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Hardy Type Spaces and Bergman Type Classes of Complex-Valued Harmonic Functions

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Abstract

The main purpose of this paper is to discuss Hardy type spaces and Bergman type classes of complex-valued harmonic functions. We first establish a Hardy-Littlewood type theorem on complex-valued harmonic functions. Next, the relationships between the Bergman type classes and the Hardy type spaces of complex-valued harmonic functions or the relationships between the Bergman type classes and the Hardy type spaces of harmonic (K, K')-elliptic mappings will be discussed, where $K \geq 1$ and $K' \geq 0$ are constants.

Keywords Bergman type class · Complex-valued harmonic function · Elliptic mapping · Hardy type space

Mathematics Subject Classification Primary 30H10 · 30H20 · 31A05; Secondary 30C62

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1 Introduction

Recently, the characterizations of Hardy type spaces and Bergman type classes of complex-valued harmonic functions have been attracted much attention of many mathematicians (one can see the references [5, 7, 10, 16–19, 22, 25, 27, 29] for more details). This paper is mainly motivated by the results given by Eenigenburg [11], Girela [14] and Kalaj [17].

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. Kalaj [17] established some elegant Riesz type estimates for complex-valued harmonic functions in the harmonic Hardy space $\mathscr{H}_H^p(\mathbb{D})$, $p \in (1, +\infty)$. As a corollary of this result, we can obtain that for a harmonic function $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$ with $g(0) = 0$ and $p \in (1, +\infty)$, f is in the harmonic Hardy space if and only if both of h and g are in the Hardy space. The first aim of this paper is to discuss the case of $p \in (0, 1] \cup \{+\infty\}$, and establish a Hardy-Littlewood type theorem on complex-valued harmonic functions, which shows that for each $p \in (0, 1] \cup \{+\infty\}$, there exist analytic functions h and g such that h and g are not in the Hardy space, but $f = h + \overline{g}$ is in the harmonic Hardy space.

Eenigenburg [11] studied several relationships between the Bergman type classes and the Hardy type spaces. The second aim of this paper is to give analogous results in the case of the harmonic Hardy type spaces. Eenigenburg [11] showed that if f is in the Hardy space $\mathscr{H}^p(\mathbb{D})$, $p \in (0, +\infty)$, then f is also in the Bergman type class $\mathscr{B}^p(\mu_p)$ and the implication $f \in \mathscr{H}^p(\mathbb{D}) \rightarrow f \in \mathscr{B}^p(\mu_p)$ is bounded, where μ_p is a measure on $\mathbb{D}$ which depends on p. In this paper, we give an analogous result for the harmonic Hardy space $\mathscr{H}_H^p(\mathbb{D})$ and the Bergman type class $\mathscr{B}^p(\mu_p)$, $p \in (1, +\infty)$. Eenigenburg [11] also showed that for $p \in [2, +\infty)$, if f is in the Bergman type class $\mathscr{B}^p(\mu)$, then f is also in the Hardy space $\mathscr{H}^p(\mathbb{D})$, where $d\mu$ is a measure on $\mathbb{D}$. In this paper, we will give an example which shows that similar result does not hold unless one considers the class of complex-valued harmonic functions under certain constraints. As a generalization of analytic mappings on $\mathbb{D}$, we consider harmonic (K, K')-elliptic mapping on $\mathbb{D}$ and give an analogous result for harmonic (K, K')-elliptic mappings. Our result also gives an improvement in the analytic case.

2 Preliminaries and Main Results

In order to state our main results, we need to recall some basic definitions and some results which motivate the present work.

For $a \in \mathbb{C}$ and $r > 0$, let $\mathbb{D}(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$ be the disk centered at a with the radius r, and for the disk center at 0, i.e. $\mathbb{D}(0, r)$, we then simply write it as $\mathbb{D}_r$. In particular, we use $\mathbb{D} =: \mathbb{D}_1$ to denote the unit disk, and let $\mathbb{T} =: \partial\mathbb{D}$ be the unit circle.

For $z = x + iy \in \mathbb{C}$, the complex formal differential operators are defined by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

For $\alpha \in [0, 2\pi]$, the directional derivative of a complex-valued harmonic function f at $z \in \mathbb{D}$ is defined by

$$\partial_\alpha f(z) = \lim_{\rho \to 0^+} \frac{f(z + \rho e^{i\alpha}) - f(z)}{\rho} = f_z(z)e^{i\alpha} + f_{\overline{z}}(z)e^{-i\alpha},$$

where $f_z =: \partial f/\partial z$, $f_{\overline{z}} =: \partial f/\partial \overline{z}$ and ρ is a positive real number such that $z+\rho e^{i\alpha} \in \mathbb{D}$. Then

$$\Lambda_f(z) =: \max\{|\partial_\alpha f(z)| : \alpha \in [0, 2\pi]\} = |f_z(z)| + |f_{\overline{z}}(z)|$$

and

$$\lambda_f(z) =: \min\{|\partial_\alpha f(z)| : \alpha \in [0, 2\pi]\} = \big||f_z(z)| - |f_{\overline{z}}(z)|\big|.$$

For a complex-valued harmonic function f defined in $\mathbb{D}$, the Jacobian of f is given by

$$J_f = |f_z|^2 - |f_{\overline{z}}|^2.$$

In particular, if f a sense-preserving harmonic function in $\mathbb{D}$, then $J_f = \Lambda_f \lambda_f$. It is well-known that every complex-valued harmonic function f defined in a simply connected domain Ω admits the canonical decomposition $f = h + \overline{g}$, where h and g are analytic with $g(0) = 0$. Recall that f is sense-preserving in Ω if $J_f > 0$ in Ω. Thus f is locally univalent and sense-preserving in Ω if and only if $J_f > 0$ in Ω, which means that $h' \neq 0$ in Ω and the second complex dilatation

$$\nu_f = \frac{\overline{f_{\overline{z}}}}{f_z} = \frac{g'}{h'}$$

has the property that $|\nu_f(z)| < 1$ in Ω (see [8, 20]).

Denote by $\mathscr{H}$ the set of all complex-valued harmonic functions of $\mathbb{D}$ into $\mathbb{C}$.

Throughout of this paper, we use the symbol C to denote the various positive constants, whose value may change from one occurrence to another.

2.1 Hardy Type Spaces

For $p \in (0, +\infty]$, the generalized Hardy space $\mathscr{H}^p_G(\mathbb{D})$ consists of all those measurable functions $f : \mathbb{D} \to \mathbb{C}$ such that, for $0 < p < +\infty$,

$$\|f\|_p =: \sup_{0<r<1} M_p(r, f) < +\infty,$$

and, for $p = +\infty$,

$$\|f\|_p =: \sup_{z\in\mathbb{D}} |f(z)|,$$

where

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{\frac{1}{p}}.$$

Let $\mathscr{H}_H^p(\mathbb{D}) = \mathscr{H}_G^p(\mathbb{D}) \cap \mathscr{H}$ be the harmonic Hardy space. The classical Hardy space $\mathscr{H}^p(\mathbb{D})$, that is, all the elements are analytic, is a subspace of $\mathscr{H}_H^p(\mathbb{D})$ (see [5, 9, 10, 17, 29–31]).

Recently, the study of $\mathscr{H}_H^p(\mathbb{D})$ has been attracted much attention of many mathematicians (see [1, 5, 7, 10, 17, 18, 22, 29]). Let's recall one of the celebrated results on $\mathscr{H}^p(\mathbb{D})$ by Riesz.

Theorem A (M. Riesz) *If a real-valued harmonic function $u \in \mathscr{H}_H^p(\mathbb{D})$ for some $p \in (1, +\infty)$, then its harmonic conjugate v is also of class $\mathscr{H}_H^p(\mathbb{D})$, where $v(0) = 0$. Furthermore, there is a constant C, depending only on p, such that*

$$M_p(r, v) \le C M_p(r, u), \ \ r \in [0, 1), \tag{2.1}$$

for all real-valued harmonic functions $u \in \mathscr{H}_H^p(\mathbb{D})$.

In 1972, Pichorides [26] improved (2.1) and obtained a sharp estimate as follows

$$\|v\|_p \le \cot\frac{\pi}{2p^*}\|u\|_p, \tag{2.2}$$

where $p^* = \max\{p, \ p/(p-1)\}$. Later, Verbitsky [28] further improved (2.2) into the following form.

$$\frac{1}{\cos\frac{\pi}{2p^*}}\|v\|_p \le \|f\|_p \le \frac{1}{\sin\frac{\pi}{2p^*}}\|u\|_p,$$

where $f = u + iv$ is analytic. For relevant studies on high-dimensional cases, see [6, 12]. As an analogy of Theorem A, Kalaj [17] established some elegant Riesz type estimates for complex-valued harmonic functions in $\mathscr{H}_H^p(\mathbb{D})$ as follows.

Theorem B ([17, Theorems 2.1 and 2.3]) *Let $p \in (1, +\infty)$ be a constant and $f = h + \overline{g} \in \mathscr{H}_H^p(\mathbb{D})$, where h and g are analytic in $\mathbb{D}$.*

(1) *If* $\mathrm{Re}(g(0)h(0)) \ge 0$, *then,*

$$\left(\int_0^{2\pi} \left(|h(e^{i\theta})|^2 + |g(e^{i\theta})|^2\right)^{\frac{p}{2}} \frac{d\theta}{2\pi}\right)^{\frac{1}{p}} \le \frac{1}{C_1(p)}\|f\|_p,$$

where $C_1(p) = \sqrt{1 - \left|\cos\frac{\pi}{p}\right|}$ *and* "Re" *denotes the real part of a complex number.*

(2) *If* $\mathrm{Re}(g(0)h(0)) \le 0$, *then,*

$$\|f\|_p \le C_2(p)\left(\int_0^{2\pi}\left(|h(e^{i\theta})|^2+|g(e^{i\theta})|^2\right)^{\frac{p}{2}}\frac{d\theta}{2\pi}\right)^{\frac{1}{p}},$$

where $C_2(p)=\sqrt{2}\max\left\{\sin\frac{\pi}{2p},\ \cos\frac{\pi}{2p}\right\}$.

In particular, for a harmonic function $f=h+\overline{g}$, where h and g are analytic in $\mathbb{D}$ with $g(0)=0$, the following result holds.

$$f\in\mathscr{H}_H^p(\mathbb{D})\quad\text{iff}\quad h,g\in\mathscr{H}^p(\mathbb{D}),\quad p\in(1,+\infty). \tag{2.3}$$

In the following, we will discuss the case of $p\in(0,1]\cup\{+\infty\}$, and establish a Hardy-Littlewood type theorem on complex-valued harmonic functions.

Theorem 2.1 *Let* $p\in(0,1]\cup\{+\infty\}$ *be a given constant. Then the following statements hold.*

(I) *If* $p\in(0,1]$ *and* $f=h+\overline{g}\in\mathscr{H}_H^p(\mathbb{D})$, *then*

$$M_p(r,h)=O\left(\left(\log\frac{1}{1-r}\right)^{\frac{1}{p}}\right)\text{ and }M_p(r,g)=O\left(\left(\log\frac{1}{1-r}\right)^{\frac{1}{p}}\right) \tag{2.4}$$

as $r\to 1^-$. *In particular, if* $p=1,\ \frac{1}{2},\ \frac{1}{3},\ \ldots$, *then the estimate of* (2.4) *is sharp.*

(II) *If* $p=1$, *then there is a function* $f=h+\overline{g}\in\mathscr{H}_H^1(\mathbb{D})$, *but* $h,\ g\notin\mathscr{H}^1(\mathbb{D})$. *Furthermore, if* $f=h+\overline{g}\in\mathscr{H}_H^1(\mathbb{D})$, *then* $h,\ g\in\mathscr{H}^q(\mathbb{D})$ *for all* $q\in(0,1)$;

(III) *If* $p=+\infty$, *then there are two unbounded analytic functions* h *and* g *such that* $f=h+\overline{g}\in\mathscr{H}_H^{+\infty}(\mathbb{D})$.

2.2 Bergman Type Classes

Let μ be a positive measure on $\mathbb{D}$ and $p>0$. We use $\mathscr{B}^p(\mu)$ to stand for the analytic Bergman class of all analytic functions f of $\mathbb{D}$ into $\mathbb{C}$ such that

$$\|f\|_{\mathscr{B}^p(\mu),s}=:\left(\int_{\mathbb{D}}|f'(z)|^p d\mu(z)\right)^{\frac{1}{p}}<+\infty.$$

Furthermore, denote by $\mathscr{B}_H^p(\mu)$ the harmonic Bergman class of all functions $f\in\mathscr{H}$ such that

$$\mathscr{I}_p(f,\mu)=:\left(\int_{\mathbb{D}}\left(\sqrt{|J_f(z)|}\right)^p d\mu(z)\right)^{\frac{1}{p}}<+\infty.$$

For a real number $r \in [0, 1)$ and a complex-valued harmonic function f defined in $\mathbb{D}$, let

$$\mathscr{A}_H(r, f) = \int_{\mathbb{D}_r} |J_f(z)| dA(z)$$

denote the area of the image of $\mathbb{D}_r$ under f, multiply covered points being counted multiply, where $dA(z) = dxdy$. In particular, if f is analytic in $\mathbb{D}$, then we let $\mathscr{A}(r, f) := \mathscr{A}_H(r, f)$. In [16], Holland and Twomey proved the following result.

Theorem C ([16, Theorem 1]) *Let f be analytic in $\mathbb{D}$. If $p \in (0, 2]$, then*

$$f \in \mathscr{H}^p(\mathbb{D}) \Rightarrow \int_0^1 \mathscr{A}^{\frac{p}{2}}(r, f) dr < +\infty; \tag{2.5}$$

while if $p \in [2, +\infty)$, then

$$\int_0^1 \mathscr{A}^{\frac{p}{2}}(r, f) dr < +\infty \Rightarrow f \in \mathscr{H}^p(\mathbb{D}). \tag{2.6}$$

For $p \in (0, 2]$ and $f \in \mathscr{H}^p(\mathbb{D})$, it follows from (2.5) and Hölder's inequality that

$$\begin{aligned} \int_0^1 \left(\int_{\mathbb{D}_r} |f'(z)|^p dA(z) \right) dr &\leq \int_0^1 \mathscr{A}^{\frac{p}{2}}(r, f) \left(\int_{\mathbb{D}_r} dA(z) \right)^{1-\frac{p}{2}} dr \\ &\leq \pi^{1-\frac{p}{2}} \int_0^1 \mathscr{A}^{\frac{p}{2}}(r, f) dr < +\infty. \end{aligned} \tag{2.7}$$

On the other hand, if $p \geq 2$ and

$$\int_0^1 \left(\int_{\mathbb{D}_r} |f'(z)|^p dA(z) \right) dr < +\infty,$$

then, by Hölder's inequality, we have

$$\mathscr{A}(r, f) \leq \left(\int_{\mathbb{D}_r} |f'(z)|^p dA(z) \right)^{\frac{2}{p}} \left(\int_{\mathbb{D}_r} dA(z) \right)^{1-\frac{2}{p}}$$

which implies that

$$\int_0^1 \mathscr{A}^{\frac{p}{2}}(r, f) dr \leq \pi^{\frac{p}{2}-1} \int_0^1 \left(\int_{\mathbb{D}_r} |f'(z)|^p dA(z) \right) dr < +\infty. \tag{2.8}$$

Let χ_r denote the characteristic function of $\mathbb{D}_r$. By Fubini's theorem, we have

$$\int_0^1 \left(\int_{\mathbb{D}_r} |f'(z)|^p dA(z) \right) dr = \int_0^1 \left(\int_{\mathbb{D}} \chi_r(z) |f'(z)|^p dA(z) \right) dr \tag{2.9}$$

$$= \int_{\mathbb{D}} (1 - |z|)|f'(z)|^p dA(z).$$

Then, by (2.5), (2.6), (2.7), (2.8) and (2.9), we obtain the following result (see [11, Corollary]).

Theorem D *Let f be analytic in $\mathbb{D}$. Then, if $p \in (0, 2]$*

$$f \in \mathscr{H}^p(\mathbb{D}) \;\Rightarrow\; f \in \mathscr{B}^p(\mu) \tag{2.10}$$

while if $p \in [2, +\infty)$,

$$f \in \mathscr{B}^p(\mu) \;\Rightarrow\; f \in \mathscr{H}^p(\mathbb{D}), \tag{2.11}$$

where $d\mu(z) = (1 - |z|)dA(z)$.

In [11], Eenigenburg improved (2.10) into the following form.

Theorem E ([11, Theorem 1]) *Let f be analytic in $\mathbb{D}$. Then, for $p \in (0, +\infty)$,*

$$f \in \mathscr{H}^p(\mathbb{D}) \;\Rightarrow\; f \in \mathscr{B}^p(\mu_p),$$

where

$$d\mu_p(z) = \begin{cases} (1 - |z|)dA(z), & p \in (0, 2], \\ (1 - |z|)^{p-1} dA(z), & p \in [2, +\infty), \end{cases}$$

and there exists a constant $C(p) > 0$, which depends only on p, such that

$$\|f\|_{\mathscr{B}^p(\mu_p),s} \le C(p)\|f\|_p, \quad f \in \mathscr{H}^p(\mathbb{D}).$$

For $p \in (0, 2)$, the following result shows that the measure $(1 - |z|)dA(z)$ can not be replaced by $(1 - |z|)^{p-1} dA(z)$.

Theorem F ([14, Theorem 1]) *Let $p \in (0, 2)$. Then there exists $f \in \mathscr{H}^p(\mathbb{D})$ such that*

$$\int_{\mathbb{D}} |f'(z)|^p (1 - |z|)^{p-1} dA(z) = +\infty.$$

By analogy with (2.5) and Theorem E, we obtain the following result for complex-valued harmonic functions by applying Theorems B and 2.1.

Proposition 2.2 *Let f be a complex-valued harmonic function in $\mathbb{D}$. Then the following statements hold.*

(I) $f \in \mathscr{H}_H^1(\mathbb{D}) \;\Rightarrow\; f \in \mathscr{B}_H^p(\mu_p)$ *for all $p \in (0, 1)$;*
(II) *For $p \in (1, +\infty)$,*

$$f \in \mathscr{H}_H^p(\mathbb{D}) \;\Rightarrow\; f \in \mathscr{B}_H^p(\mu_p),$$

and there exists a constant $C(p) > 0$*, which depends only on* p*, such that*

$$\mathscr{I}_p(f, \mu_p) \le C(p)\|f\|_p, \quad f \in \mathscr{H}_H^p(\mathbb{D}),$$

where $d\mu_p$ *is defined in Theorem E.*

Inspired by (2.11) in Theorem D, Eenigenburg proved the following result.

Theorem G ([11, Theorem 2]) *Let* **BMOA** *denote the class of analytic functions in* $\mathbb{D}$ *having boundary functions of bounded mean oscillation, and let* $\mathcal{A}$ *denote those analytic functions in* $\mathbb{D}$ *which are continuous on* $\overline{\mathbb{D}}$*. If* f *is an analytic function in* $\mathbb{D}$*, and, for* $p > 2$*,*

$$\int_0^{2\pi}\int_0^1 (1-r)|f'(re^{i\theta})|^p dr d\theta < +\infty,$$

then $f \in \mathscr{H}^{\frac{p}{3-p}}(\mathbb{D})$ *if* $p < 3$*, in* **BMOA** *if* $p = 3$*, and in* $\mathcal{A}$ *if* $p > 3$*.*

In general, (2.6) in Theorem C, or (2.11) in Theorem D does not hold for complex-valued harmonic functions defined in $\mathbb{D}$. Our example is as follows. Let $f_\zeta(z) = h_\zeta(z) + \overline{h_\zeta(z)}$ for $z \in \mathbb{D}$, where $\zeta \in \mathbb{T}$ is fixed and $h_\zeta(z) = 1/(1 - z\zeta)$. By [3, Theorem E], we have

$$\lim_{r\to 1^-}\frac{1}{2\pi}\int_0^{2\pi}|h_\zeta(re^{i\theta})|^p d\theta = \lim_{r\to 1^-}\sum_{n=0}^{+\infty}\left(\frac{\Gamma\left(n+\frac{p}{2}\right)}{n!\Gamma\left(\frac{p}{2}\right)}\right)^2 r^{2n} = +\infty,$$

which implies that $h_\zeta \notin \mathscr{H}^p(\mathbb{D})$ for $p \ge 2$, where $z = re^{i\theta} \in \mathbb{D}$ and Γ denotes the Gamma function. However, $J_{f_\zeta}(z) \equiv 0$. For $p \ge 2$, although

$$\int_0^1 \mathscr{A}_H^{\frac{p}{2}}(r, f_\zeta)dr = 0 \text{ and } \int_{\mathbb{D}}(1-|z|)\left(\sqrt{|J_{f_\zeta}(z)|}\right)^p dA(z) = 0,$$

Theorem B implies that $f_\zeta \notin \mathscr{H}_H^p(\mathbb{D})$. Consequently, (2.6) in Theorem C and (2.11) in Theorem D do not hold unless one considers the class of complex-valued harmonic functions under certain constraints.

A mapping $f : \Omega \to \mathbb{C}$ is said to be absolutely continuous on lines, ACL in brief, in the domain Ω if for every closed rectangle $R \subset \Omega$ with sides parallel to the axes x and y, f is absolutely continuous on almost every horizontal line and almost every vertical line in R. Such a mapping has, of course, partial derivatives f_x and f_y a.e. in Ω. Moreover, we say $f \in ACL^2$ if $f \in ACL$ and its partial derivatives are locally L^2 integrable in Ω.

A sense-preserving and continuous mapping f of $\mathbb{D}$ into $\mathbb{C}$ is called a (K, K')-elliptic mapping if

1. f is ACL^2 in $\mathbb{D}$ and $J_f > 0$ a.e. in $\mathbb{D}$;

2. there are constants $K \geq 1$ and $K' \geq 0$ such that

$$\Lambda_f^2 \leq K J_f + K' \text{ a.e. in } \mathbb{D}.$$

We remark that the unit disk $\mathbb{D}$ in the definition of (K, K')-elliptic mapping can be replaced by a general domain in $\mathbb{C}$. In particular, if $K' \equiv 0$, then a (K, K')-elliptic mapping is said to be K-quasiregular. It is well known that every quasiregular mapping is an elliptic mapping. But the inverse of this statement is not true. We refer to [2, 4, 7, 13, 23] for more details of elliptic mappings.

By using Theorem G, we obtain the following result for harmonic (K, K')-elliptic mappings.

Proposition 2.3 *Let $K \geq 1$ and $K' \geq 0$ be constants. Suppose that the complex-valued harmonic function $f = h + \overline{g}$ is (K, K')-elliptic, where h and g are holomorphic in $\mathbb{D}$. If $p > 2$ and if*

$$\int_0^{2\pi} \int_0^1 (1 - r) J_f\big(re^{i\theta}\big)^{p/2} dr d\theta < +\infty,$$

then $h, \ g \in \mathscr{H}^{\frac{p}{3-p}}(\mathbb{D})$ if $p < 3$, in **BMOA** *if $p = 3$, and in $\mathcal{A}$ if $p > 3$.*

By analogy with Theorem D (2.11) and Theorem G, we get a result for harmonic (K, K')-elliptic mappings as follows.

Theorem 2.4 *Suppose that the complex-valued harmonic function f is (K, K')-elliptic, where $K \geq 1$ and $K' \geq 0$ are constants. Then, for $q \in (1, +\infty)$,*

$$f \in \mathscr{B}_H^q(\tilde{\mu}_q) \Rightarrow f \in \mathscr{H}_H^q(\mathbb{D}),$$

where

$$d\tilde{\mu}_q(z) = \begin{cases} (1 - |z|)^{q-1} dA(z), & q \in (1, 2], \\ (1 - |z|)^{\frac{q}{2}} dA(z), & q \in [2, +\infty). \end{cases}$$

Moreover, if the complex-valued harmonic function f is K-quasiregular, and $q \in (1, 2]$, then there exists a constant $C(q, K) > 0$, which depends only on q and K, such that

$$\|f\|_q \leq C(q, K)(|f(0)| + \mathscr{I}_q(f, \tilde{\mu}_q)), \quad f \in \mathscr{B}_H^q(\tilde{\mu}_q).$$

By using similar reasoning as in the proof of Theorem 2.4, we get the following result which is an improvement of Theorem D (2.11) and Theorem G.

Corollary 2.5 *Let f be analytic in $\mathbb{D}$. Then, for $q \in (1, +\infty)$,*

$$f \in \mathscr{B}^q(\tilde{\mu}_q) \Rightarrow f \in \mathscr{H}^q(\mathbb{D}),$$

where $d\tilde{\mu}_q$ is defined in Theorem 2.4. Moreover, if $q \in (1,2]$, then there exists a constant $C(q) > 0$, which depends only on q, such that

$$\|f\|_q \leq C(q)(|f(0)| + \|f\|_{\mathscr{B}^q(\tilde{\mu}_q),s}), \quad f \in \mathscr{B}^q(\tilde{\mu}_q).$$

The proofs of Theorems 2.1, 2.4, Propositions 2.2 and 2.3 will be presented in Sect. 3.

3 Proofs of the Main Results

Before proving Theorem 2.1, let's recall some classical results. Let $f = U + iV$ be analytic in $\mathbb{D}$ with $V(0) = 0$. In [15, Theorem 7], Hardy and Littlewood proved that if $U \in \mathscr{H}^p_H(\mathbb{D})$ for some $0 < p \leq 1$, then V satisfies

$$M_p(r, V) \leq C\|U\|_p + C\|U\|_p \left(\log \frac{1}{1-r}\right)^{\frac{1}{p}}, \tag{3.1}$$

where C is a positive constant depending only on p.

The following results are well-known.

Lemma H (cf. [5, Lemma 5]) *Suppose that $a,\ b \in [0, +\infty)$ and $\tau \in (0, +\infty)$. Then*

$$(a+b)^\tau \leq 2^{\max\{\tau-1,0\}}(a^\tau + b^\tau).$$

Lemma I ([9, Hardy's inequality]) *If $f(z) = \sum_{n=0}^{+\infty} a_n z^n \in \mathscr{H}^1(\mathbb{D})$, then*

$$\sum_{n=0}^{+\infty} \frac{|a_n|}{n+1} \leq \pi\|f\|_1.$$

3.1 The Proof of Theorem 2.1

We first prove (I). We may assume that $h(0) = g(0) = 0$. Let $f = h + \overline{g} = u + iv \in \mathscr{H}^p_H(\mathbb{D})$, where $h = u_1 + iv_1$ and $g = u_2 + iv_2$. Then $u,\ v \in \mathscr{H}^p_H(\mathbb{D})$. Set $F = h + g$ and $\widetilde{v} = \mathrm{Im}(F)$, where "Im" denotes the imaginary part of a complex number. Since $\mathrm{Re}(F) = \mathrm{Re}(f)$, by (3.1), we see that there is a positive constant C, depending only on p, such that, for $r \in (0,1)$,

$$\begin{aligned} M_p(r, \widetilde{v}) &\leq C\|u\|_p + C\|u\|_p \left(\log \frac{1}{1-r}\right)^{\frac{1}{p}} \\ &\leq C\|f\|_p + C\|f\|_p \left(\log \frac{1}{1-r}\right)^{\frac{1}{p}}. \end{aligned} \tag{3.2}$$

It follows from Lemma H that

$$|v_2|^p = |\mathrm{Im}(g)|^p = \frac{|\widetilde{v} - v|^p}{2^p} \le \frac{|\widetilde{v}|^p + |v|^p}{2^p},$$

which, together with (3.2), implies that there is a positive constant C, depending only on p, such that,

$$\begin{aligned} M_p(r, v_2) &\le 2^{\frac{1}{p}-2} \left(M_p(r, \widetilde{v}) + M_p(r, v)\right) \\ &\le 2^{\frac{1}{p}-2} M_p(r, \widetilde{v}) + 2^{\frac{1}{p}-2}\|f\|_p \\ &\le (C+1)2^{\frac{1}{p}-2}\|f\|_p + C\|f\|_p 2^{\frac{1}{p}-2}\left(\log\frac{1}{1-r}\right)^{\frac{1}{p}}. \end{aligned} \tag{3.3}$$

Let $F_1 = h - g$ and $\widetilde{u} = \mathrm{Re}(F_1)$. Then

$$\mathrm{Re}(-iF_1) = \mathrm{Im}(F_1) = \mathrm{Im}(f) = v \in \mathscr{H}_H^p(\mathbb{D}),$$

which, together with $-\widetilde{u} = \mathrm{Im}(-iF_1)$ and (3.1), implies that there is a positive constant C, depending only on p, such that,

$$\begin{aligned} M_p(r, \widetilde{u}) = M_p(r, -\widetilde{u}) &\le C\|v\|_p + C\|v\|_p\left(\log\frac{1}{1-r}\right)^{\frac{1}{p}} \\ &\le C\|f\|_p + C\|f\|_p\left(\log\frac{1}{1-r}\right)^{\frac{1}{p}}. \end{aligned} \tag{3.4}$$

On the other hand, by Lemma H, we have

$$|u_2|^p = |\mathrm{Re}(g)|^p = \frac{|u - \widetilde{u}|^p}{2^p} \le \frac{|\widetilde{u}|^p + |u|^p}{2^p},$$

which, together with (3.4), yields that there is a positive constant C, depending only on p, such that, for $r \in (0, 1)$,

$$\begin{aligned} M_p(r, u_2) &\le 2^{\frac{1}{p}-2} \left(M_p(r, \widetilde{u}) + M_p(r, u)\right) \\ &\le (C+1)2^{\frac{1}{p}-2}\|f\|_p + C\|f\|_p 2^{\frac{1}{p}-2}\left(\log\frac{1}{1-r}\right)^{\frac{1}{p}}. \end{aligned} \tag{3.5}$$

From (3.3), (3.5) and Lemma H, we conclude that there is a positive constant C, depending only on p, such that, for $r \in (0, 1)$,

$$\begin{aligned} M_p(r, g) &\le \left(M_p^p(r, u_2) + M_p^p(r, v_2)\right)^{\frac{1}{p}} \\ &\le 2^{\frac{1}{p}-1}\left(M_p(r, u_2) + M_p(r, v_2)\right) \end{aligned} \tag{3.6}$$

$$\le (C+1)2^{\frac{2}{p}-2}\|f\|_p + C\|f\|_p 2^{\frac{2}{p}-2}\left(\log\frac{1}{1-r}\right)^{\frac{1}{p}}.$$

Since $h = f - \overline{g}$, by (3.6) and Lemma H, we see that there is a positive constant C, depending only on p, such that, for $r \in (0,1)$,

$$\begin{aligned} M_p(r,h) &\le 2^{\frac{1}{p}-1}\left(M_p(r,f) + M_p(r,g)\right) \\ &\le \left(1 + (C+1)2^{\frac{2}{p}-2}\right)2^{\frac{1}{p}-1}\|f\|_p + C\|f\|_p 2^{\frac{3}{p}-3}\left(\log\frac{1}{1-r}\right)^{\frac{1}{p}}. \end{aligned}$$

We come to prove the sharpness part. Let $k \in \{0,1,\ldots\}$ and

$$h(z) = g(z) = e^{\frac{ik\pi}{2}}\frac{1}{(1-z)^{1+k}}$$

for $z \in \mathbb{D}$. Then $f = h + \overline{h} = 2\,\mathrm{Re}(h) \in \mathscr{H}_H^{\frac{1}{1+k}}(\mathbb{D})$ (see [9, Chapter 3] or [15]). It follows from [3, Theorem E] that

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi}|h(re^{i\theta})|^{\frac{1}{1+k}}d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-re^{i\theta}|} = \sum_{n=0}^{+\infty}\left(\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!\Gamma\left(\frac{1}{2}\right)}\right)^2 r^{2n} \\ &\sim \log\frac{1}{1-r}. \end{aligned}$$

Hence the logarithmic factor in (2.4) can not be improved when $p = 1/(1+k)$.

Now, we prove (II). For $z \in \mathbb{D}$, let

$$f(z) = h(z) + \overline{h(z)},$$

where $h(z) = C\sum_{n=2}^{+\infty}\frac{z^n}{\log n}$ and C is a non-zero real constant. From Lemma I, we know that $h \notin \mathscr{H}^1(\mathbb{D})$. However, $f = 2\mathrm{Re}(h) \in \mathscr{H}^1(\mathbb{D})$, and is in fact a Poisson integral. To see this, let $z = e^{it}$, where $t \in [0, 2\pi]$. We observe that

$$\mathrm{Re}(h(e^{it})) = C\sum_{n=2}^{+\infty}\frac{\cos nt}{\log n}$$

is the Fourier series of an integrable function. Next, we show that if $f = h + \overline{g} \in \mathscr{H}_H^1(\mathbb{D})$, then $h,\ g \in \mathscr{H}^q(\mathbb{D})$ for all $q \in (0,1)$. Since $f = h + \overline{g} = u + iv \in \mathscr{H}_H^1(\mathbb{D})$, we see that $u,\ v \in \mathscr{H}_H^1(\mathbb{D})$. It follows from Kolmogorov's Theorem (see [9, Theorem 4.2]) that $\widetilde{v} = \mathrm{Im}(F) \in \mathscr{H}_H^q(\mathbb{D})$ for all $q \in (0,1)$, where F is defined in the proof of (I). Consequently,

$$|v_2| = |\mathrm{Im}(g)| = \frac{|\widetilde{v} - v|}{2} \le \frac{|\widetilde{v}| + |v|}{2}, \tag{3.7}$$

which implies that $v_2 \in \mathscr{H}_H^q(\mathbb{D})$ for all $q \in (0, 1)$. As in the proof of Case (I), we let $F_1 = h - g$ and $\widetilde{u} = \mathrm{Re}(F_1)$. Then

$$\mathrm{Re}(-iF_1) = \mathrm{Im}(F_1) = \mathrm{Im}(f) \in \mathscr{H}_H^1(\mathbb{D}),$$

which, together with Kolmogorov's Theorem, yields that $-\widetilde{u} = \mathrm{Im}(-iF_1) \in \mathscr{H}_H^q(\mathbb{D})$ for all $q \in (0, 1)$. Hence

$$|u_2| = |\mathrm{Re}(g)| = \frac{|u - \widetilde{u}|}{2} \leq \frac{|\widetilde{u}| + |u|}{2} \tag{3.8}$$

and $u_2 \in \mathscr{H}_H^q(\mathbb{D})$ for all $q \in (0, 1)$. By (3.7), (3.8) and Lemma H, we conclude that $g = u_2 + iv_2 \in \mathscr{H}^q(\mathbb{D})$ for all $q \in (0, 1)$. Hence $h = f - \overline{g}$ also belongs to $\mathscr{H}^q(\mathbb{D})$ for all $q \in (0, 1)$.

At last, we show (III). For $z \in \mathbb{D}$, let

$$f(z) = h(z) + \overline{h(z)},$$

where $h(z) = iC \log \frac{1+z}{1-z}$ and C is a positive constant. It is not difficult to know that h maps $\mathbb{D}$ onto the vertical strip $\{z \in \mathbb{C} : \ -C\pi/2 < \mathrm{Re}(z) < C\pi/2\}$. Hence $f = 2\mathrm{Re}(h) \in \mathscr{H}_H^{+\infty}(\mathbb{D})$. The proof of this theorem is completed. □

3.2 The Proof of Proposition 2.2

Since $\mathbb{D}$ is a simply connected domain, then f admits a decomposition $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$. Then we have

$$|J_f(z)| = \left| |h'(z)|^2 - |g'(z)|^2 \right| \leq J_h(z) + J_g(z).$$

Combining this inequality with Theorems B, 2.1, E and Lemma H gives the desired result. □

3.3 The Proof of Proposition 2.3

For $p > 2$, it follows from the assumptions and Lemma H that

$$|h'|^p \leq \Lambda_f^p \leq \left(KJ_f + K'\right)^{\frac{p}{2}} \leq 2^{\frac{p}{2}-1} K^{\frac{p}{2}} J_f^{\frac{p}{2}} + 2^{\frac{p}{2}-1}(K')^{\frac{p}{2}}.$$

Consequently,

$$\begin{aligned}\int_0^{2\pi}\int_0^1 (1-r)|h'(re^{i\theta})|^p dr d\theta &\leq 2^{\frac{p}{2}-1}K^{\frac{p}{2}} \int_0^{2\pi}\int_0^1 (1-r)(J_f(re^{i\theta}))^{p/2} dr d\theta \\ &\quad + 2^{\frac{p}{2}-1}(K')^{\frac{p}{2}}\pi \\ &< +\infty,\end{aligned}$$

which, together with $|g'| \leq |h'|$, gives that

$$\int_0^{2\pi} \int_0^1 (1-r)|g'(re^{i\theta})|^p dr d\theta < +\infty.$$

Combining the above two inequalities and Theorem G gives the desired result. □

The following lemma is the well-known Littlewood-Paley's inequality.

Lemma J (See [21], cf. [11, Ineq. (11)]) *Let $q \in (1,2]$ and let f be analytic in $\mathbb{D}$. Then, there is a positive constant $C(q)$, which depends only on q, such that*

$$\|f\|_q \leq C(q) \left(|f(0)|^q + \int_{\mathbb{D}} |f'(z)|^q d\tilde{\mu}_q(z) \right)^{\frac{1}{q}}, \quad f \in \mathscr{B}^q(\tilde{\mu}_q),$$

where $d\tilde{\mu}_q(z) = (1-|z|)^{q-1} dA(z)$.

The following lemma is proved in the proof of [4, Lemma 2.2].

Lemma K *Suppose that complex-valued harmonic function f is (K, K')-elliptic, where $K \geq 1$ and $K' \geq 0$ are constants. Then,*

$$|f_{\overline{z}}(z)| \leq \frac{K-1}{K+1}|f_z(z)| + \frac{\sqrt{K'}}{1+K}, \quad z \in \mathbb{D}.$$

Theorem L *Let g be a two times continuous differentiable function in $\mathbb{D}$. Then, for $r \in (0,1)$,*

$$\frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta})\, d\theta = g(0) + \frac{1}{2\pi} \int_{\mathbb{D}_r} \Delta g(z) \log \frac{r}{|z|} dA(z),$$

where

$$\Delta := 4\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the Laplacian operator (cf. [5, 24]).

3.4 The Proof of Theorem 2.4

We divide the proof of this theorem into two cases.

Case 1. $q \in (1,2]$. Since $\mathbb{D}$ is a simply connected domain, we see that f admits a decomposition $f = h + \overline{g}$, where h and g are analytic with $g(0) = 0$. By the assumptions, we have

$$|h'| \leq \Lambda_f \leq \left(K J_f + K'\right)^{\frac{1}{2}},$$

which, together with Lemma H and the assumption $f \in \mathscr{B}_H^q(\tilde{\mu}_q)$, implies that

$$\int_{\mathbb{D}} |h'(z)|^q d\tilde{\mu}_q(z) \le K^{\frac{q}{2}} \int_{\mathbb{D}} \left(\sqrt{J_f(z)}\right)^q d\tilde{\mu}_q(z) + (K')^{\frac{q}{2}} \int_{\mathbb{D}} d\tilde{\mu}_q(z) < +\infty. \tag{3.9}$$

It follows from (3.9), Lemmas H, J and K that we have

$$\|h\|_q \le C(q) \left(|h(0)|^q + \int_{\mathbb{D}} |h'(z)|^q d\tilde{\mu}_q(z)\right)^{\frac{1}{q}} < +\infty \tag{3.10}$$

and

$$\begin{aligned} \|g\|_q &\le C(q) \left(\int_{\mathbb{D}} |g'(z)|^q d\tilde{\mu}_q(z)\right)^{\frac{1}{q}} \\ &\le C(q, K, K') \left(\int_{\mathbb{D}} |h'(z)|^q d\tilde{\mu}_q(z) + \int_{\mathbb{D}} d\tilde{\mu}_q(z)\right)^{\frac{1}{q}} \\ &< +\infty, \end{aligned} \tag{3.11}$$

where $C(q)$ is a positive constant which depends only on q and $C(q, K, K')$ is a positive constant which depends only on q, K and K'. Hence, by (3.10), (3.11) and Lemma H, we obtain

$$\|f\|_q^q \le 2^{q-1} \left(\|h\|_q^q + \|g\|_q^q\right) < +\infty.$$

Next, assume that the complex-valued harmonic function f is K-quasiregular. It follows from (3.9), (3.10), Lemmas H, J and K that we have

$$\|h\|_q \le C(q) \left(|h(0)|^q + K^{\frac{q}{2}} (\mathscr{I}_q(f, \tilde{\mu}_q))^q\right)^{\frac{1}{q}} \tag{3.12}$$

and

$$\begin{aligned} \|g\|_q &\le C(q) \left(\int_{\mathbb{D}} |g'(z)|^q d\tilde{\mu}_q(z)\right)^{\frac{1}{q}} \\ &\le C(q) \left(\left(\frac{K-1}{K+1}\right)^q \int_{\mathbb{D}} |h'(z)|^q d\tilde{\mu}_q(z)\right)^{\frac{1}{q}} \\ &\le C(q) \left(\left(\frac{K-1}{K+1}\right)^q K^{\frac{q}{2}} (\mathscr{I}_q(f, \tilde{\mu}_q))^q\right)^{\frac{1}{q}}, \end{aligned} \tag{3.13}$$

Hence, by (3.12), (3.13) and Lemma H, we obtain

$$\|f\|_q^q \le 2^{q-1} \left(\|h\|_q^q + \|g\|_q^q\right) \le C(q, K) \left(|h(0)|^q + (\mathscr{I}_q(f, \tilde{\mu}_q))^q\right)^{\frac{1}{q}}.$$

Case 2. $q \in [2, +\infty)$.

For $q \in [2, +\infty)$ and $r \in (0, 1)$, elementary calculations lead to

$$\begin{aligned}\Delta\left(|f|^q\right) &= q(q-2)|f|^{q-4}|f_z\overline{f} + f\overline{f_{\bar{z}}}|^2 + 2q|f|^{q-2}\left(|f_z|^2 + |f_{\bar{z}}|^2\right)\\ &\leq q^2|f|^{q-2}\Lambda_f^2,\end{aligned}$$

which, together with Theorem J, yields that

$$\begin{aligned}M_q^q(r, f) &= |f(0)|^q + \frac{1}{2\pi}\int_{\mathbb{D}_r}\Delta\left(|f(z)|^q\right)\log\frac{r}{|z|}\,dA(z)\\ &\leq |f(0)|^q + q^2\int_0^r \rho\log\frac{r}{\rho}\Phi_f(\rho)d\rho,\end{aligned} \tag{3.14}$$

where

$$\Phi_f(\rho) = \frac{1}{2\pi}\int_0^{2\pi}\Lambda_f^2(\rho e^{i\theta})|f(\rho e^{i\theta})|^{q-2}d\theta.$$

Since f is a (K, K')-elliptic mapping, by Lemma H, we see that

$$\Lambda_f^q \leq \left(KJ_f + K'\right)^{\frac{q}{2}} \leq 2^{\frac{q}{2}-1}\left(K^{\frac{q}{2}}J_f^{\frac{q}{2}} + (K')^{\frac{q}{2}}\right). \tag{3.15}$$

By Hölder's inequality, (3.15) and Lemma H, we have

$$\begin{aligned}\Phi_f(\rho) &\leq M_q^{q-2}(\rho, f)\left(\frac{1}{2\pi}\int_0^{2\pi}\Lambda_f^q(\rho e^{i\theta})d\theta\right)^{\frac{2}{q}}\\ &\leq M_q^{q-2}(\rho, f)\left(\frac{2^{\frac{q}{2}-1}K^{\frac{q}{2}}}{2\pi}\int_0^{2\pi}\left(J_f(\rho e^{i\theta})\right)^{\frac{q}{2}}d\theta + 2^{\frac{q}{2}-1}(K')^{\frac{q}{2}}\right)^{\frac{2}{q}} \qquad (3.16)\\ &\leq M_q^{q-2}(\rho, f)\left(2^{\frac{q-2}{q}}K\left(\frac{1}{2\pi}\int_0^{2\pi}\left(J_f(\rho e^{i\theta})\right)^{\frac{q}{2}}d\theta\right)^{\frac{2}{q}} + 2^{\frac{q-2}{q}}K'\right).\end{aligned}$$

Since $|f|^q$ is subharmonic and $M_q(\rho, f)$ is increasing with respect to $\rho \in (0, r]$, by (3.14) and (3.16), we see that

$$\begin{aligned}M_q^2(r, f) \leq\ & q^2 2^{\frac{q-2}{q}}K\int_0^r \rho\log\frac{r}{\rho}\left(\frac{1}{2\pi}\int_0^{2\pi}\left(J_f(\rho e^{i\theta})\right)^{\frac{q}{2}}d\theta\right)^{\frac{2}{q}}d\rho \\ &+ q^2 2^{\frac{q-2}{q}}K'\int_0^r \rho\log\frac{r}{\rho}d\rho + |f(0)|^2.\end{aligned} \tag{3.17}$$

It is not difficult to know that

$$\int_0^r \rho \log \frac{r}{\rho} d\rho = \frac{r^2}{4}$$

and

$$\rho \log \frac{r}{\rho} \leq r - \rho$$

for $\rho \in (0, r]$, which, together with (3.17) and Hölder's inequality, imply that

$$\begin{aligned} M_q^2(r, f) &\leq q^2 2^{\frac{q-2}{q}} K \int_0^r (r-\rho) \left(\frac{1}{2\pi} \int_0^{2\pi} \left(J_f(\rho e^{i\theta}) \right)^{\frac{q}{2}} d\theta \right)^{\frac{2}{q}} d\rho \\ &\quad + |f(0)|^2 + q^2 2^{\frac{-(q+2)}{q}} K' r^2 \\ &\leq |f(0)|^2 + q^2 2^{\frac{q-2}{q}} K \left(\int_0^r d\rho \right)^{\frac{q-2}{q}} \left(\Psi_f(r) \right)^{\frac{2}{q}} + q^2 2^{\frac{-(q+2)}{q}} K' \\ &\leq |f(0)|^2 + q^2 2^{\frac{q-2}{q}} K \left(\Psi_f(r) \right)^{\frac{2}{q}} + q^2 2^{\frac{-(q+2)}{q}} K', \end{aligned} \tag{3.18}$$

where

$$\Psi_f(r) = \frac{1}{2\pi} \int_0^r (r-\rho)^{\frac{q}{2}} \left(\int_0^{2\pi} \left(J_f(\rho e^{i\theta}) \right)^{\frac{q}{2}} d\theta \right) d\rho.$$

Elementary computations give

$$\Psi_f'(r) = \frac{q}{4\pi} \int_0^r (r-\rho)^{\frac{q}{2}-1} \left(\int_0^{2\pi} \left(J_f(\rho e^{i\theta}) \right)^{\frac{q}{2}} d\theta \right) d\rho > 0. \tag{3.19}$$

Since

$$\begin{aligned} \Psi_f(1) &\leq \frac{1}{2\pi} \int_0^{\frac{1}{2}} (1-\rho)^{\frac{q}{2}} \left(\int_0^{2\pi} \left(J_f(\rho e^{i\theta}) \right)^{\frac{q}{2}} d\theta \right) d\rho \\ &\quad + \frac{1}{\pi} \int_{\frac{1}{2}}^1 \rho (1-\rho)^{\frac{q}{2}} \left(\int_0^{2\pi} \left(J_f(\rho e^{i\theta}) \right)^{\frac{q}{2}} d\theta \right) d\rho \\ &\leq \frac{1}{2\pi} \int_0^{\frac{1}{2}} (1-\rho)^{\frac{q}{2}} \left(\int_0^{2\pi} \left(J_f(\rho e^{i\theta}) \right)^{\frac{q}{2}} d\theta \right) d\rho \\ &\quad + \frac{1}{\pi} \int_{\mathbb{D}} \left(\sqrt{J_f(z)} \right)^q d\tilde{\mu}_q(z) \\ &< +\infty, \end{aligned}$$

by (3.19), we see that

$$\Psi_f(r) \leq \Psi_f(1) < +\infty. \tag{3.20}$$

Combining (3.18) and (3.20) yields the desired result for $q \in [2, +\infty)$. The proof of this theorem is finished. □

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Conflict of interest The authors declare that they have no conflict of interest.

References

1. Chen, S.L., Hamada, H., Zhu, J.-F.: Composition operators on Bloch and Hardy type spaces. Math. Z. **301**, 3939–3957 (2022)
2. Chen, S.L., Kalaj, D.: On asymptotically sharp bi-Lipschitz inequalities of quasiconformal mappings satisfying inhomogeneous polyharmonic equations. J. Geom. Anal. **31**, 4865–4905 (2021)
3. Chen, S.L., Li, P., Wang, X.T.: Schwarz-type lemma, Landau-type theorem, and Lipschitz-type space of solutions to inhomogeneous biharmonic equations. J. Geom. Anal. **29**, 2469–2491 (2019)
4. Chen, S.L., Ponnusamy, S.: On certain quasiconformal and elliptic mappings. J. Math. Anal. Appl. **486**, 1–16 (2020)
5. Chen, S.L., Ponnusamy, S., Rasila, A.: On characterizations of Bloch-type, Hardy-type, and Lipschitz-type spaces. Math. Z. **279**, 163–183 (2015)
6. Chen, S.L., Ponnusamy, S., Wang, X.: The isoperimetric type and Fejer-Riesz type inequalities for pluriharmonic mappings (in Chinese). Sci. Sin. Math. **44**, 127–138 (2014)
7. Chen, S.L., Ponnusamy, S., Wang, X.: Remarks on 'Norm estimates of the partial derivatives for harmonic mappings and harmonic quasiregular mappings'. J. Geom. Anal. **31**, 11051–11060 (2021)
8. Clunie, J.G., Sheil-Small, T.: Harmonic univalent functions. Ann. Acad. Sci. Fenn. Ser. A I Math. **9**, 3–25 (1984)
9. Duren, P.: Theory of H^p spaces, 2nd edn. Dover, Mineola (2000)
10. Duren, P.: Harmonic mappings in the plane. Cambridge University Press, Cambridge (2004)
11. Eenigenburg, P.J.: The integral means of analytic functions. Quart. J. Math. Oxford. **32**, 313–322 (1981)
12. Fefferman, C., Stein, E.M.: H^p spaces of several variables. Acta Math. **129**, 137–193 (1972)
13. Finn, R., Serrin, J.: On the Hölder continuity of quasiconformal and elliptic mappings. Trans. Amer. Math. Soc. **89**, 1–15 (1958)
14. Girela, D.: Mean growth of the derivative of certain classes of analytic functions. Math. Proc. Camb. Phil. Soc. **112**, 335–342 (1992)
15. Hardy, G.H., Littlewood, J.E.: Some properties of conjugate functions. J. Reine Angew. Math. **167**, 405–423 (1931)
16. Holland, F., Twomey, J.B.: On Hardy classes and the area function. J. London Math. Soc. **17**, 257–283 (1978)
17. Kalaj, D.: On Riesz type inequalities for harmonic mappings on the unit disk. Trans. Am. Math. Soc. **372**, 4031–4051 (2019)
18. Kayumov, I., Ponnusamy, S., Kaliraj, A.: Riesz-Fejćr inequalities for harmonic functions. Potential Anal. **52**, 105–113 (2020)
19. Kovalev, L.V., Yang, X.R.: Near-isometric duality of Hardy norms with applications to harmonic mappings. J. Math. Anal. Appl. **487**(124040), 13 (2020)

20. Lewy, H.: On the non-vanishing of the Jacobian in certain one-to-one mappings. Bull. Amer. Math. Soc. **42**, 689–692 (1936)
21. Littlewood, J.E., Paley, R.E.A.C.: Theorems on Fourier series and power series (II). Proc. London Math. Soc. **42**, 52–89 (1936)
22. Melentijević, P., Božin, V.: Sharp Riesz-Fejér inequality for harmonic Hardy spaces. Potent. Anal. **54**, 575–580 (2021)
23. Nirenberg, L.: On nonlinear elliptic partial differential equations and Hölder continuity. Commun. Pure Appl. Math. **6**, 103–156 (1953)
24. Pavlović, M.: Green's formula and the Hardy-Stein identities. Filomat **23**, 135–153 (2009)
25. M. Pavlović, *Function classes on the unit disc.* An introduction. 2nd revised and extended edition. (English) De Gruyter Studies in Mathematics 52. Berlin: De Gruyter. xv, 553 p. (2019)
26. Pichorides, S.K.: On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov. Studia Math. **44**, 165–179 (1972)
27. Shapiro, J.H.: The essential norm of a composition operator. Ann. Math. **125**, 375–404 (1987)
28. I. E. Verbitsky, Estimate of the norm of a function in a Hardy space in terms of the norms of its real and imaginary parts, Mat. Issled. 54 (1980), 16–20 (Russian); Amer. Math. Soc. Transl. Ser. 124 (1984), 11–12 (English translation)
29. Zhu, J.-F.: Norm estimates of the partial derivatives for harmonic mappings and harmonic quasiregular mappings. J. Geom. Anal. **31**, 5505–5525 (2021)
30. Zhu, K.: Operator theory in function spaces. Marcel Dekker, New York (1990)
31. Zhu, K.: Spaces of holomorphic functions in the unit ball. Springer, New York (2005)