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Boundedness of the differential transforms for the generalized Poisson operators generated by Laplacian

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Abstract

In this paper, we consider the boundedness of the differential transforms for the generalized Poisson operators associated with the Laplace operator Δ. The related results of the differential transforms for the heat semigroup are proved previously. By using the subordination formula method, we prove the boundedness of the maximal operator related to the differential transforms in weighted Lebesgue spaces. Moreover, we get some L^∞-behavior results and the local growth of the maximal operator related to the differential transforms. Also, we get some similar results of the differential transforms related to the generalized Poisson operators generated by Schrödinger operator $-\Delta + V$, where the nonnegative potential V belongs to the reverse Hölder class B_q with $q \geq n/2$.

Keywords Differential transforms · Generalized Poisson operator, Lacunary sequence · Schrödinger operator

Mathematics Subject Classification 42B20 · 42B25

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1 Introduction

Let $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ be the Laplace operator in $\mathbb{R}^n$. Consider its heat semigroup

$$e^{t\Delta}\varphi(x) = \int_{\mathbb{R}^n} W_t(x-y)\varphi(y)\mathrm{d}y, \ x \in \mathbb{R}^n, \quad t > 0,$$

where W is the Gauss–Weierstrass kernel

$$W_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}.$$

For more information related with this semigroup, see [17].

For $0 < \alpha < 1$, the generalized Poisson formula of f is given by

$$\begin{aligned} \mathcal{P}_t^\alpha f(x) &= \frac{t^{2\alpha}}{4^\alpha \Gamma(\alpha)} \int_0^{+\infty} e^{-t^2/(4s)} e^{s\Delta} f(x) \frac{\mathrm{d}s}{s^{1+\alpha}} \\ &= \frac{t^{2\alpha}}{4^\alpha \Gamma(\alpha)} \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{e^{-(t^2+|y|^2)/(4s)}}{(4\pi s)^{n/2}} f(x-y)\mathrm{d}y \frac{\mathrm{d}s}{s^{1+\alpha}}, \quad x \in \mathbb{R}^n, \ t > 0. \end{aligned} \tag{1.1}$$

It means that the generalized Poisson formula can be obtained via the heat semigroup $\{e^{t\Delta}\}_{t>0}$. In [6], Carffarelli and Silvestre studied the generalized Poisson formula to solve an extension problem. Stinga and Torrea defined this kind of Poisson formula for Hermite operator $L = -\Delta + |x|^2$ in [19]. In the case $\alpha = 1/2$, $\mathcal{P}_t^{1/2}$ is the Bochner subordinated Poisson semigroup of $e^{t\Delta}$; see [17].

Let $\{a_j\}_{j\in\mathbb{Z}}$ be an increasing sequence of positive real numbers, and $\{v_j\}_{j\in\mathbb{Z}}$ be a bounded sequence of real or complex numbers. Let $\{T_t\}_{t>0}$ be an operator sequence. We consider the differential transform series

$$\sum_{j\in\mathbb{Z}} v_j (T_{a_{j+1}} f(x) - T_{a_j} f(x)). \tag{1.2}$$

In [12], Jones and Rosenblatt studied the behavior of the series of the differences of ergodic averages and the differences of differentiation operators along lacunary sequences in the context of the L^p spaces. In [2], the authors solved these problems with a different approach, which relied heavily on the method of Calderón–Zygmund singular integrals (see [15]). In [3], the authors considered the series (1.2) with the Poisson operator related with translation semigroups $f(t-s)$.

In order to analyze the series

$$\sum_{j\in\mathbb{Z}} v_j (\mathcal{P}_{a_{j+1}}^\alpha f(x) - \mathcal{P}_{a_j}^\alpha f(x)),$$

where $\mathcal{P}_t^\alpha$ is the generalized Poisson operator defined in (1.1), we can consider the convergence of its partial sums. For each $N \in \mathbb{Z}^2,\ N = (N_1, N_2)$ with $N_1 < N_2$, we define the sum

$$\begin{aligned} T_N^\alpha f(x) &= \sum_{j=N_1}^{N_2} v_j (\mathcal{P}^\alpha_{a_{j+1}} f(x) - \mathcal{P}^\alpha_{a_j} f(x)) \\ &= \frac{1}{4^\alpha \Gamma(\alpha)} \sum_{j=N_1}^{N_2} v_j \int_{\mathbb{R}^n} \int_0^\infty \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} \frac{e^{-|y|^2/(4s)}}{(4\pi s)^{n/2}} f(x-y)\, \mathrm{d}s \mathrm{d}y \\ &= \frac{1}{4^\alpha \Gamma(\alpha)} \int_{\mathbb{R}^n} \int_0^{+\infty} \sum_{j=N_1}^{N_2} v_j \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} \frac{e^{-|y|^2/(4s)}}{(4\pi s)^{n/2}} \mathrm{d}s f(x-y) \mathrm{d}y. \end{aligned} \tag{1.3}$$

We denote the kernel of T_N^α by

$$K_N^\alpha(y) = \frac{1}{4^\alpha \Gamma(\alpha)} \int_0^{+\infty} \sum_{j=N_1}^{N_2} v_j \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} \frac{e^{-|y|^2/(4s)}}{(4\pi s)^{n/2}} \mathrm{d}s.$$

We shall also consider the maximal operators

$$T^* f(x) = \sup_N \left| T_N^\alpha f(x) \right|, \quad x \in \mathbb{R}^n,$$

where the supremum are taken over all $N = (N_1, N_2) \in \mathbb{Z}^2$ with $N_1 < N_2$. We shall consider the boundedness problem related to these operators. In [3], the authors proved the boundedness of the above operators related with the one-sided generalized Poisson type operator sequence.

Some of our results will be valid only when the sequence $\{a_j\}_{j\in\mathbb{Z}}$ is lacunary. It means that there exists a $\rho > 1$ such that $\dfrac{a_{j+1}}{a_j} \geq \rho,\ j \in \mathbb{Z}$. In particular, we shall prove the boundedness of the operators T^* in the weighted spaces $L^p(\mathbb{R}^n, \omega)$, where ω is the usual Muckenhoupt weights on $\mathbb{R}^n$. We refer the reader to the book by J. Duoandikoetxea [7, Chapter 7] for definitions and properties of the A_p classes. We have the following results:

Theorem 1 (a) *For any* $1 < p < \infty$ *and* $\omega \in A_p$*, there exists a constant* C *depending on* n, p, ρ, α *and* $\|v\|_{l^\infty(\mathbb{Z})}$ *such that*

$$\left\| T^* f \right\|_{L^p(\mathbb{R}^n,\omega)} \leq C \left\| f \right\|_{L^p(\mathbb{R}^n,\omega)},$$

for all functions $f \in L^p(\mathbb{R}^n, \omega)$.

(b) *For any* $\omega \in A_1$*, there exists a constant* C *depending on* n, ρ, α *and* $\|v\|_{l^\infty(\mathbb{Z})}$ *such that*

$$\omega\left(\{x \in \mathbb{R}^n : \left|T^* f(x)\right| > \lambda\}\right) \leq C \frac{1}{\lambda} \left\| f \right\|_{L^1(\mathbb{R}^n,\omega)}, \quad \lambda > 0,$$

for all functions $f \in L^1(\mathbb{R}^n, \omega)$.

(c) *Given* $f \in L^\infty(\mathbb{R}^n)$, *then either* $T^* f(x) = \infty$ *for all* $x \in \mathbb{R}^n$, *or* $T^* f(x) < \infty$ *for a.e.* $x \in \mathbb{R}^n$. *And in this latter case, there exists a constant* C *depending on* n, ρ, α *and* $\|v\|_{l^\infty(\mathbb{Z})}$ *such that*

$$\left\| T^* f \right\|_{BMO(\mathbb{R}^n)} \le C \, \|f\|_{L^\infty(\mathbb{R}^n)}.$$

(d) *Given* $f \in BMO(\mathbb{R}^n)$, *then either* $T^* f(x) = \infty$ *for all* $x \in \mathbb{R}^n$, *or* $T^* f(x) < \infty$ *for a.e.* $x \in \mathbb{R}^n$. *And in this latter case, there exists a constant* C *depending on* n, ρ, α *and* $\|v\|_{l^\infty(\mathbb{Z})}$ *such that*

$$\left\| T^* f \right\|_{BMO(\mathbb{R}^n)} \le C \, \|f\|_{BMO(\mathbb{R}^n)}. \tag{1.4}$$

Remark 1 From the conclusions we got in Theorem 1, for $f \in L^p(\mathbb{R}^n, \omega)$ with $\omega \in A_p$, in Theorem 8 we shall see that we can define Tf by the limit of $T_N f$ in L^p-norm

$$Tf(x) = \lim_{(N_1, N_2) \to (-\infty, +\infty)} T_N f(x), \qquad x \in \mathbb{R}^n.$$

In classical harmonic analysis, if $f = \chi_{(0,1)}$ and $\mathcal{H}$ is the Hilbert transform, it is easy to see that $\frac{1}{r}\int_{-r}^{0} \mathcal{H}(f)(x)\mathrm{d}x \sim \log\frac{e}{r}$ as $r \to 0^+$. In general, this is the growth of a singular integral applied to a bounded function at the origin. The following theorem shows that the growth of the function $T^* f$ for bounded function f at the origin is of the same order of a singular integral operator. Some related results about the local behavior of variation operators can be found in [1]. One-dimensional results about the variation of some convolutions operators can be found in [14]. And the one-dimensional results about the differential transforms of one-sided fractional Poisson type operator sequence is proved in [3]. In [4], the authors got local growth of the differential transforms of heat semigroup generated by Laplacian.

Theorem 2 (a) *Let* $\{v_j\}_{j\in\mathbb{Z}} \in l^p(\mathbb{Z})$ *for some* $1 \le p \le \infty$. *For every* $f \in L^\infty(\mathbb{R}^n)$ *with support in the unit ball* $B = B(0, 1)$, *for any ball* $B_r \subset B$ *with* $2r < 1$, *we have*

$$\frac{1}{|B_r|}\int_{B_r} \left| T^* f(x) \right| \mathrm{d}x \le C \left(\log \frac{2}{r} \right)^{1/p'} \|v\|_{l^p(\mathbb{Z})} \, \|f\|_{L^\infty(\mathbb{R}^n)}.$$

(b) *When* $1 < p < \infty$, *for any* $0 < \varepsilon < p - 1$, *there exist a* ρ*-lacunary sequence* $\{a_j\}_{j\in\mathbb{Z}}$, *a sequence* $\{v_j\}_{j\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$ *and a function* $f \in L^\infty(\mathbb{R}^n)$ *with support in the unit ball* $B = B(0, 1)$, *satisfying the following statement: for any ball* $B_r \subset B$ *with* $2r < 1$, *we have*

$$\frac{1}{|B_r|}\int_{B_r} \left| T^* f(x) \right| \mathrm{d}x \ge C \left(\log \frac{2}{r} \right)^{1/(p-\varepsilon)'} \|v\|_{l^p(\mathbb{Z})} \, \|f\|_{L^\infty(\mathbb{R}^n)}.$$

(c) *When $p=\infty$, there exist a ρ-lacunary sequence $\{a_j\}_{j\in\mathbb{Z}}$, a sequence $\{v_j\}_{j\in\mathbb{Z}}\in l^\infty(\mathbb{Z})$ and $f\in L^\infty(\mathbb{R}^n)$ with support in the unit ball $B=B(0,1)$, satisfying the following statements: for any ball $B_r\subset B$ with $2r<1$,*

$$\frac{1}{|B_r|}\int_{B_r}\left|T^*f(x)\right|\mathrm{d}x\geq C\left(\log\frac{2}{r}\right)\|v\|_{l^\infty(\mathbb{Z})}\,\|f\|_{L^\infty(\mathbb{R}^n)}.$$

In the statements above, $p'=\dfrac{p}{p-1}$, and if $p=1$, $p'=\infty$.

The statements in Theorem 2 shows that, when $1<p<\infty$, the growth of T^* is between the growth of the standard singular integral and the growth of the Hardy–Littlewood maximal operator. And when $p=\infty$, the growth of T^* is the same with the standard singular integral operator.

The organization of the paper is as follows: Sect. 2 is devoted to prove the boundedness of the maximal operators T^*. And we will give the proof of the local growth of T^*, i.e. Theorem 2, in Sect. 3. In Sect. 4, we will get some similar results in the Schrödinger setting.

Throughout this paper, the symbol C in an inequality always denotes a constant which may depend on some indices, but never on the functions f under consideration.

2 Proof of Theorem 1

In this section, we will prove Theorem 1. In order to prove Theorem 1, we need to prove the uniform boundedness of T_N^α first. By the Fourier transform, we can prove that the operators T_N^α are uniform bounded in $L^2(\mathbb{R}^n)$ for all $N\in\mathbb{Z}^2,\ N_1<N_2$. Since the kernel $K_N^\alpha(y,s)$ satisfies the size and smoothness conditions (see Theorem 5), we can deduce the L^p-boundedness results by using the Calderón–Zygmund theorem. Thus, we have the following results:

Theorem 3 *For the operator T_N^α defined in (1.3), we have the following statements.*

(a) *For any $1<p<\infty$ and $\omega\in A_p$, there exists a constant C depending on n, p, α and $\|v\|_{l^\infty(\mathbb{Z})}$ such that*

$$\left\|T_N^\alpha f\right\|_{L^p(\mathbb{R}^n,\omega)}\leq C\,\|f\|_{L^p(\mathbb{R}^n,\omega)},$$

for all functions $f\in L^p(\mathbb{R}^n,\omega)$.

(b) *For any $\omega\in A_1$, there exists a constant C depending on n, α and $\|v\|_{l^\infty(\mathbb{Z})}$ such that*

$$\omega\left(\{x\in\mathbb{R}^n:\left|T_N^\alpha f(x)\right|>\lambda\}\right)\leq C\frac{1}{\lambda}\,\|f\|_{L^1(\mathbb{R}^n,\omega)},\quad \lambda>0,$$

for all functions $f\in L^1(\mathbb{R}^n,\omega)$.

The constants C appeared above all are independent of N.

We shall use the Calderón–Zygmund theory in proving the L^p-boundedness of the differential transforms T_N^α associated with the generalized Poisson operators. We will prove the L^2-estimates first. And then, it remains to give the estimates about the kernels of the differential transforms. By a standard argument, the results in Theorem 3 will be obtained.

First, we present a lemma which will be used later.

Lemma 1 ([3, Lemma 2.1]) *Let* $0 < \alpha < 1$. *Then for any complex number* z_0 *with* $Rez_0 > 0$ *and* $|\arg z_0| \le \pi/4$, *we have*

$$\int_0^{+\infty} e^{-z_0 u} e^{-\frac{z_0}{u}} \frac{\mathrm{d}u}{u^\alpha} = z_0^{1-\alpha} \int_0^{+\infty} \frac{e^{-r} e^{-z_0^2/r}}{r^{2-\alpha}} \mathrm{d}r.$$

Now we present the uniform L^2-boundedness of the operator T_N^α in the following theorem:

Theorem 4 *There exists a constant* $C > 0$, *depending on* n, α *and* $\|v\|_{l^\infty(\mathbb{Z})}$, *such that*

$$\sup_N \|T_N^\alpha f\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^2(\mathbb{R}^n)}.$$

Proof Let $f \in L^2(\mathbb{R}^n)$. Using the Plancherel theorem, we have

$$\left\|T_N^\alpha f\right\|_{L^2(\mathbb{R}^n)} = \left\| \sum_{j=N_1}^{N_2} v_j (\mathcal{P}_{a_{j+1}}^\alpha f - \mathcal{P}_{a_j}^\alpha f) \right\|_{L^2(\mathbb{R}^n)} \le C\, \|v\|_{l^\infty(\mathbb{Z})} \left\| \sum_{j=-\infty}^{\infty} \int_{a_j}^{a_{j+1}} \left| \partial_t \widehat{\mathcal{P}_t^\alpha f} \right| \mathrm{d}t \right\|_{L^2(\mathbb{R}^n)}.$$

By using the second identity in (1.1), we have

$$\begin{aligned} \partial_t \widehat{\mathcal{P}_t^\alpha f}(\xi) &= C \partial_t \int_0^\infty e^{-r} \widehat{e^{-\frac{t^2}{4r}\Delta} f}(\xi) \frac{\mathrm{d}r}{r^{1-\alpha}} \\ &= C \partial_t \int_0^\infty e^{-r} e^{-\frac{t^2}{4r}|\xi|^2} \widehat{f}(\xi) \frac{\mathrm{d}r}{r^{1-\alpha}} \\ &= C \int_0^\infty e^{-r} t|\xi|^2 e^{-\frac{t^2}{4r}|\xi|^2} \widehat{f}(\xi) \frac{\mathrm{d}r}{r^{2-\alpha}}. \end{aligned}$$

Note that the Fourier transform above can be well defined. Then we deduce that

$$\left\|T_N^\alpha f\right\|_{L^2(\mathbb{R}^n)} \le C \left\| \widehat{f}(\xi) \int_0^\infty \left| \int_0^\infty e^{-r} t|\xi|^2 e^{-\frac{t^2}{4r}|\xi|^2} \frac{\mathrm{d}r}{r^{2-\alpha}} \right| \mathrm{d}t \right\|_{L^2(\mathbb{R}^n)}.$$

Thus again by the Plancherel theorem, the remainder is devoted to prove the uniform boundedness of the multiplier

$$\left|\widehat{K_N^\alpha}(\xi)\right| \le C \left| \int_0^\infty \left| \int_0^\infty e^{-r} t|\xi|^2 e^{-\frac{t^2}{4r}|\xi|^2} \frac{\mathrm{d}r}{r^{2-\alpha}} \right| \mathrm{d}t \right| \le C, \quad \xi \in \mathbb{R}^n.$$

Taking $z_0 = t|\xi|$, we rewrite the above inequality in

$$\left|\widehat{K_N^\alpha}(\xi)\right| \le C \int_0^\infty \left| \int_0^\infty e^{-r} z_0 e^{-\frac{z_0^2}{4r}} \frac{\mathrm{d}r}{r^{2-\alpha}} \right| \mathrm{d}z_0, \quad \xi \in \mathbb{R}^n.$$

By Lemma 1, for any $\xi \in \mathbb{R}^n$, we have

$$\int_0^\infty \left| \int_0^\infty e^{-r} z_0 e^{-\frac{z_0^2}{4r}} \frac{\mathrm{d}r}{r^{2-\alpha}} \right| \mathrm{d}z_0 = 2^{1-\alpha} \int_0^\infty \left| z_0^\alpha \int_0^\infty e^{-\frac{z_0}{2u}} e^{-\frac{z_0}{2}u} \frac{\mathrm{d}u}{u^\alpha} \right| \mathrm{d}z_0.$$

Since $|\arg z_0| \le \pi/4$, we have $|e^{-z_0/(2u)}| \le e^{-c|z_0|/u}$ and $|e^{-z_0 u/2}| \le e^{-c|z_0|u}$, where $c = \sqrt{2}/4$. Then

$$\begin{aligned}\left| \int_0^\infty z_0^\alpha \int_0^\infty e^{-z_0/u} e^{-z_0 u} \frac{\mathrm{d}u}{u^\alpha} \mathrm{d}z_0 \right| &\le \int_0^\infty |z_0|^\alpha \int_0^\infty e^{-c|z_0|/u} e^{-c|z_0|u} \frac{\mathrm{d}u}{u^\alpha} \mathrm{d}z_0 \\ &\le \int_0^\infty |z_0|^{2\alpha-1} \int_0^\infty e^{-c|z_0|^2/v} e^{-cv} \frac{\mathrm{d}v}{v^\alpha} \mathrm{d}z_0.\end{aligned}$$

Recall that $z_0 = t|\xi|$. Then, we have

$$\begin{aligned}&\int_0^\infty |z_0|^{2\alpha-1} \int_0^\infty e^{-c|z_0|^2/v} e^{-cv} \frac{\mathrm{d}v}{v^\alpha} \mathrm{d}z_0 \\ &= \int_0^\infty |\xi|^{2\alpha} t^{2\alpha-1} \int_0^\infty e^{-c(|\xi|t)^2/v} e^{-cv} \frac{\mathrm{d}v}{v^\alpha} \mathrm{d}t \\ &= \int_0^\infty \int_0^\infty (|\xi|t)^{2\alpha-1} e^{-c(|\xi|t)^2/v} d(|\xi|t) e^{-cv} \frac{\mathrm{d}v}{v^\alpha} \\ &= \int_0^\infty \int_0^\infty t^{2\alpha-1} e^{-ct^2/v} \mathrm{d}t e^{-cv} \frac{dv}{v^\alpha} \le C \int_0^\infty e^{-cv} \mathrm{d}v \le C,\end{aligned}$$

where the constants C appeared above all are independent of N. Then the proof of the theorem is complete. □

Also, we can get the kernel estimates in the following:

Theorem 5 *There exists constant $C > 0$ depending on n, α and $\|v\|_{l^\infty(\mathbb{Z})}$(not on N) such that, for any $y \ne 0$,*

(i) $|K_N^\alpha(y)| \le \dfrac{C}{|y|^n}$,

(ii) $|\nabla_y K_N^\alpha(y)| \le \dfrac{C}{|y|^{n+1}}$.

Proof *i)* This is the size condition of the kernel. We have

$$|K_N^\alpha(y)| \le C \int_0^{+\infty} \sum_{j=-\infty}^{\infty} \left| \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} \frac{e^{-|y|^2/(4s)}}{s^{n/2}} \right| \mathrm{d}s$$

$$= C \int_0^{+\infty} \sum_{j=-\infty}^{\infty} \left| a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)} \right| \frac{e^{-|y|^2/(4s)}}{s^{1+\alpha+n/2}} \mathrm{d}s.$$

Observe that

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} \left| a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)} \right| &= \sum_{j=-\infty}^{\infty} \left| \int_{a_j}^{a_{j+1}} \partial_u \left(u^{2\alpha} e^{-u^2/(4s)} \right) \mathrm{d}u \right| \\
&\leq \int_0^{+\infty} \left| (2\alpha u^{2\alpha-1} - \frac{u^{2\alpha+1}}{2s}) e^{-u^2/(4s)} \right| \mathrm{d}u \leq C \int_0^{+\infty} \left| (u^{2\alpha-1} + \frac{u^{2\alpha+1}}{2s}) e^{-u^2/(4s)} \right| \mathrm{d}u \\
&\leq C\sqrt{s} \Big(\int_0^{+\infty} (\sqrt{s})^{2\alpha-1} \left(\frac{u}{\sqrt{s}} \right)^{2\alpha-1} e^{-\frac{1}{4}(u/\sqrt{s})^2} d\frac{u}{\sqrt{s}} \\
&\quad + s^{\alpha-1/2} \int_0^{+\infty} \left(\frac{u}{\sqrt{s}} \right)^{2\alpha+1} e^{-\frac{1}{4}(u/\sqrt{s})^2} d\frac{u}{\sqrt{s}} \Big) \\
&\leq C s^{\alpha}. \qquad (2.1)
\end{aligned}$$

Then, we have

$$\begin{aligned}
|K_N^{\alpha}(y)| \leq C \int_0^{+\infty} \frac{e^{-|y|^2/(4s)}}{s^{n/2+1}} \mathrm{d}s &= C \left(\int_0^{|y|^2} + \int_{|y|^2}^{+\infty} \right) \frac{e^{-|y|^2/(4s)}}{s^{n/2+1}} \mathrm{d}s \\
&\leq C \left(\int_0^{|y|^2} \frac{1}{|y|^{n+2}} \mathrm{d}s + \int_{|y|^2}^{+\infty} \frac{1}{s^{n/2+1}} \mathrm{d}s \right) \leq \frac{C}{|y|^n}.
\end{aligned}$$

ii) It suffices to prove that for the first variable $y_1 \in \mathbb{R}$, we have

$$|\partial_{y_1} K_N^{\alpha}(y)| \leq \frac{C}{|y|^{n+1}},$$

where

$$\partial_{y_1} K_N^{\alpha}(y) = -\int_0^{+\infty} \sum_{j=N_1}^{N_2} v_j \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} \frac{y_1 e^{-|y|^2/(4s)}}{2s^{n/2+1}} \mathrm{d}s.$$

Then, by (2.1) we conclude that

$$\begin{aligned}
|\partial_{y_1} K_N^{\alpha}(y)| &\leq C \int_0^{+\infty} s^{\alpha} \frac{y_1 e^{-|y|^2/(4s)}}{s^{2+\alpha+n/2}} \mathrm{d}s \leq C \int_0^{+\infty} \frac{\frac{|y|}{\sqrt{s}} e^{-|y|^2/(4s)}}{s^{n/2+3/2}} \mathrm{d}s \\
&\leq C \int_0^{+\infty} \frac{e^{-|y|^2/(8s)}}{s^{n/2+3/2}} \mathrm{d}s = C \left(\int_0^{|y|^2} + \int_{|y|^2}^{+\infty} \right) \frac{e^{-|y|^2/(8s)}}{s^{n/2+3/2}} \mathrm{d}s \leq \frac{C}{|y|^{n+1}}.
\end{aligned}$$

The proof of the theorem is complete. □

Remark 2 If we consider an $l^\infty(\mathbb{Z}^2)$-valued operator $Q : f \mapsto \left\{T_N^\alpha f(x)\right\}_{N\in\mathbb{Z}^2}$ on the homogeneous space $(\mathbb{R}^n, d, \mathrm{d}x)$, then $T_\alpha^* f(x) = \|Qf(x)\|_{l^\infty(\mathbb{R}^n)}$, and by Theorem 5, we know that the kernel of the operator Q is an $l^\infty(\mathbb{Z}^2)$-valued Calderón–Zygmund kernel.

In the next result, we will take care of the behavior of T_N^α on $BMO(\mathbb{R}^n)$.

Theorem 6 *Let $\{a_j\}_{j\in\mathbb{Z}}$ be an increasing sequence. There exists a constant C depending on n, α and $\|v\|_{\ell^\infty(\mathbb{Z})}$(not on N) such that*

$$\left\|T_N^\alpha f\right\|_{BMO(\mathbb{R}^n)} \le C\, \|f\|_{L^\infty(\mathbb{R}^n)}\,,$$

and

$$\left\|T_N^\alpha f\right\|_{BMO(\mathbb{R}^n)} \le C\, \|f\|_{BMO(\mathbb{R}^n)}\,.$$

Proof The finiteness of T_N^α for functions in $L^\infty(\mathbb{R}^n)$ is obvious, since for each N, K_N^α is an integrable function. On the other hand, assume that $f \in BMO(\mathbb{R}^n)$. Let $B = B(x_0, r_0)$ and $B^* = B(x_0, 2r_0)$ with $x_0 \in \mathbb{R}^n$ and $r_0 > 0$. We decompose f to be

$$f = (f - f_B)\chi_{B^*} + (f - f_B)\chi_{(B^*)^c} + f_B = f_1 + f_2 + f_3.$$

By Theorem 4, we have

$$\int_{\mathbb{R}^n} \left|T_N^\alpha f_1\right|^2 \mathrm{d}x \le C\, \|f_1\|^2_{L^2(\mathbb{R}^n)} \le C \int_{B^*} |f(x) - f_B|^2\, \mathrm{d}x \le C|B|\, \|f\|^2_{BMO(\mathbb{R}^n)}\,. \tag{2.2}$$

This means that $T_N^\alpha f_1(x) < \infty$, $a.e.\ x \in \mathbb{R}^n$. And we should note that $T_N^\alpha f_3(x) \equiv 0$, since $\mathcal{P}_{a_j}^\alpha f_3 \equiv f_B$ for any $j \in \mathbb{Z}$. For $T_N^\alpha f_2$, we note that, for any $x \in B$ and $t > 0$,

$$\begin{aligned}
\mathcal{P}_t^\alpha f_2(x) &= \frac{t^{2\alpha}}{4^\alpha\Gamma(\alpha)} \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{e^{-(t^2+|x-y|^2)/(4s)}}{(4\pi s)^{n/2}} f_2(y)\mathrm{d}y \frac{\mathrm{d}s}{s^{1+\alpha}} \\
&\le C\frac{t^{2\alpha}}{4^\alpha\Gamma(\alpha)} \int_0^{+\infty} \sum_{k=1}^{+\infty} \int_{2^k r_0 < |x_0-y| \le 2^{k+1} r_0} \frac{1}{|x-y|^{n+2\alpha'}}\, |f(y) - f_B|\, \mathrm{d}y\, e^{-\frac{t^2}{4s}} \frac{\mathrm{d}s}{s^{1+\alpha-\alpha'}} \\
&\le C\frac{t^{2\alpha'}\Gamma(\alpha-\alpha')}{4^{\alpha'}\Gamma(\alpha)} \sum_{k=1}^{+\infty} \left(2^k r_0\right)^{-2\alpha'} \frac{1}{\left(2^k r_0\right)^n} \int_{|x_0-y|\le 2^{k+1} r_0} |f(y) - f_B|\, \mathrm{d}y \\
&\le C\frac{t^{2\alpha'}\Gamma(\alpha-\alpha')}{4^{\alpha'}\Gamma(\alpha)} \sum_{k=1}^{+\infty} \left(2^k r_0\right)^{-2\alpha'} (1+2k)\, \|f\|_{BMO(\mathbb{R}^n)} < \infty,
\end{aligned}$$

where $0 < \alpha' < \alpha$. So, $\mathcal{P}_t^\alpha f_2(x)$ is well defined for $x \in B$ and $t > 0$. Since $T_N^\alpha f_2(x)$ is a finite summation and x_0, r_0 is arbitrary, $T_N^\alpha f_2(x) < \infty\ a.e.\ x \in \mathbb{R}^n$. Hence, $T_N^\alpha f(x) < \infty\ a.e.\ x \in \mathbb{R}^n$.

Now, let us prove the two inequalities. Since $L^\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$, we only need to prove the inequality in the case of $f \in BMO(\mathbb{R}^n)$. Choose some $x_1 \in B(x_0, r_0)$ such that $T_N^\alpha f_2(x_1) < \infty$. Now, taking a constant $c_B = T_N^\alpha f_2(x_1)$, we can write

$$\begin{aligned}
&\frac{1}{|B|}\int_B \left|T_N^\alpha f(x) - c_B\right| \mathrm{d}x \\
&= \frac{1}{|B|}\int_B \left|T_N^\alpha (f_1+f_2+f_3)(x) - T_N^\alpha f_2(x_1)\right| \mathrm{d}x \\
&\le \frac{1}{|B|}\int_B \left|T_N^\alpha f_1(x)\right| \mathrm{d}x + \frac{1}{|B|}\int_B \left|T_N^\alpha f_2(x) - T_N^\alpha f_2(x_1)\right| \mathrm{d}x \\
&=: I_1 + I_2.
\end{aligned}$$

For the first term I_1, by Hölder's inequality and (2.2) we have

$$I_1 = \frac{1}{|B|}\int_B \left|T_N^\alpha f_1(x)\right| \mathrm{d}x \le \left(\frac{1}{|B|}\int_B \left|T_N^\alpha f_1(x)\right|^2 \mathrm{d}x\right)^{1/2} \le C\, \|f\|_{BMO(\mathbb{R}^n)}.$$

For the term I_2, by Theorem 5 ii) we have

$$\begin{aligned}
I_2 &= \frac{1}{|B|}\int_B \left|T_N^\alpha f_2(x) - T_N^\alpha f_2(x_1)\right| \mathrm{d}x \\
&= \frac{1}{|B|}\int_B \left|\int_{\mathbb{R}^n} \left(K_N^\alpha(x-y) - K_N^\alpha(x_1-y)\right) f_2(y)\mathrm{d}y\right| \mathrm{d}x \\
&\le \frac{1}{|B|}\int_B \int_{\mathbb{R}^n} \left|K_N^\alpha(x-y) - K_N^\alpha(x_1-y)\right| |f_2(y)|\, \mathrm{d}y\mathrm{d}x \\
&\le C\frac{1}{|B|}\int_B \int_{(B^*)^c} \frac{|x-x_1|}{|x-y|^{n+1}} |f(y) - f_B|\, \mathrm{d}y\mathrm{d}x \\
&\le C\frac{r_0}{|B|}\int_B \int_{(B^*)^c} \frac{1}{(|y-x_0| - |x_0-x|)^{n+1}} |f(y) - f_B|\, \mathrm{d}y\mathrm{d}x \\
&= C\frac{r_0}{|B|}\int_B \sum_{k=1}^{+\infty} \int_{2^k r_0 \le |y-x_0| < 2^{k+1} r_0} \frac{1}{(|y-x_0| - |x_0-x|)^{n+1}} |f(y) - f_B|\, \mathrm{d}y\mathrm{d}x \\
&\le C \sum_{k=1}^{+\infty} \frac{1}{2^k} \frac{1}{(2^k r_0)^n} \int_{|y-x_0| < 2^{k+1} r_0} |f(y) - f_B|\, \mathrm{d}y \\
&\le C\, \|f\|_{BMO(\mathbb{R}^n)}.
\end{aligned}$$

Hence, we deduce

$$\frac{1}{|B|}\int_B \left|T_N^\alpha f(x) - c_B\right| \mathrm{d}x \le C\, \|f\|_{BMO(\mathbb{R}^n)}.$$

Thus, we proved that $T_N^\alpha f \in BMO(\mathbb{R}^n)$ and

$$\left\| T_N^\alpha f \right\|_{BMO(\mathbb{R}^n)} \leq C \left\| f \right\|_{BMO(\mathbb{R}^n)} .$$

□

In the following, we aim to prove Theorem 1. The next proposition, parallel to Proposition 3.2 in [2](also Proposition 3.1 in [3]), shows that, without lost of generality, we may assume that

$$1 < \rho \leq \frac{a_{j+1}}{a_j} \leq \rho^2, \quad j \in \mathbb{Z}.$$

Proposition 1 *Given a ρ-lacunary sequence $\{a_j\}_{j\in\mathbb{Z}}$ and a multiplying sequence $\{v_j\}_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$, we can define a ρ-lacunary sequence $\{\eta_j\}_{j\in\mathbb{Z}}$ and $\{\omega_j\}_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ verifying the following properties:*

(i) $1 < \rho \leq \eta_{j+1}/\eta_j \leq \rho^2, \qquad \left\|\{\omega_j\}\right\|_{\ell^\infty(\mathbb{Z})} = \left\|\{v_j\}\right\|_{\ell^\infty(\mathbb{Z})}$.

(ii) *For all $N = (N_1, N_2)$ there exists $N' = (N_1', N_2')$ with $T_N^\alpha = \hat{T}_{N'}^\alpha$, where $\hat{T}_{N'}^\alpha$ is the operator defined in (1.3) with the new sequences $\{\eta_j\}_{j\in\mathbb{Z}}$ and $\{\omega_j\}_{j\in\mathbb{Z}}$.*

In order to prove Theorem 1, we need a Cotlar's type inequality. For any $M \in \mathbb{Z}^+$, let

$$T_M^* f(x) = \sup_{-M\leq N_1<N_2\leq M} |T_N f(x)|, \quad x \in \mathbb{R}^n,$$

where T_N denotes the differential transform operator related with the heat-diffusion semigroup generated by $-\Delta$. By a similar(in fact, easier) argument as in the proof of Theorem 4, we can prove that T_N is uniform bounded on $L^2(\mathbb{R}^n)$. Also, we can prove that T_N is uniform bounded in $L^p(\mathbb{R}^n, \omega)$ for $1 < p < \infty$, uniform weak-(1, 1) bounded and uniform BMO-bounded, because it is a Calderón–Zygmund operator. For these results, see [4].

Theorem 7 (See [4, Theorem 2.4]) *For each $q \in (1, +\infty)$, there exists a constant C depending on n, $\|v\|_{l^\infty(\mathbb{Z})}$ and ρ such that for every $x \in \mathbb{R}^n$ and every $M \in \mathbb{Z}^+$,*

$$T_M^* f(x) \leq C \left\{ \mathcal{M}(T_{-M,M} f)(x) + \mathcal{M}_q f(x) \right\},$$

where

$$T_{-M,M} f(x) = \sum_{j=-M}^{M} v_j (e^{a_{j+1}\Delta} f(x) - e^{a_j\Delta} f(x))$$

and

$$\mathcal{M}_q f(x) = \sup_{\varepsilon>0} \left(\frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} |f(y)|^q \, dy \right)^{\frac{1}{q}}, \quad 1 < q < \infty.$$

Now, we are in a position to prove Theorem 1.

Proof of Theorem 1 For each $\omega \in A_p$, choose $1 < q < p < \infty$ such that $\omega \in A_{p/q}$. Then, it is well known that the maximal operators $\mathcal{M}$ and $\mathcal{M}_q$ are bounded on $L^p(\mathbb{R}^n, \omega)$. On the other hand, since the operators T_N are uniformly bounded in $L^p(\mathbb{R}^n, \omega)$ with $\omega \in A_p$. Hence, by Theorem 7, we have

$$\begin{aligned}\left\|T_M^* f\right\|_{L^p(\mathbb{R}^n,\omega)} &\leq C\left(\left\|\mathcal{M}(T_{-M,M} f)\right\|_{L^p(\mathbb{R}^n,\omega)} + \left\|\mathcal{M}_q f\right\|_{L^p(\mathbb{R}^n,\omega)}\right)\\ &\leq C\left(\left\|T_{-M,M} f\right\|_{L^p(\mathbb{R}^n,\omega)} + \|f\|_{L^p(\mathbb{R}^n,\omega)}\right) \leq C\,\|f\|_{L^p(\mathbb{R}^n,\omega)}.\end{aligned}$$

Note that the constants C appeared above do not depend on M. Consequently, letting M increase to infinity, we get the proof of the L^p boundedness of the maximal operator T_Δ^*, where $T_\Delta^* f(x) = \sup_N |T_N f(x)|$.

We should note that

$$\begin{aligned}T^* f(x) &= \sup_{N\in\mathbb{Z}^2}\left|T_N^\alpha f(x)\right| = \sup_{N\in\mathbb{Z}^2}\left|\sum_{j=N_1}^{N_2} v_j(\mathcal{P}^\alpha_{a_{j+1}} f(x) - \mathcal{P}^\alpha_{a_j} f(x))\right|\\ &= C_\alpha \sup_{N\in\mathbb{Z}^2}\left|\sum_{j=N_1}^{N_2} v_j\left(\int_0^{+\infty} e^{-s}\left(e^{-\frac{a_{j+1}^2}{4s}\Delta} f(x) - e^{-\frac{a_j^2}{4s}\Delta} f(x)\right)\frac{ds}{s^{1-\alpha}}\right)\right|\\ &\leq C_\alpha \int_0^{+\infty} e^{-s}\sup_{N\in\mathbb{Z}^2}\left|\sum_{j=N_1}^{N_2} v_j\left(e^{-\frac{a_{j+1}^2}{4s}\Delta} f(x) - e^{-\frac{a_j^2}{4s}\Delta} f(x)\right)\right|\frac{ds}{s^{1-\alpha}}\\ &= C_{\alpha,\rho,v,n}\int_0^{+\infty} e^{-s}\bar{T}_\Delta^* f(x)\frac{ds}{s^{1-\alpha}},\end{aligned}$$

where the operator $\bar{T}_\Delta^*$(which is bounded on $L^p(\mathbb{R}^n, \omega)$ and the boundedness constant is not depending on s) denotes the maximal differential transform related with $\{v_j\}_{j\in\mathbb{Z}}$ and ρ^2-lacunary sequence $\left\{a_j^2/4s\right\}_{j\in\mathbb{Z}}$. Then,

$$\left\|T^* f\right\|_{L^p(\mathbb{R}^n,\omega)} \leq C_{\alpha,\rho,v,n}\int_0^{+\infty} e^{-s}\left\|\bar{T}_\Delta^* f\right\|_{L^p(\mathbb{R}^n,\omega)}\frac{ds}{s^{1-\alpha}} \leq C_{\alpha,\rho,v,n}\,\|f\|_{L^p(\mathbb{R}^n,\omega)}.$$

This completes the proof of part (a) of the theorem.

In order to prove (b), we consider the $\ell^\infty(\mathbb{Z}^2)$-valued operator $\mathcal{T} f(x) = \{T_N^\alpha f(x)\}_{N\in\mathbb{Z}^2}$. Since $\|\mathcal{T} f(x)\|_{\ell^\infty(\mathbb{Z}^2)} = T^* f(x)$, by using (a) we know that the operator $\mathcal{T}$ is bounded from $L^p(\mathbb{R}^n, \omega)$ into $L^p_{\ell^\infty(\mathbb{Z}^2)}(\mathbb{R}^n, \omega)$, for every $1 < p < \infty$ and $\omega \in A_p$. The kernel of the operator $\mathcal{T}$ is given by $\mathcal{K}^\alpha(x) = \{K_N^\alpha(x)\}_{N\in\mathbb{Z}^2}$. By Theorem 5 and the vector valued version of Theorem 7.12 in [7], we get that the operator $\mathcal{T}$ is bounded from $L^1(\mathbb{R}^n, \omega)$ into weak- $L^1_{\ell^\infty(\mathbb{Z}^2)}(\mathbb{R}^n, \omega)$ for $\omega \in A_1$. Hence, as $\|\mathcal{T} f(x)\|_{\ell^\infty(\mathbb{Z}^2)} = T^* f(x)$, we get the proof of (b).

For (c), we shall prove that if $f \in L^{\infty}(\mathbb{R}^n)$ and there exists $x_0 \in \mathbb{R}^n$ such that $T^*f(x_0) < \infty$, then $T^*f(x) < \infty$ for $a.e.$ $x \in \mathbb{R}^n$. Given $x \neq x_0$. Set $f_1 = f\chi_{B(x_0,4|x-x_0|)}$ and $f_2 = f - f_1$. Note that T^* is L^p-bounded for any $1 < p < \infty$. Then $T^*f_1(x) < \infty$, because $f_1 \in L^p(\mathbb{R}^n)$ for any $1 < p < \infty$. On the other hand, by Theorem 5 we have

$$\begin{aligned}
&\left|T_N^\alpha f_2(x) - T_N^\alpha f_2(x_0)\right| \\
&= \left|\int_{\mathbb{R}^n} K_N^\alpha(x,y) f_2(y)\mathrm{d}y - \int_{\mathbb{R}^n} K_N^\alpha(x_0,y) f_2(y)\mathrm{d}y\right| \\
&= \left|\int_{B^c(x_0,4|x-x_0|)} \left(K_N^\alpha(x,y) - K_N^\alpha(x_0,y)\right) f(y)\mathrm{d}y\right| \\
&\le C\int_{B^c(x_0,4|x-x_0|)} \frac{|x-x_0|}{|y-x_0|^{n+1}}\,|f(y)|\,\mathrm{d}y \\
&\le C\,\|f\|_{L^\infty(\mathbb{R})} < +\infty.
\end{aligned}$$

Hence

$$\left\|T_N^\alpha f_2(x) - T_N^\alpha f_2(x_0)\right\|_{l^\infty(\mathbb{Z}^2)} \le C\,\|f\|_{L^\infty(\mathbb{R}^n)}$$

and therefore $T^*f(x) = \left\|T_N^\alpha f(x)\right\|_{l^\infty(\mathbb{Z}^2)} \le C < \infty$. For the $L^\infty - BMO$ boundedness, we will prove it later.

(d) Let $x_0 \in \mathbb{R}^n$ be one point such that $T^*f(x_0) < \infty$. Set $B = B(x_0, 4|x - x_0|)$ with $x \neq x_0$. And we decompose f to be

$$f = (f - f_B)\chi_B + (f - f_B)\chi_{B^c} + f_B =: f_1 + f_2 + f_3.$$

Note that T^* is L^p-bounded for any $1 < p < \infty$. Then $T^*f_1(x) < \infty$, because $f_1 \in L^p(\mathbb{R}^n)$, for any $1 < p < \infty$. And $T^*f_3 = 0$, since $\mathcal{P}_{a_j}^\alpha f_3 = f_3$ for any $j \in \mathbb{Z}$. On the other hand, by Theorem 5 we have

$$\begin{aligned}
&\left|T_N^\alpha f_2(x) - T_N^\alpha f_2(x_0)\right| \\
&= \left|\int_{\mathbb{R}^n} K_N^\alpha(x,y) f_2(y)\mathrm{d}y - \int_{\mathbb{R}^n} K_N^\alpha(x_0,y) f_2(y)\mathrm{d}y\right| \\
&= \left|\int_{B^c} \left(K_N^\alpha(x,y) - K_N^\alpha(x_0,y)\right) f_2(y)\mathrm{d}y\right| \\
&\le C\int_{B^c} \frac{|x-x_0|}{|y-x_0|^{n+1}}\,|f(y) - f_B|\,\mathrm{d}y \\
&\le C\sum_{k=1}^{+\infty} |x-x_0| \int_{2^kB\setminus 2^{k-1}B} \frac{|f(y)-f_B|}{|y-x_0|^{n+1}}\mathrm{d}y \\
&\le C\sum_{k=1}^{+\infty} \frac{|x-x_0|}{(2^{k+1}|x-x_0|)^{n+1}} \int_{2^kB} |f(y)-f_B|\mathrm{d}y
\end{aligned}$$

$$\leq C\sum_{k=1}^{+\infty}2^{-(k+1)}\frac{1}{|2^kB|}\int_{2^kB}\left(\left|f(y)-f_{2^kB}\right|+\sum_{l=1}^{k}\left|f_{2^lB}-f_{2^{l-1}B}\right|\right)\mathrm{d}y$$
$$\leq C\sum_{k=1}^{+\infty}2^{-(k+1)}\frac{1}{|2^kB|}\int_{2^kB}\left(\left|f(y)-f_{2^kB}\right|+2k\,\|f\|_{BMO(\mathbb{R}^n)}\right)\mathrm{d}y$$
$$\leq C\sum_{k=1}^{+\infty}2^{-(k+1)}(1+2k)\,\|f\|_{BMO(\mathbb{R}^n)}$$
$$\leq C\,\|f\|_{BMO(\mathbb{R}^n)}\,,$$

where $2^kB=B(x_0,2^k\cdot 4|x-x_0|)$ for any $k\in\mathbb{N}$. Hence

$$\left\|T_N^\alpha f_2(x)-T_N^\alpha f_2(x_0)\right\|_{l^\infty(\mathbb{Z}^2)}\leq C\,\|f\|_{BMO(\mathbb{R}^n)}$$

and therefore $T^*f(x)=\left\|T_N^\alpha f(x)\right\|_{l^\infty(\mathbb{Z}^2)}\leq C<\infty$.

Now, we shall prove the estimate (1.4) for functions such that $T^*f(x)<\infty$ *a.e.* For any $r>0$ and x_0 such that $T^*f(x_0)<\infty$, consider the ball $B=B(x_0,r)$ and $f_B=\frac{1}{|B|}\int_B f(x)\mathrm{d}x$. Let

$$f=(f-f_B)\chi_{2B}+(f-f_B)\chi_{(2B)^c}+f_B=:f_1+f_2+f_3.$$

We have $T^*f_3(x)=0$. Then,

$$\frac{1}{|B|}\int_B\left|T^*f(x)-(T^*f)_B\right|\mathrm{d}x=\frac{1}{|B|}\int_B\left|\frac{1}{|B|}\int_B\left(T^*f(x)-T^*f(y)\right)\mathrm{d}y\right|\mathrm{d}x$$
$$\leq\frac{1}{|B|^2}\int_B\int_B\left|T^*f(x)-T^*f(y)\right|\mathrm{d}y\mathrm{d}x$$
$$=\frac{1}{|B|^2}\int_B\int_B\left|\left\|T_N^\alpha f(x)\right\|_{l^\infty(\mathbb{Z}^2)}-\left\|T_N^\alpha f(y)\right\|_{l^\infty(\mathbb{Z}^2)}\right|\mathrm{d}y\mathrm{d}x$$
$$\leq\frac{1}{|B|^2}\int_B\int_B\left\|T_N^\alpha f(x)-T_N^\alpha f(y)\right\|_{l^\infty(\mathbb{Z}^2)}\mathrm{d}y\mathrm{d}x$$
$$\leq\frac{1}{|B|^2}\int_B\int_B\left\|T_N^\alpha f_1(x)-T_N^\alpha f_1(y)\right\|_{l^\infty(\mathbb{Z}^2)}\mathrm{d}y\mathrm{d}x$$
$$+\frac{1}{|B|^2}\int_B\int_B\left\|T_N^\alpha f_2(x)-T_N^\alpha f_2(y)\right\|_{l^\infty(\mathbb{Z}^2)}\mathrm{d}y\mathrm{d}x$$
$$=:I+II.$$

The Hölder inequality and L^2-boundedness of T^* imply that

$$I\leq\frac{1}{|B|}\int_B\left\|T_N^\alpha f_1(x)\right\|_{l^\infty(\mathbb{Z}^2)}\mathrm{d}x+\frac{1}{|B|}\int_B\left\|T_N^\alpha f_1(y)\right\|_{l^\infty(\mathbb{Z}^2)}\mathrm{d}y$$

$$\leq \left(\frac{1}{|B|}\int_B \left\|T_N^\alpha f_1(x)\right\|_{l^\infty(\mathbb{Z}^2)}^2 \mathrm{d}x\right)^{1/2} + \left(\frac{1}{|B|}\int_B \left\|T_N^\alpha f_1(y)\right\|_{l^\infty(\mathbb{Z}^2)}^2 \mathrm{d}y\right)^{1/2}$$
$$\leq C\frac{1}{|B|^{1/2}}\left\|f_1\right\|_{L^2(\mathbb{R}^n)} \leq C\left\|f\right\|_{BMO(\mathbb{R}^n)}.$$

For II, since $x, y \in B$ and the support of f_2 is $(2B)^c$, by Theorem 5 we have

$$\begin{aligned}
&\left|T_N^\alpha f_2(x) - T_N^\alpha f_2(y)\right| \\
&= \left|\int_{\mathbb{R}^n} K_N^\alpha(x,z) f_2(z)\mathrm{d}z - \int_{\mathbb{R}^n} K_N^\alpha(y,z) f_2(z)\mathrm{d}z\right| \\
&= \left|\int_{(2B)^c} \left(K_N^\alpha(x,z) - K_N^\alpha(y,z)\right) f_2(z)\mathrm{d}z\right| \\
&\leq C\sum_{k=2}^{+\infty} r\int_{2^kB\backslash 2^{k-1}B} \frac{|f(z)-f_B|}{|z-x_0|^{n+1}}\mathrm{d}z \\
&\leq C\sum_{k=2}^{+\infty} \frac{r}{(2^kr)^{n+1}}\int_{2^kB} |f(z)-f_B|\mathrm{d}z \\
&\leq C\sum_{k=2}^{+\infty} 2^{-k}\frac{1}{\left|2^kB\right|}\int_{2^kB} \left(\left|f(z)-f_{2^kB}\right| + \sum_{l=1}^{k}\left|f_{2^lB}-f_{2^{l-1}B}\right|\right)\mathrm{d}z \\
&\leq C\sum_{k=2}^{+\infty} 2^{-k}\frac{1}{\left|2^kB\right|}\int_{2^kB} \left(\left|f(z)-f_{2^kB}\right| + 2k\left\|f\right\|_{BMO(\mathbb{R}^n)}\right)\mathrm{d}z \\
&\leq C\sum_{k=2}^{+\infty} 2^{-k}(1+2k)\left\|f\right\|_{BMO(\mathbb{R}^n)} \\
&\leq C\left\|f\right\|_{BMO(\mathbb{R}^n)},
\end{aligned}$$

where $2^kB = B(x_0, 2^kr)$. Hence, we have $II \leq C\left\|f\right\|_{BMO(\mathbb{R}^n)}$. Then by the arbitrary of x_0 and $r > 0$, we proved

$$\left\|T^*f\right\|_{BMO(\mathbb{R}^n)} \leq C\left\|f\right\|_{BMO(\mathbb{R}^n)}.$$

For the second part of (c), we can deduce it from the BMO-boundedness of T^* and the inclusion $L^\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$. This completes the proof of Theorem 1. □

From the conclusions we got in Theorem 1, we have the following result:

Theorem 8 (a) *If* $1 < p < \infty$ *and* $\omega \in A_p$, *then* $T_N^\alpha f$ *converges a.e. and in* $L^p(\mathbb{R}^n, \omega)$ *norms for all* $f \in L^p(\mathbb{R}^n, \omega)$ *as* $N = (N_1, N_2)$ *tends to* $(-\infty, +\infty)$.

(b) *If* $p = 1$ *and* $\omega \in A_1$, *then* $T_N^\alpha f$ *converges a.e. and in measure for all* $f \in L^1(\mathbb{R}^n, \omega)$ *as* $N = (N_1, N_2)$ *tends to* $(-\infty, +\infty)$.

Proof First, we shall see that if φ is a test function, then $T_N^\alpha \varphi(x)$ converges for all $x \in \mathbb{R}^n$. In order to prove this, it is enough to see that for any $N = (L, M)$ with $0 < L < M$, the series

$$A = \sum_{j=L}^{M} v_j (\mathcal{P}_{a_{j+1}}^\alpha \varphi(x) - \mathcal{P}_{a_j}^\alpha \varphi(x)) \text{ and } B = \sum_{j=-M}^{-L} v_j (\mathcal{P}_{a_{j+1}}^\alpha \varphi(x) - \mathcal{P}_{a_j}^\alpha \varphi(x))$$

converge to zero, when $L, M \to +\infty$. For A, by the mean value theorem and the ρ-lacunarity of the sequence $\{a_j\}_{j\in\mathbb{Z}}$ we have

$$\begin{aligned}
|A| &= C_\alpha \left| \sum_{j=L}^{M} v_j \left(\int_0^{+\infty} e^{-s} \left(e^{\frac{a_{j+1}^2}{4s}\Delta} \varphi(x) - e^{\frac{a_j^2}{4s}\Delta} \varphi(x) \right) \frac{ds}{s^{1-\alpha}} \right) \right| \\
&\le C_\alpha \int_0^{+\infty} e^{-s} \left| \sum_{j=L}^{M} v_j \left(e^{\frac{a_{j+1}^2}{4s}\Delta} \varphi(x) - e^{\frac{a_j^2}{4s}\Delta} \varphi(x) \right) \right| \frac{ds}{s^{1-\alpha}} \\
&= C_{n,\alpha} \int_0^{+\infty} e^{-s} \left| \sum_{j=L}^{M} v_j \left(\int_{\mathbb{R}^n} \frac{s^{n/2}}{a_{j+1}^n} e^{-\frac{s|y|^2}{a_{j+1}^2}} \varphi(x-y) \mathrm{d}y - \int_{\mathbb{R}^n} \frac{s^{n/2}}{a_j^n} e^{-\frac{s|y|^2}{a_j^2}} \varphi(x-y) \mathrm{d}y \right) \right| \frac{ds}{s^{1-\alpha}} \\
&\le C_{n,\alpha,v,\rho} \int_0^{+\infty} e^{-s} \int_{\mathbb{R}^n} \sum_{j=L}^{M} \frac{s^{n/2}}{a_j^n} |\varphi(x-y| \mathrm{d}y \, \frac{ds}{s^{1-\alpha}} \\
&\le C_{n,\alpha,v,\rho} \left(\frac{1}{a_L^n} \sum_{j=L}^{M} \frac{a_L^n}{a_j^n} \right) \int_0^{+\infty} e^{-s} \int_{\mathbb{R}^n} |\varphi(x-y| \mathrm{d}y \, \frac{ds}{s^{1-\alpha-n/2}} \\
&\le C_{n,\alpha,v,\rho} \, \|\varphi\|_{L^1(\mathbb{R}^n)} \frac{1}{a_L^n} \to 0, \quad \text{as } L, M \to +\infty.
\end{aligned}$$

For B, as the integral of the kernels are zero, we can write

$$\begin{aligned}
B &= C_{n,\alpha} \int_{\mathbb{R}^{n+1}} \sum_{j=-M}^{-L} v_j \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} \frac{e^{-|y|^n/(4s)}}{(4\pi s)^{n/2}} (\varphi(x-y) - \varphi(x)) \mathrm{d}y \mathrm{d}s \\
&= C_{n,\alpha} \int_0^1 \int_{\mathbb{R}^n} \sum_{j=-M}^{-L} v_j \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} \frac{e^{-|y|^n/(4s)}}{(4\pi s)^{n/2}} (\varphi(x-y) - \varphi(x)) \mathrm{d}y \mathrm{d}s \\
&\quad + C_{n,\alpha} \int_1^\infty \int_{\mathbb{R}^n} \sum_{j=-M}^{-L} v_j \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} \frac{e^{-|y|^n/(4s)}}{(4\pi s)^{n/2}} (\varphi(x-y) - \varphi(x)) \mathrm{d}y \mathrm{d}s \\
&=: B_1 + B_2.
\end{aligned}$$

Proceeding as in the case A, and by using the fact that φ is a test function, we have

$$|B_1| = C_{n,\alpha} \left| \int_0^1 \int_{\mathbb{R}^n} \sum_{j=-M}^{-L} v_j \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} \frac{e^{-|y|^2/(4s)}}{(4\pi s)^{n/2}} (\varphi(x-y) - \varphi(x)) \mathrm{d}y \mathrm{d}s \right|$$

$$\leq C_{n,\alpha}\Big|\int_0^1\int_{\mathbb{R}^n}\sum_{j=-M}^{-L} v_j \frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)}-a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\frac{e^{-|y|^2/(4s)}}{(4\pi s)^{n/2}}\,\|\nabla\varphi\|_{L^\infty(\mathbb{R}^{n+1})}\,|y|\mathrm{d}y\mathrm{d}s\Big|$$

$$\leq C_{n,\alpha}\,\|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)}\int_0^1\int_{\mathbb{R}^n}\sum_{j=-M}^{-L} v_j \frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)}-a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{\alpha+1/2}}\frac{e^{-|y|^2/(4s)}}{(4\pi s)^{n/2}}\mathrm{d}y\mathrm{d}s$$

$$\leq C_{n,\alpha,v,\rho}\,\|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)}\int_0^1\sum_{j=-M}^{-L}\frac{a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{\alpha+1/2}}\mathrm{d}s.$$

If $0<\alpha\leq\frac{1}{2}$, then, for any $0<\varepsilon<2\alpha$, we have

$$\begin{aligned}|B_1| &\leq C_{n,\alpha,v,\rho}\,\|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)}\sum_{j=-M}^{-L} a_j^{2\alpha-\varepsilon}\int_0^1 s^{\varepsilon/2-\alpha-1/2}\mathrm{d}s\\ &\leq C_{n,\alpha,v,\rho,\varepsilon}\,\|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)}\,a_{-L}^{2\alpha-\varepsilon}\sum_{j=-M}^{-L}\frac{a_j^{2\alpha-\varepsilon}}{a_{-L}^{2\alpha-\varepsilon}}\\ &\leq C_{n,\alpha,v,\rho,\varepsilon}\,\|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)}\,a_{-L}^{2\alpha-\varepsilon}\longrightarrow 0,\quad \text{as } L,M\to+\infty.\end{aligned}$$

If $\frac{1}{2}<\alpha<1$, then

$$\begin{aligned}|B_1| &\leq C_{n,\alpha,v,\rho}\,\|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)}\int_0^1\sum_{j=-M}^{-L}\frac{a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{\alpha+1/2}}\mathrm{d}s\\ &\leq C_{n,\alpha,v,\rho}\,\|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)}\,a_{-L}^{2\alpha-1}\sum_{j=-M}^{-L}\frac{a_j^{2\alpha-1}}{a_{-L}^{2\alpha-1}}\int_0^1\frac{1}{s^\alpha}\mathrm{d}s\\ &\leq C_{n,\alpha,v,\rho}\,\|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)}\,a_{-L}^{2\alpha-1}\longrightarrow 0,\quad \text{as } L,M\to+\infty.\end{aligned}$$

Therefore, we get

$$|B_1|\longrightarrow 0,\quad \text{as } L,M\to+\infty.$$

On the other hand,

$$\begin{aligned}|B_2| &\leq C_{n,\alpha,\rho}\,\|v\|_{l^\infty(\mathbb{Z})}\,\|\varphi\|_{L^\infty(\mathbb{R}^n)}\int_1^\infty\sum_{j=-M}^{-L}\frac{a_j^{2\alpha}}{s^{1+\alpha}}\mathrm{d}s\\ &\leq C_{n,\alpha,v}\|\varphi\|_{L^\infty(\mathbb{R}^n)}\sum_{j=-M}^{-L} a_j^{2\alpha}\int_1^\infty\frac{1}{s^{1+\alpha}}\mathrm{d}s\end{aligned}$$

$$
\leq C_{n,\alpha,v,\rho}\|\varphi\|_{L^\infty(\mathbb{R}^n)}a_{-L}^{2\alpha}\sum_{j=-M}^{-L}\frac{a_j^{2\alpha}}{a_{-L}^{2\alpha}}
$$
$$
\leq C_{n,\alpha,v,\rho}\|\varphi\|_{L^\infty(\mathbb{R}^n)}\frac{\rho^{2\alpha}}{\rho^{2\alpha}-1}a_{-L}^{2\alpha}\longrightarrow 0, \quad \text{as } L, M \to +\infty.
$$

As the set of test functions is dense in $L^p(\mathbb{R}^n)$, by Theorem 1 we get the $a.e.$ convergence for any function in $L^p(\mathbb{R}^n)$. Analogously, since $L^p(\mathbb{R}^n)\cap L^p(\mathbb{R}^n,\omega)$ is dense in $L^p(\mathbb{R}^n,\omega)$, we get the $a.e.$ convergence for functions in $L^p(\mathbb{R}^n,\omega)$ with $1\leq p<\infty$. By using the dominated convergence theorem, we can prove the convergence in $L^p(\mathbb{R}^n,\omega)$-norm for $1<p<\infty$, and also in measure. □

3 Proof of Theorem 2

The dichotomy results announced in Theorem 1, parts (c) and (d), about $L^\infty(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$ are motivated, in part, by the existence of a bounded function f such that $T^*f(x)=\infty$ as the following theorem shows. In [5], we can find some related results for the variation operators.

Theorem 9 *There exist bounded sequence $\{v_j\}_{j\in\mathbb{Z}}$, ρ-lacunary sequence $\{a_j\}_{j\in\mathbb{Z}}$ and $f\in L^\infty(\mathbb{R}^n)$ such that $T^*f(x)=\infty$ for all $x\in\mathbb{R}^n$.*

Proof We will only consider the case $n=1$. For the multi-dimensional case, it is similar just with minor modifications. Let f be the function defined by

$$
f(x)=\sum_{k\in\mathbb{Z}}(-1)^k\chi_{(-a^{2k+1},-a^{2k}]}(x), \quad x\in\mathbb{R},
$$

where $a>1$ is a real number that we shall fix it later. It is easy to see that

$$
f(a^{2j}x)=(-1)^jf(x). \tag{3.1}
$$

By changing variable, we have

$$
\begin{aligned}
\mathcal{P}_{a_j}^\alpha f(x)&=\frac{1}{4^\alpha\Gamma(\alpha)}\int_0^{+\infty}\int_{\mathbb{R}}\frac{a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\frac{e^{-|y|^2/4s}}{(4\pi s)^{1/2}}f(x-y)\mathrm{d}y\mathrm{d}s\\
&=\frac{1}{\sqrt{\pi}\Gamma(\alpha)}\int_0^{+\infty}s^\alpha e^{-s}\int_{\mathbb{R}}s^{1/2}e^{-s|y|^2}f(x-a_jy)\mathrm{d}y\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

Let $a_j=a^{2j}$. Then,

$$
\begin{aligned}
\mathcal{P}_{a_j}^\alpha f(0)&=\frac{1}{\sqrt{\pi}\Gamma(\alpha)}\int_0^{+\infty}s^\alpha e^{-s}\int_{\mathbb{R}}s^{1/2}e^{-s|y|^2}f(-a^{2j}y)\mathrm{d}y\frac{\mathrm{d}s}{s}\\
&=\frac{(-1)^j}{\sqrt{\pi}\Gamma(\alpha)}\int_0^{+\infty}s^\alpha e^{-s}\int_0^{+\infty}s^{1/2}e^{-s|y|^2}f(-y)\mathrm{d}y\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

We observe that

$$\int_0^{+\infty} s^{1/2}e^{-s|y|^2}\,|f(-y)|\,\mathrm{d}y \le \int_0^{+\infty} s^{1/2}e^{-s|y|^2}\mathrm{d}y = \int_0^{+\infty} e^{-t^2}\mathrm{d}t = \frac{\sqrt{\pi}}{2}.$$

Therefore,

$$\int_0^{+\infty} s^{\alpha}e^{-s}\int_0^{+\infty} s^{1/2}e^{-s|y|^2}\,|f(-y)|\,\mathrm{d}y\frac{\mathrm{d}s}{s} \le \frac{\sqrt{\pi}}{2}\int_0^{+\infty} s^{\alpha}e^{-s}\frac{\mathrm{d}s}{s} = \frac{\sqrt{\pi}}{2}\Gamma(\alpha) < \infty.$$

Also, we have

$$\lim_{R\to+\infty}\int_0^{+\infty} s^{\alpha}e^{-s}\int_R^{+\infty} s^{1/2}e^{-s|y|^2}f(-y)\mathrm{d}y\frac{\mathrm{d}s}{s} = 0$$

and

$$\lim_{\varepsilon\to 0^+}\int_0^{+\infty} s^{\alpha}e^{-s}\int_0^{\varepsilon} s^{1/2}e^{-s|y|^2}f(-y)\mathrm{d}y\frac{\mathrm{d}s}{s} = 0.$$

On the other hand, there exists a constant $C > 0$ such that

$$\begin{aligned}&\lim_{a\to+\infty}\int_0^{+\infty} s^{\alpha}e^{-s}\int_1^{a} s^{1/2}e^{-s|y|^2}f(-y)\mathrm{d}y\frac{\mathrm{d}s}{s}\\ &\quad= \lim_{a\to+\infty}\int_0^{+\infty} s^{\alpha}e^{-s}\int_1^{a} s^{1/2}e^{-s|y|^2}\mathrm{d}y\frac{\mathrm{d}s}{s} = C > 0.\end{aligned}$$

Hence we can choose $a > 1$ big enough such that

$$\begin{aligned}&\int_0^{+\infty} s^{\alpha}e^{-s}\int_1^{a} s^{1/2}e^{-s|y|^2}f(-y)\mathrm{d}y\frac{\mathrm{d}s}{s} = \int_0^{+\infty} s^{\alpha}e^{-s}\int_1^{a} s^{1/2}e^{-s|y|^2}\mathrm{d}y\frac{\mathrm{d}s}{s}\\ &> \int_0^{+\infty} s^{\alpha}e^{-s}\int_0^{1/a} s^{1/2}e^{-s|y|^2}\mathrm{d}y\frac{\mathrm{d}s}{s} + \int_0^{+\infty} s^{\alpha}e^{-s}\int_{a^2}^{+\infty} s^{1/2}e^{-s|y|^2}\mathrm{d}y\frac{\mathrm{d}s}{s}\\ &> \int_0^{+\infty} s^{\alpha}e^{-s}\left|\int_0^{1/a} s^{1/2}e^{-s|y|^2}f(-y)\mathrm{d}y\right|\frac{\mathrm{d}s}{s}\\ &\quad+ \int_0^{+\infty} s^{\alpha}e^{-s}\left|\int_{a^2}^{+\infty} s^{1/2}e^{-s|y|^2}f(-y)\mathrm{d}y\right|\frac{\mathrm{d}s}{s}.\end{aligned}$$

In other words, with the $a > 1$ fixed above, there exists constant $C_1 > 0$ such that

$$\int_0^{+\infty} s^{\alpha}e^{-s}\int_0^{+\infty} s^{1/2}e^{-s|y|^2}f(-y)\mathrm{d}y\frac{\mathrm{d}s}{s} = C_1. \tag{3.2}$$

Hence

$$\left|\mathcal{P}^{\alpha}_{a_{j+1}}f(0) - \mathcal{P}^{\alpha}_{a_j}f(0)\right| = \frac{2C_1}{\sqrt{\pi}\,\Gamma(\alpha)} > 0.$$

Therefore, we have

$$\sum_{j\in\mathbb{Z}}\left|\mathcal{P}^{\alpha}_{a_{j+1}}f(0)-\mathcal{P}^{\alpha}_{a_j}f(0)\right|=\infty.$$

By using (3.1) and changing variable we get

$$\begin{aligned}\mathcal{P}^{\alpha}_{a_j}f(x)&=\frac{1}{\sqrt{\pi}\Gamma(\alpha)}\int_0^{+\infty}s^{\alpha}e^{-s}\int_{\mathbb{R}}s^{1/2}e^{-s|y|^2}f(x-a^{2j}y)\mathrm{d}y\frac{\mathrm{d}s}{s}\\&=\frac{(-1)^j}{\sqrt{\pi}\Gamma(\alpha)}\int_0^{+\infty}s^{\alpha}e^{-s}\int_{\mathbb{R}}s^{1/2}e^{-s|y|^2}f\left(\frac{x}{a^{2j}}-y\right)\mathrm{d}y\frac{\mathrm{d}s}{s}.\end{aligned}$$

Then

$$\begin{aligned}&\mathcal{P}^{\alpha}_{a_{j+1}}f(t)-\mathcal{P}^{\alpha}_{a_j}f(t)\\&=\frac{(-1)^{j+1}}{\sqrt{\pi}\Gamma(\alpha)}\Big\{\int_0^{+\infty}s^{\alpha}e^{-s}\int_{\mathbb{R}}s^{1/2}e^{-s|y|^2}f\left(\frac{x}{a^{2(j+1)}}-y\right)\mathrm{d}y\frac{\mathrm{d}s}{s}\\&\quad+\int_0^{+\infty}s^{\alpha}e^{-s}\int_{\mathbb{R}}s^{1/2}e^{-s|y|^2}f\left(\frac{x}{a^{2j}}-y\right)\mathrm{d}y\frac{\mathrm{d}s}{s}\Big\}.\end{aligned}\tag{3.3}$$

By the dominated convergence theorem, we know that

$$\begin{aligned}&\lim_{h\to 0}\int_0^{+\infty}s^{\alpha}e^{-s}\int_{\mathbb{R}}s^{1/2}e^{-s|y|^2}f\left(h-y\right)\mathrm{d}y\frac{\mathrm{d}s}{s}\\&=\int_0^{+\infty}s^{\alpha}e^{-s}\int_{\mathbb{R}}s^{1/2}e^{-s|y|^2}f\left(-y\right)\mathrm{d}y\frac{\mathrm{d}s}{s}\\&=\int_0^{+\infty}s^{\alpha}e^{-s}\int_0^{+\infty}s^{1/2}e^{-s|y|^2}f\left(-y\right)\mathrm{d}y\frac{\mathrm{d}s}{s}\\&=C_1>0,\end{aligned}$$

where C_1 is the constant appeared in (3.2). So, there exists $0<\eta_0<1$, such that, for $|h|<\eta_0$,

$$\int_0^{+\infty}s^{\alpha}e^{-s}\int_{\mathbb{R}}s^{1/2}e^{-s|y|^2}f\left(h-y\right)\mathrm{d}y\frac{\mathrm{d}s}{s}\geq\frac{C_1}{2}.$$

Then, for each $x\in\mathbb{R}$, we can choose $j\in\mathbb{Z}$ such that $\frac{|x|}{a^j}<\eta_0$ (there are infinite j satisfying this condition), and we have

$$\begin{aligned}&\int_0^{+\infty}s^{\alpha}e^{-s}\int_{\mathbb{R}}s^{1/2}e^{-s|y|^2}f\left(\frac{x}{a^{2(j+1)}}-y\right)\mathrm{d}y\frac{\mathrm{d}s}{s}\\&\quad+\int_0^{+\infty}s^{\alpha}e^{-s}\int_{\mathbb{R}}s^{1/2}e^{-s|y|^2}f\left(\frac{x}{a^{2j}}-y\right)\mathrm{d}y\frac{\mathrm{d}s}{s}\geq C_1>0.\end{aligned}$$

Choosing $v_j = (-1)^{j+1},\ j \in \mathbb{Z}$, by (3.3) we have, for any $x \in \mathbb{R}$,

$$
\begin{aligned}
T^* f(x) &\geq \sum_{\left|\frac{x}{a^j}\right|<\eta_0} (-1)^{j+1}\big(\mathcal{P}^\alpha_{a_{j+1}} f(x) - \mathcal{P}^\alpha_{a_j} f(x)\big)\\
&= \frac{1}{\sqrt{\pi}\Gamma(\alpha)} \sum_{\left|\frac{x}{a^j}\right|<\eta_0} \left\{ \int_0^{+\infty} s^\alpha e^{-s} \int_{\mathbb{R}} s^{1/2} e^{-s|y|^2} f\left(\frac{x}{a^{2(j+1)}} - y\right) \mathrm{d}y \frac{\mathrm{d}s}{s}\right.\\
&\quad \left. + \int_0^{+\infty} s^\alpha e^{-s} \int_{\mathbb{R}} s^{1/2} e^{-s|y|^2} f\left(\frac{x}{a^{2j}} - y\right) \mathrm{d}y \frac{\mathrm{d}s}{s}\right\}\\
&\geq \frac{1}{\sqrt{\pi}\Gamma(\alpha)} \sum_{\left|\frac{x}{a^j}\right|<\eta_0} C_1 = \infty.
\end{aligned}
$$

We complete the proof of Theorem 9. □

At the end of this section, we will give the proof of Theorem 2.

Proof of Theorem 2. First, we prove the theorem in the case $1 < p < \infty$. Since $2r < 1$, we know that $B \backslash B_{2r} \neq \emptyset$. Let $f(x) = f_1(x) + f_2(x)$, where $f_1(x) = f(x)\chi_{B_{2r}}(x)$ and $f_2(x) = f(x)\chi_{B\backslash B_{2r}}(x)$. Then

$$
\left|T^* f(x)\right| \leq \left|T^* f_1(x)\right| + \left|T^* f_2(x)\right|.
$$

By Theorem 1,

$$
\begin{aligned}
\frac{1}{|B_r|}\int_{B_r} \left|T^* f_1(x)\right| \mathrm{d}x &\leq \left(\frac{1}{|B_r|}\int_{B_r} \left|T^* f_1(x)\right|^2 \mathrm{d}x\right)^{1/2}\\
&\leq C\left(\frac{1}{|B_r|}\int_{\mathbb{R}} |f_1(x)|^2 \mathrm{d}x\right)^{1/2} \leq C\,\|f\|_{L^\infty(\mathbb{R}^n)}.
\end{aligned}
$$

We also know that, for any $j \in \mathbb{Z}$,

$$
\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}^n} \left|\frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}}\right| \frac{e^{-|x-y|^2/4s}}{(4\pi s)^{n/2}} \mathrm{d}y\mathrm{d}s\\
&\leq \int_0^{+\infty}\int_{\mathbb{R}^n} \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} + a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} \frac{e^{-|x-y|^2/4s}}{(4\pi s)^{n/2}} \mathrm{d}y\mathrm{d}s = 2\cdot 4^\alpha\Gamma(\alpha).
\end{aligned}
\tag{3.4}
$$

Then, by Hölder's inequality, (3.4) and Fubini's Theorem, for $1 < p < \infty$ and any $N = (N_1, N_2)$, we have

$$
\begin{aligned}
&\left|\sum_{j=N_1}^{N_2} v_j\left(\mathcal{P}^\alpha_{a_{j+1}} f_2(x)-\mathcal{P}^\alpha_{a_j} f_2(x)\right)\right|\\
&\le C\sum_{j=N_1}^{N_2}\left|v_j\int_0^{+\infty}\frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)}-a_j^{2\alpha}e^{-a_j^2/(4s))}}{s^{1+\alpha}}\int_{\mathbb{R}^n}\frac{e^{-|x-y|^2/4s}}{(4\pi s)^{n/2}}f_2(y)\,\mathrm{d}y\mathrm{d}s\right|\\
&\le C\,\|v\|_{l^p(\mathbb{Z})}\left(\sum_{j=N_1}^{N_2}\left(\int_0^{+\infty}\left|\frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)}-a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\right|\int_{\mathbb{R}^n}\frac{e^{-|x-y|^2/4s}}{(4\pi s)^{n/2}}\,|f_2(y)|\,\mathrm{d}y\mathrm{d}s\right)^{p'}\right)^{1/p'}\\
&\le C\,\|v\|_{l^p(\mathbb{Z})}\Bigg(\sum_{j=N_1}^{N_2}\left\{\int_0^{+\infty}\int_{\mathbb{R}^n}\left|\frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)}-a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\right|\frac{e^{-|x-y|^2/4s}}{(4\pi s)^{n/2}}\,|f_2(y)|^{p'}\,\mathrm{d}y\mathrm{d}s\right\}\\
&\qquad\times\left\{\int_0^{+\infty}\int_{\mathbb{R}^n}\left|\frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)}-a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\right|\frac{e^{-|x-y|^2/4s}}{(4\pi s)^{n/2}}\,\mathrm{d}y\mathrm{d}s\right\}^{p'/p}\Bigg)^{1/p'}\\
&\le C\,\|v\|_{l^p(\mathbb{Z})}\left(\sum_{j=N_1}^{N_2}\int_0^{+\infty}\int_{\mathbb{R}^n}\left|\frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)}-a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\right|\frac{e^{-|x-y|^2/4s}}{(4\pi s)^{n/2}}\,|f_2(y)|^{p'}\,\mathrm{d}y\mathrm{d}s\right)^{1/p'}\\
&\le C\,\|v\|_{l^p(\mathbb{Z})}\left(\int_0^{+\infty}\int_{\mathbb{R}^n}\sum_{j=-\infty}^{+\infty}\left|\frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)}-a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\right|\frac{e^{-|x-y|^2/4s}}{(4\pi s)^{n/2}}\,|f_2(y)|^{p'}\,\mathrm{d}y\mathrm{d}s\right)^{1/p'}\\
&\le C\,\|v\|_{l^p(\mathbb{Z})}\left(\int_0^{+\infty}\int_{\mathbb{R}^n}\frac{1}{s}\frac{e^{-|x-y|^2/4s}}{(4\pi s)^{n/2}}\,|f_2(y)|^{p'}\,\mathrm{d}y\mathrm{d}s\right)^{1/p'}.
\end{aligned}
$$

For $x\in B\backslash B_{2r}$ and $y\in B_r$, we have $r\le|x-y|\le 2$. Then, by integration with polar coordinates we get

$$
\begin{aligned}
\frac{1}{|B_r|}\int_{B_r}\left|T^*f_2(x)\right|\mathrm{d}x &\le C\frac{1}{|B_r|}\int_{B_r}\left(\int_0^{+\infty}\int_{\mathbb{R}^n}\frac{1}{s}\frac{e^{-|x-y|^2/4s}}{(4\pi s)^{n/2}}\,|f_2(y)|^{p'}\,\mathrm{d}y\mathrm{d}s\right)^{1/p'}\mathrm{d}t\\
&\le C\frac{\|f\|_{L^\infty(\mathbb{R}^n)}}{|B_r|}\int_{B_r}\left(\int_0^{+\infty}\int_{S^{n-1}}\int_{r\le|t|\le 2}\frac{1}{s}\frac{e^{-t^2/4s}}{(4\pi s)^{n/2}}\,\mathrm{d}t d_{\omega_{n-1}}\mathrm{d}s\right)^{1/p'}\mathrm{d}t\\
&\sim\left(\log\frac{2}{r}\right)^{1/p'}\|f\|_{L^\infty(\mathbb{R}^n)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{1}{|B_r|}\int_{B_r}\left|T^*f(x)\right|\mathrm{d}x &\le C\left(1+\left(\log\frac{2}{r}\right)^{1/p'}\right)\|f\|_{L^\infty(\mathbb{R}^n)}\\
&\le C\left(\log\frac{2}{r}\right)^{1/p'}\|f\|_{L^\infty(\mathbb{R}^n)}.
\end{aligned}
$$

For the case $p=1$ and $p=\infty$, the proof is similar and easier. Then we get the proof of (a).

For (b), we only consider the case $n = 1$. It is similar in the multi-dimensional case. When $1 < p < \infty$, for any $0 < \varepsilon < p - 1$, let

$$f(x) = \sum_{k=-\infty}^{0} (-1)^k \chi_{(-a^{2k}, -a^{2k-1}]}(x) \text{ and } a_j = a^{2j},$$

with $a > 1$ being fixed later. Then, the support of f is contained in $[-1, 0)$, and $\{a_j\}_{j\in\mathbb{Z}}$ is a ρ-lacunary sequence with $\rho = a^2 > 1$. We observe that

$$\left|\int_0^{+\infty} s^\alpha e^{-s} \int_0^{+\infty} s^{1/2} e^{-s|y|^2} f(-y)\mathrm{d}y \frac{\mathrm{d}s}{s}\right| \leq \int_0^{+\infty} s^\alpha e^{-s} \int_0^{+\infty} s^{1/2} e^{-s|y|^2} \mathrm{d}y \frac{\mathrm{d}s}{s}$$
$$= \frac{\sqrt{\pi}}{2}\Gamma(\alpha) < \infty.$$

Hence

$$\lim_{R\to+\infty} \int_0^{+\infty} s^\alpha e^{-s} \int_R^{+\infty} s^{1/2} e^{-s|y|^2} f(-y)\mathrm{d}y = 0$$

and

$$\lim_{\varepsilon\to 0^+} \int_0^{+\infty} s^\alpha e^{-s} \int_0^{\varepsilon} s^{1/2} e^{-s|y|^2} f(-y)\mathrm{d}y \frac{\mathrm{d}s}{s} = 0.$$

Also there exists a constant $C > 0$ such that

$$\lim_{a\to+\infty} \int_0^{+\infty} s^\alpha e^{-s} \int_{a^{-1}}^{1} s^{1/2} e^{-s|y|^2} f(-y)\mathrm{d}y \frac{\mathrm{d}s}{s}$$
$$= \lim_{a\to+\infty} \int_0^{+\infty} s^\alpha e^{-s} \int_{a^{-1}}^{1} s^{1/2} e^{-s|y|^2} \mathrm{d}y \frac{\mathrm{d}s}{s} = C.$$

So, we can choose $a > 1$ big enough such that

$$\int_0^{+\infty} s^\alpha e^{-s} \int_{a^{-1}}^{1} s^{1/2} e^{-s|y|^2} f(-y)\mathrm{d}y \frac{\mathrm{d}s}{s} = \int_0^{+\infty} s^\alpha e^{-s} \int_{a^{-1}}^{1} s^{1/2} e^{-s|y|^2} \mathrm{d}y \frac{\mathrm{d}s}{s}$$
$$\geq 10 \left(\int_0^{+\infty} s^\alpha e^{-s} \int_0^{1/a^2} s^{1/2} e^{-s|y|^2} \mathrm{d}y + \int_0^{+\infty} s^\alpha e^{-s} \int_{a-1}^{+\infty} s^{1/2} e^{-s|y|^2} \mathrm{d}y \frac{\mathrm{d}s}{s} \right)$$
$$> 10 \left(\left| \int_0^{+\infty} s^\alpha e^{-s} \int_0^{1/a^2} s^{1/2} e^{-s|y|^2} f(-y)\mathrm{d}y \frac{\mathrm{d}s}{s} \right| \right.$$
$$\left. + \left| \int_0^{+\infty} s^\alpha e^{-s} \int_{a-1}^{+\infty} s^{1/2} e^{-s|y|^2} f(-y)\mathrm{d}y \frac{\mathrm{d}s}{s} \right| \right).$$

Therefore, there exists a constant $C_1 > 0$ such that

$$\int_0^{+\infty} s^{\alpha} e^{-s} \int_0^{+\infty} s^{1/2} e^{-s|y|^2} f(-y) \mathrm{d}y \frac{\mathrm{d}s}{s} = C_1 > 0 \tag{3.5}$$

and

$$0 < \int_0^{+\infty} s^{\alpha} e^{-s} \int_0^{1/a^2} s^{1/2} e^{-s|y|^2} \mathrm{d}y \frac{\mathrm{d}s}{s} + \int_0^{+\infty} s^{\alpha} e^{-s} \int_{a-1}^{+\infty} s^{1/2} e^{-s|y|^2} \mathrm{d}y \frac{\mathrm{d}s}{s} \leq \frac{C_1}{9}. \tag{3.6}$$

On the other hand, by the dominated convergence theorem, we have

$$\lim_{h\to 0} \int_0^{+\infty} s^{\alpha} e^{-s} \int_{\mathbb{R}} s^{1/2} e^{-s|y|^2} f(h-y) \mathrm{d}y \frac{\mathrm{d}s}{s} = \int_0^{+\infty} s^{\alpha} e^{-s} \int_0^{+\infty} s^{1/2} e^{-s|y|^2} f(-y) \mathrm{d}y \frac{\mathrm{d}s}{s} = C_1 > 0,$$

where C_1 is the constant appeared in (3.5). So, there exists $0 < \eta_0 < 1$, such that, for $|h| < \eta_0$,

$$\int_0^{+\infty} s^{\alpha} e^{-s} \int_{\mathbb{R}} s^{1/2} e^{-s|y|^2} f(h-y) \mathrm{d}y \frac{\mathrm{d}s}{s} \geq \frac{1}{2} \int_0^{+\infty} s^{\alpha} e^{-s} \int_0^{+\infty} s^{1/2} e^{-s|y|^2} f(-y) \mathrm{d}y \frac{\mathrm{d}s}{s} = \frac{C_1}{2}. \tag{3.7}$$

It can be checked that

$$f(a^{2j}x) = (-1)^j f(x) + (-1)^j \sum_{k=1}^{-j} (-1)^k \chi_{(-a^{2k}, -a^{2k-1}]}(x)$$

when $j \leq 0$. We will always assume $j \leq 0$ in the following. By changing variable,

$$\begin{aligned} \mathcal{P}_{a_j}^{\alpha} f(x) &= \frac{1}{\sqrt{\pi}\Gamma(\alpha)} \int_0^{+\infty} s^{\alpha} e^{-s} \int_{\mathbb{R}} s^{1/2} e^{-s|y|^2} f\left(x - a^{2j} y\right) \mathrm{d}y \frac{\mathrm{d}s}{s} \\ &= \frac{(-1)^j}{\sqrt{\pi}\Gamma(\alpha)} \int_0^{+\infty} s^{\alpha} e^{-s} \int_{\mathbb{R}} s^{1/2} e^{-s|y|^2} \\ &\left\{ f\left(\frac{x}{a^{2j}} - y\right) + \sum_{k=1}^{-j} (-1)^k \chi_{(-a^{2k}, -a^{2k-1}]}\left(\frac{x}{a^{2j}} - y\right) \right\} \mathrm{d}y \frac{\mathrm{d}s}{s}. \end{aligned}$$

Then

$$
\begin{aligned}
&\mathcal{P}^{\alpha}_{a_{j+1}} f(x) - \mathcal{P}^{\alpha}_{a_j} f(x)\\
&= \frac{(-1)^{j+1}}{\sqrt{\pi}\Gamma(\alpha)} \Big\{ \int_0^{+\infty} s^{\alpha} e^{-s} \int_{\mathbb{R}} s^{1/2} e^{-s|y|^2} f\left(\frac{x}{a^{2j+2}} - y\right) \mathrm{d}y \frac{\mathrm{d}s}{s}\\
&+ \int_0^{+\infty} s^{\alpha} e^{-s} \int_{\mathbb{R}} s^{1/2} e^{-s|y|^2} f\left(\frac{x}{a^{2j}} - y\right) \mathrm{d}y \frac{\mathrm{d}s}{s}\\
&+ \int_0^{+\infty} s^{\alpha} e^{-s} \int_{\mathbb{R}} s^{1/2} e^{-s|y|^2} \sum_{k=1}^{-j-1} (-1)^k \chi_{(-a^{2k}, -a^{2k-1}]} \left(\frac{x}{a^{2j+2}} - y\right) \mathrm{d}y \frac{\mathrm{d}s}{s}\\
&+ \int_0^{+\infty} s^{\alpha} e^{-s} \int_{\mathbb{R}} s^{1/2} e^{-s|y|^2} \sum_{k=1}^{-j} (-1)^k \chi_{(-a^{2k}, -a^{2k-1}]} \left(\frac{x}{a^{2j}} - y\right) \mathrm{d}y \frac{\mathrm{d}s}{s} \Big\}. \quad (3.8)
\end{aligned}
$$

For given η_0 as above, let $2r < 1$ such that $r < \eta_0^2$ and $r \sim a^{2J_0}\eta_0$ for a certain negative integer J_0. If $J_0 \leq j \leq 0$, we have $\frac{r}{a^{2j}} < \eta_0$. And, for any $-r \leq x \leq r$ we have

$$
-1 \cdot \chi_{[a-1,+\infty)}(y) \leq \sum_{k=1}^{-j-1} (-1)^k \chi_{(-a^{2k}, -a^{2k-1}]} \left(\frac{x}{a^{2j+2}} - y\right) \leq \chi_{[a-1,+\infty)}(y)
$$

and

$$
-1 \cdot \chi_{[a-1,+\infty)}(y) \leq \sum_{k=1}^{-j} (-1)^k \chi_{(-a^{2k}, -a^{2k-1}]} \left(\frac{x}{a^{2j}} - y\right) \leq \chi_{[a-1,+\infty)}(y).
$$

Hence, for the third and fourth integrals in (3.8), by (3.6) we have

$$
\begin{aligned}
&\int_0^{+\infty} s^{\alpha} e^{-s} \int_{\mathbb{R}} s^{1/2} e^{-s|y|^2} \sum_{k=1}^{-j-1} (-1)^k \chi_{(-a^{2k}, -a^{2k-1}]} \left(\frac{x}{a^{2j+2}} - y\right) \mathrm{d}y \frac{\mathrm{d}s}{s}\\
&\qquad + \int_0^{+\infty} s^{\alpha} e^{-s} \int_{\mathbb{R}} s^{1/2} e^{-s|y|^2} \sum_{k=1}^{-j} (-1)^k \chi_{(-a^{2k}, -a^{2k-1}]} \left(\frac{x}{a^{2j}} - y\right) \mathrm{d}y \frac{\mathrm{d}s}{s}\\
&\qquad\qquad \geq (-2) \int_0^{+\infty} s^{\alpha} e^{-s} \int_{a-1}^{+\infty} s^{1/2} e^{-s|y|^2} \mathrm{d}y \frac{\mathrm{d}s}{s} \geq -\frac{2C_1}{9}. \quad (3.9)
\end{aligned}
$$

So, for any $x \in [-r, r]$ and $J_0 \leq j \leq 0$, combining (3.8), (3.7) and (3.9), we have

$$
\left| \mathcal{P}^{\alpha}_{a_{j+1}} f(x) - \mathcal{P}^{\alpha}_{a_j} f(x) \right| \geq C_{\alpha} \cdot \left(C_1 - \frac{2C_1}{9} \right) = C \cdot C_1 > 0.
$$

We choose the sequence $\{v_j\}_{j\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$ given by $v_j = (-1)^{j+1}(-j)^{-\frac{1}{p-\varepsilon}}$, then for $N = (J_0, 0)$, we have

$$\begin{aligned}
&\frac{1}{2r}\int_{[-r,r]} \left|T^* f(x)\right| \mathrm{d}x \\
&\geq \frac{1}{2r}\int_{[-r,r]} \left|T_N^\alpha f(x)\right| \mathrm{d}x \geq \frac{1}{\sqrt{\pi}\Gamma(\alpha)}\frac{1}{2r}\int_{[-r,r]} \sum_{j=J_0}^{0} \left(C\cdot C_1\cdot(-j)^{-\frac{1}{p-\varepsilon}}\right)\mathrm{d}x \\
&\geq C_{p,\varepsilon,\alpha}\cdot C_1\cdot(-J_0)^{\frac{1}{(p-\varepsilon)'}} \sim \left(\log\frac{2}{r}\right)^{\frac{1}{(p-\varepsilon)'}}.
\end{aligned}$$

For (c), let $v_j = (-1)^{j+1}$, $a_j = a^{2j}$ with $a > 1$ and $0 < \eta_0 < 1$ fixed in the proof of (b). Consider the same function f as in (b). Then, $\|v\|_{l^\infty(\mathbb{Z})} = 1$ and $\|f\|_{L^\infty(\mathbb{R})} = 1$. By the same argument as in (b), with $N = (J_0, 0)$ and $0 < \alpha < 1$, we have

$$\begin{aligned}
\frac{1}{2r}\int_{[-r,r]} \left|T^* f(x)\right| \mathrm{d}x &\geq \frac{1}{2r}\int_{[-r,r]} \left|T_N^\alpha f(x)\right| \mathrm{d}x \\
&\geq \frac{1}{\sqrt{\pi}\Gamma(\alpha)}\frac{1}{2r}\int_{[-r,r]} \sum_{j=J_0}^{0} C_1 \mathrm{d}t \geq \frac{C_1}{\sqrt{\pi}\Gamma(\alpha)}\cdot(-J_0) \sim \log\frac{2}{r}.
\end{aligned}$$

□

4 Boundedness of the differential transforms related to Schrödinger operator $-\Delta + V$

In this section, we would consider the differential transforms related with the generalized Poisson operators generated by the Schrödinger operator $\mathcal{L} = -\Delta + V$ in $\mathbb{R}^n$ with $n \geq 3$, where the nonnegative potential V belongs to the reverse Hölder class RH_q with $q \geq n/2$, that is, there exists $C > 0$, such that

$$\left(\frac{1}{|B|}\int_B V(x)^q \mathrm{d}x\right)^{\frac{1}{q}} \leq \frac{C}{|B|}\int_B V(x)\mathrm{d}x,$$

for every ball B in $\mathbb{R}^n$. Associated with this potential, Z. Shen defines the critical radii function in [16] as

$$\rho(x) := \sup\left\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,\mathrm{d}y \leq 1\right\}, \quad x \in \mathbb{R}^n. \tag{4.1}$$

We will abuse ρ in this article, and it should be easy to distinct the ρ-lacunary with $\rho(x)$ for the reader. For more information related with Schödinger operators, see [8, 16].

Lemma 2 (See [16, Lemma 1.4]) *There exist $c > 0$ and $k_0 \geq 1$ such that for all $x, y \in \mathbb{R}^n$*

$$c^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq c\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}}.$$

In particular, there exists a positive constant $C_1 < 1$ such that

$$\textit{if} \quad |x-y| \leq \rho(x) \quad \textit{then} \quad C_1\rho(x) < \rho(y) < C_1^{-1}\rho(x).$$

Let $\{\mathcal{T}_t\}_{t>0}$ be the heat–diffusion semigroup associated with $\mathcal{L}$:

$$\mathcal{T}_t f(x) \equiv e^{-t\mathcal{L}} f(x) = \int_{\mathbb{R}^n} e^{-t\mathcal{L}}(x, y) f(y)\,\mathrm{d}y, \qquad f \in L^2(\mathbb{R}^n),\ x \in \mathbb{R}^n,\ t > 0.$$

Lemma 3 (See [9, 11]) *For every $M > 0$ there exists a constant C_M such that*

$$0 \leq e^{-t\mathcal{L}}(x, y) \leq C_M t^{-n/2} e^{-\frac{|x-y|^2}{5t}} \left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-M}, \quad x, y \in \mathbb{R}^n,\ t > 0.$$

Lemma 4 (See [9, Proposition 2.16]) *There exists a nonnegative Schwartz class function ω on $\mathbb{R}^n$ such that*

$$\left|e^{-t\mathcal{L}}(x, y) - W_t(x-y)\right| \leq \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0} \omega_t(x-y), \quad x, y \in \mathbb{R}^n,\ t > 0,$$

where W_t is the Gauss–Weierstrass kernel, $\omega_t(x-y) := t^{-n/2}\omega\left((x-y)/\sqrt{t}\right)$ and

$$\delta_0 := 2 - \frac{n}{q} > 0. \tag{4.2}$$

Lemma 5 (See [10, Proposition 4.11]) *For every $0 < \delta < \delta_0$, there exists a constant $c > 0$ such that for every $M > 0$ there exists a constant $C > 0$ such that for $|x-y| < \sqrt{t}$ we have*

$$\left|e^{-t\mathcal{L}}(x, z) - e^{-t\mathcal{L}}(y, z)\right| \leq C\left(\frac{|x-y|}{\sqrt{t}}\right)^{\delta} t^{-n/2}\, e^{-c|x-z|^2/t}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(z)}\right)^{-M}.$$

Lemma 6 (See [9, Proposition 2.17]) *For every $0 < \delta < \min\{1, \delta_0\}$,*

$$\left|\left(e^{-t\mathcal{L}}(x, z) - e^{t\Delta}(x-z)\right) - \left(e^{-t\mathcal{L}}(y, z) - e^{t\Delta}(y-z)\right)\right| \leq C\left(\frac{|x-y|}{\rho(z)}\right)^{\delta} \omega_t(x-z),$$

for all $x, z \in \mathbb{R}^n$ and $t > 0$, with $|x-y| < C\rho(x)$ and $|x-y| < \frac{1}{4}|x-z|$.

In fact, going through the proof of [9] we see that $\omega(x) = e^{-|x|^2}$.

Then by (1.1), we can define the generalized Poisson operators associated with the Schrödinger operator $\mathcal{L}$ as follows:

$$\tilde{\mathcal{P}}_t^\alpha f(x) = \frac{t^{2\alpha}}{4^\alpha \Gamma(\alpha)} \int_0^{+\infty} \int_{\mathbb{R}^n} e^{-\frac{t^2}{4s}} e^{-s\mathcal{L}}(x, y) f(y) \mathrm{d}y \frac{\mathrm{d}s}{s^{1+\alpha}}.$$

Let $\{a_j\}_{j\in\mathbb{Z}}$ be an increasing sequence of positive real numbers, and $\{v_j\}_{j\in\mathbb{Z}}$ be a bounded sequence of real or complex numbers. We shall consider the differential transform series

$$\sum_{j\in\mathbb{Z}} v_j (\tilde{\mathcal{P}}_{a_{j+1}}^\alpha f(x) - \tilde{\mathcal{P}}_{a_j}^\alpha f(x)).$$

For each $N \in \mathbb{Z}^2,\ N = (N_1, N_2)$ with $N_1 < N_2$, we define the sum

$$\tilde{T}_N^\alpha f(x) = \sum_{j=N_1}^{N_2} v_j (\tilde{\mathcal{P}}_{a_{j+1}}^\alpha f(x) - \tilde{\mathcal{P}}_{a_j}^\alpha f(x)), \quad x \in \mathbb{R}^n. \tag{4.3}$$

Then, we have the following formula:

$$\begin{aligned}
\tilde{T}_N^\alpha f(x) &= \sum_{j=N_1}^{N_2} v_j (\tilde{\mathcal{P}}_{a_{j+1}}^\alpha f(x) - \tilde{\mathcal{P}}_{a_j}^\alpha f(x)) \\
&= \frac{1}{4^\alpha \Gamma(\alpha)} \sum_{j=N_1}^{N_2} v_j \int_{\mathbb{R}^n} \int_0^\infty \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} e^{-s\mathcal{L}}(x, y) f(y)\, \mathrm{d}s\mathrm{d}y \\
&= \frac{1}{4^\alpha \Gamma(\alpha)} \int_{\mathbb{R}^n} \int_0^{+\infty} \sum_{j=N_1}^{N_2} v_j \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} e^{-s\mathcal{L}}(x, y)\, \mathrm{d}s f(y)\mathrm{d}y.
\end{aligned}$$

We denote the kernel of $\tilde{T}_N^\alpha$ by

$$\tilde{K}_N^\alpha(x, y) = \frac{1}{4^\alpha \Gamma(\alpha)} \int_0^{+\infty} \sum_{j=N_1}^{N_2} v_j \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} e^{-s\mathcal{L}}(x, y)\mathrm{d}s.$$

By Lemma 3, we can prove the following theorem as in the proof of Theorem 5, which indicate that the kernel $\tilde{K}_N^\alpha$ is an $\ell^\infty(\mathbb{Z}^2)$-valued Calderón–Zygmund kernel.

Theorem 10 *For any $x, y \in \mathbb{R}^n$, $x \neq y$, and $M > 0$, there exists constants C depending on n, M, α and $\|v\|_{l^\infty(\mathbb{Z})}$ such that*

(i) $\left|\tilde{K}_N^\alpha(x, y)\right| \leq \dfrac{C}{|x-y|^n}\left(1 + \dfrac{|x-y|}{\rho(x)} + \dfrac{|x-y|}{\rho(y)}\right)^{-M}$,

(ii) $|\tilde{K}_N^\alpha(x,y) - \tilde{K}_N^\alpha(x,z)| + |\tilde{K}_N^\alpha(y,x) - \tilde{K}_N^\alpha(z,x)| \le C\frac{|y-z|^\delta}{|x-y|^{n+\delta}}$, *whenever* $|x-y| > 2\,|y-z|$, *for any* $0 < \delta < 2 - \frac{n}{q}$.

Proof *(i)* For any $M > 0$, we have

$$|\tilde{K}_N^\alpha(x,y)| \le C\,\|v\|_{l^\infty(\mathbb{Z})}\int_0^{+\infty}\sum_{j=-\infty}^{\infty}\left|\frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}e^{-s\mathcal{L}}(x,y)\right|\mathrm{d}s$$

$$= C\int_0^{+\infty}\sum_{j=-\infty}^{\infty}\left|a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha}e^{-a_j^2/(4s)}\right|\frac{e^{-s\mathcal{L}}(x,y)}{s^{1+\alpha}}\mathrm{d}s.$$

Then, by (2.1) and Lemma 3 we have

$$|\tilde{K}_N^\alpha(x,y)| \le C\int_0^{+\infty}\frac{e^{-s\mathcal{L}}(x,y)}{s}\mathrm{d}s = C\left(\int_0^{|x-y|^2} + \int_{|x-y|^2}^{+\infty}\right)\frac{e^{-|x-y|^2/(cs)}}{s^{n/2+1}}\mathrm{d}s$$
$$\le C\left(\int_0^{|x-y|^2}\frac{1}{|x-y|^{n+2}}\mathrm{d}s + \int_{|x-y|^2}^{+\infty}\frac{1}{s^{n/2+1}}\mathrm{d}s\right) \le \frac{C}{|x-y|^n},$$

and

$$|\tilde{K}_N^\alpha(x,y)| \le C\int_0^{+\infty}\frac{e^{-s\mathcal{L}}(x,y)}{s}\mathrm{d}s$$
$$= C\left(\int_0^{|x-y|^2} + \int_{|x-y|^2}^{+\infty}\right)s^{-n/2-1}e^{-|x-y|^2/(cs)}\left(\frac{\sqrt{s}}{\rho(x)}\right)^{-M}\mathrm{d}s$$
$$\le C\left(\int_0^{|x-y|^2}\frac{\rho^M(x)}{|x-y|^{n+M+2}}\mathrm{d}s + \int_{|x-y|^2}^{+\infty}\frac{\rho^M(x)}{s^{n/2+M/2+1}}\mathrm{d}s\right)$$
$$\le \frac{C}{|x-y|^n}\left(\frac{|x-y|}{\rho(x)}\right)^{-M}.$$

Then, together with the symmetry of $e^{-s\mathcal{L}}(x,y)$, we have

$$\tilde{K}_N^\alpha(x,y) \le \frac{C}{|x-y|^n}\left(1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)}\right)^{-M}.$$

(ii) It can be proved with the same method as in $i)$ with Proposition 3.10 in [4]. □

Theorem 11 *For the operator $\tilde{T}_N^\alpha$ defined in (4.3), we have the following statements:*

(a) *For any $1 < p < \infty$ and $\omega \in A_p$, there exists a constant $C > 0$ depending on n, p, α and $\|v\|_{l^\infty(\mathbb{Z})}$ such that*

$$\left\|\tilde{T}_N^\alpha f\right\|_{L^p(\mathbb{R}^n,\omega)} \le C\,\|f\|_{L^p(\mathbb{R}^n,\omega)},$$

for all functions $f \in L^p(\mathbb{R}^n, \omega)$.

(b) *For any* $\omega \in A_1$, *there exists a constant* $C > 0$ *depending on* n, α *and* $\|v\|_{l^\infty(\mathbb{Z})}$ *such that*

$$\omega\left(\{x \in \mathbb{R}^n : \left|\tilde{T}_N^\alpha f(x,t)\right| > \lambda\}\right) \le C\frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n,\omega)}, \quad \lambda > 0,$$

for all functions $f \in L^1(\mathbb{R}^n, \omega)$.

The constants C appeared above all are independent with N.

Proof For any $f \in L^p(\mathbb{R}^n, \omega)(1 \le p < +\infty)$, we have

$$\begin{aligned}\left|\tilde{T}_N^\alpha f(x)\right| &= \left|\int_{\mathbb{R}^n} \tilde{K}_N^\alpha(x,y)f(y)\mathrm{d}y\right| \\ &\le \left|\int_{|x-y|\le\rho(x)} \tilde{K}_N^\alpha(x,y)f(y)\mathrm{d}y\right| + \left|\int_{|x-y|>\rho(x)} \tilde{K}_N^\alpha(x,y)f(y)\mathrm{d}y\right| \\ &\le \left|\int_{|x-y|\le\rho(x)} \left(\tilde{K}_N^\alpha(x,y) - K_N^\alpha(x-y)\right) f(y)\mathrm{d}y\right| \\ &+ \left|\int_{|x-y|\le\rho(x)} K_N^\alpha(x,y)f(y)\mathrm{d}y\right| \\ &\quad + \left|\int_{|x-y|>\rho(x)} \tilde{K}_N^\alpha(x,y)f(y)\mathrm{d}y\right| \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

For I_1, by (2.1) and Lemma 4, we get

$$\begin{aligned}I_1 &= C\left|\int_{|x-y|\le\rho(x)}\int_0^{+\infty}\sum_{j=N_1}^{N_2} v_j \frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\left(e^{-s\mathcal{L}}(x,y) - W_s(x-y)\right)\mathrm{d}sf(y)\mathrm{d}y\right| \\ &\le C\|v\|_{l^\infty(\mathbb{Z})}\int_{|x-y|\le\rho(x)}\int_0^{+\infty}\sum_{j=-\infty}^{+\infty}\left|\frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\right|\left(\frac{\sqrt{s}}{\rho(x)}\right)^{\delta_0}\frac{e^{-c\frac{|x-y|^2}{4s}}}{s^{n/2}}\mathrm{d}s|f(y)|\mathrm{d}y \\ &\le C\int_{|x-y|\le\rho(x)}\int_0^{+\infty}\frac{1}{s}\left(\frac{\sqrt{s}}{\rho(x)}\right)^{\delta_0}\frac{e^{-c\frac{|x-y|^2}{4s}}}{s^{n/2}}\mathrm{d}s|f(y)|\mathrm{d}y \\ &= C\int_{|x-y|\le\rho(x)}\int_0^{\rho^2(x)}\frac{1}{s}\left(\frac{\sqrt{s}}{\rho(x)}\right)^{\delta_0}\frac{e^{-c\frac{|x-y|^2}{4s}}}{s^{n/2}}\mathrm{d}s|f(y)|\mathrm{d}y \\ &\quad + C\int_{|x-y|\le\rho(x)}\int_{\rho^2(x)}^{+\infty}\frac{1}{s}\left(\frac{\sqrt{s}}{\rho(x)}\right)^{\delta_0}\frac{e^{-c\frac{|x-y|^2}{4s}}}{s^{n/2}}\mathrm{d}s|f(y)|\mathrm{d}y \\ &=: I_{11} + I_{12}.\end{aligned}$$

For the term I_{11}, we have

$$I_{11} = C\int_{|x-y|\le\rho(x)}\int_0^{\rho^2(x)} s^{\frac{\delta_0-n}{2}-1}\rho(x)^{-\delta_0}e^{-c\frac{|x-y|^2}{4s}}\mathrm{d}s|f(y)|\mathrm{d}y$$

$$
\begin{aligned}
&\leq C\rho(x)^{-\delta_0}\int_0^{\rho^2(x)} s^{\frac{\delta_0}{2}-1}\cdot\frac{1}{s^{n/2}}\int_{\mathbb{R}^n} e^{-c\frac{|x-y|^2}{4s}}|f(y)|\mathrm{d}y\mathrm{d}s\\
&\leq C\sup_{s>0}\frac{1}{s^{n/2}}\int_{\mathbb{R}^n} e^{-c\frac{|x-y|^2}{4s}}|f(y)|\mathrm{d}y\\
&\leq C\mathcal{M}(f)(x),\quad x\in\mathbb{R}^n,
\end{aligned}
$$

where $\mathcal{M}$ denotes the classical Hardy–Littlewood maximal function. For the term I_{12}, since $\delta_0 = 2-\frac{n}{q}$(see (4.2)) and $n\geq 3$, we have

$$
\begin{aligned}
I_{12} &\leq C\int_{|x-y|\leq\rho(x)}\int_{\rho^2(x)}^{+\infty} s^{\frac{\delta_0-n}{2}-1}\rho(x)^{-\delta_0}\mathrm{d}s|f(y)|\mathrm{d}y\\
&\leq C\rho(x)^{-n}\int_{|x-y|\leq\rho(x)}|f(y)|\mathrm{d}y\\
&\leq C\mathcal{M}(f)(x),\quad x\in\mathbb{R}^n.
\end{aligned}
$$

Hence, we have

$$
I_1\leq C\mathcal{M}(f)(x),\quad x\in\mathbb{R}^n.
$$

For I_2, we can write

$$
\begin{aligned}
I_2 = \left|\int_{|x-y|\leq\rho(x)} K_N^{\alpha}(x,y)f(y)\mathrm{d}y\right| &\leq \left|\int_{\mathbb{R}^n} K_N^{\alpha}(x,y)f(y)\mathrm{d}y\right|\\
&+\sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} K_N^{\alpha}(x,y)f(y)\mathrm{d}y\right|.
\end{aligned}
$$

Now, let us consider the operator defined by

$$
\begin{aligned}
T: L^2(\mathbb{R}^n) &\longrightarrow L^2(\mathbb{R}^n)\\
f &\mapsto Tf(x)=\int_{\mathbb{R}^n} K_N^{\alpha}(x,y)f(y)\mathrm{d}y.
\end{aligned}
$$

From Theorem 4, we know that T is bounded on $L^2(\mathbb{R}^n)$. And T is a Calderón–Zygmund operator associated with the kernel $K_N^{\alpha}(x,y)$(see Theorem 5). Then, by proving a Cotlar's inequality as in [18, p. 34, Proposition 2] and the argument in [18, p. 36, Corollary 2], we can prove that the maximal operator $T_N^{\alpha,*}$ defined by

$$
T_N^{\alpha,*}f(x)=\sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} K_N^{\alpha}(x,y)f(y)\mathrm{d}y\right|
$$

is bounded on $L^p(\mathbb{R}^n,\omega)$, for every $1 < p < \infty$, and from $L^1(\mathbb{R}^n,\omega)$ in to $L^{1,\infty}(\mathbb{R}^n,\omega)$. Combining this fact with Theorem 3, we conclude that

$$\|I_2\|_{L^p(\mathbb{R}^n,\omega)} \le C\,\|f\|_{L^p(\mathbb{R}^n,\omega)}, \quad 1 < p < \infty$$

and

$$\|I_2\|_{L^{1,\infty}(\mathbb{R}^n,\omega)} \le C\,\|f\|_{L^1(\mathbb{R}^n,\omega)}.$$

For the last term I_3, by (2.1) and Lemma 3 with $M > 1$,

$$\begin{aligned}
I_3 &= \left|\int_{|x-y|>\rho(x)} \tilde{K}_N^{\alpha}(x,y)f(y)\mathrm{d}y\right| \\
&= C\left|\int_{|x-y|>\rho(x)}\int_0^{+\infty}\sum_{j=N_1}^{N_2} v_j \frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}} e^{-s\mathcal{L}}(x,y)\mathrm{d}s f(y)\mathrm{d}y\right| \\
&\le C\,\|v\|_{l^\infty(\mathbb{Z})}\int_{|x-y|>\rho(x)}\int_0^{+\infty}\sum_{j=-\infty}^{+\infty}\left|\frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\right| e^{-s\mathcal{L}}(x,y)\mathrm{d}s|f(y)|\mathrm{d}y \\
&\le C\int_{|x-y|>\rho(x)}\int_0^{+\infty}\frac{1}{s^{\frac{n}{2}+1}}e^{-\frac{|x-y|^2}{5s}}\left(1+\frac{\sqrt{s}}{\rho(x)}+\frac{\sqrt{s}}{\rho(y)}\right)^{-M}\mathrm{d}s|f(y)|\mathrm{d}y \\
&= C\int_{|x-y|>\rho(x)}\int_0^{\rho^2(x)}\frac{1}{s^{\frac{n}{2}+1}}e^{-\frac{|x-y|^2}{5s}}\left(1+\frac{\sqrt{s}}{\rho(x)}+\frac{\sqrt{s}}{\rho(y)}\right)^{-M}\mathrm{d}s|f(y)|\mathrm{d}y \\
&\quad + C\int_{|x-y|>\rho(x)}\int_{\rho^2(x)}^{+\infty}\frac{1}{s^{\frac{n}{2}+1}}e^{-\frac{|x-y|^2}{5s}}\left(1+\frac{\sqrt{s}}{\rho(x)}+\frac{\sqrt{s}}{\rho(y)}\right)^{-M}\mathrm{d}s|f(y)|\mathrm{d}y \\
&=: I_{31} + I_{32}.
\end{aligned}$$

For the term I_{31},

$$\begin{aligned}
I_{31} &\le C\int_{|x-y|>\rho(x)}\int_0^{\rho^2(x)}\frac{1}{s^{\frac{n}{2}+1}}e^{-\frac{|x-y|^2}{5s}}\mathrm{d}s|f(y)|\mathrm{d}y \\
&\le C\int_0^{\rho^2(x)}\frac{1}{s}e^{-c_1\frac{\rho^2(x)}{s}}\left(\frac{1}{s^{n/2}}\int_{\mathbb{R}^n}e^{-c_2\frac{|x-y|^2}{s}}|f(y)|\mathrm{d}y\right)\mathrm{d}s \\
&\le C\sup_{s>0}\frac{1}{s^{n/2}}\int_{\mathbb{R}^n}e^{-c_2\frac{|x-y|^2}{s}}|f(y)|\mathrm{d}y \\
&\le C\mathcal{M}(f)(x), \quad x\in\mathbb{R}^n.
\end{aligned}$$

For the other term I_{32}, by changing variable we have

$$\begin{aligned}
I_{32} &\le C\int_{|x-y|>\rho(x)}\int_{\rho^2(x)}^{+\infty}\frac{1}{s^{\frac{n}{2}+1}}e^{-\frac{|x-y|^2}{5s}}\left(1+\frac{\sqrt{s}}{\rho(x)}\right)^{-M}\mathrm{d}s|f(y)|\mathrm{d}y \\
&= C\int_{|x-y|>\rho(x)}|f(y)|\int_1^{+\infty}\frac{1}{(u\rho(x))^n}(1+u)^{-M}e^{-c\frac{|x-y|^2}{u^2\rho^2(x)}}\frac{\mathrm{d}u}{u}\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&\le C\frac{1}{\rho^n(x)}\int_{|x-y|>\rho(x)}|f(y)|\int_1^{+\infty}(1+u)^{-M}\left(\frac{u\rho(x)}{|x-y|}\right)^{n+1}\frac{\mathrm{d}u}{u^{n+1}}\mathrm{d}y\\
&\le C\frac{1}{\rho^n(x)}\int_{|x-y|>\rho(x)}|f(y)|\left(\frac{\rho(x)}{|x-y|}\right)^{n+1}\mathrm{d}y\\
&= C\frac{1}{\rho^n(x)}\sum_{k=0}^{+\infty}\int_{2^k\rho(x)<|x-y|\le 2^{k+1}\rho(x)}|f(y)|\left(\frac{\rho(x)}{|x-y|}\right)^{n+1}\mathrm{d}y\\
&\le C\sum_{k=0}^{+\infty}\frac{1}{2^k}\frac{1}{(2^k\rho(x))^n}\int_{|x-y|\le 2^{k+1}\rho(x)}|f(y)|\mathrm{d}y\\
&\le C\mathcal{M}(f)(x),\quad x\in\mathbb{R}^n.
\end{aligned}
$$

Hence, we get

$$
I_3\le C\mathcal{M}(f)(x),\quad x\in\mathbb{R}^n.
$$

Then, combining the above estimates of I_1, I_2, I_3 and the L^p-boundedness of $\mathcal{M}$, we conclude that $\tilde{T}_N^\alpha$ is a bounded operator on $L^p(\mathbb{R}^n,\omega)$ for every $1<p<\infty$, and from $L^1(\mathbb{R}^n,\omega)$ into $L^{1,\infty}(\mathbb{R}^n,\omega)$. We should note that the constants C appeared in the above estimates all are independent with $N=(N_1,N_2)$. Thus, the proof of the theorem is complete. □

We shall also analyze the behavior in L^∞ and $BMO_{\mathcal{L}}$. The space $BMO_{\mathcal{L}}(\mathbb{R}^n)$, introduced in [8], is defined as the set of functions f such that

$$
\begin{cases}
\dfrac{1}{|B|}\displaystyle\int_B\left|f(z)-\frac{1}{|B|}\int_B f\right|\mathrm{d}z\le C_1,\ \text{for all } B=B_R(x):R\le\rho(x),\\
\dfrac{1}{|B|}\displaystyle\int_B|f|\le C_2,\ \text{for all } B=B_R(x):R>\rho(x),
\end{cases}
$$

where $\rho(x)$ is the critical radii associated with $\mathcal{L}$, see (4.1). The norm $\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}$ is defined as $\min\{C_1,C_2\}$.

Theorem 12 *Given $f\in BMO_{\mathcal{L}}(\mathbb{R}^n)$, then there exists a constant C depending only on n, α, ρ and $\|v\|_{l^\infty(\mathbb{Z})}$ such that*

$$
\left\|\tilde{T}_N^\alpha f\right\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}\le C\,\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)},
$$

for all functions $f\in BMO_{\mathcal{L}}(\mathbb{R}^n)$.

Proof We first show that, if $f\in BMO_{\mathcal{L}}(\mathbb{R}^n)$, then $\tilde{T}_N^\alpha f$ is finite almost everywhere. This information is contained in the lemma as follows:

Lemma 7 *Given $f\in BMO_{\mathcal{L}}(\mathbb{R}^n)$ and any $x_0\in\mathbb{R}^n$, $C_0\ge 1$, then $\tilde{T}_N^\alpha f(x)<\infty$ at almost every $x\in B=B(x_0,C_0\rho(x_0))$.*

Proof Let us split the function f to be

$$f = (f - f_B)\chi_{B^*} + (f - f_B)\chi_{B^{*c}} + f_B =: f_1 + f_2 + f_3,$$

where $B^* = B(x_0, 2C_0\rho(x_0))$. Since $f \in BMO_{\mathcal{L}}(\mathbb{R}^n)$, $f_1 \in L^1(\mathbb{R}^n)$. By Theorem 11 (b), we know that $\tilde{T}_N^\alpha f_1(x) < \infty$ $a.e.$ $x \in B$. For $\tilde{T}_N^\alpha f_2$, we note that, for any $x \in B$ and $t > 0$,

$$\begin{aligned}
\tilde{\mathcal{P}}_t^\alpha f_2(x) &\le \frac{t^{2\alpha}}{4^\alpha \Gamma(\alpha)} \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{e^{-(t^2+|x-y|^2)/(4s)}}{(4\pi s)^{n/2}} f_2(y)\mathrm{d}y \frac{\mathrm{d}s}{s^{1+\alpha}} \\
&\le C \frac{t^{2\alpha}}{4^\alpha \Gamma(\alpha)} \int_0^{+\infty} \sum_{k=1}^{+\infty} \int_{2^k C_0\rho(x_0) < |x_0-y| \le 2^{k+1}C_0\rho(x_0)} \frac{1}{|x-y|^{n+2\alpha'}} |f(y) - f_B| \,\mathrm{d}y\, e^{-\frac{t^2}{4s}} \frac{\mathrm{d}s}{s^{1+\alpha-\alpha'}} \\
&\le C \frac{t^{2\alpha'}\Gamma(\alpha-\alpha')}{4^{\alpha'}\Gamma(\alpha)} \sum_{k=1}^{+\infty} (2^k C_0\rho(x_0))^{-2\alpha'} \frac{1}{(2^k C_0\rho(x_0))^n} \int_{|x_0-y|\le 2^{k+1}C_0\rho(x_0)} |f(y) - f_B| \,\mathrm{d}y \\
&\le C \frac{t^{2\alpha'}\Gamma(\alpha-\alpha')}{4^{\alpha'}\Gamma(\alpha)} \sum_{k=1}^{+\infty} (2^k C_0\rho(x_0))^{-2\alpha'} (1+2k) \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} < \infty,
\end{aligned}$$

where $0 < \alpha' < \alpha$. So, $\tilde{\mathcal{P}}_t^\alpha f_2(x)$ is well defined for $x \in B$ and $t > 0$. Since $\tilde{T}_N^\alpha f_2(x)$ is a finite summation and x_0, r_0 is arbitrary, $\tilde{T}_N^\alpha f_2(x) < \infty$ $a.e.$ $x \in \mathbb{R}^n$. And, we should note that

$$\begin{aligned}
\left|\tilde{\mathcal{P}}_t^\alpha f_3(x)\right| &\le \frac{t^{2\alpha}}{4^\alpha \Gamma(\alpha)} \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{e^{-(t^2+|x-y|^2)/(4s)}}{(4\pi s)^{n/2}} f_B \mathrm{d}y \frac{\mathrm{d}s}{s^{1+\alpha}} \\
&\le C \cdot f_B \le C \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} .
\end{aligned}$$

So, $\tilde{T}_N^\alpha f_3(x) < \infty$. Hence, $\tilde{T}_N^\alpha f(x) < \infty$ at almost every $x \in B = B(x_0, C_0\rho(x_0))$. This completes the proof of the lemma. □

Assume that $f \in BMO_{\mathcal{L}}(\mathbb{R}^n)$. Our goal is to show that $\tilde{T}_N^\alpha f \in BMO_{\mathcal{L}}(\mathbb{R}^n)$. By Theorem 10, we know that the operator $\tilde{T}_N^\alpha$ is a γ-Schrödinger-Calderón–Zygmund operator with $\gamma = 0$ appeared in [13]. By Theorem 1.2 in [13], we can prove the BMO-boundedness of $\tilde{T}_N^\alpha$ by checking a condition related with $\tilde{T}_N^\alpha 1$:

$$\log\left(\frac{\rho(x_0)}{t}\right) \frac{1}{|B|} \int_B |\tilde{T}_N^\alpha 1(x) - (\tilde{T}_N^\alpha 1)_B| \,\mathrm{d}x \le C \tag{4.4}$$

for every ball $B = B(x_0, t)$, $x_0 \in \mathbb{R}^n$ and $0 < t \le \frac{1}{2}\rho(x_0)$. In fact, since

$$|\tilde{T}_N^\alpha 1(x) - (\tilde{T}_N^\alpha 1)_B| \le \frac{1}{|B|} \int_B \left|\tilde{T}_N^\alpha 1(x) - \tilde{T}_N^\alpha 1(y)\right| \,\mathrm{d}y,$$

we only need to prove that, with some $0 < \delta < \delta_0$,

$$\left|\tilde{T}_N^{\alpha}1(x) - \tilde{T}_N^{\alpha}1(y)\right| \le C\left(\frac{t}{\rho(x_0)}\right)^{\delta}, \quad x, y \in B, \tag{4.5}$$

and then, (4.4) follows.

We shall note that, when $x, y \in B$, $\rho(x) \sim \rho(y) \sim \rho(x_0)$. First, we have

$$\begin{aligned}
&\left|\tilde{T}_N^{\alpha}1(x) - \tilde{T}_N^{\alpha}1(y)\right| \\
&= C\left|\int_{\mathbb{R}^n}\int_0^{+\infty}\sum_{j=N_1}^{N_2} v_j \frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\left(e^{-s\mathcal{L}}(x,z) - e^{-s\mathcal{L}}(y,z)\right)\,\mathrm{d}s\mathrm{d}z\right| \\
&\le C\left|\int_{\mathbb{R}^n}\int_0^{4t^2}\sum_{j=N_1}^{N_2} v_j \frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\left(e^{-s\mathcal{L}}(x,z) - e^{-s\mathcal{L}}(y,z)\right)\,\mathrm{d}s\mathrm{d}z\right| \\
&\quad + C\left|\int_{\mathbb{R}^n}\int_{4t^2}^{\rho^2(x_0)}\sum_{j=N_1}^{N_2} v_j \frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\left(e^{-s\mathcal{L}}(x,z) - e^{-s\mathcal{L}}(y,z)\right)\,\mathrm{d}s\mathrm{d}z\right| \\
&\quad + C\left|\int_{\mathbb{R}^n}\int_{\rho^2(x_0)}^{+\infty}\sum_{j=N_1}^{N_2} v_j \frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\left(e^{-s\mathcal{L}}(x,z) - e^{-s\mathcal{L}}(y,z)\right)\,\mathrm{d}s\mathrm{d}z\right| \\
&=: I + II + III.
\end{aligned}$$

For the term I, since $\int_{\mathbb{R}^n} W_s(x,z)\mathrm{d}z = \int_{\mathbb{R}^n} W_s(y,z)\mathrm{d}z = 1$ and by Lemma 4, we get

$$\begin{aligned}
I &\le C\left|\int_{\mathbb{R}^n}\int_0^{4t^2}\sum_{j=N_1}^{N_2} v_j \frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\left(e^{-s\mathcal{L}}(x,z) - W_s(x,z)\right)\,\mathrm{d}s\mathrm{d}z\right| \\
&\quad + C\left|\int_{\mathbb{R}^n}\int_0^{4t^2}\sum_{j=N_1}^{N_2} v_j \frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\left(e^{-s\mathcal{L}}(y,z) - W_s(y,z)\right)\,\mathrm{d}s\mathrm{d}z\right| \\
&\le C\int_0^{4t^2}\sum_{j=N_1}^{N_2}\left|v_j \frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\right|\left(\frac{\sqrt{s}}{\rho(x)}\right)^{\delta_0}\int_{\mathbb{R}^n}\omega_s(x-z)\,\mathrm{d}z\mathrm{d}s \\
&\quad + C\int_0^{4t^2}\sum_{j=N_1}^{N_2}\left|v_j \frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\right|\left(\frac{\sqrt{s}}{\rho(y)}\right)^{\delta_0}\int_{\mathbb{R}^n}\omega_s(y-z)\,\mathrm{d}z\mathrm{d}s \\
&\le C\int_0^{4t^2}\frac{1}{s}\left(\frac{\sqrt{s}}{\rho(x)}\right)^{\delta_0}\mathrm{d}s \le C\left(\frac{t}{\rho(x_0)}\right)^{\delta_0}.
\end{aligned}$$

For II, we have

$$II = C\left|\int_{\mathbb{R}^n}\int_{4t^2}^{\rho^2(x_0)}\sum_{j=N_1}^{N_2} v_j \frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}\left(e^{-s\mathcal{L}}(x,z) - e^{-s\mathcal{L}}(y,z)\right)\,\mathrm{d}s\mathrm{d}z\right|$$

$$
\begin{aligned}
&= C\Big(\int_{|x-z|>c\rho(x)}\int_{4t^2}^{\rho^2(x_0)}\frac{1}{s}\left|e^{-s\mathcal{L}}(x,z)-e^{-s\mathcal{L}}(y,z)\right|\,\mathrm{d}s\mathrm{d}z\\
&\quad+\int_{4|x-y|<|x-z|\le c\rho(x)}\int_{4t^2}^{\rho^2(x_0)}\frac{1}{s}\left|e^{-s\mathcal{L}}(x,z)-W_s(x,z)+W_s(y,z)-e^{-s\mathcal{L}}(y,z)\right|\,\mathrm{d}s\mathrm{d}z\\
&\quad+\int_{|x-z|\le 4|x-y|}\int_{4t^2}^{\rho^2(x_0)}\frac{1}{s}\left|e^{-s\mathcal{L}}(x,z)-W_s(x,z)+W_s(y,z)-e^{-s\mathcal{L}}(y,z)\right|\,\mathrm{d}s\mathrm{d}z\Big)\\
&=: C\,(II_1+II_2+II_3)\,.
\end{aligned}
$$

Since $|x-y|\le 2t\le\sqrt{s}$, by Lemma 5 we have

$$
\begin{aligned}
II_1 &\le C\int_{|x-z|>c\rho(x)}\int_{4t^2}^{\rho^2(x_0)}\frac{1}{s}\left(\frac{|x-y|}{\sqrt{s}}\right)^{\delta}\omega_s(x-z)\,\mathrm{d}s\mathrm{d}z\\
&\le C\int_{|x-z|>c\rho(x)}\frac{|x-y|^{\delta}}{|x-z|^{2+\delta}}\int_{4t^2}^{\rho^2(x_0)}\omega_s(x-z)\,\mathrm{d}s\mathrm{d}z\\
&\le C\int_{|x-z|>c\rho(x)}\frac{|x-y|^{\delta}}{|x-z|^{n+2+\delta}}\int_{4t^2}^{\rho^2(x_0)}1\,\mathrm{d}s\mathrm{d}z\\
&\le C\int_{|x-z|>c\rho(x)}\frac{|x-y|^{\delta}}{|x-z|^{n+2+\delta}}\rho^2(x_0)\mathrm{d}z\\
&\le C\left(\frac{t}{\rho(x_0)}\right)^{\delta}.
\end{aligned}
$$

For the term II_2, in this case, $|x-y|<c\rho(x)$ and $|x-y|<\dfrac{|x-z|}{4}$. By Lemma 6 we get

$$
\begin{aligned}
II_2 &\le C\int_{4|x-y|<|x-z|\le c\rho(x)}\int_{4t^2}^{\rho^2(x_0)}\frac{1}{s}\left(\frac{|x-y|}{\rho(z)}\right)^{\delta}\omega_s(x-z)\,\mathrm{d}s\mathrm{d}z\\
&\le C\int_{4|x-y|<|x-z|\le c\rho(x)}\left(\frac{|x-y|}{\rho(z)}\right)^{\delta}\frac{1}{|x-z|^{n+2}}\int_{4t^2}^{\rho^2(x_0)}1\,\mathrm{d}s\mathrm{d}z\\
&\le C\left(\frac{|x-y|}{\rho(x_0)}\right)^{\delta}\int_{4|x-y|<|x-z|\le c\rho(x)}\frac{\rho^2(x_0)}{|x-z|^{n+2}}\mathrm{d}z\\
&\le C\left(\frac{|x-y|}{\rho(x_0)}\right)^{\delta}\le C\left(\frac{t}{\rho(x_0)}\right)^{\delta}.
\end{aligned}
$$

For the term II_3, by Lemma 4, we have

$$
\begin{aligned}
II_3 &\le C\int_{|x-z|\le 4|x-y|}\int_{4t^2}^{\rho^2(x_0)}\frac{1}{s}\left[\left(\frac{\sqrt{s}}{\rho(x)}\right)^{\delta_0}\omega_s(x-z)+\left(\frac{\sqrt{s}}{\rho(y)}\right)^{\delta_0}\omega_s(y-z)\right]\mathrm{d}s\mathrm{d}z\\
&\le C\Big(\int_{|x-z|\le 4|x-y|}\int_{4t^2}^{\rho^2(x_0)}\frac{1}{s}\left(\frac{\sqrt{s}}{\rho(x)}\right)^{\delta_0}\omega_s(x-z)\,\mathrm{d}s\mathrm{d}z
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{|y-z|\le 5|x-y|}\int_{4t^2}^{\rho^2(x_0)}\frac{1}{s}\left(\frac{\sqrt{s}}{\rho(y)}\right)^{\delta_0}\omega_s(y-z)\,\mathrm{d}s\mathrm{d}z\Big)\\
&\le C\int_{4t^2}^{\rho^2(x_0)}\int_{|\xi|\le 5\frac{|x-y|}{\sqrt{s}}}\frac{1}{s}\left(\frac{\sqrt{s}}{\rho(x)}\right)^{\delta_0}\omega_s(\xi)\,d\xi\,\mathrm{d}s\\
&\le C\int_{4t^2}^{+\infty}\frac{1}{s}\left(\frac{\sqrt{s}}{\rho(x_0)}\right)^{\delta_0}\left(\frac{|x-y|}{\sqrt{s}}\right)^{n}\mathrm{d}s\\
&\le C\frac{|x-y|^n}{\rho(x_0)^{\delta_0}}\,t^{\delta_0-n}\le C\left(\frac{t}{\rho(x_0)}\right)^{\delta_0}.
\end{aligned}
$$

Then, we get

$$
II\le C\left(\frac{t}{\rho(x_0)}\right)^{\delta}
$$

with some $0<\delta<\delta_0$.

We shall treat the latest term III. In this case, since $s\ge\rho^2(x_0)>4t^2$, then $\sqrt{s}>2t>|x-y|$. By Lemma 5, we have, for $0<\delta<\delta_0$,

$$
\begin{aligned}
III&\le C\int_{\rho^2(x_0)}^{+\infty}\frac{1}{s}\left(\frac{|x-y|}{\sqrt{s}}\right)^{\delta}\int_{\mathbb{R}^n}\omega_s(x-z)\mathrm{d}z\,\mathrm{d}s\\
&\le C\int_{\rho^2(x_0)}^{+\infty}\frac{1}{s}\left(\frac{|x-y|}{\sqrt{s}}\right)^{\delta}\mathrm{d}s\le C\left(\frac{|x-y|}{\rho(x_0)}\right)^{\delta}\le C\left(\frac{t}{\rho(x_0)}\right)^{\delta}.
\end{aligned}
$$

Combining the above estimates for I, II and III, we have proved (4.5). Hence, we get the estimation (4.4) and the BMO-boundedness of $\tilde{T}_N^{\alpha}$. This completes the proof of Theorem 12. □

As in the Laplacian case, we also can consider the maximal operator

$$
\tilde{T}^*f(x)=\sup_N\left|\tilde{T}_N^{\alpha}f(x)\right|,\quad x\in\mathbb{R}^n,
$$

where the supremum are taken over all $N=(N_1,N_2)\in\mathbb{Z}^2$ with $N_1<N_2$.

Now we present our results as follows:

Theorem 13 (a) *For any $1<p<\infty$ and $\omega\in A_p$, there exists a constant C depending on n, p, ρ, α and $\|v\|_{l^\infty(\mathbb{Z})}$ such that*

$$
\left\|\tilde{T}^*f\right\|_{L^p(\mathbb{R}^n,\omega)}\le C\,\|f\|_{L^p(\mathbb{R}^n,\omega)},
$$

for all functions $f\in L^p(\mathbb{R}^n,\omega)$.

(b) *For any* $\lambda > 0$ *and* $\omega \in A_1$*, there exists a constant* C *depending on* n, ρ, α *and* $\|v\|_{l^\infty(\mathbb{Z})}$ *such that*

$$\omega\left(\{x \in \mathbb{R}^n : \left|\tilde{T}^* f(x)\right| > \lambda\}\right) \le C\frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n,\omega)},$$

for all functions $f \in L^1(\mathbb{R}^n, \omega)$.

(c) *There exists a constant* C *depending on* n, ρ, α *and* $\|v\|_{l^\infty(\mathbb{Z})}$ *such that*

$$\left\|\tilde{T}^* f\right\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \le C\|f\|_{L^\infty(\mathbb{R}^n)},$$

for any $f \in L^\infty(\mathbb{R}^n)$.

(d) *There exists a constant* C *depending on* n, ρ, α *and* $\|v\|_{l^\infty(\mathbb{Z})}$ *such that*

$$\left\|\tilde{T}^* f\right\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \le C\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)},$$

for all functions $f \in BMO_{\mathcal{L}}(\mathbb{R}^n)$.

With a similar argument as in the proof of Theorem 1, we can prove a Cotlar's type inequality in the Schrödinger setting. And then, all the statements in Theorem 13 can be gotten just with minor changes. We omit the proof at here.

Data availability Not applicable

Declarations

Conflict of interest The authors declare that there is no conflict of interest regarding the publication of this article.

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