^t S_{tu} g_R(y_u) d\mathbf{w}_u, \tag{2.27}$$

with mildly Gubinelli derivative  $\mathcal{T}_R(w, y, y)' = g_R(y)$ . Because of the estimates derived in the previous lemmas, we easily have the following result.

**Remark 2.6** For the convenience of argument for the fixed point of Lyapunov–Perron operator in the following Sect. 4, here we define operator  $\mathcal{T}_R$  with no initial value.

**Theorem 2.4** *The following estimate holds true*

$$\begin{aligned} &\left\| \int_0^\cdot S_{\cdot u} (f_R(y_u) - f_R(v_u)) du + \int_0^\cdot S_{\cdot u} (g_R(y_u) - g_R(v_u)) d\mathbf{w}_u, g_R(y) - g_R(v) \right\|_{\mathcal{D}^{2\gamma, \eta}_w} \\ &\leq (C_{f, \chi, |w|_\gamma} R + C_{g, \chi, |w|_\gamma} C(R)(1 + |w|_\gamma + |w|^2_{2\gamma})(1 + |w|_\gamma)^2) \|y - v, (y - v)'\|_{\mathcal{D}^{2\gamma, \eta}_w}. \end{aligned} \tag{2.28}$$

Furthermore, the mapping  $\mathcal{T}_R : \mathcal{D}^{2\gamma, \eta}_w([0, 1]; \mathcal{H}) \rightarrow \mathcal{D}^{2\gamma, \eta}_w([0, 1]; \mathcal{H})$  has a fixed-point.

Return to our consideration, in order to reduce the Lipschitz constants of  $f$  and  $g$  by using  $\chi_R$ , the next goal is to characterize  $R$  as required. As already seen we have to choose  $R$  as small as possible. Since in our discussions, it is always required that  $R \leq 1$  and  $C(R)$  is strictly increasing in  $R$ . As is often encountered in the theory of stochastic dynamical systems [24], since all the estimates depend on the random input,



it is meaningful to employ a cut-off technique for a random variable, i.e.  $R = R(w)$ . Such an argument will also be used here as follows.

We fix  $K > 0$  and regard to (2.28), set  $\tilde{R}(w)$  be the unique solution of

$$C_{f,\chi,|w|_\gamma} \tilde{R}(w) + C_{g,\chi,|w|_\gamma} (\tilde{R}(w))(1 + |w|_\gamma + |w|^2|_{2\gamma})(1 + |w|_\gamma)^2 = K \quad (2.29)$$

and set

$$R(w) := \min\{\tilde{R}(w), 1\}. \quad (2.30)$$

This means that if  $R(w) = 1$ , we apply the cut-off procedure for  $\|y, y'\|_{\mathcal{D}_w^{2\gamma,\eta}} \leq 1/2$  or else if  $R(w) < 1$  for  $\|y, y'\|_{\mathcal{D}_w^{2\gamma,\eta}} \leq R(w)/2$ .

In the following sections, we work with a modified equation of (2.2), where the drift and diffusion coefficients  $f$  and  $g$  are replaced by  $f_{R(w)}$  and  $g_{R(w)}$ . For notational simplicity, the  $w$ -dependence of  $R$  will be omitted whenever there is no confusion.

According to (2.29), we have

**Lemma 2.9** *Let  $(y, y')$  and  $(v, v') \in \mathcal{D}_w^{2\gamma,\eta}([0, 1]; \mathcal{H})$ , we have*

$$\begin{aligned} & \left\| \mathcal{T}_R(w, y, y') - \mathcal{T}_R(w, v, v'), (\mathcal{T}_R(w, y, y') - \mathcal{T}_R(w, v, v'))' \right\|_{\mathcal{D}_w^{2\gamma,\eta}} \\ & \leq K \|y - v, (y - v)'\|_{\mathcal{D}_w^{2\gamma,\eta}}. \end{aligned} \quad (2.31)$$

### 3 Random dynamical system

In this section we will analyze the dynamics of REEs (2.2). Firstly we recall some basic concepts and results on the random dynamical systems theory [1, 3], which allow us to study invariant manifolds for (2.2).

**Definition 3.1** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $\theta : \mathbb{R} \times \Omega \rightarrow \Omega$  be a family of  $\mathbb{P}$ -preserving transformations (i.e.,  $\theta_t \mathbb{P} = \mathbb{P}$  for  $t \in \mathbb{R}$ ) with following properties:

- the mapping  $(t, \omega) \mapsto \theta_t \omega$  is  $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$ -measurable, where  $\mathcal{B}(\cdot)$  denotes the Borel sigma-algebra;
- $\theta_0 = I_\Omega$ ;
- $\theta_{t+s} = \theta_t \circ \theta_s$  for all  $t, s \in \mathbb{R}$ .

Then the quadruple  $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$  is called a metric dynamical system.

In our evolution setting, the construction of metric dynamical system depends on the construction of shift map  $\Theta$ . According to [24] we know that shifts act quite naturally on rough paths. For a  $\gamma$ -Hölder rough path  $\mathbf{w} = (w, w^2)$  and  $t, \tau \in \mathbb{R}$ , let us define the time-shift  $\Theta_\tau \mathbf{w} = (\theta_\tau w, \tilde{\theta}_\tau w^2)$  by

$$\begin{aligned} \theta_\tau w_t & := w_{t+\tau} - w_\tau, \\ \tilde{\theta}_\tau w_{t,s}^2 & := w_{t+\tau,s+\tau}^2. \end{aligned}$$

Note that  $\delta(\theta_\tau w)_{t,s} = w_{t+\tau} - w_{s+\tau}$ . Furthermore, the shift leaves the path space invariant:

**Lemma 3.1** [24] *Let  $T_1, T_2, \tau \in \mathbb{R}$ , and  $\mathbf{w} = (w, w^2)$  be a  $\gamma$ -Hölder rough path on  $[T_1, T_2]$  for  $\gamma \in (\frac{1}{3}, \frac{1}{2}]$ . Then the time-shift  $\Theta_\tau \mathbf{w} = (\theta_\tau w, \tilde{\theta}_\tau w^2)$  is also an  $\gamma$ -Hölder rough path on  $[T_1 - \tau, T_2 - \tau]$ .*

According to [3], we consider the following concept:

**Definition 3.2** [3] *Let  $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$  be a metric dynamical system. We call  $\mathbf{w} = (w, w^2)$  a rough path cocycle if the identity*

$$\mathbf{w}_{t+s, s}(\omega) = \mathbf{w}_{t, 0}(\theta_s \omega)$$

holds true for every  $\omega \in \Omega, s \in \mathbb{R}$  and  $t > 0$ .

The previous definitions imply that one can use a space of paths as a probability space  $\Omega$ . As example 3.5 in [24], fractional Brownian motion  $\mathbf{B}^H = (B^H, \mathbb{B}^H)$  represents a rough path cocycle, by the same construction of path-space  $(\Omega_{B^H}, \mathcal{F}_{B^H}, \mathbb{P}_{B^H})$  of fractional Brownian motion (for further details see [3]), we have the abstract definition of metric dynamical systems for our problem modelling the underlying rough driving process. Now we also need to define the dynamical system structure of the solution operators of our rough evolution equations (2.2). Meanwhile, we recall the classical definition of random dynamical system [1].

**Definition 3.3** *A random dynamical system  $\varphi$  on  $\mathcal{H}$  over a metric dynamical system  $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$  is a measurable mapping*

$$\varphi : [0, \infty) \times \Omega \times \mathcal{H} \rightarrow \mathcal{H}, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x),$$

such that:

- $\varphi(0, \omega, \cdot) = I_{\mathcal{H}}$  for all  $\omega \in \Omega$ ;
- $\varphi(t + \tau, \omega, x) = \varphi(t, \theta_\tau \omega, \varphi(\tau, \omega, x))$ , for all  $x \in \mathcal{H}, t, \tau \in [0, \infty), \omega \in \Omega$ ;
- $\varphi(t, \omega, \cdot) : \mathcal{H} \rightarrow \mathcal{H}$  is continuous for all  $t \in [0, \infty)$  and all  $\omega \in \Omega$ .

Now one can hope that the solution operators of (2.2) generate random dynamical systems. As for all we know, the rough integral given in (2.6) is pathwise, no exceptional sets occur. For completeness, we give a proof of this fact, see [24].

**Lemma 3.2** *Let  $\mathbf{w}$  be a rough path cocycle, then the solution operator*

$$t \mapsto \varphi(t, w, \xi) = y_t = S_t \xi + \int_0^t S_{tu} f(y_u) du + \int_0^t S_{tu} g(y_u) d\mathbf{w}_u,$$

for any  $t \in [0, \infty)$  of the REE (2.2) generates a random dynamical system over the metric dynamical system  $(\Omega_w, \mathcal{F}_w, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ .

**Proof** The proof is analogous to [22] and [24] Lemma 3.7. The difficulty is to check the cocycle property for the solution operator. Here we just prove the cocycle property. Firstly, we easily check that if  $(y, y') \in \mathcal{D}_w^{2\gamma, \eta}([T_1 + \tau, T_2 + \tau]; \mathcal{H})$  then  $(y_{\cdot + \tau}, y'_{\cdot + \tau}) \in$

$\mathcal{D}_{\theta_\tau}^{2\gamma,\eta}([T_1, T_2]; \mathcal{H})$ , here  $T_1, T_2 \in \mathbb{R}$  with  $T_1 < T_2$ . The  $\gamma$ -Hölder continuity of  $y_{\cdot+\tau}$  and  $y'_{\cdot+\tau}$  is obvious. For the remainder we have

$$\begin{aligned} \|R_{t,s}^{y_{\cdot+\tau}}\|_{\mathcal{H}_{-2\gamma}} &= \|\hat{\delta}y_{t+\tau,s+\tau} - S_{ts}y'_{s+\tau}\delta w_{t+\tau,s+\tau}\|_{\mathcal{H}_{-2\gamma}} \\ &= \|\hat{\delta}y_{t+\tau,s+\tau} - S_{(t+\tau)-(s+\tau)}y'_{s+\tau}\delta w_{t+\tau,s+\tau}\|_{\mathcal{H}_{-2\gamma}} \\ &= \|R_{t+\tau,s+\tau}^y\|_{\mathcal{H}_{-2\gamma}} \leq |R^y|_{2\gamma,-2\gamma}|t-s|^{2\gamma}. \end{aligned}$$

Next, we will obtain the shift property of rough integral. Let  $\mathcal{P}$  be a partition of  $[\tau, t + \tau]$ , then we have

$$\begin{aligned} &\int_{\tau}^{t+\tau} S_{t+\tau-u}g(y_u)d\mathbf{w}_u \\ &= \lim_{|\mathcal{P}'| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}'} (S_{t+\tau-u}g(y_u)\delta w_{v,u} + S_{t+\tau-u}Dg(y_u)y'_{u'}w_{v,u}^2) \\ &= \lim_{|\mathcal{P}'| \rightarrow 0} \sum_{[u',v'] \in \mathcal{P}'} (S_{t-u'}g(y_{u'+\tau})\delta w_{v'+\tau,u'+\tau} + S_{t-u'}Dg(y_{u'+\tau})y'_{u'+\tau}w_{v'+\tau,u'+\tau}^2) \\ &= \lim_{|\mathcal{P}'| \rightarrow 0} \sum_{[u',v'] \in \mathcal{P}'} (S_{t-u'}g(y_{u'+\tau})\delta(\theta_\tau w)_{v',u'} + S_{t-u'}Dg(y_{u'+\tau})y'_{u'+\tau}\tilde{\theta}_\tau w_{v',u'}^2) \\ &= \int_0^t S_{t-u'}g(y_{u'+\tau})d\Theta_\tau \mathbf{w}_{u'}, \end{aligned}$$

where  $\mathcal{P}'$  is a partition of  $[0, t]$  given by  $\mathcal{P}' := \{[s - \tau, t - \tau] : [s, t] \in \mathcal{P}\}$ . The proof of the cocycle property and measurability of solution operators is similar to [22] and [24], here we omit.  $\square$

The next concept of tempered random variables [1] is of fundamental importance in the study of local random invariant manifolds.

**Definition 3.4** A random variable  $\tilde{R} : \Omega \rightarrow (0, \infty)$  is called tempered from above, with respect to a metric dynamical system  $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ , if

$$\limsup_{t \rightarrow \pm\infty} \frac{\ln^+ \tilde{R}(\theta_t \omega)}{|t|} = 0, \quad \text{for all } \omega \in \Omega, \tag{3.1}$$

where  $\ln^+ a := \max\{\ln a, 0\}$ . A random variable is called tempered from below if  $1/\tilde{R}$  is tempered from above. A random variable is tempered if and only if is tempered from above and from below.

The temperedness reflexes the subexponential growth of the mapping  $t \mapsto \tilde{R}(\theta_t \omega)$ , according to [1] Proposition 4.1.3, a sufficient condition for temperedness is

$$\mathbb{E} \sup_{t \in [0,1]} \tilde{R}(\theta_t \omega) < \infty. \tag{3.2}$$

Moreover, if the random variable  $\tilde{R}$  is tempered from below with  $t \rightarrow \tilde{R}(\theta_t \omega)$  continuous for all  $\omega \in \Omega$ , then for every  $\varepsilon > 0$  there exists a constant  $C[\varepsilon, \omega] > 0$  such that

$$\tilde{R}(\theta_t \omega) \geq C[\varepsilon, \omega]e^{-\varepsilon|t|}, \tag{3.3}$$

for any  $\omega \in \Omega$ .

According to [24] Lemma 3.9 and Lemma 3.10, we can assume that  $\mathbf{w} = (w, w^2)$  is a rough path cocycle such that the random variables

$$R_1(w) = |w|_\gamma \quad \text{and} \quad R_2(w^2) = |w^2|_{2\gamma}$$

are tempered from above. These will be necessary for the proof of the existence for a local unstable manifold. One needs to ensure that for initial values belonging to a ball with a sufficiently small tempered from below radius, the corresponding trajectories remain within such a ball (for further details, refer to [9, 10, 14, 24]). By previous discussions, we easily obtain the result below:

**Lemma 3.3** *The random variable  $R(w)$  in (2.30) is tempered from below.*

### 4 Local unstable manifolds for REEs

In this section, we will study the existence of local unstable manifolds for (2.2) by the Lyapunov–Perron method which is similar to the one employed in [14, 22] and [24]. However, here we want to connect the theory of random invariant manifolds for REEs as in [6, 9, 14, 22] to rough paths theory.

Firstly, as in [14] and [10], we assume that the spectrum  $\sigma(A)$  of linear operator  $A$  only consists of a countable number of eigenvalues, and it splits as

$$\sigma(A) = \{\lambda_k, k \in \mathbb{N}\} = \sigma_u \cup \sigma_s, \tag{4.1}$$

with both  $\sigma_u$  and  $\sigma_s$  nonempty, and

$$\sigma_u \subset \{z \in \mathbb{C} : \text{Re}z > 0\} \quad \text{and} \quad \sigma_s \subset \{z \in \mathbb{C} : \text{Re}z < 0\},$$

where  $\mathbb{C}$  denotes the set of complex numbers and  $\sigma_u = \{\lambda_k, \dots, \lambda_N\}$  for some  $N > 0$ . Denote the corresponding eigenvectors for  $\{\lambda_k, k \in \mathbb{N}\}$  by  $\{e_1, \dots, e_N, e_{N+1}, \dots\}$ , furthermore, assume that the eigenvectors form an orthonormal basis of  $\mathcal{H}$ . Thus there is an invariant orthogonal decomposition  $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s$  with  $\dim \mathcal{H}_u = N$ , such that for the restrictions which are  $A_u = A|_{\mathcal{H}_u}$ ,  $A_s = A|_{\mathcal{H}_s}$ , one has  $\sigma_u = \{z \in \sigma(A_u)\}$  and  $\sigma_s = \{z \in \sigma(A_s)\}$ . Moreover,  $e^{A_u t}$  is a group of linear operators on  $\mathcal{H}_u$ , and there exist projections  $\pi^u$  and  $\pi^s$ , such that  $\pi^u + \pi^s = I_{\mathcal{H}}$ ,  $A_u = \pi^u A$  and  $A_s = \pi^s A$ . Furthermore, we assume that the projections  $\pi^u$  and  $\pi^s$  commute with  $A$ . Additionally, suppose that there are constants  $0 < \beta < \alpha$  such that

$$\|e^{tA_u} x\| \leq e^{\alpha t} \|x\|, \quad t \leq 0, \tag{4.2}$$

$$\|e^{tA_s} x\| \leq e^{-\beta t} \|x\|, \quad t \geq 0. \tag{4.3}$$

**Definition 4.1** If a random set  $\mathcal{M}^u(w)$ , which is invariant respect to random dynamical system  $\varphi$  (i.e.  $\varphi(t, w, \mathcal{M}^u(w)) \subset \mathcal{M}^u(\theta_t w)$  for  $t \in \mathbb{R}$  and  $w \in \Omega_w$ ), can be

represented as

$$\mathcal{M}^u(w) = \{\xi + h^u(\xi, w) : \xi \in \mathcal{H}_u\}, \tag{4.4}$$

where  $h^u(\xi, W) : \mathcal{H}_u \rightarrow \mathcal{H}_s$  is Lipschitz continuous. Then we call  $\mathcal{M}^u(w)$  an unstable manifold.

**Definition 4.2** There exists a random neighborhood  $\mathcal{U}(w) \subset \mathcal{H}_u$  of 0, if a random set  $\mathcal{M}_{loc}^u(w)$ , which is invariant respect to random dynamical system  $\varphi$  (i.e.  $\varphi(t, w, \mathcal{M}_{loc}^u(w)) \subset \mathcal{M}_{loc}^u(\theta_t w)$  for  $t \in \mathbb{R}$  and  $w \in \Omega_w$ ), can be represented as

$$\mathcal{M}_{loc}^u(w) = \{\xi + h^u(\xi, w) : \xi \in \mathcal{U}(w)\} \text{ and } 0 \in \mathcal{M}_{loc}^u(w), \tag{4.5}$$

where  $h^u(\xi, W) : \mathcal{U}(w) \rightarrow \mathcal{H}_s$  is Lipschitz continuous. Then we call  $\mathcal{M}_{loc}^u(w)$  a local unstable manifold.

By proving the existence of a global unstable manifold for a modified equation of (2.2) with cut-off over a random neighborhood of 0, we obtain a local unstable manifold  $\mathcal{M}_{loc}^u(W)$  for (2.2), namely (4.5) holds true when  $\xi$  belongs to a random ball of  $\mathcal{H}_u$  with a tempered radius.

Here, we employ the Lyapunov–Perron method which is similar with [22] and [24]. As well, in our case, the continuous-time Lyapunov–Perron mapping for (2.2) is presented by (compare with [10] and [24])

$$\begin{aligned} J(w, y)[\tau] := & S_\tau^u \xi^u + \int_0^\tau S_{\tau u}^u \pi^u f(y_u) du + \int_0^\tau S_{\tau u}^u \pi^u g(y_u) d\mathbf{w}_u \\ & + \int_{-\infty}^\tau S_{\tau u}^s \pi^s f(y_u) du + \int_{-\infty}^\tau S_{\tau u}^s \pi^s g(y_u) d\mathbf{w}_u \end{aligned} \tag{4.6}$$

for  $\tau \leq 0$ . Thanks to the presence of the rough integral we couldn't directly deal with (4.6), we need to track  $|w|_\gamma$  and  $|w^2|_\gamma$  that appear in (2.14) on a finite-time horizon. Similar to [14] and [22], we derive an appropriate discretized Lyapunov–Perron mapping and prove that it has a fixed-point in a suitable function space. The local unstable manifold will be developed for the discrete-time random dynamical system and will be shown that it holds true for the original continuous-time one, as in [14].

Analogous to [14, 22] and [24], we only need to deal with rough integral on time-interval  $[0, 1]$ . Let  $w \in \Omega_w$ ,  $t \in [0, 1]$  and  $i \in \mathbb{Z}^-$ , replacing  $\tau$  by  $t + i - 1$  in (4.6), we have

$$\begin{aligned} J(w, y)[t + i - 1] = & S_{t+i-1}^u \xi^u \\ & - \sum_{k=0}^{i+1} S_{t+i-1-k}^u \left( \int_0^1 S_{1-u}^u \pi^u f(y_{u+k-1}) du \right. \\ & \left. + \int_0^1 S_{1-u}^u \pi^u g(y_{u+k-1}) d\Theta_{k-1} \mathbf{w}_u \right) \\ & - \int_t^1 S_{t-u}^u \pi^u f(y_{u+i-1}) du - \int_t^1 S_{t-u}^u \pi^u g(y_{u+i-1}) d\Theta_{i-1} \mathbf{w}_u \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{k=-\infty}^{i-1} S_{t+i-1-k}^s \left( \int_0^1 S_{1-u}^s \pi^s f(y_{u+k-1}) du \right. \\
 &+ \left. \int_0^1 S_{1-u}^s \pi^s g(y_{u+k-1}) d\Theta_{k-1} \mathbf{w}_u \right) \\
 &+ \int_0^t S_{t-u}^s \pi^s f(y_{u+i-1}) du + \int_0^t S_{t-u}^s \pi^s g(y_{u+i-1}) d\Theta_{i-1} \mathbf{w}_u, \tag{4.7}
 \end{aligned}$$

by applying (4.7), we will give the structure of the discrete Lyapunov–Perron mapping. for  $(y, y') \in \mathcal{D}_w^{2\gamma, \eta}([0, 1]; \mathcal{H})$ , we denote

$$\mathcal{T}^{s/u}(w, y, y')[\cdot] = \int_0^\cdot S_{t-u}^{s/u} \pi^{s/u} f(y_u) du + \int_0^\cdot S_{t-u}^{s/u} \pi^{s/u} g(y_u) d\mathbf{w}_u, \tag{4.8}$$

$$\tilde{\mathcal{T}}^u(w, y, y')[\cdot] = \int_0^1 S_{t-u}^u \pi^u f(y_u) du + \int_0^1 S_{t-u}^u \pi^u g(y_u) d\mathbf{w}_u, \tag{4.9}$$

where  $(\mathcal{T}^{s/u}(w, y, y')[\cdot])' = (\tilde{\mathcal{T}}^u(w, y, y')[\cdot])' = g(y)$ . Meanwhile, in our evolution setting, we directly deal with solutions of the REEs (2.2). It is an essential problem that we need to find an appropriate space for the fixed-point argument. For this, similar to [22] and [24] we introduce the following function space which helps us incorporate the discretized version of (4.6).

Let  $\delta = \frac{\alpha-\beta}{2} > 0$ , we denote  $BC_\delta(\mathcal{D}_w^{2\gamma, \eta})$  as the space of a sequence of mildly controlled rough paths  $\mathbf{y} := (y^{i-1}, (y^{i-1})')_{i \in \mathbb{Z}^-}$  with  $y_0^{i-1} = y_1^{i-2}$ , where  $(y^{i-1}, (y^{i-1})') \in \mathcal{D}_w^{2\gamma, \eta}([0, 1]; \mathcal{H})$ , if

$$\|\mathbf{y}\|_{BC_\delta(\mathcal{D}_w^{2\gamma, \eta})} := \sup_{i \in \mathbb{Z}^-} e^{-\delta(i-1)} \|y^{i-1}, (y^{i-1})'\|_{\mathcal{D}_w^{2\gamma, \eta}([0,1]; \mathcal{H})} < \infty. \tag{4.10}$$

In the following, for notational simplicity, we denote  $\tilde{y}[i-1, t] = \tilde{y}_t^{i-1}$  for  $t \in [0, 1]$  and  $\tilde{y}[\tau] = \tilde{y}[i-1, t]$  if  $\tau = t + i - 1$ .

Next, we modify (2.2) by the cut-off function given in Sect. 2, i.e. we replace  $f$  and  $g$  by  $f_R$  respectively  $g_R$ . According to (4.7), it is reasonable to introduce the discrete Lyapunov–Perron transform  $J_d(w, \mathbf{y}, \xi)$  for a sequence of mildly controlled rough paths as the pair  $J_d(w, \mathbf{y}, \xi) := (J_d^1(w, \mathbf{y}, \xi), J_d^2(w, \mathbf{y}, \xi))$ , where  $\mathbf{y} \in BC_\delta(\mathcal{D}_w^{2\gamma, \eta})$  and  $\xi \in \mathcal{H}$ , the precise structure is given below. The dependence of  $J_d$  on the cut-off parameter  $R$  is indicated by the subscript  $R$ . For  $t \in [0, 1]$ ,  $w \in \Omega_w$  and  $i \in \mathbb{Z}^-$ , we define

$$\begin{aligned}
 J_{R,d}^1(w, \mathbf{y}, \xi)[i-1, t] &= S_{t+i-1}^u \xi^u - \sum_{k=0}^{i+1} S_{t+i-1-k}^u \left( \int_0^1 S_{1-u}^u \pi^u f_R(y_u^{k-1}) du \right. \\
 &+ \left. \int_0^1 S_{1-u}^u \pi^u g_R(y_u^{k-1}) d\Theta_{k-1} \mathbf{w}_u \right) \\
 &- \int_t^1 S_{t-u}^u \pi^u f_R(y_u^{i-1}) du - \int_t^1 S_{t-u}^u \pi^u g_R(y_u^{i-1}) d\Theta_{i-1} \mathbf{w}_u
 \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{k=-\infty}^{i-1} S_{t+i-1-k}^s \left( \int_0^1 S_{1-u}^s \pi^s f_R(y_u^{k-1}) du \right. \\
 &+ \left. \int_0^1 S_{1-u}^s \pi^s g_R(y_u^{k-1}) d\Theta_{k-1} \mathbf{w}_u \right) \\
 &+ \int_0^t S_{t-u}^s \pi^s f_R(y_u^{i-1}) du + \int_0^t S_{t-u}^s \pi^s g_R(y_u^{i-1}) d\Theta_{i-1} \mathbf{w}_u,
 \end{aligned} \tag{4.11}$$

Moreover,  $J_{R,d}^2(w, \mathbf{y}, \xi)$  is denoted as the mildly Gubinelli derivative of  $J_{R,d}^1(w, \mathbf{y}, \xi)$ , i.e.  $J_{R,d}^2(w, \mathbf{y}, \xi)[i - 1, t] := (J_{R,d}^1(w, \mathbf{y}, \xi)[i - 1, t])'$ . Notice that one can easily obtain  $\xi^u = \pi^u J_{R,d}^1(w, \mathbf{y}, \xi)[-1, 1]$  by setting  $i = 0$  and  $t = 1$ .

In the following, we will prove that (4.11) maps  $\mathbf{y} \in BC_\delta(\mathcal{D}_w^{2\gamma,\eta})$  into itself and is a contractive mapping.

**Theorem 4.1** *In our setting, if  $K$  satisfies the gap condition*

$$K \left( \frac{e^{\beta+\delta}(Ce^{-\delta} + 1)}{1 - e^{-(\beta+\delta)}} + \frac{(e^{-(\alpha-\delta)} - 1)(Ce^{-\delta} + e^{\alpha-\delta})}{1 - e^{\alpha-\delta}} \right) \leq \frac{1}{2}, \tag{4.12}$$

then, the mapping  $J_{R,d} : \Omega \times BC_\delta(\mathcal{D}_w^{2\gamma,\eta}) \rightarrow BC_\delta(\mathcal{D}_w^{2\gamma,\eta})$  possesses a unique fixed-point

$\Gamma \in BC_\delta(\mathcal{D}_w^{2\gamma,\eta})$ . Also, the mapping  $\xi^u \rightarrow \Gamma(\xi^u, w) \in BC_\delta(\mathcal{D}_w^{2\gamma,\eta})$  is Lipschitz continuous.

**Proof** Let  $\mathbf{y} := (y^{i-1}, (y^{i-1})')_{i \in \mathbb{Z}^-}$  and  $\mathbf{v} := (v^{i-1}, (v^{i-1})')_{i \in \mathbb{Z}^-} \in BC_\delta(\mathcal{D}_w^{2\gamma,\eta})$  with  $\pi^u y_1^{-1} = \pi^u v_1^{-1} = \xi^u$ . Firstly, we give several estimates which is essential for the proof. According to Lemma 2.5, we easily have

$$\|S_{\cdot+i-1}^u \xi^u, 0\|_{BC_\delta(\mathcal{D}_w^{2\gamma,\eta})} \leq Ce^{(\alpha-\delta)(i-1)} \|\xi^u, 0\|_{\mathcal{D}_w^{2\gamma,\eta}} \leq Ce^{(\alpha-\delta)(i-1)} \|\xi^u\|, \tag{4.13}$$

the above expression keeps bounded for  $i \in \mathbb{Z}^-$ . Denote

$$\Lambda = \mathcal{T}_R^s(\theta_{k-1}w, y^{k-1}, (y^{k-1})')[1] - \mathcal{T}_R^s(\theta_{k-1}w, v^{k-1}, (v^{k-1})')[1],$$

from (2.14), one has

$$\|\Lambda\|_{\mathcal{H}} \leq \|y^{k-1} - v^{k-1}, (y^{k-1} - v^{k-1})'\|_{\mathcal{D}_w^{2\gamma,\eta}}$$

by (4.13), we have

$$\begin{aligned}
 &\|S_{\cdot+i-1-k}^s \Lambda, (S_{\cdot+i-1-k}^s \Lambda)'\|_{\mathcal{D}_w^{2\gamma,\eta}} \\
 &= \|S_{\cdot+i-1-k}^s \Lambda, 0\|_{\mathcal{D}_w^{2\gamma,\eta}} \leq Ce^{-\beta(i-1-k)} \|\Lambda\|_{\mathcal{H}} \\
 &\leq Ce^{-\beta(i-1-k)} \|y^{k-1} - v^{k-1}, (y^{k-1} - v^{k-1})'\|_{\mathcal{D}_w^{2\gamma,\eta}}.
 \end{aligned} \tag{4.14}$$

Similarly, denote

$$\tilde{\Lambda} = \mathcal{T}_R^u(\theta_{k-1}w, y^{k-1}, (y^{k-1})')[1] - \mathcal{T}_R^u(\theta_{k-1}w, v^{k-1}, (v^{k-1})')[1],$$

we easily have

$$\|S_{\cdot+i-1-k}^u \tilde{\Lambda}, (S_{\cdot+i-1-k}^u \tilde{\Lambda})'\|_{\mathcal{D}_w^{2\gamma,\eta}} \leq C e^{\alpha(i-1-k)} \|y^{k-1} - v^{k-1}, (y^{k-1} - v^{k-1})'\|_{\mathcal{D}_w^{2\gamma,\eta}}. \tag{4.15}$$

Next, for the stable part of (4.11), due to (2.31), (4.14) and the norm of  $BC_\delta(\mathcal{D}_w^{2\gamma,\eta})$ , we have

$$\begin{aligned} & \sum_{k=-\infty}^{i-1} e^{-\delta(i-1)} \left\| S_{\cdot+i-1-k}^s (\mathcal{T}_R^s(\theta_{k-1}w, y^{k-1}, (y^{k-1})')[1] - \mathcal{T}_R^s(\theta_{k-1}w, v^{k-1}, (v^{k-1})')[1]), \right. \\ & \quad \left. (S_{\cdot+i-1-k}^s (\mathcal{T}_R^s(\theta_{k-1}w, y^{k-1}, (y^{k-1})')[1] - \mathcal{T}_R^s(\theta_{k-1}w, v^{k-1}, (v^{k-1})')[1]))' \right\|_{\mathcal{D}_w^{2\gamma,\eta}} \\ & + e^{-\delta(i-1)} \left\| \mathcal{T}_R^s(\theta_{i-1}w, y^{i-1}, (y^{i-1})')[\cdot] - \mathcal{T}_R^s(\theta_{i-1}w, v^{i-1}, (v^{i-1})')[\cdot], \right. \\ & \quad \left. (\mathcal{T}_R^s(\theta_{i-1}w, y^{i-1}, (y^{i-1})')[\cdot] - \mathcal{T}_R^s(\theta_{i-1}w, v^{i-1}, (v^{i-1})')[\cdot])' \right\|_{\mathcal{D}_w^{2\gamma,\eta}} \\ & \leq \sum_{k=-\infty}^{i-1} e^{-\delta(i-1)} C e^{-\beta(i-1-k)} K \|y^{k-1} - v^{k-1}, (y^{k-1} - v^{k-1})'\|_{\mathcal{D}_w^{2\gamma,\eta}} \\ & + e^{-\delta(i-1)} K \|y^{i-1} - v^{i-1}, (y^{i-1} - v^{i-1})'\|_{\mathcal{D}_w^{2\gamma,\eta}} \\ & \leq \sum_{k=-\infty}^{i-1} e^{-\delta(i-1)} C e^{-\beta(i-1-k)} e^{\delta(k-1)} K e^{-\delta(k-1)} \|y^{k-1} - v^{k-1}, (y^{k-1} - v^{k-1})'\|_{\mathcal{D}_w^{2\gamma,\eta}} \\ & + e^{-\delta(i-1)} K \|y^{i-1} - v^{i-1}, (y^{i-1} - v^{i-1})'\|_{\mathcal{D}_w^{2\gamma,\eta}} \\ & \leq \sum_{k=-\infty}^{i-1} e^{-(\beta+\delta)(i-1-k)} C e^{-\delta} K e^{-\delta(k-1)} \|y^{k-1} - v^{k-1}, (y^{k-1} - v^{k-1})'\|_{\mathcal{D}_w^{2\gamma,\eta}} \\ & + e^{-(\beta+\delta)(i-1-i)} e^{-\delta(i-1)} K \|y^{i-1} - v^{i-1}, (y^{i-1} - v^{i-1})'\|_{\mathcal{D}_w^{2\gamma,\eta}} \\ & \leq \sum_{k=-\infty}^i e^{-(\beta+\delta)(i-1-k)} K (C e^{-\delta} + 1) e^{-\delta(k-1)} \|y^{k-1} - v^{k-1}, (y^{k-1} - v^{k-1})'\|_{\mathcal{D}_w^{2\gamma,\eta}} \\ & \leq \frac{K e^{\beta+\delta} (C e^{-\delta} + 1)}{1 - e^{-(\beta+\delta)}} \|\mathbf{y} - \mathbf{v}\|_{BC_\delta(\mathcal{D}_w^{2\gamma,\eta})}. \end{aligned}$$

Similarly, for the unstable part, according to (2.31), (4.15) and the norm of  $BC_\delta(\mathcal{D}_w^{2\gamma,\eta})$ , we have

$$\begin{aligned} & \sum_{k=0}^{i+1} e^{-\delta(i-1)} \left\| S_{\cdot+i-1-k}^u (\mathcal{T}_R^u(\theta_{k-1}w, y^{k-1}, (y^{k-1})')[1] - \mathcal{T}_R^u(\theta_{k-1}w, v^{k-1}, (v^{k-1})')[1]), \right. \\ & \quad \left. (S_{\cdot+i-1-k}^u (\mathcal{T}_R^u(\theta_{k-1}w, y^{k-1}, (y^{k-1})')[1] - \mathcal{T}_R^u(\theta_{k-1}w, v^{k-1}, (v^{k-1})')[1]))' \right\|_{\mathcal{D}_w^{2\gamma,\eta}} \end{aligned}$$



$$\begin{aligned}
 & +e^{-\delta(i-1)} \left\| \tilde{T}_R^u(\theta_{i-1}w, y^{i-1}, (y^{i-1})')[\cdot] - \tilde{T}_R^u(\theta_{i-1}w, v^{i-1}, (v^{i-1})')[\cdot], \right. \\
 & \left. \left( \tilde{T}_R^u(\theta_{i-1}w, y^{i-1}, (y^{i-1})')[\cdot] - \tilde{T}_R^u(\theta_{i-1}w, v^{i-1}, (v^{i-1})')[\cdot] \right)' \right\|_{\mathcal{D}_w^{2\gamma, \eta}} \\
 & \leq \sum_{k=0}^{i+1} e^{-\delta(i-1)} C e^{\alpha(i-1-k)} K \|y^{k-1} - v^{k-1}, (y^{k-1} - v^{k-1})'\|_{\mathcal{D}_w^{2\gamma, \eta}} \\
 & \quad + e^{-\delta(i-1)} K \|y^{i-1} - v^{i-1}, (y^{i-1} - v^{i-1})'\|_{\mathcal{D}_w^{2\gamma, \eta}} \\
 & \leq \sum_{k=0}^{i+1} e^{-\delta(i-1)} C e^{\alpha(i-1-k)} e^{\delta(k-1)} K e^{-\delta(k-1)} \|y^{k-1} - v^{k-1}, (y^{k-1} - v^{k-1})'\|_{\mathcal{D}_w^{2\gamma, \eta}} \\
 & \quad + e^{-\delta(i-1)} K \|y^{i-1} - v^{i-1}, (y^{i-1} - v^{i-1})'\|_{\mathcal{D}_w^{2\gamma, \eta}} \\
 & \leq \sum_{k=0}^{i+1} e^{(\alpha-\delta)(i-1-k)} C e^{-\delta} K e^{-\delta(k-1)} \|y^{k-1} - v^{k-1}, (y^{k-1} - v^{k-1})'\|_{\mathcal{D}_w^{2\gamma, \eta}} \\
 & \quad + e^{(\alpha-\delta)(i-1-i)} e^{\alpha-\delta} e^{-\delta(i-1)} K \|y^{i-1} - v^{i-1}, (y^{i-1} - v^{i-1})'\|_{\mathcal{D}_w^{2\gamma, \eta}} \\
 & \leq \sum_{k=0}^i e^{(\alpha-\delta)(i-1-k)} K (C e^{-\delta} + e^{\alpha-\delta}) e^{-\delta(k-1)} \|y^{k-1} - v^{k-1}, (y^{k-1} - v^{k-1})'\|_{\mathcal{D}_w^{2\gamma, \eta}} \\
 & \leq \frac{K(e^{-(\alpha-\delta)} - 1)(C e^{-\delta} + e^{\alpha-\delta})}{1 - e^{\alpha-\delta}} \|\mathbf{y} - \mathbf{v}\|_{BC_\delta(\mathcal{D}_w^{2\gamma, \eta})}.
 \end{aligned}$$

Combining previous estimates, we obtain that

$$\left\| J_{R,d}(w, \mathbf{y}, \xi) - J_{R,d}(w, \mathbf{v}, \xi) \right\|_{BC_\delta(\mathcal{D}_w^{2\gamma, \eta})} \leq \frac{1}{2} \|\mathbf{y} - \mathbf{v}\|_{BC_\delta(\mathcal{D}_w^{2\gamma, \eta})}.$$

When  $\mathbf{v} \equiv 0$ , we easily have that  $J_{R,d}$  maps  $BC_\delta(\mathcal{D}_w^{2\gamma, \eta})$  into itself. Applying fixed-point argument, we deduce that  $J_{R,d}(w, \mathbf{y}, \xi^u)$  has a unique fixed-point  $\Gamma(\xi^u, w) \in BC_\delta(\mathcal{D}_w^{2\gamma, \eta})$  for every  $\xi^u \in \mathcal{H}_u$ , meanwhile, for  $\xi_1^u, \xi_2^u \in \mathcal{H}_u$ , we have

$$\begin{aligned}
 & \left\| \Gamma(\xi_1^u, w) - \Gamma(\xi_2^u, w) \right\|_{BC_\delta(\mathcal{D}_w^{2\gamma, \eta})} \\
 & = \left\| J_{R,d}(w, \Gamma(\xi_1^u, w), \xi_1^u) - J_{R,d}(w, \Gamma(\xi_2^u, w), \xi_2^u) \right\|_{BC_\delta(\mathcal{D}_w^{2\gamma, \eta})} \\
 & \leq \left\| J_{R,d}(w, \Gamma(\xi_1^u, w), \xi_1^u) - J_{R,d}(w, \Gamma(\xi_1^u, w), \xi_2^u) \right\|_{BC_\delta(\mathcal{D}_w^{2\gamma, \eta})} \\
 & \quad + \left\| J_{R,d}(w, \Gamma(\xi_1^u, w), \xi_2^u) - J_{R,d}(w, \Gamma(\xi_2^u, w), \xi_2^u) \right\|_{BC_\delta(\mathcal{D}_w^{2\gamma, \eta})} \\
 & \leq \|S_{\cdot+i-1}^u(\xi_1^u - \xi_2^u), 0\|_{BC_\delta(\mathcal{D}_w^{2\gamma, \eta})} + \frac{1}{2} \left\| \Gamma(\xi_1^u, w) - \Gamma(\xi_2^u, w) \right\|_{BC_\delta(\mathcal{D}_w^{2\gamma, \eta})} \\
 & \leq C e^{(\alpha-\delta)} \|\xi_1^u - \xi_2^u\| + \frac{1}{2} \left\| \Gamma(\xi_1^u, w) - \Gamma(\xi_2^u, w) \right\|_{BC_\delta(\mathcal{D}_w^{2\gamma, \eta})},
 \end{aligned}$$

which implies that  $\Gamma(\xi^u, w)$  is Lipschitz continuous. □

At last, as similar discussion have taken place in [14, 22] and [24], we derive a local unstable manifold for our REEs (2.2). The proof of following results is identical to the one of [22] and [24], we omit here. In the following we denote  $B_{\mathcal{H}_u}(0, \rho(w))$  as a ball in  $\mathcal{H}_u$ , which is centered at 0 and has a random radius  $\rho(w)$ .

**Lemma 4.1** *The local unstable manifold of (2.2) is given by the graph of a Lipschitz function i.e.*

$$\mathcal{M}_{loc}^u(w) = \{\xi + h^u(\xi, w) : \xi \in B_{\mathcal{H}_u}(0, \rho(w))\}, \tag{4.16}$$

where,  $\rho(w)$  is a tempered from below random variable and

$$h^u(\xi, w) := \pi^s \Gamma(\xi, w)[-1, 1]|_{B_{\mathcal{H}_u}(0, \rho(w))},$$

that is

$$\begin{aligned} h^u(\xi, w) &= \sum_{k=-\infty}^0 S_{-k}^s \int_0^1 S_{1-u}^s \pi^s f(\Gamma(\xi, w)[k-1, u]) du \\ &+ \sum_{k=-\infty}^0 S_{-k}^s \int_0^1 S_{1-u}^s \pi^s g(\Gamma(\xi, w)[k-1, u]) d\Theta_{k-1} \mathbf{w}_u. \end{aligned}$$

According to previous analysis, we easily obtain:

**Theorem 4.2** *The local unstable manifold of (2.2) is given by the graph of a Lipschitz function i.e.*

$$\mathcal{M}_{loc}^u(w) = \{\xi + h^u(\xi, w) : \xi \in B_{\mathcal{H}_u}(0, \hat{\rho}(w))\},$$

where,  $\hat{\rho}(w)$  is a tempered from below random variable and

$$h^u(\xi, w) := \int_{-\infty}^0 S_{-u}^s \pi^s f(y_u) du + \int_{-\infty}^0 S_{-u}^s \pi^s g(y_u) d\mathbf{w}_u.$$

### 4.1 Example

Consider the 2mth order parabolic partial equation

$$\left\{ \begin{aligned} &dy(u, x) = (L_{2m} y_u(x) + \mu y_u(x) + f(y_u(x))) du + g(y_u(x)) dw_u(x), u \in [0, T], \\ &y(0) = \xi \in \mathcal{O}, \\ &\frac{\partial y}{\partial \nu}(u, x) = 0, (u, x) \in (0, T) \times \partial \mathcal{O}, k = 0, 1, \dots, m-1. \end{aligned} \right.$$

where  $\frac{\partial}{\partial \nu}$  stands for the normal derivative,  $\mathcal{O}$  is a bounded domain in  $\mathbb{R}^d$  with a smooth boundary,

$$-L_{2m} = \sum_{|\kappa| \leq 2m} a_\kappa(x) D^\kappa$$

is a uniformly elliptic operator with  $a_\kappa \in C^\infty(\bar{\mathcal{O}})$  and  $w$  is a  $\gamma$ -Hölder continuous path with  $1/3 < \gamma \leq 1/2$ .

We can consider the above equation as (2.2) in the space  $\mathcal{H} = L^2(\mathcal{O})$ . Let  $A = L_{2m} + \mu$ ,  $\text{Dom}(A) = H^{2m}(\mathcal{O}) \cap H_0^m(\mathcal{O})$  if  $2m > \frac{d}{2}$ , thus we have that  $\mathcal{H}_{-2\gamma} = H_0^{2m-2\gamma}(\mathcal{O})$  and the requirement about  $2m$  to such that  $2m > \frac{d}{2} + 4\gamma$ . As we all know that  $A$  has a compact resolvent and has countably many eigenvalues  $\lambda_j$  of finite multiplicity, that tend to  $-\infty$  when  $j \rightarrow \infty$ . In additional, the associated eigenfunctions  $\{e_j\}_{j \in \mathbb{N}}$  form an orthogonal basis of  $\mathcal{H}$ . Set  $\mu > 0$  sufficiently large such that there exists  $j^* \in \mathbb{N}$

$$\lambda_{j^*+1} \leq -\beta < 0 < \alpha \leq \lambda_{j^*}.$$

Let  $\mathcal{H}_u = \text{span}(e_j : \lambda_j \geq \alpha)$  and  $\mathcal{H}_s$  be its orthogonal complement space in  $\mathcal{H}$ . i.e.  $\mathcal{H}$  has an invariant splitting  $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s$ . Meanwhile, the nonlinear terms  $f$  and  $g$  satisfy our assumptions in (2.2).

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