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Sharp Approximations for Complete *p*-Elliptic Integral of the Second Kind by Weighted Power Means

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Abstract

In this paper, the well-known double inequality for the complete elliptic integral $E(r)$ of the second kind, which gives sharp approximations of $E(r)$ by power means (or Hölder means), is extended to the complete p-elliptic integral $E_p(r)$ of the second kind, and thus sharp approximations of $E_p(r)$ by weighted power means are obtained. This result confirmed the truth of Conjecture I by Barnard, Ricards and Tiedeman in the case when $a = b = 1/p \in (0, 1/2)$ and $c = 1$ and also provides a new method to prove the above double inequality of $E(r)$.

Keywords Gaussian hypergeometric function · Complete p-elliptic integral · Weighted power mean

Mathematics Subject Classification Primary 33E05 · Secondary 26E60

1 Introduction

For $1 < p < \infty$ and $x \in [0, 1]$, the generalized sine function $\sin_p x$ is defined as the inverse function of

$$\operatorname{arcsin}_p x := \int_0^x \frac{dt}{(1 - t^p)^{\frac{1}{p}}},$$

which can be extended to a function of half-period π_p on $(-\infty, \infty)$ as follows

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$$\pi_p := 2\arcsin_p 1 = 2\int_0^1 \frac{\mathrm{d}t}{(1-t^p)^{\frac{1}{p}}} = \frac{2\pi}{p\sin(\pi/p)} = \frac{2}{p}B\left(\tfrac{1}{p}, 1-\tfrac{1}{p}\right),$$

where

$$B(a,b) := \int_0^1 t^{a-1}(1-t)^{b-1}\mathrm{d}t = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (a, b > 0)$$

is the beta function and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$ is the gamma function. Clearly, $\sin_p = \sin$ and $\pi_p = \pi$ in the case when $p = 2$.

For $r \in (0,1)$, the well-known Legendre's complete elliptic integrals of the first and second kinds [1–4] are, respectively, defined by

$$\begin{cases} K(r) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-r^2\sin^2\theta}} = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-r^2t^2)}}, \\ K(0) = \pi/2, \quad K(1^-) = \infty \end{cases}$$

and

$$\begin{cases} E(r) = \int_0^{\pi/2} \sqrt{1-r^2\sin^2\theta}\,\mathrm{d}\theta = \int_0^1 \sqrt{\frac{1-r^2t^2}{1-t^2}}\mathrm{d}t, \\ E(0) = \pi/2, \quad E(1^-) = 1. \end{cases}$$

It is natural to try to apply generalized trigonometric functions to Legendre's complete elliptic integrals. In [5], Takeuchi introduced the complete p-elliptic integrals of the first and second kind defined as

$$K_p(r) = \int_0^{\pi_p/2} \frac{\mathrm{d}\theta}{(1-r^p\sin_p^p\theta)^{1-1/p}} = \int_0^1 \frac{\mathrm{d}t}{(1-t^p)^{1/p}(1-r^pt^p)^{1-1/p}}$$

and

$$E_p(r) = \int_0^{\pi_p/2} (1-r^p\sin_p^p\theta)^{1/p}\mathrm{d}\theta = \int_0^1 \left(\frac{1-r^pt^p}{1-t^p}\right)^{1/p}\mathrm{d}t \tag{1.1}$$

for $1 < p < \infty$ and $r \in (0,1)$, respectively, where $K_p(0) = E_p(0) = \pi_p/2$, $K_p(1^-) = \infty$ and $E_p(1^-) = 1$. It is clear that for $p = 2$, K_p and E_p reduce to K and E, respectively. Moreover, the complete p-elliptic integrals have the following expressions

$$K_p(r) = \frac{\pi_p}{2}F\left(1-\tfrac{1}{p}, \tfrac{1}{p}; 1; r^p\right), \quad E_p(r) = \frac{\pi_p}{2}F\left(-\tfrac{1}{p}, \tfrac{1}{p}; 1; r^p\right) \tag{1.2}$$

(cf. [5, Proposition 2.8]), where $F(a, b; c; x)$ is the Gaussian hypergeometric function [6]

$$F(a, b; c; x) := {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n \quad (|x| < 1) \tag{1.3}$$

for complex parameters a, b, c with $c \neq 0, -1, -2, \cdots$, while $(a)_0 = 1$ for $a \neq 0$ and the Pochhammer symbol $(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \Gamma(n+a)/\Gamma(a)$ for $n \in \mathbb{N}$. The behavior of the hypergeometric function near $x = 1$ in the three cases $a + b < c$, $a + b = c$ and $a + b > c$, respectively, is given by

$$\begin{cases} F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \\ B(a, b)F(a, b; c; x) + \log(1 - x) = R(a, b) + O((1 - x)\log(1 - x)), \\ F(a, b; c; x) = (1 - x)^{c-a-b} F(c - a, c - b; c; x), \end{cases} \tag{1.4}$$

which can be found in the literature [6, Theorems 1.19 and 1.48], where

$$R(a, b) = -\psi(a) - \psi(b) - 2\gamma, \quad \psi(x) = \Gamma'(x)/\Gamma(x)$$

and $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) = 0.57721\cdots$ is the Euler–Mascheroni constant. For more information on these and related functions, we refer the reader to [6–9] and, for recently obtained related results, to [10–23] and the references contained therein.

The study of this paper begins with the following elegant inequality

$$\frac{\pi}{2} M_\alpha\left(1, \sqrt{1 - r^2}\right) < E(r) < \frac{\pi}{2} M_\beta\left(1, \sqrt{1 - r^2}\right) \tag{1.5}$$

for all $r \in (0, 1)$ with the best possible constants $\alpha = 3/2$ and $\beta = (\log 2)/\log \frac{\pi}{2}$, where $M_q(x, y)$ is the q-th power mean of x and y defined by $M_q(x, y) = \left[(x^q + y^q)/2\right]^{1/q}$ for $q \neq 0$ and $M_0(x, y) = \sqrt{xy}$. The first inequality in (1.5) was conjectured by Vuorinen [24] and proved in 1997 by Qiu and Shen [25, Theorem 2] (see also [26, Theorem 1.1] by different methods), while the second inequality in (1.5) was established in 2000 by Qiu [27, Corollary (1)] (see also [28, Theorem 22]).

In light of inequality (1.5), the following questions are natural:

Question 1.1 (1) Can (1.5) be extended to the case of complete p-elliptic integral?

(2) Can we use $M_s(x, y; w)$ to approximate to the complete p-elliptic integral of the second kind, and if yes, what are the best possible constants s in the lower and upper bounds? Here and hereafter,

$$M_s(x, y; w) = \begin{cases} \left[(1 - w)x^s + wy^s\right]^{1/s}, & s \neq 0, \\ x^{1-w} y^w, & s = 0 \end{cases}$$

is the w-weighted power mean of order s.

The main purpose of this paper is to give the answer to Question 1.1 by proving our following theorem.

Theorem 1.2 *For $p > 1$ and $s > 0$, let*

$$\sigma := \sigma(p) = \frac{p+1}{2}, \quad \tau := \tau(p) = \frac{\log(\frac{p}{p-1})}{\log \frac{\pi_p}{2}},$$
$$\varrho := \varrho(p) = \frac{2}{\pi_p(1-\frac{1}{p})^{1/\sigma}} \tag{1.6}$$

and $\eta := \varrho(2) - 1 = 2^{5/3}\pi - 1 \approx 0.01057$, and define the function Q_s on $(0, 1)$ by

$$Q_s(x) = \frac{F(-\frac{1}{p}, \frac{1}{p}; 1; x)}{M_s\big(1, (1-x)^{1/p}; \frac{1}{p}\big)} = \frac{F(-\frac{1}{p}, \frac{1}{p}; 1; x)}{\left[1 - \frac{1}{p} + \frac{(1-x)^{s/p}}{p}\right]^{1/s}}$$

and $f(x) \equiv Q_\sigma(x)$, $g(x) \equiv Q_\tau(x)$. Then we have the following conclusions:

(1) If $p \geq 2$, then the function f is strictly increasing and convex from $(0, 1)$ onto $(1, \varrho)$. In particular, for $p \geq 2$ and $x, r \in (0, 1)$,

$$M_\sigma\left(1, (1-x)^{1/p}; \tfrac{1}{p}\right) < F\left(-\tfrac{1}{p}, \tfrac{1}{p}; 1; x\right)$$
$$< \big[1 + (\varrho - 1)x\big] M_\sigma\left(1, (1-x)^{1/p}; \tfrac{1}{p}\right), \tag{1.7}$$
$$\frac{\pi}{2} M_{3/2}\left(1, \sqrt{1-r^2}\right) < E(r) < \frac{\pi}{2}(1 + \eta r^2) M_{3/2}\left(1, \sqrt{1-r^2}\right). \tag{1.8}$$

Moreover, if $1 < p < 2$, then f is neither increasing nor decreasing on $(0, 1)$.

(2) For $p \geq 2$, there exists a number $x^* \in (0, 1)$ such that g is strictly decreasing on $(0, x^*]$ and strictly increasing on $[x^*, 1)$, with $g(0) = g(1) = 1$.

(3) If $p \geq 2$, then the double inequality

$$M_\alpha\big(1, (1-x)^{1/p}; \tfrac{1}{p}\big) < F\left(-\tfrac{1}{p}, \tfrac{1}{p}; 1; x\right) < M_\beta\left(1, (1-x)^{1/p}; \tfrac{1}{p}\right) \tag{1.9}$$

holds for all $x \in (0, 1)$ if and only if $\alpha \leq \sigma$ and $\beta \geq \tau$, and the inequality

$$F\left(-\tfrac{1}{p}, \tfrac{1}{p}; 1; x\right) < \big[1 + (\varrho - 1)x\big] M_s\left(1, (1-x)^{1/p}; \tfrac{1}{p}\right) \tag{1.10}$$

holds for all $x \in (0, 1)$ if and only if $s \geq \sigma$.

Taking $x = r^p$ in Theorem 1.2 and letting $r' = (1-r^p)^{1/p}$, we immediately obtain the following.

Corollary 1.3 *For $p \geq 2$, let σ and τ be as in Theorem 1.2. Then the double inequality*

$$\frac{\pi_p}{2} M_\alpha\big(1, r'; \tfrac{1}{p}\big) < E_p(r) < \frac{\pi_p}{2} M_\beta\big(1, r'; \tfrac{1}{p}\big) \tag{1.11}$$

holds for all $r \in (0, 1)$ *if and only if* $\alpha \le \sigma$ *and* $\beta \ge \tau$*, and the inequality*

$$E_p(r) < \frac{\pi_p}{2}\left[1 + (\varrho - 1)r^p\right] M_s\left(1, r'; \tfrac{1}{p}\right) \tag{1.12}$$

holds for all $r \in (0, 1)$ *if and only if* $s \ge \sigma$.

Observe that for $p = 2$, the double inequality (1.11) coincides with (1.5). The proof of Theorem 1.2 given in Sect. 4 requires several properties of π_p and the Riemann zeta function $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ for $\Re(z) > 1$ or the Bernoulli numbers B_n defined by the power series expansion

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!} \quad (|z| < 2\pi), \tag{1.13}$$

which will be revealed in Sect. 2, and some properties of $F(a, b; c; x)$ presented in Sect. 3.

Throughout this paper, we denote the set of positive integers by $\mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and keep in mind the definitions of σ, τ and ϱ given in (1.6).

2 Some Properties of the Riemann Zeta Function and π_p

In this section, we prove several lemmas, which present several properties of π_p and the Riemann zeta function needed in the proof of our main results stated in Sect. 1.

Let us recall the following well-known formulas listed in [29, 23.2.1, 23.2.16 & 4.3.70-4.3.71]

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \quad \text{for} \quad n \in \mathbb{N}, \tag{2.1}$$

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2\zeta(2n)}{\pi^{2n}} x^{2n-1} \quad \text{and} \quad \frac{1}{x^2} \log \frac{x}{\sin x} = \sum_{n=0}^{\infty} \frac{\zeta(2n+2)}{(n+1)\pi^{2n+2}} x^{2n} \tag{2.2}$$

for $|x| < \pi$. By (2.1) and [30, 23.1], the first few values of Riemann zeta function are

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450},$$
$$\zeta(10) = \frac{\pi^{10}}{93555}. \tag{2.3}$$

The following lemma is a useful tool for dealing with the monotonicity of the ratio of power series. The first part of Lemma 2.2 is first established by Biernacki and Krzyz [31], while the second part comes from Yang et al. [32, Theorem 2.1]. But we cite the latest version of the second part [33, Lemma 2], where the authors have corrected a bug in the previous version [32, Theorem 2.1].

Lemma 2.1 ([33]). *Suppose that the power series* $A(t)=\sum_{n=0}^{\infty}a_nt^n$ *and* $B(t)=\sum_{n=0}^{\infty}b_nt^n$ *have the radius of convergence* $r>0$ *with* $b_n>0$ *for all* $n\in\mathbb{N}_0$. *Let* $H_{A,B}=(A'/B')B-A$. *Then the following statements hold true:*

(i) If the non-constant sequence $\{a_n/b_n\}_{n=0}^{\infty}$ *is increasing (decreasing) for all* $n\geq 0$, *then* $A(t)/B(t)$ *is strictly increasing (decreasing) on* $(0,r)$;

(ii) If for certain $m\in\mathbb{N}$, *the sequences* $\{a_k/b_k\}_{0\leq k\leq m}$ *and* $\{a_k/b_k\}_{k\geq m}$ *both are non-constant, and they are increasing (decreasing) and decreasing (increasing), respectively. Then* $A(t)/B(t)$ *is strictly increasing (decreasing) on* $(0,r)$ *if and only if* $H_{A,B}(r^-)\geq(\leq)0$. *If* $H_{A,B}(r^-)<(>)0$, *then there exists* $t_0\in(0,r)$ *such that* $A(t)/B(t)$ *is strictly increasing (decreasing) on* $(0,t_0)$ *and strictly decreasing (increasing) on* (t_0,r).

By (2.1), the double inequality for the ratio $|B_{2n+2}|/|B_{2n}|$ obtained in [34, Theorem 1.1] can derive the following lower and upper bounds of $\zeta(2n+2)/\zeta(2n)$.

Lemma 2.2 ([34, Theorem 1.1]). *For* $n\in\mathbb{N}$, *the double inequality holds*

$$\frac{2^{2n+1}-4}{2^{2n+1}-1}<\frac{\zeta(2n+2)}{\zeta(2n)}<\frac{2^{2n+2}-4}{2^{2n+2}-1}. \tag{2.4}$$

The following two lemmas present some properties of $\zeta(2n)$ for $n\in\mathbb{N}$, and properties of π_p, respectively.

Lemma 2.3 *For* $n\in\mathbb{N}$, *let*

$$a_n=\frac{2n+1}{n+1}\zeta(2n+2),\quad b_n=\frac{2n+1}{6n+1}a_{n+1},\quad c_n=\frac{n+1}{6n+7}\zeta(2n+4),$$
$$d_n=\frac{(n+2)(2n-1)\zeta(2n+2)+n(2n+3)\zeta(2n+4)}{(n+2)(3n-1)}.$$

Then the following statements hold:

(1) The sequence $\{a_n\}$ *is strictly increasing for* $n\in\mathbb{N}$ *with* $a_1=\pi^4/64$ *and* $\lim_{n\to\infty}a_n=2$;

(2) The sequence $\{b_n\}$ *is strictly decreasing for* $1\leq n\leq 6$ *and strictly increasing for* $n\geq 6$ *with* $b_1=\pi^6/1323$ *and* $\lim_{n\to\infty}b_n=2/3$;

(3) The sequence $\{c_n\}$ *is strictly increasing for* $n\in\mathbb{N}$ *with* $c_1=2\pi^6/12285$ *and* $\lim_{n\to\infty}c_n=1/6$;

(4) The sequence $\{d_n\}$ *is strictly decreasing for* $1\leq n\leq 3$ *and strictly increasing for* $n\geq 3$ *with* $d_1=\pi^4(1+10\pi^2/63)/180$ *and* $\lim_{n\to\infty}d_n=4/3$.

Proof Due to binomial expansion theorem, it can be easily established the following inequality which will be often used in the proof of Lemma 2.3

$$4^n=(1+3)^n=\sum_{k=0}^{n}C_n^k3^k\geq 1+3n+\frac{9n(n-1)}{2}+\frac{27n(n-1)(n-2)}{6}$$
$$+\cdots, \tag{2.5}$$

for $n \in \mathbb{N}_0$, where C_n^k is a binomial coefficient. Clearly, the first item of each sequence and the limits can be obtained from (2.1) and (2.4).

(1) By Lemma 2.2 and (2.5), we obtain

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)(2n+3)\zeta(2n+4)}{(n+2)(2n+1)\zeta(2n+2)} > \frac{(n+1)(2n+3)(2^{2n+3}-4)}{(n+2)(2n+1)(2^{2n+3}-1)}\\ &= \frac{8\cdot 4^n - 6n^2 - 15n - 10}{(n+2)(2n+1)(2^{2n+3}-1)} + 1\\ &\geq \frac{8[1+3n+9n(n-1)/2] - 6n^2 - 15n - 10}{(n+2)(2n+1)(2^{2n+3}-1)} + 1\\ &= \frac{3(6+n+10n^2)}{(n+2)(2n+1)(2^{2n+3}-1)} + 1 > 1,\end{aligned}$$

which yields the monotonicity of $\{a_n\}$.

(2) Lemma 2.3(2) will be true if we can prove $b_{n+1}/b_n < 1$ for $1 \leq n \leq 5$ and $b_{n+1}/b_n > 1$ for $n \geq 6$. By Lemma 2.2, we obtain

$$\begin{aligned}\frac{b_{n+1}}{b_n} &= \frac{(n+2)(2n+5)(6n+1)\zeta(2n+6)}{(n+3)(2n+1)(6n+7)\zeta(2n+4)}\\ &< \frac{(n+2)(2n+5)(6n+1)(2^{2n+6}-4)}{(n+3)(2n+1)(6n+7)(2^{2n+6}-1)}\\ &= \frac{t_1(n)}{(n+3)(2n+1)(6n+7)(2^{2n+6}-1)} + 1\end{aligned} \tag{2.6}$$

for $1 \leq n \leq 5$ and

$$\begin{aligned}\frac{b_{n+1}}{b_n} &> \frac{(n+2)(2n+5)(6n+1)(2^{2n+5}-4)}{(n+3)(2n+1)(6n+7)(2^{2n+5}-1)}\\ &= \frac{t_2(n)}{(n+3)(2n+1)(6n+7)(2^{2n+5}-1)} + 1\end{aligned} \tag{2.7}$$

for $n \geq 6$, where $t_1(n) = 64(2n-11)4^n - 36n^3 - 168n^2 - 209n - 19$ and $t_2(n) = 32(2n-11)4^n - 36n^3 - 168n^2 - 209n - 19$. Moreover, it can be easily from (2.5) proved that

$$\begin{aligned}t_1(n) &\leq 64(2n-11)(1+3n) - 36n^3 - 168n^2 - 209n - 19\\ &= -3[241 + (731 - 72n)n + 12n^3] < 0 \qquad (1 \leq n \leq 5),\\ t_2(n) &\geq 32(2n-11)\left[1 + 3n + \frac{9n(n-1)}{2} + \frac{27n(n-1)(n-2)}{6}\right]\\ &\quad - 36n^3 - 168n^2 - 209n - 19\\ &= 7111 + (n-6)[1247 + 672n + 36n^2(8n-13)] > 0 \qquad (n \geq 6).\end{aligned}$$

This in conjunction with (2.6) and (2.7) gives the desired result of (2).

(3) As in the proof of (1), by (2.4) and (2.5), the monotonicity of $\{c_n\}$ follows easily from

$$\begin{aligned}\frac{c_{n+1}}{c_n} &= \frac{(n+2)(6n+7)\zeta(2n+6)}{(n+1)(6n+13)\zeta(2n+4)} > \frac{(n+2)(6n+7)(2^{2n+5}-4)}{(n+1)(6n+13)(2^{2n+5}-1)}\\ &= \frac{32\cdot 4^n - 18n^2 - 57n - 43}{(n+1)(6n+13)(2^{2n+5}-1)} + 1\\ &> \frac{32[1+3n+9n(n-1)/2] - 18n^2 - 57n - 43}{(n+1)(6n+13)(2^{2n+5}-1)} + 1\\ &= \frac{126n^2 + 183n - 11}{(n+1)(6n+13)(2^{2n+5}-1)} + 1 > 1 \quad (n \ge 1).\end{aligned}$$

(4) Numerical experiment results show

$$\begin{aligned} d_1 &= \frac{\pi^4(63+10\pi^2)}{11340} \approx 1.38895 > d_2 = \frac{\pi^6(60+7\pi^2)}{94500} \approx 1.31326\\ &> d_3 = \frac{\pi^8(55+6\pi^2)}{831600} \approx 1.30322\\ &< d_4 = \frac{\pi^{10}(143325+15202\pi^2)}{21070924875} \approx 1.30383. \end{aligned} \tag{2.8}$$

Moreover, it can be proved from Lemma 2.2 that

$$\begin{aligned}\frac{d_{n+1}}{d_n} &= \frac{(n+2)(3n-1)[(2n^2+7n+3)+(2n^2+7n+5)\zeta(2n+6)/\zeta(2n+4)]}{(n+3)(3n+2)[n(2n+3)+(2n^2+3n-2)\zeta(2n+2)/\zeta(2n+4)]}\\ &> \frac{p_4(n)2^{2n+3} + 90n^4 + 465n^3 + 677n^2 + 112n - 172}{(n+3)(3n+2)(2^{2n+5}-1)[2^{2n+4}(2n^2+3n-1)-(10n^2+15n-2)]} + 1,\end{aligned} \tag{2.9}$$

where $t_3(n) = 64(n+1)(n-2)4^n - 90n^4 - 465n^3 - 640n^2 + 49n + 242.$

Therefore, the desired result of (4) can be derived from (2.8) and (2.9) together with

$$\begin{aligned} t_3(n) &> 64(n+1)(n+2)\left[1+3n+\frac{9n(n-1)}{2}+\frac{27n(n-1)(n-2)}{6}\right]\\ &\quad - 90n^4 - 465n^3 - 640n^2 + 49n + 242.\\ &= 3\left[16466 + (n-4)(96n^4 + 66n^3 + 269n^2 + 1108n + 4107)\right]\\ &> 0 \quad (n \ge 4). \end{aligned}$$

□

Lemma 2.4 *Let* $a = \frac{3}{\pi^2} \approx 0.30396$*,* $b = \frac{\log 2}{\log \frac{\pi}{2}} - \frac{12}{\pi^2} \approx 0.31907$ *and*

$$\varphi(x) = \left(\frac{2a}{x} + b\right) \log \frac{\pi x}{\sin(\pi x)} - \log \frac{1}{1-x},$$
$$\phi(x) = \log \frac{1}{1-x} - a\left(1 + \frac{2}{x}\right) \log \frac{\pi x}{\sin(\pi x)}$$

for $x \in (0, \frac{1}{2}]$*. Then we have the following statements:*

(i) There exists a number $x_1 \in (0, \frac{1}{2})$ *such that the function* $\varphi(x)$ *is strictly increasing on* $(0, x_1]$ *and strictly decreasing on* $[x_1, \frac{1}{2}]$ *with* $\varphi(0^+) = \varphi(\frac{1}{2}) = 0$*. Moreover, the function* $\varphi_1(x) = \varphi(x)/x^2$ *is strictly decreasing from* $(0, \frac{1}{2}]$ *onto* $[0, (b-a)/2a)$ *and the function* $\varphi_2(x) = \varphi(x)/(1-4x^2)$ *is strictly increasing from* $(0, \frac{1}{2}]$ *onto* $\left(0, \frac{1-b}{2} - 2a(1 - \log \frac{\pi}{2})\right]$*.*

(ii) The function $\phi_1(x) = \phi(x)/x^3$ *is strictly increasing from* $(0, \frac{1}{2}]$ *onto* $\left(\frac{1}{3} - \frac{1}{10a}, 8(b-a) \log \frac{\pi}{2}\right]$*.*

In particular, for $p \in [2, \infty)$*, we have*

$$\frac{76\sigma}{75} \le \frac{3p}{5} + \frac{8}{25} < 2ap + a + \frac{1}{p^3}\left(\frac{1}{3} - \frac{1}{10a}\right) < \tau$$
$$\le 2ap + b < \frac{76p}{125} + \frac{8}{25} \le \frac{96p}{125} \tag{2.10}$$

with the equality in each instance if and only $p = 2$*. Moreover, the constants* a *and* b*, and the coefficient* $2a$ *in the the second and third inequalities in* (2.10) *are all best possible.*

Proof (i) Clearly, $\varphi(0^+) = \varphi(\frac{1}{2}) = 0$. By differentiation and (2.2), we obtain

$$\varphi_3(x) := \frac{\varphi'(x)}{x} = \frac{1}{x}\left\{\left(\frac{2a}{x} + b\right)\left[\frac{1}{x} - \pi \cot(\pi x)\right] - \frac{2a}{x^2} \log \frac{\pi x}{\sin(\pi x)} - \frac{1}{1-x}\right\}$$
$$= \sum_{n=0}^{\infty} \left[2b\zeta(2n+2) - 1\right] x^{2n} + \sum_{n=0}^{\infty} (2aa_{n+1} - 1) x^{2n+1} \tag{2.11}$$

and

$$\varphi_3'(x) = \frac{1}{x^2}\left\{\frac{6a}{x^2} \log \frac{\pi x}{\sin(\pi x)} - 2\left(\frac{4a}{x} + b\right)\left[\frac{1}{x} - \pi \cot(\pi x)\right]\right.$$
$$\left. + \pi\left(\frac{2a}{x} + b\right) \frac{2\pi x - \sin(2\pi)}{2\sin^2(\pi x)} + \frac{1-2x}{(1-x)^2}\right\}$$
$$= \frac{\pi^2}{10} - 1 + \sum_{n=0}^{\infty} (2n+3)(2aa_{n+2} - 1) x^{2n+2}$$

$$+2\sum_{n=0}^{\infty}(n+1)\big[2b\zeta(2n+4)-1\big]x^{2n+1} \tag{2.12}$$

with

$$\begin{aligned}\varphi_3(0^+)&=\frac{b\pi^2}{3}-1\approx 0.04971 \quad \text{and}\\ \varphi_3(\tfrac{1}{2})&=4\left[b-1-4a(\log\tfrac{\pi}{2}-1)\right]\approx -0.05652,\end{aligned} \tag{2.13}$$

where a_n is given as in Lemma 2.3. It follows from Lemma 2.3(1) that

$$2aa_{n+2}-1\geq 2aa_2-1=\frac{2\pi^4}{189}-1\approx 0.030784>0 \tag{2.14}$$

for $n\in\mathbb{N}_0$. Hence by (2.12) and (2.14),

$$\begin{aligned}\varphi_3'(x)\leq{}&\frac{\pi^2}{10}-1+\frac{1}{2}\sum_{n=0}^{\infty}(2n+3)(2aa_{n+2}-1)x^{2n+1}\\ &+2\sum_{n=0}^{\infty}(n+1)\big[2b\zeta(2n+4)-1\big]x^{2n+1}\\ ={}&\frac{\pi^2}{10}-1+\sum_{n=0}^{4}(6n+7)\left(ab_{n+1}+4bc_n-\tfrac{1}{2}\right)x^{2n+1}\\ &+\sum_{n=5}^{\infty}(6n+7)\left(ab_{n+1}+4bc_n-\tfrac{1}{2}\right)x^{2n+1}\end{aligned}$$

for $x\in(0,\frac{1}{2}]$, where b_n and c_n are given as in Lemma 2.3. According to this with Lemma 2.3 (2)–(3), it follows that

$$\begin{aligned}\varphi_3'(x)<{}&\frac{\pi^2}{10}-1+\left(ab_1+4bc_4-\tfrac{1}{2}\right)\sum_{n=0}^{4}(6n+7)x^{2n+1}\\ &+\left[\tfrac{2(a+b)}{3}-\tfrac{1}{2}\right]\sum_{n=5}^{\infty}(6n+7)x^{2n+1}<0,\end{aligned}$$

since $ab_1+4bc_4-\frac{1}{2}\approx -0.07321<0$ and $\frac{2(a+b)}{3}-\frac{1}{2}\approx -0.08464<0$. This implies that $\varphi_3(x)$ is strictly decreasing on $(0,\frac{1}{2}]$. Hence the result for φ follows from (2.11) and (2.13).

Furthermore, since

$$\frac{\varphi'(x)}{(x^2)'}=\frac{\varphi_3(x)}{2} \quad \text{and} \quad \frac{\varphi'(x)}{(1-4x^2)'}=-\frac{\varphi_3(x)}{8},$$

the desired results for φ_1 and φ_2 follow from the monotonicity of φ_3 together with the L'Hôpital Monotone Rule [6, Theorem1.25].
(ii) Differentiation gives

$$\phi_2(x) := 3\frac{\phi'(x)}{(x^3)'} = \frac{1}{x^2}\left\{\frac{1}{1-x} + \frac{2a}{x^2}\log\frac{\pi x}{\sin(\pi x)} - a\left(\frac{2}{x}+1\right)\left[\frac{1}{x} - \pi\cot(\pi x)\right]\right\}$$
$$= \sum_{n=0}^{\infty}[1-2a\zeta(2n+4)]\,x^{2n+1} - \sum_{n=0}^{\infty}(2aa_{n+1}-1)x^{2n} \tag{2.15}$$

and

$$\phi_2'(x) = \frac{1}{x^3}\left\{a\left(\frac{10}{x}+3\right)\left[\frac{1}{x} - \pi\cot(\pi x)\right] - \frac{8a}{x^2}\log\frac{\pi x}{\sin(\pi x)}\right.$$
$$\left. -\pi a\left(\frac{2}{x}+1\right)\frac{2\pi x - \sin(2\pi)}{2\sin^2(\pi x)} - \frac{2-3x}{(1-x)^2}\right\}$$
$$= \sum_{n=0}^{\infty}(2n+1)\big[1-2a\zeta(2n+4)\big]x^{2n} - 2\sum_{n=0}^{\infty}(n+1)(2aa_{n+2}-1)x^{2n+1}. \tag{2.16}$$

It follows from (2.14) and (2.16) that

$$\phi_2'(x) \geq \sum_{n=0}^{\infty}(2n+1)\big[1-2a\zeta(2n+4)\big]x^{2n} - \sum_{n=0}^{\infty}(n+1)(2aa_{n+2}-1)x^{2n}$$
$$= \sum_{n=0}^{\infty}(3n+2)(1-2ad_{n+1})x^{2n}, \tag{2.17}$$

where d_n is given in Lemma 2.3.
By Lemma 2.3(4), we obtain

$$1-2ad_{n+1} \geq \min\big\{1-2ad_1,\,1-2ad_\infty\big\} = 1 - \frac{\pi^2}{30} - \frac{\pi^4}{189} \approx 0.15562 > 0$$

for $n \in \mathbb{N}_0$. This in conjunction with (2.17) implies ϕ_2 is strictly increasing on $(0, \frac{1}{2}]$. Hence the monotonicity of ϕ_1 follows from (2.15) and the L'Hôpital Monotone Rule [6, Theorem1.25].

To this end, by substituting $x = \frac{1}{p}$, the second and third inequalities in (2.10) can be derived immediately from Lemma 2.4(i) and (ii). The first inequality of (2.10) can be obtained from

$$l(p) = 2ap + a + \frac{1}{p^3}\left(\frac{1}{3} - \frac{1}{10a}\right) - \left(\frac{3p}{5} + \frac{8}{25}\right),$$
$$l(2) = \frac{15}{\pi^2} - \frac{\pi^2}{240} - \frac{887}{600} \approx 0.00036107,$$

$$l'(p) = \left(\frac{1}{\pi^2} - \frac{1}{10}\right)\left(6 - \frac{\pi^2}{p^4}\right) \geq \left(\frac{1}{\pi^2} - \frac{1}{10}\right)\left(6 - \frac{\pi^2}{2^4}\right) \approx 0.00711213.$$

The last inequality is clear by numerical results. □

3 Some Properties of the Gaussian Hypergeometric Functions

We will show, in this section, some properties of the Gaussian hypergeometric function $F(a, b; c; x)$, which are also needed in the proof of Theorem 1.2. The technique tool is to give a recurrence relation of maclaurin's coefficients for the product of power function and hypergeometric function, which has been proved by Yang in [35] that the coefficients of the function $x \mapsto (1-\theta x)^p F(a, b; c; x)$ satisfy a 3-order recurrence relation for $\theta \in [-1, 1]$, and in particular they satisfy a 2-order recurrence relation for $\theta = 1$.

As a special case of [35, Corollary 2], we state it in the following lemma.

Lemma 3.1 *For $p \in (1, \infty)$ and $s \in (0, \infty)$, defined the function f_s on $(0, 1)$ by*

$$f_s(x) = (1-x)^{-\frac{s}{p}} F\big(1 - \tfrac{1}{p}, 1 + \tfrac{1}{p}; 2; x\big) = \sum_{n=0}^{\infty} u_n x^n.$$

Then $u_0 = 1$, $u_1 = (2ps + p^2 - 1)/(2p^2)$ and for $n \in \mathbb{N}$, the coefficients u_n satisfy the recurrence relation

$$u_{n+1} = \alpha_n u_n - \beta_n u_{n-1}, \tag{3.1}$$

where

$$\alpha_n = \frac{2n^2 + (3 + 2s/p)n + 2s/p + 1 - 1/p^2}{(n+1)(n+2)}, \quad \beta_n = \frac{(n + s/p)^2 - 1/p^2}{(n+1)(n+2)}.$$

Moreover, we have $u_n > 0$ for all $n \geq 0$.

Lemma 3.2 *For $p \in [2, \infty)$ and $s \in (0, \infty)$, let u_n be defined as in Lemma 3.1 and*

$$v_n = \frac{(2 - \frac{1}{p})_n (1 + \frac{1}{p})_n}{(2)_n n!}.$$

Then we have the following conclusions:

(i) If $s = \sigma$, then $u_0/v_0 = u_1/v_1$ and the sequence $\{u_n/v_n\}$ is strictly decreasing for $n \in \mathbb{N}$.
(ii) If $s = \tau$, then for each $p \in [2, \infty)$, there exists an integer $n_0 = n_0(p) \in \{2, 3, 4, 5, 6\}$ such that the sequence $\{u_n/v_n\}$ is increasing for $0 \leq n \leq n_0$ and decreasing for $n \geq n_0$.

Proof In order to obtain the monotonicity of $\{u_n/v_n\}$, it suffices to consider the sign of

$$D_n = D_n(s) = u_{n+1} - \frac{v_{n+1}}{v_n}u_n = u_{n+1} - \frac{(n+2-1/p)(n+1+1/p)}{(n+1)(n+2)}u_n, \tag{3.2}$$

due to $v_n > 0$ for $n \in \mathbb{N}_0$.

By (3.1) and (3.2), D_n can be written as

$$D_n = \alpha_n u_n - \beta_n u_{n-1} - \frac{v_{n+1}}{v_n}u_n = \tilde{\alpha}_n u_n - \beta_n u_{n-1}, \tag{3.3}$$

where

$$\tilde{\alpha}_n = \alpha_n - \frac{v_{n+1}}{v_n} = \frac{pn^2 + 2(1+n)s - (p+1)}{p(n+1)(n+2)}.$$

Let us first analyze the sign of

$$\Delta_n(s) = \tilde{\alpha}_{n+1}(\alpha_n - \tilde{\alpha}_n) - \beta_{n+1} = -\frac{\tilde{\Delta}_n(s)}{p^3(n+1)(n+2)^2(n+3)}, \tag{3.4}$$

where

$$\tilde{\Delta}_n(s) = (1+p+np)\left[p^2(n+2)-1\right] - 2(n+2)\left[p-1+p^2(n+1)\right]s + p(n+1)(n+2)s^2$$

can be regarded as a quadratic function of s. More precisely, $\tilde{\Delta}_n(s)$ is a upward opening parabola satisfying with $\tilde{\Delta}_n(0) = (1+p+np)\left[p^2(n+2)-1\right] > 0$ and its symmetric axis $x = \frac{2(n+2)\left[p-1+p^2(n+1)\right]}{2p(n+1)(n+2)} = p + \frac{p-1}{p(n+1)} > p$ for $n \geq 0$, which makes easily for us to know $\tilde{\Delta}_n(s)$ is strictly decreasing for $s \in (0, p)$.

Taking $s = \sigma$ into (3.4), we obtain

$$\Delta_n(\sigma) = -\frac{(p-1)\left[p(p-1)n + 2(p+1)(p-2)\right]}{4p^3(n+2)^2(n+3)} < 0 \tag{3.5}$$

for $n \geq 1$. On the other hand, for $n \geq 1$, inequality (2.10) and the monotonicity of $s \mapsto \tilde{\Delta}_n(s)$ on $(0, p)$ lead to the following estimation

$$\tilde{\Delta}_n(\tau) > \tilde{\Delta}_n\left(\frac{76p}{125} + \frac{8}{25}\right) = \frac{7}{25} + \frac{223p}{625} - \frac{2918p^2}{3125} + \frac{4802p^3}{15625} + \frac{p(49p-40)^2}{15625}n^2 + \frac{n}{15625}\left[3(p-2)^2(2401p+4559) + 24071(p-2) + 3434\right]$$

$$\begin{aligned}&\geq \frac{7}{25}+\frac{223p}{625}-\frac{2918p^2}{3125}+\frac{4802p^3}{15625}+\frac{p(49p-40)^2}{15625}\\&\quad+\frac{1}{15625}\Big[3(p-2)^2(2401p+4559)+24071(p-2)+3434\Big]\\&=\frac{1}{15625}\Big[3(p-2)^2(4802p+7993)+43642(p-2)+5743\Big]>0,\end{aligned}$$

which in conjunction with (3.4) implies

$$\Delta_n(\tau)<0 \quad \text{for } n\in\mathbb{N}. \tag{3.6}$$

Based on the above preparation, we are now in a position to study the monotonicity of u_n/v_n by investigating the sign of D_n.
(i) In the case of $s=\sigma$, it can be obtained from Lemma 3.1 and (3.2) that

$$\begin{cases} D_0=0, \quad D_1=-\frac{(p^2-1)(p-2)}{24p^3}\leq 0,\\ D_2=-\frac{(p^2-1)[6+(p-2)(1+5p+12p^2)]}{288p^5}<0. \end{cases} \tag{3.7}$$

Assume that $D_n<0$ for $n\geq 2$, that is, by (3.3),

$$\tilde{\alpha}_n u_n<\beta_n u_{n-1}. \tag{3.8}$$

We now show $D_{n+1}<0$ for $n\geq 2$.

Clearly, $\tilde{\alpha}_n>0$ and $\beta_n>0$. If $\tilde{\alpha}_{n+1}\alpha_n-\beta_{n+1}\leq 0$, then it follows easily from (3.1) that

$$\begin{aligned}D_{n+1}&=\tilde{\alpha}_{n+1}u_{n+1}-\beta_{n+1}u_n=\tilde{\alpha}_{n+1}(\alpha_n u_n-\beta_n u_{n-1})-\beta_{n+1}u_n\\&=(\tilde{\alpha}_{n+1}\alpha_n-\beta_{n+1})u_n-\tilde{\alpha}_{n+1}\beta_n u_{n-1}<0.\end{aligned}$$

If $\tilde{\alpha}_{n+1}\alpha_n-\beta_{n+1}>0$, then by (3.5) and the assumption (3.8), we obtain

$$\begin{aligned}D_{n+1}&=(\tilde{\alpha}_{n+1}\alpha_n-\beta_{n+1})u_n-\tilde{\alpha}_{n+1}\beta_n u_{n-1}\\&<(\tilde{\alpha}_{n+1}\alpha_n-\beta_{n+1})\frac{\beta_n}{\tilde{\alpha}_n}u_{n-1}-\tilde{\alpha}_{n+1}\beta_n u_{n-1}\\&=\frac{\beta_n}{\tilde{\alpha}_n}\Big[\tilde{\alpha}_{n+1}(\alpha_n-\tilde{\alpha}_n)-\beta_{n+1}\Big]u_{n-1}=\frac{\Delta_n(\sigma)\beta_n u_{n-1}}{\tilde{\alpha}_n}<0.\end{aligned}$$

Hence by mathematical induction, $D_n<0$ for all $n\geq 2$ and we conclude by (3.7) that $D_n\leq 0$ for $n\in\mathbb{N}_0$, with equality if and only if $n=0$ or ($n=1$ and $p=2$). This completes the proof of (i).
(ii) In the case of $s=\tau$, we will divide into three cases to complete the proof.

Case 1: $n=0,1$. From (2.10) and Lemma 2.3 we clearly see that

$$D_0=\frac{\tau-\sigma}{p}>0,$$

$$D_1(\tau) = \frac{2\tau(3p\tau - p - 2) - (p^2 - 1)(2p + 1)}{12p^3} > D_1\left(\frac{3p}{5} + \frac{8}{25}\right)$$
$$= \frac{353 + (p - 2)(100p^2 + 265p + 264)}{7500p^3} > 0.$$

Case 2: $n = 2, 3, 4$. In this case, we will prove that $D_n > 0$ if $D_{n+1} \geq 0$.
If $D_{n+1} \geq 0$, then it follows from (3.1) and (3.3) that $(\tilde{\alpha}_{n+1}\alpha_n - \beta_{n+1})u_n \geq \tilde{\alpha}_{n+1}\beta_n u_{n-1}$, so that $\tilde{\alpha}_{n+1}\alpha_n - \beta_{n+1} > 0$. Combining this with (3.3) and (3.6), we obtain

$$D_n = \tilde{\alpha}_n u_n - \beta_n u_{n-1} \geq \tilde{\alpha}_n \cdot \frac{\tilde{\alpha}_{n+1}\beta_n u_{n-1}}{\tilde{\alpha}_{n+1}\alpha_n - \beta_{n+1}} - \beta_n u_{n-1} = -\frac{\Delta_n(\tau)\beta_n u_{n-1}}{\tilde{\alpha}_{n+1}\alpha_n - \beta_{n+1}} > 0.$$

In conclusion, it can be easily seen that for $2 \leq n \leq 5$, only the following possible signs of D_n can be happened:

$$\begin{cases} D_5 \geq 0 \Rightarrow D_4 > 0,\ D_3 > 0,\ D_2 > 0, \\ D_5 < 0 \Rightarrow \begin{cases} D_4 \geq 0 \Rightarrow D_3 > 0,\ D_2 > 0, \\ D_4 < 0 \Rightarrow \begin{cases} D_3 \geq 0 \Rightarrow D_2 > 0, \\ D_3 < 0 \Rightarrow \begin{cases} D_2 \geq 0, \\ D_2 < 0. \end{cases} \end{cases} \end{cases} \end{cases}$$

Case 3: $n \geq 6$. In this case, we shall show $D_n < 0$ for $n \geq 6$ by mathematical induction.
By Lemma 3.1 and (3.2), $D_6(\tau)$ can be written explicitly as

$$D_6(\tau) = -\frac{1}{203212800p^{13}} \sum_{k=0}^{7} C_k(p)\tau^k, \tag{3.9}$$

where

$$C_0(p) = (7p + 1)\prod_{k=1}^{6}(k^2p^2 - 1),$$
$$C_1(p) = 14p^3(p - 2)\big(751680p^8 + 1684800p^7 + 2865600p^6 + 5548752p^5 + 11144348p^4 + 22337716p^3 + 44684772p^2 + 89364525p + 178727517\big) + 14\big(357455244p^3 + 70p^2 - 3p - 1\big),$$
$$C_2(p) = -42p\big[50400p^{10} - 83520p^9 - 266648p^8 + 61500p^7 + 116160p^6 - 11595p^5 - 15649p^4 + 750p^3 + 790p^2 - 15p - 13\big],$$
$$C_3(p) = -840p^2\big[9 + 5p - 400p^2 - 160p^3 + 5297p^4 + 116853p^5 + 28p^5(p - 2)(762p^2 + 1434p + 2063)\big],$$

$$C_4(p) = -4200p^3(3360p^6 - 150p^5 - 2216p^4 + 51p^3 + 313p^2 - 3p - 11),$$
$$C_5(p) = -5040p^4(924p^4 - 18p^3 - 374p^2 + 3p + 25)$$
$$C_6(p) = -5040p^5(140p^2 - p - 27), \quad C_7(p) = -40320p^6.$$

Clearly, $C_1(p) > 0$ and $C_k(p) < 0$ $(3 \le k \le 7)$ for $p \ge 2$. Since

$$\begin{aligned}\Big[C_2(p)x^2 + C_3(p)x^3\Big]' &= x\Big[2C_2(p) + 3C_3(p)x\Big] \le 2x\Big[C_2(p) + C_3(p)\Big]\\ &= -2x\Big[p^3(p-2)\big(50400p^6 + 444000p^5 + 570952p^4\\ &\quad + 752604p^3 + 1647868p^2 + 3390081p + 6761313\big)\\ &\quad + 13515376p^3 + 890p^2 + 165p - 13\Big] < 0,\end{aligned}$$

the function $x \mapsto C_2(p)x^2 + C_3(p)x^3$ is strictly decreasing on $[\frac{2}{3}, \infty)$. Hence by (2.10),

$$\begin{aligned}\sum_{k=0}^{7} C_k(p)\tau^k &= C_0(p) + C_1(p)\tau + C_2(p)\tau^2 + C_3(p)\tau^3 + \sum_{k=4}^{7} C_k(p)\tau^k\\ &> C_0(p) + C_1(p)\left(\frac{3p}{5} + \frac{8}{25}\right) + \sum_{k=2}^{7} C_k(p)\left(\frac{76p}{125} + \frac{8}{25}\right)^k\\ &= \frac{955885494544287392363 67 + (p-2)\big[\theta_1(p) + 48p^8(p-2)\theta_2(p)\big]}{95367431640625}\\ &> 0,\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}\theta_1(p) &= 4779427489315370067 2871 + 238971354881875681098 73p\\ &\quad + 1194855693679703112524 9p^2 + 597442743575996575793 7p^3\\ &\quad + 2987550877507058074281p^4 + 1489664993280209701203p^5\\ &\quad + 737767438296634655289p^6 + 416386484287891444832p^7\\ &\quad + 332696031888772422664 0p^8,\\ \theta_2(p) &= 31651285140980996919 + 16430140617613083499p\\ &\quad + 11955793840667518488p^2 + 5523472440264886632p^3.\end{aligned}$$

Hence $D_6(\tau) < 0$ follows from (3.9) and (3.10).

Next, we assume that $D_n < 0$ for $n \ge 6$. In other words, the inequality (3.8) is valid again. If $\tilde{\alpha}_{n+1}\alpha_n - \beta_{n+1} \le 0$, then $D_{n+1} = (\tilde{\alpha}_{n+1}\alpha_n - \beta_{n+1})u_n - \tilde{\alpha}_{n+1}\beta_n u_{n-1} < 0$. If $\tilde{\alpha}_{n+1}\alpha_n - \beta_{n+1} > 0$, then it follows from (3.6) and

(3.8) that

$$D_{n+1} < (\tilde{\alpha}_{n+1}\alpha_n - \beta_{n+1})\frac{\beta_n}{\tilde{\alpha}_n}u_{n-1} - \tilde{\alpha}_{n+1}\beta_n u_{n-1}$$
$$= \frac{\beta_n}{\tilde{\alpha}_n}\Big[\tilde{\alpha}_{n+1}(\alpha_n - \tilde{\alpha}_n) - \beta_{n+1}\Big]u_{n-1} = \frac{\Delta_n(\tau)\beta_n u_{n-1}}{\tilde{\alpha}_n} < 0.$$

Hence by mathematical induction, $D_n < 0$ for all $n \geq 6$.

By the discussion in *Cases 1-3*, we conclude that for each $p \in [2, \infty)$, there exists an integer $n_0 = n_0(p) \in \{2, 3, 4, 5, 6\}$ such that the sequence $\{u_n/v_n\}$ is increasing for $0 \leq n \leq n_0$ and decreasing for $n \geq n_0$. □

Proposition 3.3 *For $p \geq 2$, let $f_s(x)$ be defined as in Lemma 3.1 and*

$$h(x) = F\left(2 - \tfrac{1}{p}, 1 + \tfrac{1}{p}; 2; x\right).$$

Then the following statements are true:

- *(i) The function $\Phi_1(x) = f_\sigma(x)/h(x)$ is strictly decreasing from $(0, 1)$ onto $(0, 1)$ if and only if $p \geq 2$;*
- *(ii) There exists $x_0 \in (0, 1)$ such that $\Phi_2(x) = f_\tau(x)/h(x)$ is strictly increasing on $(0, x_0)$ and strictly decreasing on $(x_0, 1)$ with $\Phi_2(0) = 1$ and $\Phi_2(1^-) = 0$.*

Proof (i) In terms of power series, we can rewrite as

$$\Phi_1(x) = \frac{(1-x)^{-\frac{\sigma}{p}}F\left(1 - \frac{1}{p}, 1 + \frac{1}{p}; 2, ; x\right)}{F\left(2 - \frac{1}{p}, 1 + \frac{1}{p}; 2; x\right)} = \frac{\sum_{n=0}^{\infty} u_n x^n}{\sum_{n=0}^{\infty} v_n x^n},$$

where u_n and v_n are given as in Lemmas 3.1 and 3.2, respectively.

Clearly, by (1.4), $\Phi_1(0) = u_0/v_0 = 1$ and $\Phi_1(1^-) = 0$. Hence for $p \in [2, \infty)$, it can be easily seen from Lemma 2.1 and Lemma 3.2(i) that Φ_1 is strictly decreasing from $(0, 1)$ onto itself.

Conversely, the necessary condition of Proposition 3.3(i) requires us to satisfy

$$\lim_{x\to 0^-}\frac{\Phi_1'(x)}{x} = \lim_{x\to 0^-}\frac{f_\sigma'(x)h(x) - f_\sigma(x)h'(x)}{xh^2(x)}$$
$$= \lim_{x\to 0^-}\frac{1}{x}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty}(k+1)(u_{k+1}v_{n-k} - u_{n-k}v_{k+1})\right]x^n$$
$$= \lim_{x\to 0^-}\left[2(u_2 - v_2) + o(x)\right] = 2(u_2 - v_2) = -\frac{(p^2-1)(p-2)}{12p^3} \leq 0,$$

since $u_1 = v_1$ for $s = \sigma$. This yields $p \geq 2$ and completes the proof of (i).
(ii) For $s = \tau$, it can be computed from (1.4) and $\tau < p$ that

$$H_{f_\tau,h}(1^-) = \lim_{x\to 1^-}\left(\frac{f_\tau'}{h'}h - f_\tau\right)$$

$$= \lim_{x\to 1^-} \left\{ \frac{(1-x)^{-\tau/p}\left[(p^2-1)\widehat{F}_1(x)+2p\tau F_0(x)\right]F_1(x)}{(p+1)(2p-1)F_2(x)} -(1-x)^{-\tau/p}F_0(x)\right\}$$
$$= \lim_{x\to 1^-}(1-x)^{-\tau/p}\left[\frac{p^2-1}{2p^2}\widehat{F}_1(x)-\left(1-\frac{\tau}{p}\right)F_0(x)\right]=-\infty, \tag{3.11}$$

where

$$F_0(x)=F\left(1-\tfrac{1}{p},1+\tfrac{1}{p};2;x\right),\quad F_1(x)=F\left(1-\tfrac{1}{p},\tfrac{1}{p};2;x\right),$$
$$\widehat{F}_1(x)=F\left(1-\tfrac{1}{p},1+\tfrac{1}{p};3;x\right),\quad F_2(x)=F\left(1-\tfrac{1}{p},\tfrac{1}{p};3;x\right).$$

Hence the piecewise monotonicity of $\Phi_2(x)$ follows from Lemma 2.1, Lemma 3.2(2) and (3.11). The limiting values of Φ_2 are clear. □

4 Proof of Theorem 1.2

In this section, we shall prove Theorem 1.2 stated in Sect. 1.

Proof of Theorem 1.2 Let f_s be defined as in Lemma 3.1, and h, Φ_1, Φ_2 be given as in Proposition 3.3. By differentiation,

$$Q_s'(x)=\frac{(1-x)^{s/p}q_s(x)}{p^2\left[1-\frac{1}{p}+\frac{1}{p}(1-x)^{s/p}\right]^{1+1/s}}, \tag{4.1}$$

where

$$q_s(x)=\frac{F\left(-\frac{1}{p},\frac{1}{p};1;x\right)}{1-x}-\left[\left(1-\frac{1}{p}\right)(1-x)^{-s/p}+\frac{1}{p}\right]F\left(1-\frac{1}{p},1+\frac{1}{p};2;x\right).$$

By (1.4), it can be easily seen that

$$\frac{F\left(-\frac{1}{p},\frac{1}{p};1;x\right)}{1-x}-\frac{1}{p}F\left(1-\frac{1}{p},1+\frac{1}{p};2;x\right)$$
$$=F\left(1-\frac{1}{p},1+\frac{1}{p};1;x\right)-\frac{1}{p}F\left(1-\frac{1}{p},1+\frac{1}{p};2;x\right)$$
$$=\sum_{n=0}^{\infty}\frac{(1-\frac{1}{p})_n(1+\frac{1}{p})_n}{(n!)^2}x^n-\sum_{n=0}^{\infty}\frac{(1-\frac{1}{p})_n(1+\frac{1}{p})_n}{p(2)_n n!}x^n$$
$$=\sum_{n=0}^{\infty}\frac{(1-\frac{1}{p})_n(n+1-\frac{1}{p})(1+\frac{1}{p})_n}{n!(2)_n}x^n$$

$$= \sum_{n=0}^{\infty} \frac{(1-\frac{1}{p})(2-\frac{1}{p})_n(1+\frac{1}{p})_n}{n!(2)_n} x^n = \left(1-\frac{1}{p}\right) h(x).$$

According to this, we can simplify $q_s(x)$ as follows

$$\begin{aligned} q_s(x) &= \left(1-\frac{1}{p}\right)\left[h(x) - (1-x)^{-s/p} F\left(1-\frac{1}{p}, 1+\frac{1}{p}; 2; x\right)\right] \\ &= \left(1-\frac{1}{p}\right) f_s(x) \left[\frac{h(x)}{f_s(x)} - 1\right]. \end{aligned} \tag{4.2}$$

(1) For $s = \sigma$, it follows from (4.1) and (4.2) that

$$f'(x) = \frac{(1-\frac{1}{p})}{p^2\left[1-\frac{1}{p}+\frac{1}{p}(1-x)^{\sigma/p}\right]^{1+1/\sigma}} \cdot F\left(1-\frac{1}{p}, 1+\frac{1}{p}; 2; x\right) \cdot \left[\frac{1}{\Phi_1(x)} - 1\right],$$

which is a product of three positive and strictly increasing functions on (0, 1) by Proposition 3.3. Hence the monotonicity and convexity of f follow.

In particular, by the L'Hôpital Monotone Rule [6, Theorem1.25], the convexity of f shows

$$\frac{f(x) - f(0)}{x} = \frac{f(x) - 1}{x}$$

is strictly increasing on (0, 1). So we obtain

$$\frac{f(x)-1}{x} < \lim_{x \to 1^-}\left[\frac{f(x)-1}{x}\right] = \frac{2}{\pi_p(1-\frac{1}{p})^{1/\sigma}} = \varrho$$

for $x \in (0, 1)$. This together with $f(x) > 1$ gives the inequality (1.7).

(2) Similarly, for $s = \tau$, the piecewise monotonicity property of g follows from (4.1), (4.2) and Proposition 3.3(ii).

Clearly, $g(0) = 1$. By the definition of τ, it can be easily verified that

$$g(1^-) = \frac{2}{\pi_p(1-\frac{1}{p})^{1/\tau}} = 1.$$

(3) If $\alpha \le \sigma$ and $\beta \ge \tau$, then the double inequality (1.8) holds by parts (1) and (2). Conversely, the necessary conditions of Theorem 1.2(3) require us to satisfy

$$\lim_{x\to 0^+} \frac{F\left(-\frac{1}{p}, \frac{1}{p}; 1; x\right) - \left[1 - \frac{1}{p} + \frac{(1-x)^{\alpha/p}}{p}\right]^\alpha}{x^2} \ge 0 \tag{4.3}$$

and

$$\lim_{x\to 1^-} \left\{F\left(-\frac{1}{p}, \frac{1}{p}; 1; x\right) - \left[1 - \frac{1}{p} + \frac{(1-x)^{\beta/p}}{p}\right]^\beta\right\} \le 0. \tag{4.4}$$

By Taylor's series expansion, we obtain

$$F\left(-\frac{1}{p}, \frac{1}{p}; 1; x\right) = 1 - \frac{x}{p^2} - \frac{(p^2-1)x^2}{4p^4} + o(x^2),$$
$$\left[1 - \frac{1}{p} + \frac{(1-x)^{\alpha/p}}{p}\right]^\alpha = 1 - \frac{x}{p^2} - \frac{(p-1)(p+1-\alpha)x^2}{2p^4} + o(x^2),$$

which yields

$$\lim_{x\to 0^+} \frac{F\left(-\frac{1}{p}, \frac{1}{p}; 1; x\right) - \left[1 - \frac{1}{p} + \frac{(1-x)^{\alpha/p}}{p}\right]^\alpha}{x^2}$$
$$= \lim_{x\to 0^+} \frac{1}{x^2}\left[\frac{(p-1)(p+1-2\alpha)x^2}{4p^4} + o(x^2)\right] = \frac{(p-1)(p+1-2\alpha)}{4p^4}.$$

Combining this with (4.3) gives $\alpha \le (p+1)/2 = \sigma$. On the other hand, it can be easily seen from (1.4) that

$$\lim_{x\to 1^-} \left\{F\left(-\frac{1}{p}, \frac{1}{p}; 1; x\right) - \left[1 - \frac{1}{p} + \frac{(1-x)^{\beta/p}}{p}\right]^\beta\right\} = \frac{2}{\pi_p} - \left(\frac{p-1}{p}\right)^{1/\beta}.$$

Hence by (4.4) yields

$$\beta \ge \left[\log\left(\frac{p}{p-1}\right)\right] / \log(\pi_p/2) = \tau.$$

This completes the proof of Theorem 1.2. □

5 Concluding Remark

(1) In the study of the hypergeometric mean $[F(-a, b; c; x)]^{1/a}$ with $c \ge b > 0$, Richards proved in [36, Theorem 1] that the inequality

$$[F(-a, b; c; x)]^{1/a} > \left[\left(1 - \frac{b}{c}\right) + \frac{b}{c}(1-x)^\lambda\right]^{1/\lambda} \tag{5.1}$$

holds for all $x \in (0, 1)$ if and only if $\lambda \le (a + c)/(1 + c)$ provided that

$$b > 0, \quad a \le 1 \quad \text{and} \quad c \ge \max\{1 - 2a, 2b\}. \tag{5.2}$$

Our parameters $a = b = 1/p \in (0, 1/2]$ and $c = 1$ satisfy clearly the conditions in (5.1) and

$$\frac{a + c}{1 + c} = \frac{p + 1}{2p} = \frac{\sigma}{p} = a\sigma,$$

so that in this case, the first inequality (1.9) coincides with the inequality (5.1). It is worth pointing out that the method used in this paper is completely different from that used in [36, Theorem 1].

(2) In [37, Section 3] Barnard et al. proposed two conjectures on the inequalities involving the hypergeometric mean, one of which was stated as follows.

Conjecture 5.1 ([37, Conjection I]) *Let $a \le 1$, $c > b > 0$ and $c > b - a$.*

- *Suppose $c \ge \max\{1 - 2a, 2b\}$. Then*

$$[F(-a, b; c; x)]^{1/a} < \left[\left(1 - \tfrac{b}{c}\right) + \tfrac{b}{c}(1 - x)^{\mu}\right]^{1/\mu} \tag{5.3}$$

  for all $x \in (0, 1)$ if $\mu \ge \left[a \log(1 - \frac{b}{c})\right] / \log\left[\frac{\Gamma(c+a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(c+a)}\right]$ (sharp).
- *Suppose $c \le \min\{1 - 2a, 2b\}$. Then the inequality (1.10) reverses if $\mu \le \left[a \log(1 - \frac{b}{c})\right] / \log\left[\frac{\Gamma(c+a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(c+a)}\right]$ (sharp).*

Our Theorem 1.2 is related to Conjecture 5.1. As a matter of fact, it can be easily seen that the second inequality in (1.9) implies that inequality (5.3) holds in the case when $a = b = 1/p \in (0, 1/2]$, $c = 1$ and $\left[a \log(1 - \frac{b}{c})\right] / \log\left[\frac{\Gamma(c+a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(c+a)}\right] = a\tau$.

(3) For $p \ge 2$, let $\varrho, \sigma, \tau, f, g$ be defined as in Theorem 1.2 and

$$\delta_1 = \frac{(p^2 - 1)(p - 1)(p - 2)}{72p^6}, \ \delta_2 = \frac{2}{\pi_p} - (1 - \tfrac{1}{p})^{1/\sigma},$$
$$\delta_3 = \frac{(p - 1)(\tau - \sigma)}{2p^4}, \ \delta_4 = \frac{(1 - \frac{1}{p})^{1/\tau - 1}}{p\tau}$$

and define the functions G_1, G_2, G_3, G_4 on $(0, 1)$ by

$$G_1(x) = \frac{F\left(-\frac{1}{p}, \frac{1}{p}; 1; x\right) - M_\sigma\left(1, (1 - x)^{1/p}; 1/p\right)}{x^3}, \quad G_2(x) = \frac{f(x) - 1}{x^3},$$
$$G_3(x) = \frac{M_\tau\left(1, (1 - x)^{1/p}; 1/p\right) - F\left(-\frac{1}{p}, \frac{1}{p}; 1; x\right)}{x^2(1 - x)^{\tau/p}}, \quad G_4(x) = \frac{1 - g(x)}{x^2(1 - x)^{\tau/p}}.$$

Our computation seems to show that the following conjectures are true.

Conjecture 5.2 *(i) The function* G_1 *is absolutely monotone on* $(0, 1)$ *with* $G_1(0^+) = \delta_1$ *and* $G_1(1) = \delta_2$*, and* G_2 *is strictly increasing and convex from* $(0, 1)$ *onto* $(\delta_1, \varrho - 1)$*;*

(ii) The functions G_3 *and* G_4 *are both strictly increasing and convex from* $(0, 1)$ *onto* (δ_3, δ_4).

If these conjectures are true, then the inequalities in (1.7)–(1.10) and, correspondingly, Corollary 1.3 and even (1.5) can be improved.

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Declarations

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