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Einstein-Type Metrics on Almost Kenmotsu Manifolds

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Abstract
This article aims to classify the Einstein-type metrics on Kenmotsu and almost Kenmotsu manifolds. In Kenmotsu case, we find that it is T-Einstein. Also, if the manifold is complete and the scalar curvature remains invariant along the Reeb vector field, then either, it is isometric to the hyperbolic space $\mathbb{H}^{2n+1}(1)$ or, the warped product $\widetilde{M} \times_\gamma \mathbb{R}$, provided $\zeta\psi \neq \psi$. Next, we investigate non-Kenmotsu $(\kappa, \mu)'$-almost Kenmotsu manifolds obeying the Einstein-type metrics and give some classification. Finally, we establish that if (ψ, g) is a non-trivial solution of Einstein-type metrics with smooth function ψ which is constant along the Reeb vector field on almost Kenmotsu 3-H-manifold, then either, it is locally isometric to the hyperbolic space $\mathbb{H}^3(1)$ or, the Riemannian product $\mathbb{H}^2(4) \times \mathbb{R}$. Finally, we construct several non-trivial examples to verify our main results.

Keywords Static space · Kenmotsu manifold · Hyperbolic space · Warped product · Einstein-type metrics

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1 Introduction and Preliminaries

In general relativity, obtaining the global solutions to Einstein's field equations, is an important topic for both mathematics and physics. One such special solution is the

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static space-time which is closely connected to the general relativity's cosmic no-hair conjecture (see [7]). Recently, the authors in (cf. [18, 25, 27]) studied a generalized version of static space-time that contains several well circulated critical point equations that occur as solutions of the Euler-Lagrange equations on a compact manifold for curvature functionals.

Definition 1.1 [25] A smooth Riemannian manifold $(\mathbf{M}^n, g)$ is named an Einstein-type manifold if $\psi : \mathbf{M}^n \to \mathbb{R}$ solves

$$\psi Ric = \nabla^2 \psi + \sigma g, \tag{1.1}$$

where ψ is a non-constant smooth function. Here, σ, Ric and $\nabla^2 \psi$ indicate a smooth function, the Ricci tensor and the Hessian of ψ, respectively. Moreover, taking the trace of (1.1) yields

$$r\psi = \Delta\psi + n\sigma, \tag{1.2}$$

$\Delta\psi$ being the Laplacian of ψ and r denotes the scalar curvature.

As highlighted by the authors (cf. [18, 25]), the above stated two equations generalize numerous fascinating geometric equations such as static perfect fluid equation (cf. [11, 19, 20]), Miao–Tam equation (cf. [3, 21, 22]) and critical point equation (cf. [2, 27]), Einstein equation [13] and static vacuum equation [1] with null and non-null cosmological constant.

The interesting idea of Einstein-type manifolds is characterized in many papers (cf. [8, 18]). Leandro [18] classified Einstein-type manifold under the assumptions of zero-radial Weyl curvature and harmonic Weyl curvature. As a physical application, Leandro proved that,

> There are no multiple black holes in static vacuum Einstein equation with null cosmological constant having zero radial Weyl curvature and divergence free Weyl tensor of order four.

Catino et al. [8] investigated it under Bach-flat condition. The critical point equation, Miao–Tam equation and Fischer–Marsden equation on Kenmotsu and almost Kenmotsu manifold (briefly, *akm*) were studied by many authors in [9, 33]. Kumara et al. [17] characterized the static perfect fluid space-time metrics on *akm*.

Tanno's classification theorem was used to classify almost contact metric manifolds with constant sectional curvature k of a plane section containing the Reeb vector field ζ. According to Tano's classification, if k is positive, the manifold is a homogeneous Sasakian manifold. If k is zero, the manifold is a global Riemannian product of a line or, a circle with a Kahler manifold of constant holomorphic sectional curvature. If k is negative, the manifold is a warped product space $R \times_f C^{2n}$, known as a Kenmotsu manifold. Kenmotsu manifolds have important geometrical properties and almost Kenmotsu manifolds are an extension of Kenmotsu manifolds. These two structures namely Kenmotsu and almost Kenmotsu are totally different from Sasakian and K-contact structures. Sasakian and K-contact structures are equivalent on three dimensional Riemannian manifolds, but these two structures (Kenmotsu and almost

Kenmotsu) are not equivalent to Sasakian and K-contact structures. Recently, Patra and Ghosh [25] considered the Einstein-type equation within the context of contact manifolds.

Motivated by the above studies we examine Kenmotsu and *akm* manifolds admitting a smooth non-trivial function ψ satisfying the Einstein-type equation (1.1) and we acquire totally different results from the results as obtained by the authors in [25].

This article is organized as follows:

In Sect. 3, we show that if a complete Kenmotsu manifold whose Reeb vector field leaves the scalar curvature invariant and admits a non-trivial function ψ satisfying the Eq. (1.1), then either, it is isometric to the hyperbolic space $\mathbb{H}^{2n+1}(1)$ or, to the warped product $\widetilde{M} \times_\gamma \mathbb{R}$, provided $\zeta\psi \neq \psi$. Section 4 is concerned with the investigation of *akm*. Firstly, we obtain classification of $(\kappa, \mu)'$-*akm* with $h' \neq 0$ admitting non-trivial ψ satisfying the Eq. (1.1). Then we prove that if $\mathbf{M}^3(F, \zeta, T, g)$ is an almost Kenmotsu 3-H-manifold with $h' \neq 0$ and (ψ, g) is a non-trivial solution of the Eq. (1.1) with smooth function ψ which is constant along the Reeb vector field, then it is locally isometric to a non-unimodular Lie group with a left invariant *akm*.

1.1 Almost Kenmotsu Manifolds

According to Blair [4], an almost contact manifold is a smooth manifold $\mathbf{M}^{2n+1}$ with a 1-form T which is known as contact form, a unit vector field ζ with $T(\zeta) = 1$, named the Reeb vector field and a skew-symmetric $(1, 1)$-tensor field F of rank $2n$ satisfying the following relations:

$$F^2 Z_1 = -Z_1 + T(Z_1)\zeta, \quad T \circ F = 0, \tag{1.3}$$

for all vector field Z_1 on $\mathbf{M}^{2n+1}$. The smooth manifold $\mathbf{M}^{2n+1}$ together with the almost contact structure (T, ζ, F) is known as an almost contact manifold. If the global 1-form T is such that $T \wedge (dT)^n \neq 0$ everywhere on $\mathbf{M}^{2n+1}$, then it is called contact. The metric g is a Riemannian metric, also called associated metric of T, that is, $T(Z_1) = g(Z_1, \zeta)$, for Z_1 on $\mathbf{M}^{2n+1}$ and $\mathbf{M}^{2n+1}(T, \zeta, F, g)$ is named an almost contact metric manifold. Also, from (1.3) we see that

$$g(FZ_1, FZ_2) = g(Z_1, Z_2) - T(Z_1)T(Z_2), \quad F(\zeta) = 0, \tag{1.4}$$

for all Z_1, Z_2 on $\mathbf{M}^{2n+1}$. On $\mathbf{M}^{2n+1}$, two self-adjoint operators h and l are defined by $h = \frac{1}{2}\mathcal{L}_\zeta F$ and $l = K(\cdot, \zeta)\zeta$ on $\mathbf{M}^{2n+1}$ obeying $h\zeta = h'\zeta = 0$, $Tr.h = Tr.h' = 0$, $\quad hH = -Fh$ where $h' = h \cdot F$ and K is the Riemannian curvature tensor of g.

If the manifold $\mathbf{M}^{2n+1}$ obeys $dT = 0$ and $d\Phi = 2T \wedge \Phi$, where the fundamental 2-form Φ of the manifold is defined by $\Phi(Z_1, Z_2) = g(Z_1, FZ_2)$ for any Z_1, Z_2 on $\mathbf{M}^{2n+1}$, then $\mathbf{M}^{2n+1}(T, \zeta, F, g)$ is called an *akm* (see [14]).

Further, on an *akm* the subsequent formula holds [12]:

$$\nabla_{Z_1}\zeta = -F^2 Z_1 - FhZ_1, \tag{1.5}$$

for any Z_1 on $\mathbf{M}^{2n+1}$.

A normal *akm* is called a Kenmotsu manifold (see [16]). Moreover, an *akm* is a Kenmotsu manifold if and only if

$$(\nabla_{Z_1} F)Z_2 = g(FZ_1, Z_2)\zeta - T(Z_2)FZ_1,$$

for any Z_1, Z_2 on $\mathbf{M}^{2n+1}$ [14, Theorem 2.1]. A Kenmotsu manifold is locally a warped product (briefly, WP) $I \times_{ce^t} N^{2n}$ of open interval I and Kählerian manifold N^{2n}, where t is the coordinate of I and c is some positive constant (see [16]). On a Kenmotsu manifold, the following relations are true [16]:

$$\nabla_{Z_1}\zeta = Z_1 - T(Z_1)\zeta, \tag{1.6}$$

$$K(Z_1, Z_2)\zeta = T(Z_1)Z_2 - T(Z_2)Z_1, \tag{1.7}$$

$$L\zeta = -2n\zeta, \tag{1.8}$$

for any Z_1, Z_2 on $\mathbf{M}^{2n+1}$. Here L is the Ricci operator associated with the (0, 2)-type Ricci tensor Ric written by $Ric(Z_1, Z_2) = g(LZ_1, Z_2)$ for all Z_1, Z_2 on $\mathbf{M}^{2n+1}$. In *akm* the distribution $\mathcal{D} = ker(T)$ is integrable and its integral submanifold is an almost Kähler manifold. $\mathbf{M}^{2n+1}$ is called a Kenmotsu manifold if the integral submanifolds of $\mathcal{D}$ are Kähler. Therefore, a 3-dimensional *akm* is Kenmotsu if and only if $h = 0$.

Generalizing the concept of κ-nullity distribution on almost contact metric manifold $\mathbf{M}^{2n+1}$, Blair et al. [6] presented (κ, μ)-nullity distribution, which is given for each p on $\mathbf{M}^{2n+1}$ and $\kappa, \mu \in \mathbb{R}$ by

$$N_p(\kappa, \mu) = \{Z_3 \in T_p M^{2n+1} | K(Z_1, Z_2)Z_3 = \kappa(g(Z_2, Z_3)Z_1 - g(Z_1, Z_3)Z_2) + \mu(g(Z_2, Z_3)hZ_1 - g(Z_1, Z_3)hZ_2)\}.$$

Dileo and Pastore [12] classified *akm* obeying (κ, μ)-nullity condition and a modified form of nullity condition, named $(\kappa, \mu)'$-nullity condition. An *akm* $\mathbf{M}^{2n+1}(F, \zeta, T, g)$ is called $(\kappa, \mu)'$-*akm* if ζ belongs to the $(\kappa, \mu)'$-nullity distribution, that is,

$$K(Z_1, Z_2)\zeta = \kappa[T(Z_2)Z_1 - T(Z_1)Z_2] + \mu[T(Z_2)h'Z_1 - T(Z_1)h'Z_2], \tag{1.9}$$

for all Z_1, Z_2 on $\mathbf{M}^{2n+1}$ and $\kappa, \mu \in \mathbb{R}$. Moreover, if both κ and μ are smooth functions in (1.9), then $\mathbf{M}^{2n+1}$ is named a generalized $(\kappa, \mu)'$-*akm* (see [12, 24, 32]). On $(\kappa, \mu)'$-*akm*, it was proved that $\kappa \leq -1$ [12]. Moreover, if $\kappa = -1$, then $h' = 0$. On generalized $(\kappa, \mu)'$-*akm* with $h \neq 0$, the subsequent relations hold [24, Proposition 3.1]:

$$h^2 = (1+\kappa)F^2, \quad h'^2 = (1+\kappa)F^2, \tag{1.10}$$

$$L\zeta = 2n\kappa\zeta. \tag{1.11}$$

Wang and Liu [32] showed that for $(\kappa, \mu)'$-*akm*, the Ricci operator L of $\mathbf{M}^{2n+1}$ can be written as

$$LZ_1 = -2nZ_1 + 2n(1+\kappa)T(Z_1)\zeta - 2nh'Z_1, \tag{1.12}$$

for any Z_1 on $\mathbf{M}^{2n+1}$. Moreover, we have $r = 2n(\kappa - 2n)$ and $\mu = -2$.

Also, we recall the results obtained by Kanai [15].

Lemma 1.2 [15] *Suppose that* (M, g) *is a complete Riemannian manifold of dimension* $n(\geq 2)$ *and that* $k < 0$. *Then there is a non-trivial function* f *on* M *with a critical point which satisfies*

$$Hessf + kfg = 0$$

if and only if (M, g) *is the simply connected complete Riemannian manifold* $(\mathbb{H}^n, -(1/k)g_0)$ *of constant curvature* k, *where* g_0 *is the canonical metric on the hyperbolic space of constant curvature* -1.

Lemma 1.3 [15] *Let* (M, g) *and* k *be as Lemma 1.2. Then there is a function* f *on* M *without critical points which satisfies*

$$Hessf + kfg = 0$$

if and only if (M, g) *is the warped product* $(\widetilde{M}, \widetilde{g})_\xi \times (\mathbb{R}, g_0)$ *of a complete Riemannian manifold* $(\widetilde{M}, \widetilde{g})$ *and the real line* $(\mathbb{R}, g_0)$ *warped by a function* $\xi : \mathbb{R} \to \mathbb{R}$ *such that* $\ddot{\xi} + k\xi = 0, \xi > 0$, *where* g_0 *denotes the canonical metric on* $\mathbb{R}$; $g_0 = dt^2$.

2 Kenmotsu Manifolds Satisfying Einstein-Type Equations

Before proceeding to the main result, we construct an example of a Kenmotsu manifold admitting a non-trivial smooth function ψ which is the solution of the Eq. (1.1).

Example Let (N^{2n}, J, g_0) be a Kähler manifold and $(\mathbf{M}^{2n+1}, g) = (\mathbb{R} \times_{\bar{\sigma}} N, dt^2 + \bar{\sigma}^2 g_0)$ be the WP. If we set $T = dt, \zeta = \frac{\partial}{\partial t}$ and the tensor field F is defined on $\mathbb{R} \times_{\bar{\sigma}} N$ by $FZ_1 = JZ_1$ for any Z_1 on N and $FZ_1 = 0$ if Z_1 is tangent to $\mathbb{R}$, then the WP $\mathbb{R} \times_{\bar{\sigma}} N, \bar{\sigma}^2 = ce^{2t}$ with the structure (F, ζ, T, g) is a Kenmotsu manifold [16]. Specifically, if we set $N = \mathbb{CH}^{2n}$, then N is Einstein and the Ricci tensor of $\mathbf{M}^{2n+1}$ becomes $Ric = -2ng$. Further, we set a smooth function as $\psi(t) = ke^t, k > 0$. Hence, it is very easy to verify that $\psi(t)$ solves the Eq. (1.1) for $\sigma = -(2n+1)ke^t$.

Definition 2.1 An almost contact metric manifold $\mathbf{M}^{2n+1}$ is said to be T-Einstein manifold [29] if the Ricci tensor of $\mathbf{M}^{2n+1}$ satisfies

$$LZ = \alpha Z + \beta T(Z)\zeta,$$

for the vector field Z on $\mathbf{M}^{2n+1}$. Here, α and β are smooth functions on $\mathbf{M}^{2n+1}$.

Next, we establish the following:

Theorem 2.2 *If* (g, ψ) *is a non-trivial solution of Eq.* (1.1) *in a Kenmotsu manifold* $(\mathbf{M}^{2n+1}, F, \zeta, T, g)$, *then it is* T*-Einstein manifold, provided* $\zeta\psi \neq \psi$. *Moreover, if* $\mathbf{M}^{2n+1}$ *is complete and the Reeb vector field leaves the scalar curvature invariant, then we have*

1. *If ψ has a critical point which satisfies* (1.1)*, then M is isometric to the hyperbolic space* $\mathbb{H}^{2n+1}(1)$.
2. *If ψ is without critical points which satisfies* (1.1)*, then M is isometric to the warped product $\widetilde{M} \times_{\gamma} \mathbb{R}$ of a complete Riemannian manifold $\widetilde{M}^{2n}$ and the real line $\mathbb{R}$ with warped function $\gamma : \mathbb{R} \to \mathbb{R}$ such that $\ddot{\gamma} - \gamma = 0, \gamma > 0$.*

Proof Executing the covariant derivative of (1.1) along Z_2, we obtain

$$\nabla_{Z_2}\nabla_{Z_1}D\psi = (Z_2\psi)LZ_1 + \psi(\nabla_{Z_2}L)Z_1 - (Z_2\sigma)Z_1. \tag{2.1}$$

In consequence of (2.1), we get the curvature tensor as follows:

$$\begin{aligned} K(Z_1, Z_2)D\psi = (Z_1\psi)LZ_2 - (Z_2\psi)LZ_1 + \psi\{(\nabla_{Z_1}L)Z_2 \\ - (\nabla_{Z_2}L)Z_1\} + (Z_2\sigma)Z_1 - (Z_1\sigma)Z_2, \end{aligned} \tag{2.2}$$

for any Z_1, Z_2 on $\mathbf{M}^{2n+1}$. Executing the covariant derivative of (1.8) and using (1.6), we acquire

$$(\nabla_{Z_1}L)\zeta = -LZ_1 - 2nZ_1. \tag{2.3}$$

Now taking an inner product of (2.2) with ζ and inserting last expression along with (1.8), we obtain

$$\begin{aligned} g(K(Z_1, Z_2)D\psi, \zeta) = 2n\{(Z_2\psi)T(Z_1) - (Z_1\psi)T(Z_2)\} \\ + (Z_2\sigma)T(Z_1) - (Z_1\sigma)T(Z_2). \end{aligned} \tag{2.4}$$

Taking an inner product of (1.7) with $D\psi$ and combining it with (2.4), we provide

$$(2n + 1)\{D\psi - (\zeta\psi)\zeta\} + D\sigma - (\zeta\sigma)\zeta = 0. \tag{2.5}$$

Contracting (2.2) infers

$$4nD\sigma - \psi Dr - 2rD\psi = 0. \tag{2.6}$$

Taking trace of (2.3) and then using it in the inner product of (2.6) with ζ, we acquire

$$4n(\zeta\sigma) + 2\psi(r + 2n(2n + 1)) - 2r(\zeta\psi) = 0. \tag{2.7}$$

Replacing Z_2 by ζ in (2.2) and after that taking inner product with Z_2, we have

$$\begin{aligned} g(K(Z_1, \zeta)D\psi, Z_2) = -2n(Z_1\psi)T(Z_2) - (\zeta\psi)Ric(Z_1, Z_2) \\ + \psi\{Ric(Z_1, Z_2) + 2ng(Z_1, Z_2)\} \\ + (\zeta\sigma)g(Z_1, Z_2) - (Z_1\sigma)T(Z_2). \end{aligned} \tag{2.8}$$

As a consequence of taking inner product of (1.7) with $D\psi$ and combining it with (2.8), we get

$$g(Z_1, (2n+1)D\psi + D\sigma)T(Z_2) - (2n\psi + \zeta\psi + \zeta\sigma)g(Z_1, Z_2) \\ = (\psi - \zeta\psi)Ric(Z_1, Z_2). \tag{2.9}$$

Combining (2.6), (2.7) and (2.9) give

$$(\psi - \zeta\psi)\left\{\left(\frac{r}{2n}+1\right)Z_1 - \left(\frac{r}{2n}+2n+1\right)T(Z_1)\zeta\right\} = (\psi - \zeta\psi)LZ_1, \tag{2.10}$$

for any Z_1 on $\mathbf{M}^{2n+1}$. Hence, $\mathbf{M}^{2n+1}$ is T-Einstein or, $\zeta\psi = \psi$.

Let ζ leave the scalar curvature r invariant, that is, $\zeta r = 0$ implies $r = -2n(2n+1)$. In view of this, (2.10) gives $LZ_1 = -2nZ_1$. Utilizing the fact that r is constant in (2.6), we get $\sigma = -(2n+1)\psi + k$, where k indicates a constant. In consequence of last equation and $LZ_1 = -2nZ_1$ in (1.1), we infer

$$\nabla_{Z_1} D\psi = (\psi - k)Z_1.$$

By applying Kanai's theorems [15], that is, Lemmas 1.2 and 1.3 we can conclude that if ψ has a critical point which satisfies (1.1), then $\mathbf{M}^{2n+1}$ is isometric to the hyperbolic space $\mathbb{H}^{2n+1}(1)$ or, if ψ is without critical points which satisfies (1.1), then $\mathbf{M}^{2n+1}$ is isometric to the warped product $\widetilde{M} \times_\gamma \mathbb{R}$ of a complete Riemannian manifold $\widetilde{M}^{2n}$ and the real line $\mathbb{R}$ with warped function $\gamma : \mathbb{R} \to \mathbb{R}$ such that $\ddot{\gamma} - \gamma = 0, \gamma > 0$. This completes the proof. □

Remark 2.3 From (2.10), we see that either, $\mathbf{M}^{2n+1}$ is T-Einstein or, $\psi - \zeta\psi = 0$. Suppose $\psi - \zeta\psi = 0$, then Kenmotsu maifold is locally isometric to the warped product $(-\epsilon, \epsilon) \times_{ce^t} N$, where N is a Kähler manifold of dimension $2n$ and $(-\epsilon, \epsilon)$ is an open interval [16]. Using the local parametrization: $\zeta = \frac{\partial}{\partial t}$ (where $t \in (-\epsilon, \epsilon)$) we get

$$\frac{\partial \psi}{\partial t} = \psi,$$

whose solution is $\psi = ce^t$, where c is a constant. Therefore, assuming $\zeta\psi \neq \psi$ in Theorem 2.2, implies $\mathbf{M}^{2n+1}$ is T-Einstein.

3 Almost Kenmotsu Manifolds Satisfying Einstein-Type Equation

Making use of Eq. (1.12) and the result by Dileo and Pastore [12, Theorem 4.2], we can prove the subsequent:

Theorem 3.1 *Let $\mathbf{M}^{2n+1}(F, T, \zeta, g)$ be a $(\kappa, \mu)'$-akm with the condition $h' \neq 0$. If (g, ψ) is a non-trivial solution of the Eq. (1.1), then $\mathbf{M}^3$ is locally isometric to the Riemannian product $\mathbb{H}^2(4) \times \mathbb{R}$ and $\mathbf{M}^{2n+1}$ is locally isometric to the WP*

$$\mathbb{H}^{n+1}(\alpha) \times_{\bar{\psi}} \mathbb{R}^n, \quad \mathbb{B}^{n+1}(\alpha') \times_{\bar{\psi}'} \mathbb{R}^n,$$

for $n > 1$. Here, $\mathbb{H}^{n+1}(\alpha)$ and $\mathbb{B}^{n+1}(\alpha')$ are the hyperbolic space of constant curvature $\alpha = -\frac{2}{n} - \frac{1}{n^2} - 1$ and space of constant curvature $\alpha' = -\frac{1}{n^2} + \frac{2}{n} - 1$, respectively. Also, $\bar{\psi} = c_1 e^{(1-\frac{1}{n})t}$ and $\bar{\psi}' = c_1' e^{(1-\frac{1}{n})t}$, where c_1, c_1' are positive constants.

Proof We first replace Z_1 by ζ in (2.2), then take its inner product with ζ and utilizing (1.12), infer that

$$g(K(\zeta, Z_2)D\psi, \zeta) = 2n\kappa\{(\zeta\psi)T(Z_2) - (Z_2\psi)\} + (Z_2\sigma) - (\zeta\sigma)T(Z_2). \tag{3.1}$$

Also, we replace Z_1 by ζ in (1.9) and after that, taking inner product with $D\psi$ gives

$$g(K(\zeta, Z_2)\zeta, D\psi) = \kappa\{(\zeta\psi)T(Z_2) - (Z_2\psi)\} - \mu h'(Z_2\psi). \tag{3.2}$$

Since the scalar curvature $r = 2n(\kappa - 2n)$ is constant, in view of Eq. (3.2) the Eq. (2.6) becomes $4nD\sigma - 2rD\psi = 0$. Combining (3.1) and (3.2) with the last expression, we get

$$2n(\kappa + 1)\{(\zeta\psi)\zeta - D\psi\} = \mu h' D\psi. \tag{3.3}$$

Operating (3.3) by h' and using (1.10) yield

$$-2n(\kappa + 1)h' D\psi = \mu(\kappa + 1)\{-D\psi + (\zeta\psi)\zeta\}.$$

Then combining the last equation with (3.3), we obtain

$$\{\mu^2(\kappa + 1) + 4n^2(\kappa + 1)^2\}F^2 D\psi = 0. \tag{3.4}$$

Therefore, we have to discuss the following two cases: $F^2 D\psi = 0$ or, $F^2 D\psi \neq 0$.
Case-I $F^2 D\psi \neq 0$, then (3.4) gives $\kappa = -1 - \frac{\mu^2}{4n^2}$. Since $\mu = -2$, we get $\kappa = -1 - \frac{1}{n^2}$. By using Dileo and Pastore [12, Theorem 4.2] we can conclude that $\mathbf{M}^3$ is locally isometric to the Riemannian product $\mathbb{H}^2(4) \times \mathbb{R}$ and $\mathbf{M}^{2n+1}$ is locally isometric to the WPs

$$\mathbb{H}^{n+1}(\alpha) \times_{\bar{\psi}} \mathbb{R}^n, \quad \mathbb{B}^{n+1}(\alpha') \times_{\bar{\psi}'} \mathbb{R}^n,$$

for $n > 1$.
Case-II $F^2 D\psi = 0$ implies $D\psi = (\zeta\psi)\zeta$. Taking the covariant derivative and using (1.1) and (1.5), we get

$$\psi L Z_1 - \sigma Z_1 = Z_1(\zeta\psi)\zeta + (\zeta\psi)(Z_1 - T(Z_1)\zeta - FhZ_1). \tag{3.5}$$

Replacing Z_1 by ζ in (3.5) gives $\zeta(\zeta\psi) = 2n\kappa\psi - \sigma$. In view of this in the contraction of (3.5), we obtain $\zeta\psi = -2n\psi - \sigma$.

Comparing (3.5) with (1.12), then operating the obtained result by F gives $(2n\psi + (\zeta\psi))hZ_1 = 0$. Making use of $\zeta\psi = -2n\psi - \sigma$ and the fact that $h \neq 0$, we see that $\sigma = 0$. In consequence, (2.6) becomes $(\kappa - 2n)D\psi = 0$. As $\kappa < -1$, we get $D\psi = 0$, that is, ψ is constant, a contradiction. This completes the proof. □

Let us consider a generalized $(\kappa, \mu)'$-*akm* of dimension three with $\kappa < -1$. If we assume that κ is invariant along ζ, then from Proposition 3.2 in [24] we have $\zeta(\kappa) = -2(1+\kappa)(\mu+2)$ which implies $\mu = -2$. Moreover, from [28, Lemma 3.3], we have $h'(grad\mu) = grad\kappa - \zeta(\kappa)\zeta$ which implies that κ is constant under our assumption. Therefore $\mathbf{M}^3$ becomes a $(\kappa, -2)'$-*akm*. By applying Theorem 3.1, we can conclude the following:

Corollary 3.2 *Let $\mathbf{M}^3(F, T, \zeta, g)$ be a generalized $(\kappa, \mu)'$-akm with $\kappa < -1$ which is invariant along ζ. If (g, ψ) is a non-trivial solution of the Eq. (1.1) then $\mathbf{M}^3$ is locally isometric to the Riemannian product $\mathbb{H}^2(4) \times \mathbb{R}$.*

Next, we investigate 3-dimensional *akm* admitting a non-trivial solution to the Eq. (1.1). Suppose $\mathcal{U}_1$ is the open subset of a 3-dimensional *akm* $\mathbf{M}^3$ such that $h \neq 0$ and $\mathcal{U}_2$, the open subset of $\mathbf{M}^3$ is defined by $\mathcal{U}_2 = \{p \in \mathbf{M}^3 : h = 0$ in a neighbourhood of $p\}$. Hence, $\mathcal{U}_1 \cup \mathcal{U}_2$ is dense and open subset of $\mathbf{M}^3$ and there exists a local orthonormal basis $\{e_1 = e, e_2 = Fe, e_3 = \zeta\}$ of three smooth unit eigenvectors of h for any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$. On $\mathcal{U}_1$ we may set $he_1 = \vartheta e_1$ and $he_2 = -\vartheta e_2$, where ϑ is a positive function.

Lemma 3.3 [10] *On $\mathcal{U}_1$ we have*

$$\begin{aligned}
&\nabla_\zeta \zeta = 0, \quad \nabla_\zeta e = aHe, \quad \nabla_\zeta Fe = -ae, \\
&\nabla_e \zeta = e - \vartheta Fe, \quad \nabla_e e = -\zeta - bHe, \quad \nabla_e He = \vartheta\zeta + be, \\
&\nabla_{Fe} \zeta = -\vartheta e + Fe, \quad \nabla_{Fe} e = \vartheta\zeta + cHe, \quad \nabla_{Fe} Fe = -\zeta - ce,
\end{aligned}$$

where a, b, c are smooth functions.

From Lemma 3.3, the Poisson brackets for $\{e_1 = e, e_2 = Fe, e_3 = \zeta\}$ are as follows:

$$[e_3, e_1] = (a+\vartheta)e_2 - e_1, [e_1, e_2] = be_1 - ce_2, [e_2, e_3] = (a-\vartheta)e_1 + e_2. \tag{3.6}$$

Moreover, applying Lemma 3.3 in the subsequent Jacobi identity

$$[[\zeta, e], Fe] + [[Fe, \zeta], e] + [[e, Fe], \zeta] = 0$$

gives that

$$e(\vartheta) - e(a) - b - \zeta(b) + c(\vartheta - a) = 0, \tag{3.7}$$

$$Fe(\vartheta) + Fe(a) - c - \zeta(c) + b(\vartheta + a) = 0. \tag{3.8}$$

In regard of Lemma 3.3 we also get the following lemma.

Lemma 3.4 [10] *The Ricci operator L with respect to the local basis* $\{\zeta, e, Fe\}$ *on* $\mathcal{U}_1$ *can be written as*

$$\begin{aligned} L\zeta &= -2(\vartheta^2+1)\zeta - (Fe(\vartheta)+2\vartheta b)e - (e(\vartheta)+2\vartheta c)Fe, \\ Le &= -(Fe(\vartheta)+2\vartheta b)\zeta - (A+2\vartheta a)e + (\zeta(\vartheta)+2\vartheta)Fe, \\ LHe &= -(e(\vartheta)+2\vartheta c)\zeta + (\zeta(\vartheta)+2\vartheta)e - (A-2\vartheta a)Fe, \end{aligned} \tag{3.9}$$

where we set $A = e(c) + b^2 + c^2 + Fe(b) + 2$ *for simplicity.*

Before proceeding to the main result, we recollect few basic notion of harmonic vector fields. Perrone [26] characterized the harmonicity of an *akm*. Let $(\mathbf{M}^n, g)$ be a Riemannian manifold and $(T^1\mathbf{M}, g_s)$ its unit tangent sphere bundle furnished with the well-known standard Sasakian metric g_s. If $\mathbf{M}$ is compact, then the energy $E(\mathbf{V})$ is defined as the energy of the corresponding map $\mathbf{V}$ from $(\mathbf{M}, g)$ into $(T^1\mathbf{M}, g_s)$ by

$$E(\mathbf{V}) = \frac{1}{2}\int_{\mathbf{M}} ||dV||^2 dv_g = \frac{m}{2} Vol(\mathbf{M}, g) + \frac{1}{2}\int_{\mathbf{M}} ||\nabla \mathbf{V}||^2 dv_g,$$

where E indicates the energy function and ∇ being the Levi–Civita connection of g. A unit vector field $\mathbf{V}$ is named harmonic if it is a critical point for E defined on the set of all unit vector fields $\Psi^1(\mathbf{M})$, that is,

$$\bar{\Delta}\mathbf{V} - ||\nabla \mathbf{V}||^2 \mathbf{V} = 0,$$

where $\bar{\Delta}$ indicates the rough Laplacian, that is, $\bar{\Delta} V = -tr\nabla^2 V$. The critical point condition still specifies a harmonic vector field even though $\mathbf{M}$ is non-compact. A Kenmotsu 3-manifold's Reeb vector field is always harmonic. Now, we give the subsequent definition.

Definition 3.5 [31] An almost Kenmotsu 3-manifold with harmonic Reeb vector field or equivalently, the Reeb vector field is an eigenvector of the Ricci operator, is called almost Kenmotsu 3-H-manifold.

Now, we state and prove the following:

Theorem 3.6 *Let* $\mathbf{M}^3(F, \zeta, T, g)$ *be an almost Kenmotsu 3-H-manifold equipped with* $h' \neq 0$. *If* (ψ, g) *is a non-trivial solution of the Eq.* (1.1) *with smooth function* ψ *which is constant along the Reeb vector field, then it is locally isometric to a non-unimodular Lie group with a left invariant almost Kenmotsu structure.*

Proof Under our hypothesis, from the first argument of Lemma 3.4, we obtain

$$e(\vartheta) = -2\vartheta c, \quad Fe(\vartheta) = -\vartheta b. \tag{3.10}$$

Taking the inner product of (1.1) with the vector filed Z_2, the Eq. (1.1) can be rewritten as:

$$g(\nabla_{Z_1} D\psi, Z_2) = \psi Ric(Z_1, Z_2) - \sigma g(Z_1, Z_2), \tag{3.11}$$

for all Z_1, Z_2 on $\mathbf{M}^3$. Since a smooth function ψ is constant along the Reeb vector field ζ, we can write

$$D\psi = \psi_1 e + \psi_2 Fe,$$

for smooth functions ψ_1, ψ_2 on $\mathbf{M}^3$.

Replacing Z_1 and Z_2 by ζ in (3.11), then making use of Lemmas 3.3 and 3.4 we get

$$\sigma = -2\psi(\vartheta^2 + 1). \tag{3.12}$$

Substituting $Z_1 = e$ and $Z_2 = \zeta$ in (3.11) and using Lemmas 3.4, 3.3 yield

$$\vartheta\psi_2 - \psi_1 = 0. \tag{3.13}$$

Similarly, taking $Z_1 = Fe$ and $Z_2 = \zeta$ in (3.11) gives

$$\vartheta\psi_1 - \psi_2 = 0. \tag{3.14}$$

Combining (3.13) and (3.14), we get $(\vartheta^2 - 1)\psi_1 = 0$. If $\psi_1 = 0$, then from (3.14) we see that $\psi_2 = 0$ which implies $D\psi = 0$, that is, ψ is constant, a contradiction. Therefore, we must have $\vartheta^2 = 1$ which implies ϑ is constant. Since ϑ is a positive function, we get $\vartheta = 1$. Making use of the fact that $\vartheta = 1$ in (3.10) gives $b = c = 0$. Moreover, equation (3.13) implies $\psi_1 = \psi_2$.

Now consider the following open set:

$$\mathcal{O} = \{p \in \mathcal{U}_1 : \psi_1 = \psi_2 \neq 0 \text{ in a neighborhood of } p\}$$

According to Poincare's lemma $d^2\psi = 0$, that is, the relation

$$g(\nabla_{Z_1} D\psi, Z_2) = g(\nabla_{Z_2} D\psi, Z_1) \tag{3.15}$$

holds for any vector fields Z_1, Z_2 in $\mathbf{M}^3$. Letting $Z_1 = \zeta$ and $Z_2 = e$ in (3.15) and by using Lemma 3.3, we obtain

$$\zeta(\psi_1) = a\psi_2. \tag{3.16}$$

Also, taking $Z_1 = \zeta$ and $Z_2 = Fe$ in (3.15) gives $\zeta(\psi_2) = -a\psi_1$ and by combining it with (3.16), we get $2a\psi_1 = 0$, which further implies that $a = 0$ in $\mathcal{O}$.

Making use of the fact that $a = b = c = 0$ and $\vartheta = 1$ along with Lemma 3.3, we obtain

$$[e, Fe] = 0, \quad [Fe, \zeta] = Fe - e, \quad [\zeta, e] = Fe - e.$$

According to Milnor's theorem [23], we can conclude that $\mathbf{M}^3$ is locally isometric to a non-unimodular Lie group with a left invariant almost Kenmotsu structure, which completes the proof. □

Applying Wang's Theorems [31] and 3.6, we can now establish the following:

Corollary 3.7 *Let $\mathbf{M}^3(F, \zeta, T, g)$ be an almost Kenmotsu 3-H-manifold. If (ψ, g) is a non-trivial solution of the Eq. (1.1) with smooth function ψ which is constant along the Reeb vector field, then either, it is locally isometric to the hyperbolic space $\mathbb{H}^3(1)$ or, the Riemannian product $\mathbb{H}^2(4) \times \mathbb{R}$.*

Proof We shall establish the theorem via the subsequent cases:
Case i Let $h = 0$, then $\mathbf{M}^3$ is a Kenmotsu manifold. The Ricci operator of $\mathbf{M}^3$ is written by (see [10])

$$L = \left(\frac{r}{2} + 1\right) id - \left(\frac{r}{2} + 3\right) T \otimes \zeta. \tag{3.17}$$

Replacing Z_1 by ζ in (1.1), then taking the inner product of it with ζ and using (1.8), we get $\zeta(\zeta\psi) = -2\psi + \sigma$. If $\zeta\psi = 0$, last equation becomes $\sigma = 2\psi$ which further implies $\zeta\sigma = 0$. In conclusion, for $n = 1$ Eq. (2.7) becomes $r = -6$, that is, scalar curvature is constant, which together with (3.17) implies that $L = -2id$. Clearly $\mathbf{M}^3$ is conformally flat.
Case ii $h \neq 0$ on some open subset of $\mathbf{M}^3$. By the proof of Theorem 3.6, we see that $a = b = c = 0$ and $\vartheta = 1$. Using this in Lemma 3.4, we get

$$\begin{aligned} L\zeta &= -4\zeta, \\ Le &= 2Fe - 2e, \\ LHe &= 2e - 2Fe. \end{aligned}$$

Also, the scalar curvature is constant, that is, $r = -8$. Since r is constant and by the last equations, it is easy to see that $\mathbf{M}^3$ is conformally flat.

By applying Wang's theorem [31, Theorem 1.6], we conclude that either, it is locally isometric to the hyperbolic space $\mathbb{H}^3(1)$ or, the Riemannian product $\mathbb{H}^2(4) \times \mathbb{R}$, which completes the proof. □

Under the assumptions of Theorem 3.6, for non-Kenmotsu almost Kenmotsu 3-H-manifold, $\nabla_\zeta h = 0$. Also, it is known that a *akm* of dimension 3 is Kenmotsu if and only if h vanishes (see [12]). In regard of Corollary 3.3 [31] and Corollary 3.7, we can write:

Corollary 3.8 *Let $\mathbf{M}^3(F, \zeta, T, g)$ be an almost Kenmotsu 3-H-manifold. If (ψ, g) is a non-trivial solution of the Eq. (1.1) with smooth function ψ which is constant along the Reeb vector field, then either, it is locally isometric to the WP $\mathbb{R} \times_{ce^t} N(\kappa)$ ($N(\kappa)$: space of constant curvature κ) or, the Riemannian product $\mathbb{H}^2(4) \times \mathbb{R}$.*

Example In a strictly almost Kähler Einstein manifold $(\mathbf{M}, J, \bar{g})$, we set $T = dt$, $\zeta = \frac{\partial}{\partial t}$ and the tensor field F is defined on $\mathbb{R} \times_\psi N$ by $FZ_1 = JZ_1$ for a vector field Z_1 on $\mathbf{M}$ and $FZ_1 = 0$ if Z_1 is tangent to $\mathbb{R}$. Consider a metric $g = g_0 + \bar{\sigma}^2\bar{g}$, where $\bar{\sigma}^2 = ce^{2t}$, g_0 indicates the Euclidean metric on $\mathbb{R}$ and c denotes a positive constant. Then it is easy to verify that the WP $\mathbb{R} \times_{\bar{\sigma}} \mathbf{M}$, $\bar{\sigma}^2 = ce^{2t}$, with the structure (F, ζ, T, g) is an *akm* [12]. Since $\mathbf{M}$ is Einstein, we have $S = -2ng$. If we set a smooth function $\psi(x, t) = t^2$, then ψ solves the Eq. (1.1) for $\sigma = -2nt^2 - 2$.

Definition 3.9 A 3-dimensional *akm* is named a (κ, μ, ν)-*akm* if the Reeb vector field obeys the (κ, μ, ν)-nullity condition, that is,

$$K(Z_1, Z_2)\zeta = \kappa(T(Z_2)Z_1 - T(Z_1)Z_2) + \mu(T(Z_2)hZ_1 - T(Z_1)hZ_2) + \nu(T(Z_2)h'Z_1 - T(Z_1)h'Z_2),$$

for any Z_1, Z_2 and κ, μ and ν indicate smooth functions.

Example Let G^3 be a non-unimodular Lie group admitting a left invariant local orthonormal frame fields $\{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3\}$ obeying (see [23]):

$$[\mathfrak{v}_2, \mathfrak{v}_3] = 0, \quad [\mathfrak{v}_1, \mathfrak{v}_2] = \alpha\mathfrak{v}_2 + \beta\mathfrak{v}_3, \quad [\mathfrak{v}_1, \mathfrak{v}_3] = \gamma\mathfrak{v}_2 + (2-\alpha)\mathfrak{v}_3, \tag{3.18}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$. We define g on G by $g(\mathfrak{v}_i, \mathfrak{v}_j) = \delta_{ij}$ for $1 \leq i, j \leq 3$. Take $\zeta = -\mathfrak{v}_1$ and denote its dual 1-form by T. Also, we define a (1, 1) tensor field F by $F(\zeta) = 0$, $F(\mathfrak{v}_2) = \mathfrak{v}_3$ and $F(\mathfrak{v}_3) = -\mathfrak{v}_2$. We can easily verify that (G, F, ζ, T, g) admits a left invariant almost Kenmotsu structure. From [30, Theorem 3.2], we get that G has a (κ, μ, ν)-almost Kenmotsu structure where

$$\kappa = -\alpha^2 + 2\alpha - \frac{1}{4}(\beta+\gamma)^2 - 2, \mu = \beta - \gamma, \nu = -2.$$

Moreover, from [30], we have

$$he_2 = (\alpha - 1)\mathfrak{v}_3 - \frac{1}{2}(\beta+\gamma)\mathfrak{v}_2, he_3 = \frac{1}{2}(\beta+\gamma)\mathfrak{v}_3 + (\alpha-1)\mathfrak{v}_2. \tag{3.19}$$

The Ricci operator is determined as follows (see [31]):

$$L\zeta = \left(\frac{1}{2}(\beta-\gamma)^2 - \alpha^2 - \beta^2 - (\alpha-2)^2 - \gamma^2\right)\zeta.$$

Clearly, taking $\alpha = \beta = \gamma = 1$ in the above expressions shows that G is a non-Kenmotsu $(\kappa, -2)'$-*akm* with $\kappa = -2$. In view of this, we get $L\zeta = -4\zeta$ and the scalar curvature as $r = -8$ (from Lemma 1.12). We define a function $\psi = e^{-2t}$, $t \geq 0$. Then by Laplace transformation, we get $\Delta\psi = \frac{1}{s+2}$, where s is a complex number. In view of this in (1.2) gives $\sigma = -8e^{-2t}$. Then it is easy to verify that ψ is a non-trivial solution of Einstein-type metrics (1.1). Moreover, using the result of Dileo and Pastore [12, Theorem 4.2], we state that G is locally isometric to $\mathbb{H}^2(4) \times \mathbb{R}$, the Riemannian product. Hence, Theorem 3.1 is verified.

Next, we produce an example of almost Kenmotsu 3-H-manifold of dimension three (for details see [31]).

Example Consider a cylindrical coordinates (r, θ, z) of $\mathbb{R}^3$. On $\mathbf{M}^3$ which is a simply connected domain of $\mathbb{R}^3$ excluding the origin, we define an almost Kenmotsu structure as (see [5]):

$$\zeta = \frac{2}{\gamma}\frac{\partial}{\partial r}, T = \frac{\gamma}{2}dr, g = \frac{\gamma^2}{4}(dr^2 + r^2 d\theta^2 + dz^2),$$
$$F\left(\frac{\partial}{\partial z}\right) = \frac{1}{r}\frac{\partial}{\partial \theta}, F\left(\frac{\partial}{\partial r}\right) = 0, F\left(\frac{\partial}{\partial \theta}\right) = -r\frac{\partial}{\partial z},$$

where $\gamma = \frac{1}{c_1\sqrt{r}-r}$, $\sqrt{r} > c_1 > 0$ or $\sqrt{r} < c_1$, c_1 being a constant. If we set $e_1 = \frac{2}{\gamma r}\frac{\partial}{\partial \theta}$ and $e_2 = Fe_1 = -\frac{2}{\gamma}\frac{\partial}{\partial z}$, then in [31] it is showed that ζ is an eigenvector of the Ricci operator. Therefore $\mathbf{M}^3$ is an almost Kenmotsu 3-H-manifold.

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