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An Existence Result to Some Local and Nonlocal $p(u)-$Laplacian Problem Defined on $\mathbb{R}^N$

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Abstract
In this paper, we are concerned with an elliptic equation defined on $\mathbb{R}^N$, $N \geq 1$, and involving the $p(u)-$Laplacian. When $p(u) = p(u(x))$, $x \in \mathbb{R}^N$, i.e., when p depends on the variable $x \in \mathbb{R}^N$ (through the unknown solution u), we say that we are dealing with the local case of the problem. In this case the $p(u)-$Laplacian can be considered as a new class of operators with variable exponents. When $p(u) = p(\alpha(u))$ where α is a scalar function of the unknown solution u, we say that we are dealing with the nonlocal case of the problem. In the present work, the issue of the existence of nontrivial solution in the both cases is addressed.

Keywords $p(u)-$Laplacian · Schauder's fixed point theorem · Variable exponent · Approximation · Existence result

Mathematics Subject Classification 35A01 · 35A25 · 35A35 · 35D30 · 35J15 · 35J60

1 Introduction and Statement of Main Results

When $p(u) = p(u(x))$, $x \in \mathbb{R}^N$, $N \geq 1$, then the problems involving the $p(u)-$Laplacian represent a new class of equations with variable exponents whose interest has been confirmed during last decades. Actually, this kind of nonlinear partial differential equations has many applications in various branches of modern physics. Foremost among these is the mathematical modeling of electrorheological fluids which have the property that their viscosity changes when exposed to an electric field. We

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can also mention the quasi-Newtonian fluids, the thermostor problem, the motion (of flow) of a compressible or incompressible fluid in a porous media, image restoration, or the phenomenon of elasticity. For the applied aspect of the study of problems with variable exponents, we refer to [5, 12, 18, 19, 24].

But, the application of some numerical techniques to restore digital images has proved that considering the case of variable exponents depending on the solution u (or its derivatives) considerably reduces the noise of the restored image u. See [8, 9, 20]. The same situation is observed when treating the problem of thermistor which describes the electric current in a conductor that may change its properties in dependence of temperature (see [4]).

When dealing with problems involving an exponent depending on the solution, many obstacles mainly related to the theoretical well-posedeness of the problem itself arise. Actually, comparing with similar ones defined in some classical functional spaces (such as Sobolev space with constant p or variable exponent $p(x)$), such problems are not easy to study because their weak formulations cannot be written as equations in terms of duality in a fixed Banach space. This observation can explain the small number of works devoted to the study of elliptic and parabolic equations involving an exponent of the type $p(u)$ with local and nonlocal dependence of p on u. The first one is due to B. Andreianov, M. Bendahmane and S. Ouaro who have considered the problem

$$\begin{cases} u - \operatorname{div}\left(|\nabla u|^{p(u)-2}\,\nabla u\right) = f, \text{ in } \Omega, \\ \qquad\qquad u = 0, \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is some bounded domain of $\mathbb{R}^N$, $N \geq 2$, $f \in L^1(\Omega)$ and $p : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous such that $p^- = \inf_{s\in\mathbb{R}} p(s) > N$. In [1] and under the key restriction $p^- > N$, B. Andreianov, M. Bendahmane and S. Ouaro proved that (1.1) is well-posed in $L^1(\Omega)$ and, using some approximation method, they can establish the existence of so-called narrow and broad weak solution. These kinds of solution are suitable to the case when the source f is only integrable. The version of the problem (1.1) with homogeneous Neumann boundary conditions has been treated in [17].

Recently, M. Chipot and H.B. de Oliveira proposed in [13] a new simple approach to deal with a problem very similar to (1.1). More precisely, M. Chipot and H.B. de Oliveira studied the problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(u)-2}\,\nabla u\right) = f, \text{ in } \Omega, \\ \qquad\qquad u = 0, \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$ with smooth boundary, $p : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that $p^- > N$, and $f \in W^{-1,(p^-)'}(\Omega)$. The approach in [13] is mainly based on a perturbation of the problem (1.2) and the use of the Schauder's fixed point theorem to solve the approximated problem. Finally, a process of passage to the limit in the spirit of [25] is carried out to prove the existence of a weak solution u of the problem (1.2) in the sense that $u \in W_0^{1,p(u)}(\Omega)$ and satisfies

$$\int_\Omega |\nabla u|^{p(u)-2} \nabla u \nabla v \mathrm{d}x = \langle f, v \rangle, \ \forall\, v \in W_0^{1,p(u)}(\Omega).$$

The nonlocal version of (1.2) has been also considered in [13]. More precisely, the authors studied the problem

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p(b(u))-2} \nabla u\right) = f, \ \text{in } \Omega, \\ u = 0, \ \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where p is merely bounded continuous and satisfies that $1 < p^- < p(s), \ \forall\, s \in \mathbb{R}$, and $b : W_0^{1,p^-}(\Omega) \to \mathbb{R}$ sends bounded sets of $W_0^{1,p^-}(\Omega)$ into bounded sets of $\mathbb{R}$. Using the Browder's fixed point theorem applied to some compact interval of $\mathbb{R}$, M. Chipot and H.B. de Oliveira proved that (1.3) has at least one weak solution u in the sense that $u \in W_0^{1,p(b(u))}(\Omega)$ and satisfies

$$\int_\Omega |\nabla u|^{p(b(u))-2} \nabla u \nabla v \mathrm{d}x = \langle f, v \rangle, \ \forall\, v \in W_0^{1,p(b(u))}(\Omega).$$

This work has been completed in [23] where the authors treated the case when $f \in L^1(\Omega)$ for which they prove the existence of an entropy solution. It seems that the work of M. Chipot and H.B. de Oliveira had given a new impulse to the study of problems involving exponents depending on the unknown solution. In [2], S. Antontsev and S. Shmarev studied the homogeneous Dirichlet problem for the parabolic equation

$$u_t - \mathrm{div}\left(|\nabla u|^{p[u]-2} \nabla u\right) = f, \text{in } Q_T = \Omega \times]0, T[,$$

where $\Omega \subset \mathbb{R}^N, \ N \geq 2$, is a smooth domain, $p[u] = p(l(u)), \ p$ is a given differentiable function such that $\frac{2N}{N+2} < p^- \leq p^+ < 2$, and $\sup_{s\in\mathbb{R}} |p'(s)| < +\infty; \ l(u) = \int_\Omega |u(x,t)|^\alpha \,\mathrm{d}x, \ \alpha \in [1, 2]$, and $f \in L^{(p^-)'}(Q_T)$. A result of existence and uniqueness of a solution $u \in C^0\left([0, T]; L^2(\Omega)\right), \ |\nabla u|^{p[u]} \in L^\infty\left(0, T; L^1(\Omega)\right), \ u_t \in L^2(Q_T)$ has been proved. This result has been extended in [3] to the case when the source f is replaced by the nonlinear term $f((x,t), u, l(u))$. In [4], S. Antontsev, S. Shmarev and I. Kuzentsov treated the case when the exponent p is depending on the gradient of u, i.e., when $p[u] = p(l(|\nabla u|))$. More recently, in [10] C. Allalou, K. Hilal and S.A. Temghart have followed almost the same procedure as in [13] to treat the equation

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p(u)-2} \nabla u\right) = f + g(u)\, |\nabla u|^{p(u)-1}, \ \text{in } \Omega, \\ u = 0, \ \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N, \ N \geq 2$ with smooth boundary, f is given data and $p : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that $p^- > N$.

The case of unbounded domain has been considered for the first time in [6] where S. Aouaoui and A.E. Bahrouni studied the equation

$$-\text{div}(w_1(x)\,|\nabla u|^{p(u)-2}\,\nabla u) + w_0(x)\,|u|^{p(u)-2}\,u = f(x,u),\ x \in \mathbb{R}^N,\ N \geq 2,$$

where $p : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that $N < p^- < p^+ < +\infty$; $w_0, w_1 \in L^1(\mathbb{R}^N)$ and f is a Carathéodory function having a polynomial growth with exponent lower than $p^- - 1$. A result of the existence of a nontrivial solution has been established for the cases of local and nonlocal dependence of the exponent p on the unknown solution. The introduction of the weights w_1 and w_2 and assuming that they are both integrable allowed us to overcome the obstacle of constant functions not being integrable over an unbounded domain of $\mathbb{R}^N$. Moreover, in contrast to [13, 23], the source f is now a nonlinear term depending not only explicitly on $x \in \mathbb{R}^N$ but also on the unknown value $u(x)$. In [6], we used the Galerkin method to prove the existence of the solution for the approximated problems and this for the local problem as well as for the nonlocal one. Finally, we have to mention [7] where a local one-dimensional equation (i.e a differential equation) involving the weighted $p(u)-$Laplacian has been treated.

In the present work, we remove the weights and by this way, we are in presence of the pure "unbounded domain version" of (1.2). Knowing that the presence of the weights in [6] has been crucial to prove the existence of a nontrivial solution, obtaining such a solution after removing them can be regarded as a more challenging task. For instance, the boundedness of the approximated solution in $W^{1,p^-}(\mathbb{R}^N)$ cannot be obtained. So, comparing to [6], many necessary changes are introduced. The main idea of the proof is to use a double approximating schemes as well as some a priori estimates (for example we establish a priori estimate in $L^\infty(\mathbb{R}^N)$). The passage to the limit in the approximated problems needs some sophisticated arguments which gives more interest to the problems considered in this article.

In this paper, we are concerned with two kinds of nonlinear problems. First, we treat the following nonlinear equation:

$$-\text{div}\left(|\nabla u|^{p(u)-2}\,\nabla u\right) + |u|^{p(u)-2}\,u = f(x,u) + h(x),\ x \in \mathbb{R}^N,\ N \geq 2, \tag{1.4}$$

where $p : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that

$$N < p^- = \inf_{s\in\mathbb{R}} p(s) < p^+ = \sup_{s\in\mathbb{R}} p(s) < +\infty.$$

This equation is taken under the following assumptions:
(H_1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$|f(x,s)| \leq g(x)\,|s|^t\,,\ \text{a.e. } x \in \mathbb{R}^N,\ \forall\, s \in \mathbb{R},$$

where $0 < t < p^- - 1,\ g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\ g(x) \geq 0$ a.e. $x \in \mathbb{R}^N$. We assume that $f(x,s) = 0$ a.e. $x \in \mathbb{R}^N,\ \forall\, s \leq 0$.

(H_2) $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\ h \neq 0,\ h(x) \geq 0$ a.e. $x \in \mathbb{R}^N$.

Definition 1.1 A function $u : \mathbb{R}^N \to \mathbb{R}$ is said to be a weak solution to the equation (1.4) if it satisfies that

$$u \in W^{1,p(u)}(\mathbb{R}^N)=\left\{v \in L^1_{loc}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |v|^{p(u)}\,\mathrm{d}x<+\infty,\ \int_{\mathbb{R}^N} |\nabla v|^{p(u)}\,\mathrm{d}x<+\infty\right\},$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla u|^{p(u)-2}\,\nabla u \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} |u|^{p(u)-2}\,uv\mathrm{d}x\\ &= \int_{\mathbb{R}^N} f(x,u)v\mathrm{d}x + \int_{\mathbb{R}^N} hv\mathrm{d}x,\ \forall\, v \in W^{1,p(u)}(\mathbb{R}^N).\end{aligned}$$

The first main result in this work is given by the following theorem.

Theorem 1.2 *Assume that* (H_1) *and* (H_2) *hold. Then, there exists at least one non-negative and nontrivial weak solution to the* Eq. (1.4) *in the sense of* Definition 1.1.

The second part of this work is devoted to the study of the nonlocal version of (1.4). More precisely, we are concerned with the problem:

$$\begin{aligned}&-\mathrm{div}\left(|\nabla u|^{p(\alpha(u))-2}\,\nabla u\right) + |u|^{p(\alpha(u))-2}\,u\\ &= f(x,u) + h(x),\ \text{in } \mathbb{R}^N,\ N \geq 2,\end{aligned} \tag{1.5}$$

where $p : \mathbb{R} \to \mathbb{R}$ is some continuous function such that $1 < p^- < p^+ < +\infty$, $\alpha : W^{1,p^-}_{loc}(\mathbb{R}^N) \to \mathbb{R}$ is a continuous function, i.e., α satisfies the following property: for all $(u_n)_n \subset W^{1,p^-}_{loc}(\mathbb{R}^N)$ and $u \in W^{1,p^-}_{loc}(\mathbb{R}^N)$ such that $u_n \to u$ strongly in $W^{1,p^-}_{loc}(\mathbb{R}^N)$ (i.e., $u_n \to u$ strongly in $W^{1,p^-}(K)$ for all compact set K of $\mathbb{R}^N$), $\alpha(u_n) \to \alpha(u)$. For example, one can choose

$$\alpha(u) = \|u\|_{W^{1,p^-}(\Omega)},\ \alpha(u) = |\nabla u|_{L^{p^-}(\Omega)},\ \text{or}\ \alpha(u) = |u|_{L^{p^-}(\Omega)},$$

where Ω is a bounded domain of $\mathbb{R}^N$. Concerning the terms f and h, we keep the same assumptions (H_1) and (H_2).

Definition 1.3 A function $u : \mathbb{R}^N \to \mathbb{R}$ is said to be a weak solution of (1.5) if $u \in W^{1,p(\alpha(u))}(\mathbb{R}^N)$ and

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u|^{p(\alpha(u))-2}\,\nabla u\nabla v\mathrm{d}x + \int_{\mathbb{R}^N} |u|^{p(\alpha(u))-2}\,uv\mathrm{d}x &= \int_{\mathbb{R}^N} f(x,u)v\mathrm{d}x\\ &+ \int_{\mathbb{R}^N} hv\mathrm{d}x,\ \forall\, v \in W^{1,p(\alpha(u))}(\mathbb{R}^N).\end{aligned}$$

In contrast to the previous problem (1.4),

$$W^{1,p(\alpha(u))}(\mathbb{R}^N)=\left\{u\in L^{p(\alpha(u))}(\mathbb{R}^N),\ \nabla u\in\left(L^{p(\alpha(u))}(\mathbb{R}^N)\right)^N\right\}$$

is a classical Sobolev space. The second result in the present paper is given by the following theorem:

Theorem 1.4 *Under the assumptions* (H_1) *and* (H_2), *the problem* (1.5) *has at least one weak solution in the sense of* Definition 1.3.

2 Preliminaries

Denote by $L^0(\mathbb{R}^N)$ the space of all $\mathbb{R}-$valued measurable functions on $\mathbb{R}^N$, and

$$C_+(\mathbb{R}^N)=\left\{v\in C(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N),\ \inf_{x\in\mathbb{R}^N}v(x)>1\right\}.$$

For $q\in C_+(\mathbb{R}^N)$, set $q^+=\sup_{x\in\mathbb{R}^N}q(x)$, and $q^-=\inf_{x\in\mathbb{R}^N}q(x)$, and we introduce the variable exponent Lebesgue space

$$L^{q(\cdot)}(\mathbb{R}^N)=\left\{u\in L^0(\mathbb{R}^N),\ \int_{\mathbb{R}^N}|u|^{q(x)}\,\mathrm{d}x<+\infty\right\}.$$

This space becomes a Banach, reflexive and separable space with respect to the Luxemburg norm,

$$|u|_{L^{q(\cdot)}(\mathbb{R}^N)}=\inf\left\{\lambda>0,\ \int_{\mathbb{R}^N}\left|\frac{u(x)}{\lambda}\right|^{q(x)}\mathrm{d}x\le 1\right\}.$$

The following Hölder's inequality holds,

$$\left|\int_{\mathbb{R}^N}uv\mathrm{d}x\right|\le 2\,|u|_{L^{q(\cdot)}(\mathbb{R}^N)}\,|v|_{L^{q'(\cdot)}(\mathbb{R}^N)}\,, \tag{2.1}$$

for any $u\in L^{q(\cdot)}(\mathbb{R}^N)$ and $v\in L^{q'(\cdot)}(\mathbb{R}^N)$, where $q'\in C_+(\mathbb{R}^N)$ is such that $\frac{1}{q'(x)}+\frac{1}{q(x)}=1,\ \forall\,x\in\mathbb{R}^N$. Moreover, we have

$$\min\left\{|u|^{q^-}_{L^{q(\cdot)}(\mathbb{R}^N)},\ |u|^{q^+}_{L^{q(\cdot)}(\mathbb{R}^N)}\right\}\le\int_{\mathbb{R}^N}|u|^{q(x)}\,\mathrm{d}x\le\max\left\{|u|^{q^-}_{L^{q(\cdot)}(\mathbb{R}^N)},\ |u|^{q^+}_{L^{q(\cdot)}(\mathbb{R}^N)}\right\}. \tag{2.2}$$

Now, fix a measurable function $u:\mathbb{R}^N\to\mathbb{R}$ and set $q=p(u)$. Hence, $W^{1,p(u)}(\mathbb{R}^N)=W^{1,q(\cdot)}(\mathbb{R}^N)$. This space is equipped with the well known Luxemburg norm

$$\|u\|_{W^{1,q(\cdot)}(\mathbb{R}^N)} = \inf\left\{\lambda > 0,\ \int_{\mathbb{R}^N}\left(\frac{|\nabla u|^{q(x)} + |u|^{q(x)}}{\lambda^{q(x)}}\right)\mathrm{d}x \le 1\right\}.$$

It is known that $(W^{1,q(\cdot)}(\mathbb{R}^N), \|\cdot\|_{W^{1,q(\cdot)}(\mathbb{R}^N)})$ becomes a Banach, reflexive and separable space.

If $v \in W^{1,q(\cdot)}(\mathbb{R}^N),\ (v_n)_n \subset W^{1,q(\cdot)}(\mathbb{R}^N)$, then the following relations hold true.

$$\begin{aligned}
&\min\left\{\|v\|^{q^-}_{W^{1,q(\cdot)}(\mathbb{R}^N)},\ \|v\|^{q^+}_{W^{1,q(\cdot)}(\mathbb{R}^N)}\right\}\\
&\le \int_{\mathbb{R}^N}\left(|\nabla v|^{q(x)} + |v|^{q(x)}\right)\mathrm{d}x\\
&\le \max\left\{\|v\|^{q^-}_{W^{1,q(\cdot)}(\mathbb{R}^N)},\ \|v\|^{q^+}_{W^{1,q(\cdot)}(\mathbb{R}^N)}\right\},\\
&\|v_n - v\|_{W^{1,q(\cdot)}(\mathbb{R}^N)} \to 0 \Leftrightarrow \int_{\mathbb{R}^N}\left(|\nabla(v_n - v)|^{q(x)} + |v_n - v|^{q(x)}\right)\mathrm{d}x\\
&\to 0,\ n \to +\infty.
\end{aligned}$$

For more details, we can refer to [11, 14, 15].

Proposition 1 *Let Ω be a bounded Lipschitz domain. Assume that $u \in W^{1,p(u)}(\Omega)$. Then $D(\overline{\Omega}) = \left\{v|_{\overline{\Omega}},\ v \in D(\mathbb{R}^N)\right\}$ is dense in $W^{1,p(u)}(\Omega)$.*

Proof Since Ω is bounded, then $u \in W^{1,p^-}(\Omega)$. Since $p^- > 1$, then $u \in C^{0,\ 1-\frac{N}{p^-}}(\overline{\Omega})$ and there exists a constant $C > 0$ depending on p^- and N such that

$$|u(x) - u(y)| \le C\,\|u\|_{W^{1,p^-}(\Omega)}\,|x-y|^{1-\frac{N}{p^-}},\ \forall\, x, y \in \overline{\Omega}.$$

By hypothesis, there exists a constant $L > 0$ such that

$$|p(u(x)) - p(u(y))| \le L\,|u(x) - u(y)|,\ \forall\, x, y \in \overline{\Omega}.$$

Thus,

$$|p(u(x)) - p(u(y))| \le LC\,\|u\|_{W^{1,p^-}(\Omega)}\,|x-y|^{1-\frac{N}{p^-}},\ \forall\, x, y \in \overline{\Omega}.$$

Hence, there exists a constant $C' > 0$ such that

$$\begin{gathered}
|p(u(x)) - p(u(y))| \le \frac{-C'}{\log\ |x-y|},\\
\forall\, x, y \in \overline{\Omega},\ |x-y| \le \frac{1}{2},
\end{gathered}$$

i.e the variable exponent $p(u)$ is log-Hölder continuous. By [14, Theorem 9.1.7], we deduce that $D(\overline{\Omega})$ is dense in $W^{1,p(u)}(\Omega)$. □

3 Proof of Theorem 1.2

Set $X = W^{1,p^+}(\mathbb{R}^N) \cap W^{1,p^-}(\mathbb{R}^N)$. We naturally equip the space X with the norm

$$\|u\|_X = \|u\|_{W^{1,p^+}(\mathbb{R}^N)} + \|u\|_{W^{1,p^-}(\mathbb{R}^N)}, \; u \in X.$$

Lemma 1 *For each $\epsilon > 0$, there exists a function $u_\epsilon \in X$ such that*

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_\epsilon|^{p(u_\epsilon)-2} \nabla u_\epsilon \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} |u_\epsilon|^{p(u_\epsilon)-2} u_\epsilon v \mathrm{d}x \\
&\quad + \epsilon \left(\int_{\mathbb{R}^N} |\nabla u_\epsilon|^{p^+-2} \nabla u_\epsilon \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} |u_\epsilon|^{p^+-2} u_\epsilon v \mathrm{d}x \right) \\
&\quad + \epsilon \left(\int_{\mathbb{R}^N} |\nabla u_\epsilon|^{p^--2} \nabla u_\epsilon \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} |u_\epsilon|^{p^--2} u_\epsilon v \mathrm{d}x \right) \\
&= \int_{\mathbb{R}^N} f(x, u_\epsilon) v \mathrm{d}x + \int_{\mathbb{R}^N} h v \mathrm{d}x, \; \forall \, v \in X.
\end{aligned}$$

Proof Let $\epsilon > 0$ fixed. For $w : \mathbb{R}^N \to \mathbb{R}$ a measurable function, define the operator $A_w : X \to X^*$ by

$$\begin{aligned}
\langle A_w u, v \rangle &= \int_{\mathbb{R}^N} |\nabla u|^{p(w)-2} \nabla u \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} |u|^{p(w)-2} u v \mathrm{d}x \\
&\quad + \epsilon \left(\int_{\mathbb{R}^N} |\nabla u|^{p^+-2} \nabla u \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} |u|^{p^+-2} u v \mathrm{d}x \right) \\
&\quad + \epsilon \left(\int_{\mathbb{R}^N} |\nabla u|^{p^--2} \nabla u \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} |u|^{p^--2} u v \mathrm{d}x \right), \; u, v \in X.
\end{aligned}$$

Observe that A_w is well defined. In fact, for $u, v \in X$, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} |\nabla u|^{p(w)-2} \nabla u \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} |u|^{p(w)-2} u v \mathrm{d}x \right| \\
&\le \int_{\mathbb{R}^N} |\nabla u|^{p(w)-1} |\nabla v| \, \mathrm{d}x + \int_{\mathbb{R}^N} |u|^{p(w)-1} |v| \, \mathrm{d}x \\
&\le \int_{\mathbb{R}^N} |\nabla u|^{p^--1} |\nabla v| \, \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^{p^+-1} |\nabla v| \, \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} |u|^{p^--1} |v| \, \mathrm{d}x + \int_{\mathbb{R}^N} |u|^{p^+-1} |v| \, \mathrm{d}x \\
&\le |\nabla u|_{L^{p^-}(\mathbb{R}^N)}^{p^--1} |\nabla v|_{L^{p^-}(\mathbb{R}^N)} + |\nabla u|_{L^{p^+}(\mathbb{R}^N)}^{p^+-1} |\nabla v|_{L^{p^+}(\mathbb{R}^N)} \\
&\quad + |u|_{L^{p^-}(\mathbb{R}^N)}^{p^--1} |v|_{L^{p^-}(\mathbb{R}^N)} + |u|_{L^{p^+}(\mathbb{R}^N)}^{p^+-1} |v|_{L^{p^+}(\mathbb{R}^N)}.
\end{aligned}$$

Hence, for u fixed in X, the linear mapping $v \longmapsto \langle A_w u, v \rangle$ is in the topological dual X^*. Clearly, A_w is coercive and continuous. Moreover, A_w is strictly monotone, i.e.,

$$\langle A_w u_1 - A_w u_2, u_1 - u_2 \rangle > 0, \ \forall\, u_1, u_2 \in X, \ u_1 \neq u_2.$$

On the other hand, for $w \in L^{p^-}(\mathbb{R}^N)$ and $v \in X$, by Hölder's inequality we have

$$\left| \int_{\mathbb{R}^N} f(x, w) v \mathrm{d}x \right| \leq \int_{\mathbb{R}^N} g(x) \left| w \right|^t \left| v \right| \mathrm{d}x \leq |g|_{L^{\frac{p^-}{p^- - t - 1}}(\mathbb{R}^N)} |w|^t_{L^{p^-}(\mathbb{R}^N)} |v|_{L^{p^-}(\mathbb{R}^N)} .$$

Also,

$$\left| \int_{\mathbb{R}^N} h v \mathrm{d}x \right| \leq |h|_{L^{\frac{p^-}{p^- - 1}}(\mathbb{R}^N)} |v|_{L^{p^-}(\mathbb{R}^N)} , \ \forall\, v \in X.$$

Thus, $(f(\cdot, w) + h) \in X^*$. By the virtue of the Minty-Browder's theorem (see [21, Theorem 26.A]), we deduce that there exists a unique element $u_w \in X$ such that

$$A_w(u_w) = f(\cdot, w) + h \text{ in } X^*.$$

In other words,

$$\langle A_w u_w, v \rangle = \int_{\mathbb{R}^N} (f(x, w) + h) v \mathrm{d}x, \ \forall\, v \in X. \tag{3.1}$$

Taking $v = u_w$ in (3.1), we infer

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_w|^{p(w)} \mathrm{d}x + \int_{\mathbb{R}^N} |u_w|^{p(w)} \mathrm{d}x \\
&\quad + \epsilon \left(\int_{\mathbb{R}^N} |\nabla u_w|^{p^+} \mathrm{d}x + \int_{\mathbb{R}^N} |u_w|^{p^+} \mathrm{d}x \right) \\
&\quad + \epsilon \left(\int_{\mathbb{R}^N} |\nabla u_w|^{p^-} \mathrm{d}x + \int_{\mathbb{R}^N} |u_w|^{p^-} \mathrm{d}x \right) \\
&= \int_{\mathbb{R}^N} f(x, w) u_w \mathrm{d}x + \int_{\mathbb{R}^N} h u_w \mathrm{d}x \\
&\leq \int_{\mathbb{R}^N} g \left| w \right|^t \left| u_w \right| \mathrm{d}x + \int_{\mathbb{R}^N} h u_w \mathrm{d}x.
\end{aligned}$$

Using Young's inequality, it comes

$$\begin{aligned}
&\epsilon \left(\|u_w\|^{p^+}_{W^{1,p^+}(\mathbb{R}^N)} + \|u_w\|^{p^-}_{W^{1,p^-}(\mathbb{R}^N)} \right) \\
&\leq \frac{\epsilon}{2} |u_w|^{p^-}_{L^{p^-}(\mathbb{R}^N)} + c_{1,\epsilon} |h|^{\frac{p^-}{p^- - 1}}_{L^{\frac{p^-}{p^- - 1}}(\mathbb{R}^N)} + c_{2,\epsilon} \int_{\mathbb{R}^N} g^{\frac{p^-}{p^- - 1}} |w|^{\frac{tp^-}{p^- - 1}} \mathrm{d}x,
\end{aligned}$$

where $c_{1,\epsilon}$ and $c_{2,\epsilon}$ are two positive constants depending on ϵ but not on w. Hence,

$$\frac{\epsilon}{2}\left(\|u_w\|^{p^+}_{W^{1,p^+}(\mathbb{R}^N)}+\|u_w\|^{p^-}_{W^{1,p^-}(\mathbb{R}^N)}\right)\leq c_{1,\epsilon}\,|h|^{\frac{p^-}{p^--1}}_{L^{\frac{p^-}{p^--1}}(\mathbb{R}^N)} +c_{2,\epsilon}\int_{\mathbb{R}^N} g^{\frac{p^-}{p^--1}}\,|w|^{\frac{tp^-}{p^--1}}\,\mathrm{d}x. \quad (3.2)$$

Set

$$\frac{tp^-}{p^--1}=\theta,\ \text{and}\ g_1(x)=(g(x))^{\frac{p^-}{p^--1}},\ x\in\mathbb{R}^N.$$

Now, we claim that $W^{1,p^-}(\mathbb{R}^N)$ is compactly embedded into the weighted Lebesgue space

$$L^{\theta}_{g_1}(\mathbb{R}^N)=\left\{u:\mathbb{R}^N\to\mathbb{R}\ \text{measurable},\ \int_{\mathbb{R}^N} g_1(x)\,|u(x)|^{\theta}\,\mathrm{d}x<+\infty\right\},$$

equipped with the norm $u\longmapsto |u|_{L^{\theta}_{g_1}(\mathbb{R}^N)}=\left(\int_{\mathbb{R}^N} g_1(x)\,|u(x)|^{\theta}\,\mathrm{d}x\right)^{\frac{1}{\theta}}$. For that aim, take a sequence $(u_n)_n\subset W^{1,p^-}(\mathbb{R}^N)$ such that $u_n\rightharpoonup 0$ weakly in $W^{1,p^-}(\mathbb{R}^N)$. We show that, up to a subsequence, $u_n\to 0$ strongly in $L^{\theta}_{g_1}(\mathbb{R}^N)$. Observe that the sequence $(|u_n|^{\theta})_n$ is bounded in $L^{\frac{p^-}{\theta}}(\mathbb{R}^N)$ and, up to a subsequence, is weakly convergent to 0 in $L^{\frac{p^-}{\theta}}(\mathbb{R}^N)$. Since $g\in L^{\frac{p^-}{p^--1-t}}(\mathbb{R}^N)$, then $g_1\in L^{\frac{p^-}{p^--\theta}}(\mathbb{R}^N)$ which is the topological dual of $L^{\frac{p^-}{\theta}}(\mathbb{R}^N)$ which leads to

$$\int_{\mathbb{R}^N} g_1\,|u_n|^{\theta}\,\mathrm{d}x\to 0,\ n\to+\infty.$$

Let $C_1>0$ be a positive constant such that

$$|u|_{L^{\theta}_{g_1}(\mathbb{R}^N)}\leq C_1\,\|u\|_{W^{1,p^-}(\mathbb{R}^N)},\ \forall\, u\in W^{1,p^-}(\mathbb{R}^N). \quad (3.3)$$

Set

$$\mathcal{K}_\epsilon=\left\{w\in L^{\theta}_{g_1}(\mathbb{R}^N),\ |w|_{L^{\theta}_{g_1}(\mathbb{R}^N)}\leq\alpha_\epsilon\right\},$$

where α_ϵ is some positive constant to be fixed later. Define the mapping $T_\epsilon:\mathcal{K}_\epsilon\to\mathcal{K}_\epsilon$ by $T_\epsilon w=u_w$ given by (3.1). We choose $\alpha_\epsilon>0$ such that $T_\epsilon(\mathcal{K}_\epsilon)\subset\mathcal{K}_\epsilon$. By (3.2), there exist two positive constants $c_{3,\epsilon}$ and $c_{4,\epsilon}$, independent of w, such that

$$\|u_w\|_{W^{1,p^-}(\mathbb{R}^N)}\leq\left(c_{3,\epsilon}\,|h|^{\frac{p^-}{p^--1}}_{L^{\frac{p^-}{p^--1}}(\mathbb{R}^N)}+c_{4,\epsilon}\,|w|^{\theta}_{L^{\theta}_{g_1}(\mathbb{R}^N)}\right)^{\frac{1}{p^-}}.$$

If $w \in \mathcal{K}_\epsilon$, then

$$\|u_w\|_{W^{1,p^-}(\mathbb{R}^N)} \leq \left(c_{3,\epsilon} \, |h|^{\frac{p^-}{p^- -1}}_{L^{\frac{p^-}{p^- -1}}(\mathbb{R}^N)} + c_{4,\epsilon} \alpha_\epsilon^\theta \right)^{\frac{1}{p^-}}. \tag{3.4}$$

By (3.3) and (3.4), it yields

$$|u_w|_{L^\theta_{g_1}(\mathbb{R}^N)} \leq C_1 \left(c_{3,\epsilon} \, |h|^{\frac{p^-}{p^- -1}}_{L^{\frac{p^-}{p^- -1}}(\mathbb{R}^N)} + c_{4,\epsilon} \alpha_\epsilon^\theta \right)^{\frac{1}{p^-}}, \ \forall \, w \in \mathcal{K}_\epsilon. \tag{3.5}$$

Since $\frac{\theta}{p^-} < 1$, then if we choose $\alpha_\epsilon > 0$ large enough, we get

$$C_1 \left(c_{3,\epsilon} \, |h|^{\frac{p^-}{p^- -1}}_{L^{\frac{p^-}{p^- -1}}(\mathbb{R}^N)} + c_{4,\epsilon} \alpha_\epsilon^\theta \right)^{\frac{1}{p^-}} \leq \alpha_\epsilon.$$

In view of (3.5), we infer

$$|T_\epsilon(w)|_{L^\theta_{g_1}(\mathbb{R}^N)} = |u_w|_{L^\theta_{g_1}(\mathbb{R}^N)} \leq \alpha_\epsilon, \ \forall \, w \in \mathcal{K}_\epsilon.$$

Furthermore, since $W^{1,p^-}(\mathbb{R}^N)$ is compactly embedded into $L^\theta_{g_1}(\mathbb{R}^N)$, it immediately follows that $T_\epsilon(\mathcal{K}_\epsilon)$ is relatively compact. In the next step of the proof, we show that the mapping T_ϵ is continuous. To prove this, let us assume that $(w_n)_n$ is a sequence of $L^\theta_{g_1}(\mathbb{R}^N)$ and w is a function in $L^\theta_{g_1}(\mathbb{R}^N)$ such that $w_n \to w$ strongly in $L^\theta_{g_1}(\mathbb{R}^N)$. By (3.2), we know that the sequence $(u_{w_n})_n$ is bounded in X. Thus, there exists $u \in X$ such that, up to a subsequence, $u_{w_n} \rightharpoonup u$ weakly in X, $u_{w_n}(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$, and $u_{w_n} \to u$ strongly in $L^\theta_{g_1}(\mathbb{R}^N)$. By monotonicity of the p^-−Laplacian and the p^+−Laplacian, we have

$$\left\langle A_{w_n}(u_{w_n}) - A_{w_n}(v), u_{w_n} - v \right\rangle \geq 0, \ \forall \, v \in X. \tag{3.6}$$

Taking $w = w_n$ and $(u_{w_n} - v)$ as test function in (3.1), then by (3.6), it yields

$$\begin{aligned}
\int_{\mathbb{R}^N} (f(x, w_n) + h)(u_{w_n} - v) \mathrm{d}x &\geq \int_{\mathbb{R}^N} |\nabla v|^{p(w_n)-2} \nabla v \nabla (u_{w_n} - v) \mathrm{d}x \\
&+ \int_{\mathbb{R}^N} |v|^{p(w_n)-2} v (u_{w_n} - v) \mathrm{d}x \\
&+ \epsilon \int_{\mathbb{R}^N} |\nabla v|^{p^+-2} \nabla v \nabla (u_{w_n} - v) \mathrm{d}x + \epsilon \int_{\mathbb{R}^N} |v|^{p^+-2} v (u_{w_n} - v) \mathrm{d}x \\
&+ \epsilon \int_{\mathbb{R}^N} |\nabla v|^{p^--2} \nabla v \nabla (u_{w_n} - v) \mathrm{d}x + \epsilon \int_{\mathbb{R}^N} |v|^{p^--2} v (u_{w_n} - v) \mathrm{d}x.
\end{aligned} \tag{3.7}$$

By the weak convergence of $(u_{w_n})_n$ to u in X, we get

$$\begin{aligned}&\epsilon\int_{\mathbb{R}^N}|\nabla v|^{p^+-2}\nabla v\nabla(u_{w_n}-v)\mathrm{d}x+\epsilon\int_{\mathbb{R}^N}|v|^{p^+-2}v(u_{w_n}-v)\mathrm{d}x\\&+\epsilon\int_{\mathbb{R}^N}|\nabla v|^{p^--2}\nabla v\nabla(u_{w_n}-v)\mathrm{d}x+\epsilon\int_{\mathbb{R}^N}|v|^{p^--2}v(u_{w_n}-v)\mathrm{d}x\\&\to\\&\epsilon\int_{\mathbb{R}^N}|\nabla v|^{p^+-2}\nabla v\nabla(u-v)\mathrm{d}x+\epsilon\int_{\mathbb{R}^N}|v|^{p^+-2}v(u-v)\mathrm{d}x\\&+\epsilon\int_{\mathbb{R}^N}|\nabla v|^{p^--2}\nabla v\nabla(u-v)\mathrm{d}x+\epsilon\int_{\mathbb{R}^N}|v|^{p^--2}v(u-v)\mathrm{d}x,\end{aligned}\tag{3.8}$$

and

$$\int_{\mathbb{R}^N}h(u_{w_n}-v)\mathrm{d}x\to\int_{\mathbb{R}^N}h(u-v)\mathrm{d}x.\tag{3.9}$$

Moreover, by the strong convergence of $(w_n)_n$ to w in $L^\theta_{g_1}(\mathbb{R}^N)$, one can easily see that

$$\int_{\mathbb{R}^N}f(x,w_n)(u_{w_n}-v)\mathrm{d}x\to\int_{\mathbb{R}^N}f(x,w)(u-v)\mathrm{d}x.\tag{3.10}$$

Next, observe that

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}\left(|\nabla v|^{p(w_n)-2}\nabla v-|\nabla v|^{p(w)-2}\nabla v\right)\nabla(u_{w_n}-v)\mathrm{d}x\right|\\\le&\int_{\mathbb{R}^N}\left|\left(|\nabla v|^{p(w_n)-2}\nabla v-|\nabla v|^{p(w)-2}\nabla v\right)\nabla(u_{w_n}-v)\right|\mathrm{d}x\\\le&\int_{|\nabla v|\ge1}\left|\left(|\nabla v|^{p(w_n)-2}\nabla v-|\nabla v|^{p(w)-2}\nabla v\right)\nabla(u_{w_n}-v)\right|\mathrm{d}x\\&\quad+\int_{|\nabla v|<1}\left|\left(|\nabla v|^{p(w_n)-2}\nabla v-|\nabla v|^{p(w)-2}\nabla v\right)\nabla(u_{w_n}-v)\right|\mathrm{d}x\\\le&\left(\int_{|\nabla v|\ge1}\left||\nabla v|^{p(w_n)-2}\nabla v-|\nabla v|^{p(w)-2}\nabla v\right|^{\frac{p^+}{p^+-1}}\right)^{\frac{p^+-1}{p^+}}\left(\int_{\mathbb{R}^N}\left|\nabla(u_{w_n}-v)\right|^{p^+}\mathrm{d}x\right)^{\frac{1}{p^+}}\\&\quad+\left(\int_{|\nabla v|<1}\left||\nabla v|^{p(w_n)-2}\nabla v-|\nabla v|^{p(w)-2}\nabla v\right|^{\frac{p^-}{p^--1}}\right)^{\frac{p^--1}{p^-}}\left(\int_{\mathbb{R}^N}\left|\nabla(u_{w_n}-v)\right|^{p^-}\mathrm{d}x\right)^{\frac{1}{p^-}}.\end{aligned}\tag{3.11}$$

For $x\in\mathbb{R}^N$ such that $|\nabla v(x)|\ge1$, we have $\left(|\nabla v(x)|^{p(w_n)-1}\right)^{\frac{p^+}{p^+-1}}\le|\nabla v(x)|^{p^+}$, $\forall\ n\ge1$. Since $w_n(x)\to w(x)$ a.e. $x\in\mathbb{R}^N$, then one can apply the Lebesgue's dominated convergence theorem to prove that

$$\int_{|\nabla v|\ge1}\left||\nabla v|^{p(w_n)-2}\nabla v-|\nabla v|^{p(w)-2}\nabla v\right|^{\frac{p^+}{p^+-1}}\mathrm{d}x\to0,\ n\to+\infty.$$

Similarly,

$$\int_{|\nabla v|<1} \left| |\nabla v|^{p(w_n)-2} \nabla v - |\nabla v|^{p(w)-2} \nabla v \right|^{\frac{p^-}{p^- -1}} \mathrm{d}x \to 0, \; n \to +\infty.$$

Taking the boundedness of the sequence $(u_{w_n})_n$ in $W^{1,p^-}(\mathbb{R}^N)$ and in $W^{1,p^+}(\mathbb{R}^N)$ into account, we immediately deduce from (3.11) that

$$\int_{\mathbb{R}^N} \left(|\nabla v|^{p(w_n)-2} \nabla v - |\nabla v|^{p(w)-2} \nabla v \right) \nabla(u_{w_n} - v)\mathrm{d}x \to 0, \; n \to +\infty. \tag{3.12}$$

The weak convergence of $(u_{w_n})_n$ to u in X together with (3.12) implies that

$$\int_{\mathbb{R}^N} |\nabla v|^{p(w_n)-2} \nabla v \nabla(u_{w_n} - v)\mathrm{d}x \to \int_{\mathbb{R}^N} |\nabla v|^{p(w)-2} \nabla v \nabla(u - v)\mathrm{d}x. \tag{3.13}$$

Similarly,

$$\int_{\mathbb{R}^N} |v|^{p(w_n)-2} v(u_{w_n} - v)\mathrm{d}x \to \int_{\mathbb{R}^N} |v|^{p(w)-2} v(u - v)\mathrm{d}x. \tag{3.14}$$

Combining (3.14), (3.13), (3.10), (3.9), (3.8) with (3.7), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} (f(x,w) + h)(u - v)\mathrm{d}x \geq & \int_{\mathbb{R}^N} |\nabla v|^{p(w)-2} \nabla v \nabla(u - v)\mathrm{d}x \\
& + \int_{\mathbb{R}^N} |v|^{p(w)-2} v(u - v)\mathrm{d}x \\
& +\epsilon \int_{\mathbb{R}^N} |\nabla v|^{p^+-2} \nabla v \nabla(u - v)\mathrm{d}x \\
& +\epsilon \int_{\mathbb{R}^N} |v|^{p^+-2} v(u - v)\mathrm{d}x \\
& +\epsilon \int_{\mathbb{R}^N} |\nabla v|^{p^--2} \nabla v \nabla(u - v)\mathrm{d}x \\
& +\epsilon \int_{\mathbb{R}^N} |v|^{p^--2} v(u - v)\mathrm{d}x, \; \forall\, v \in X.
\end{aligned} \tag{3.15}$$

Let $z \in X$ and $t \in \mathbb{R}$. Taking $v = u - tz$ in (3.15), it yields

$$\begin{aligned}
t \int_{\mathbb{R}^N} (f(x,w) + h)z\mathrm{d}x \geq & \; t \int_{\mathbb{R}^N} |\nabla u - t\nabla z|^{p(w)-2} \nabla(u - tz)\nabla z\mathrm{d}x \\
& +t \int_{\mathbb{R}^N} |u - tz|^{p(w)-2} (u - tz)z\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\epsilon t \int_{\mathbb{R}^N} |\nabla u - t\nabla z|^{p^+-2} \nabla(u-tz)\nabla z \mathrm{d}x \\
&+\epsilon t \int_{\mathbb{R}^N} |u - tz|^{p^+-2} (u-tz)z \mathrm{d}x \\
&+\epsilon t \int_{\mathbb{R}^N} |\nabla u - t\nabla z|^{p^--2} \nabla(u-tz)\nabla z \mathrm{d}x \\
&+\epsilon t \int_{\mathbb{R}^N} |u - tz|^{p^--2} (u-tz)z \mathrm{d}x.
\end{aligned}
$$

Dividing by $t > 0$ and then tending t to 0^+ in that last inequality, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^N} (f(x,w)+h)z \mathrm{d}x \geq & \int_{\mathbb{R}^N} |\nabla u|^{p(w)-2} \nabla u \nabla z \mathrm{d}x + \int_{\mathbb{R}^N} |u|^{p(w)-2} uz \mathrm{d}x \\
&+\epsilon \int_{\mathbb{R}^N} |\nabla u|^{p^+-2} \nabla u \nabla z \mathrm{d}x + \epsilon \int_{\mathbb{R}^N} |u|^{p^+-2} uz \mathrm{d}x \\
&+\epsilon \int_{\mathbb{R}^N} |\nabla u|^{p^--2} \nabla u \nabla z \mathrm{d}x + \epsilon \int_{\mathbb{R}^N} |u|^{p^--2} uz \mathrm{d}x.
\end{aligned} \tag{3.16}
$$

Plainly, inequality (3.16) is also valid for $(-z)$ instead of z. Therefore,

$$
\begin{aligned}
\int_{\mathbb{R}^N} (f(x,w)+h)z \mathrm{d}x = & \int_{\mathbb{R}^N} |\nabla u|^{p(w)-2} \nabla u \nabla z \mathrm{d}x + \int_{\mathbb{R}^N} |u|^{p(w)-2} uz \mathrm{d}x \\
&+\epsilon \int_{\mathbb{R}^N} |\nabla u|^{p^+-2} \nabla u \nabla z \mathrm{d}x + \epsilon \int_{\mathbb{R}^N} |u|^{p^+-2} uz \mathrm{d}x \\
&+\epsilon \int_{\mathbb{R}^N} |\nabla u|^{p^--2} \nabla u \nabla z \mathrm{d}x + \epsilon \int_{\mathbb{R}^N} |u|^{p^--2} uz \mathrm{d}x, \ \forall z \in X.
\end{aligned}
$$

Consequently, $u = u_w$ which ends the proof of the continuity of the mapping T_ϵ. Now, one can use the Schauder's fixed point theorem (see [22, Theorem 2.A]) to deduce the existence of $\widetilde{w} \in \mathcal{K}_\epsilon$ such that $T_\epsilon(\widetilde{w}) = u_{\widetilde{w}} = \widetilde{w}$. Hence,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_{\widetilde{w}}|^{p(u_{\widetilde{w}})-2} \nabla u_{\widetilde{w}} \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} |u_{\widetilde{w}}|^{p(u_{\widetilde{w}})-2} u_{\widetilde{w}} v \mathrm{d}x \\
&\quad +\epsilon \left(\int_{\mathbb{R}^N} |\nabla u_{\widetilde{w}}|^{p^+-2} \nabla u_{\widetilde{w}} \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} |u_{\widetilde{w}}|^{p^+-2} u_{\widetilde{w}} v \mathrm{d}x \right) \\
&\quad +\epsilon \left(\int_{\mathbb{R}^N} |\nabla u_{\widetilde{w}}|^{p^--2} \nabla u_{\widetilde{w}} \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} |u_{\widetilde{w}}|^{p^--2} u_{\widetilde{w}} v \mathrm{d}x \right) \\
&\quad = \int_{\mathbb{R}^N} f(x, u_{\widetilde{w}}) v \mathrm{d}x + \int_{\mathbb{R}^N} hv \mathrm{d}x, \ \forall v \in X.
\end{aligned}
$$

This ends the proof of Lemma 1. □

Choosing $\epsilon = \frac{1}{n},\ n \geq 1$, in Lemma 1 we deduce that there exists $u_n \in X$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)-2} \nabla u_n \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} |u_n|^{p(u_n)-2} u_n v \mathrm{d}x \\
&\quad + \frac{1}{n}\left(\int_{\mathbb{R}^N} |\nabla u_n|^{p^+-2} \nabla u_n \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} |u_n|^{p^+-2} u_n v \mathrm{d}x\right) \\
&\quad + \frac{1}{n}\left(\int_{\mathbb{R}^N} |\nabla u_n|^{p^--2} \nabla u_n \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} |u_n|^{p^--2} u_n v \mathrm{d}x\right) \\
&= \int_{\mathbb{R}^N} f(x, u_n) v \mathrm{d}x + \int_{\mathbb{R}^N} h v \mathrm{d}x, \ \forall\, v \in X. \qquad (3.17)
\end{aligned}$$

Since $h \geq 0$ and $f(x, u_n(x)) = 0$ for a.e. $x \in \mathbb{R}^N$ such that $u_n(x) \leq 0$, by taking $v = u_n^- = \min(u_n, 0)$ as test function in (3.17), we can easily see that $u_n(x) \geq 0$ a.e. $x \in \mathbb{R}^N$.

Lemma 2 *There exists $M > 0$ independent of n such that $u_n(x) \leq M$ a.e $x \in \mathbb{R}^N,\ \forall\, n \geq 1$.*

Proof Let $M \geq 1$ be a real number. Taking $v = (u_n - M)^+ = \max(u_n - M, 0)$ as test function in (3.17) and having in mind that

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)-2} \nabla u_n \nabla (u_n - M)^+ \mathrm{d}x = \int_{\mathbb{R}^N} \left|\nabla (u_n - M)^+\right|^{p(u_n)} \mathrm{d}x \geq 0,$$

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p^+-2} \nabla u_n \nabla (u_n - M)^+ \mathrm{d}x = \int_{\mathbb{R}^N} \left|\nabla (u_n - M)^+\right|^{p^+} \mathrm{d}x \geq 0,$$

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p^--2} \nabla u_n \nabla (u_n - M)^+ \mathrm{d}x = \int_{\mathbb{R}^N} \left|\nabla (u_n - M)^+\right|^{p^-} \mathrm{d}x \geq 0,$$

and that $u_n \geq 0$, it yields

$$\begin{aligned}
\int_{\mathbb{R}^N} u_n^{p^--1} (u_n - M)^+ \mathrm{d}x &\leq \int_{\mathbb{R}^N} u_n^{p(u_n)-1} (u_n - M)^+ \mathrm{d}x \\
&\leq \int_{\mathbb{R}^N} (g u_n^t + h)(u_n - M)^+ \mathrm{d}x \\
&\leq \frac{t}{p^- - 1} \int_{\mathbb{R}^N} u_n^{p^--1} (u_n - M)^+ \mathrm{d}x \\
&\quad + \left(1 - \frac{t}{p^- - 1}\right) \int_{\mathbb{R}^N} g^{\frac{p^--1}{p^--1-t}} (u_n - M)^+ \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} h (u_n - M)^+ \mathrm{d}x.
\end{aligned}$$

Hence,

$$\left(1-\frac{t}{p^{-}-1}\right)\int_{\mathbb{R}^N}\left(u_n^{p^{-}-1}-M^{p^{-}-1}\right)(u_n-M)^{+}\mathrm{d}x$$
$$\leq\left(1-\frac{t}{p^{-}-1}\right)\int_{\mathbb{R}^N}\left(g^{\frac{p^{-}-1}{p^{-}-1-t}}+\frac{h}{1-\frac{t}{p^{-}-1}}-M^{p^{-}-1}\right)(u_n-M)^{+}\mathrm{d}x. \tag{3.18}$$

Choosing $M>1$ large enough such that

$$|g|_{L^{\infty}(\mathbb{R}^N)}^{\frac{p^{-}-1}{p^{-}-1-t}}+\frac{|h|_{L^{\infty}(\mathbb{R}^N)}}{1-\frac{t}{p^{-}-1}}\leq M^{p^{-}-1},$$

it follows from (3.18) that

$$\int_{\mathbb{R}^N}\left(u_n^{p^{-}-1}-M^{p^{-}-1}\right)(u_n-M)^{+}\mathrm{d}x\leq 0,$$

which implies that $u_n\leq M$. This ends the proof of Lemma 2. □

The completion of the proof of theorem 1.2

Taking $v=u_n$ as test function in (3.17), we get

$$\int_{\mathbb{R}^N}|\nabla u_n|^{p(u_n)}\,\mathrm{d}x+\int_{\mathbb{R}^N}u_n^{p(u_n)}\mathrm{d}x+\frac{1}{n}\|u_n\|_{W^{1,p^{+}}(\mathbb{R}^N)}^{p^{+}}+\frac{1}{n}\|u_n\|_{W^{1,p^{-}}(\mathbb{R}^N)}^{p^{-}}$$
$$=\int_{\mathbb{R}^N}f(x,u_n)u_n\mathrm{d}x+\int_{\mathbb{R}^N}hu_n\mathrm{d}x,\ \forall\, n\geq 1. \tag{3.19}$$

By Young's inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^N}(f(x,u_n)+h)u_n\mathrm{d}x\right|&\leq\int_{\mathbb{R}^N}gu_n^{t+1}\mathrm{d}x+\int_{\mathbb{R}^N}hu_n\mathrm{d}x\\&\leq\int_{u_n\leq 1}gu_n^{t+1}\mathrm{d}x\\&\quad+\int_{u_n\geq 1}gu_n^{t+1}\mathrm{d}x+\int_{u_n\leq 1}hu_n\mathrm{d}x+\int_{u_n\geq 1}hu_n\mathrm{d}x\\&\leq|g|_{L^1(\mathbb{R}^N)}+\frac{1}{4}\int_{u_n\geq 1}u_n^{p^{-}}\mathrm{d}x\\&\quad+c_1\int_{u_n\geq 1}g^{\frac{p^{-}}{p^{-}-t-1}}\mathrm{d}x+|h|_{L^1(\mathbb{R}^N)}\\&\quad+\frac{1}{4}\int_{u_n\geq 1}u_n^{p^{-}}\mathrm{d}x+c_2\int_{u_n\geq 1}h^{\frac{p^{-}}{p^{-}-1}}\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\le |g|_{L^1(\mathbb{R}^N)} + |h|_{L^1(\mathbb{R}^N)} \\
&\quad + \frac{1}{2}\int_{u_n \ge 1} u_n^{p(u_n)} \mathrm{d}x \\
&\quad + c_1 \int_{\mathbb{R}^N} g^{\frac{p^-}{p^- - t - 1}} \mathrm{d}x + c_2 \int_{\mathbb{R}^N} h^{\frac{p^-}{p^- - 1}} \mathrm{d}x \\
&\le c_3 + \frac{1}{2}\int_{\mathbb{R}^N} u_n^{p(u_n)} \mathrm{d}x.
\end{aligned}
$$

Putting that last inequality in (3.19), we obtain

$$
\int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)} \,\mathrm{d}x + \int_{\mathbb{R}^N} u_n^{p(u_n)} \mathrm{d}x + \frac{1}{n} \|u_n\|^{p^+}_{W^{1,p^+}(\mathbb{R}^N)} + \frac{1}{n} \|u_n\|^{p^-}_{W^{1,p^-}(\mathbb{R}^N)} \le c_4, \ \forall\, n \ge 1. \tag{3.20}
$$

By Lemma 2, we get

$$
\int_{\mathbb{R}^N} u_n^{p^+} \mathrm{d}x = \int_{\mathbb{R}^N} u_n^{p(u_n)} u_n^{p^+ - p(u_n)} \mathrm{d}x \le M^{p^+} \int_{\mathbb{R}^N} u_n^{p(u_n)} \mathrm{d}x \le c_4 M^{p+}, \ \forall\, n \ge 1.
$$

Thus, $(u_n)_n$ is bounded in $L^{p^+}(\mathbb{R}^N)$ and by consequence there exists $u \in L^{p^+}(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in $L^{p^+}(\mathbb{R}^N)$. Now, for $k \in \mathbb{N},\ k \ge 1$, set $\Omega_k = \left\{x \in \mathbb{R}^N,\ |x| < k\right\}$. We have

$$
\begin{aligned}
\int_{\Omega_k} |\nabla u_n|^{p^-} \,\mathrm{d}x &= \int_{\Omega_k \cap \{|\nabla u_n| \ge 1\}} |\nabla u_n|^{p^-} \,\mathrm{d}x + \int_{\Omega_k \cap \{|\nabla u_n| < 1\}} |\nabla u_n|^{p^-} \,\mathrm{d}x \\
&\le \int_{\Omega_k} |\nabla u_n|^{p(u_n)} \,\mathrm{d}x + |\Omega_k| \\
&\le c_4 + |\Omega_k|\,, \ \forall\, n \ge 1.
\end{aligned}
$$

It follows that, for every $k > 0$, there exists a subsequence $(u_{\varphi_k(n)})_n$ of $(u_n)_n$ and $v_k \in W^{1,p^-}(\Omega_k)$ such that $u_{\varphi_k(n)} \rightharpoonup v_k$ weakly in $W^{1,p^-}(\Omega_k)$. In particular, $u_{\varphi_k(n)} \to v_k$ in $D'(\Omega_k)$. But we know that $u_n \rightharpoonup u$ weakly in $L^{p^+}(\mathbb{R}^N)$. Thus, we immediately deduce that $u|_{\Omega_k} = v_k$. In particular, $u \in W^{1,p^-}_{loc}(\mathbb{R}^N)$. Now, by standard diagonal argument, we can extract from $(u_n)_n$ a subsequence (independent of k), still denoted by $(u_n)_n$, such that $u_n \rightharpoonup u$ weakly in $W^{1,p^-}(\Omega_k),\ \forall\, k \ge 1$ and $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$. Consequently, $u(x) \ge 0$ a.e. $x \in \mathbb{R}^N$. Now, we claim that

$$
\int_{\mathbb{R}^N} |\nabla u|^{p(u)} \,\mathrm{d}x + \int_{\mathbb{R}^N} u^{p(u)} \mathrm{d}x < +\infty. \tag{3.21}
$$

For that aim, set $q_n(x) = p(u_n(x))$ and $q(x) = p(u(x))$. For $k > 0$, set $w_k = q \min\left\{u^{q-1},\ k^{q-1}\right\}$. By the virtue of Young's inequality, it yields

$$u_n w_k \leq u_n^{q_n} + \frac{q_n - 1}{q_n^{q_n'}} w_k^{q_n'},\ \forall\, k > 0,\ \forall\, n \geq 1,$$

where $q_n' = \frac{q_n}{q_n - 1}$. Let $\zeta \in D(\mathbb{R}^N)$ be such that $0 \leq \zeta \leq 1$. Thus,

$$\int_{\mathbb{R}^N} \zeta u_n w_k \mathrm{d}x \leq \int_{\mathbb{R}^N} \zeta u_n^{q_n} \mathrm{d}x + \int_{\mathbb{R}^N} \frac{q_n - 1}{q_n^{q_n'}} \zeta w_k^{q_n'} \mathrm{d}x,\ \forall\, k > 0,\ \forall\, n \geq 1.$$

Tending n to $+\infty$ (using the Lebesgue's dominated convergence theorem) and having (3.20) in mind, we get

$$\int_{\mathbb{R}^N} \zeta u w_k \mathrm{d}x \leq c_4 + \int_{\mathbb{R}^N} \frac{q-1}{q^{q'}} \zeta w_k^{q'} \mathrm{d}x.$$

Consequently,

$$\begin{aligned} &\int_{u \leq k} q \zeta u^q \mathrm{d}x + \int_{u > k} q k^{q-1} \zeta u \mathrm{d}x \\ &\leq c_4 + \int_{u \leq k} (q-1) \zeta u^q \mathrm{d}x + \int_{u > k} (q-1) k^q \zeta \mathrm{d}x. \end{aligned}$$

We infer,

$$\int_{u \leq k} \zeta u^q \mathrm{d}x + \int_{u > k} k^q \zeta \mathrm{d}x \leq c_4.$$

Passing to the limit as k tends to $+\infty$ in that last inequality, we obtain

$$\int_{\mathbb{R}^N} \zeta u^q \mathrm{d}x \leq c_4.$$

Since ζ is arbitrary in $\left\{v \in D(\mathbb{R}^N),\ 0 \leq v \leq 1\right\}$, we immediately deduce that

$$\int_{\mathbb{R}^N} u^q \mathrm{d}x \leq c_4.$$

In order to prove that

$$\int_{\mathbb{R}^N} |\nabla u|^q \, \mathrm{d}x \leq c_4,$$

one can proceed exactly as previously by considering the vector

$$W_k = \begin{cases} q \nabla u \, |\nabla u|^{q-2},\ \text{if } |\nabla u| \leq k, \\ q k^{q-1} \frac{\nabla u}{|\nabla u|},\ \text{if } |\nabla u| > k. \end{cases}$$

Hence, the claim (3.21) holds. In particular, we find again that $u \in W^{1,p^-}_{loc}(\mathbb{R}^N)$.

Let $v \in X$ and $\phi \in X$ be such that $\phi \geq 0$ and $\operatorname{supp}(\phi)$ is compact. Taking $(u_n - v)\phi$ as test function in (3.17), we infer

$$\begin{aligned}
&\int_{\mathbb{R}^N} f(x,u_n)(u_n-v)\phi \mathrm{d}x + \int_{\mathbb{R}^N} h(u_n-v)\phi \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \phi\, |\nabla u_n|^{p(u_n)-2} \nabla u_n \nabla(u_n-v)\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)-2} \nabla u_n \nabla \phi (u_n-v)\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} u_n^{p(u_n)-1}(u_n-v)\phi \mathrm{d}x + \frac{1}{n}\int_{\mathbb{R}^N} \phi\, |\nabla u_n|^{p^+-2} \nabla u_n \nabla(u_n-v)\mathrm{d}x \\
&\quad + \frac{1}{n}\int_{\mathbb{R}^N} |\nabla u_n|^{p^+-2} \nabla u_n \nabla \phi(u_n-v)\mathrm{d}x + \frac{1}{n}\int_{\mathbb{R}^N} u_n^{p^+-1}(u_n-v)\phi \mathrm{d}x \\
&\quad + \frac{1}{n}\int_{\mathbb{R}^N} \phi\, |\nabla u_n|^{p^--2} \nabla u_n \nabla(u_n-v)\mathrm{d}x + \frac{1}{n}\int_{\mathbb{R}^N} |\nabla u_n|^{p^--2} \nabla u_n \nabla \phi(u_n-v)\mathrm{d}x \\
&\quad + \frac{1}{n}\int_{\mathbb{R}^N} u_n^{p^--1}(u_n-v)\phi \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \phi \left(|\nabla u_n|^{p(u_n)-2} \nabla u_n - |\nabla v|^{p(u_n)-2} \nabla v \right) \nabla(u_n-v)\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \phi\, |\nabla v|^{p(u_n)-2} \nabla v \nabla(u_n-v)\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)-2} \nabla u_n \nabla \phi(u_n-v)\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \left(u_n^{p(u_n)-1} - |v|^{p(u_n)-2} v \right) \phi(u_n-v)\mathrm{d}x + \int_{\mathbb{R}^N} |v|^{p(u_n)-2} v\phi(u_n-v)\mathrm{d}x \\
&\quad + \frac{1}{n}\int_{\mathbb{R}^N} \phi \left(|\nabla u_n|^{p^+-2} \nabla u_n - |\nabla v|^{p^+-2} \nabla v \right) \nabla(u_n-v)\mathrm{d}x \\
&\quad + \frac{1}{n}\int_{\mathbb{R}^N} \phi\, |\nabla v|^{p^+-2} \nabla v \nabla(u_n-v)\mathrm{d}x + \frac{1}{n}\int_{\mathbb{R}^N} |\nabla u_n|^{p^+-2} \nabla u_n \nabla\phi(u_n-v)\mathrm{d}x \\
&\quad + \frac{1}{n}\int_{\mathbb{R}^N} \left(u_n^{p^+-1} - |v|^{p^+-2} v \right) \phi(u_n-v)\mathrm{d}x + \frac{1}{n}\int_{\mathbb{R}^N} |v|^{p^+-2} v\phi(u_n-v)\mathrm{d}x \\
&\quad + \frac{1}{n}\int_{\mathbb{R}^N} \phi \left(|\nabla u_n|^{p^--2} \nabla u_n - |\nabla v|^{p^--2} \nabla v \right) \nabla(u_n-v)\mathrm{d}x \\
&\quad + \frac{1}{n}\int_{\mathbb{R}^N} \phi\, |\nabla v|^{p^--2} \nabla v \nabla(u_n-v)\mathrm{d}x + \frac{1}{n}\int_{\mathbb{R}^N} |\nabla u_n|^{p^--2} \nabla u_n \nabla\phi(u_n-v)\mathrm{d}x \\
&\quad + \frac{1}{n}\int_{\mathbb{R}^N} \left(u_n^{p^--1} - |v|^{p^--2} v \right) \phi(u_n-v)\mathrm{d}x + \frac{1}{n}\int_{\mathbb{R}^N} |v|^{p^--2} v\phi(u_n-v)\mathrm{d}x.
\end{aligned} \tag{3.22}$$

Forgetting the nonnegative terms in the right-hand side of the identity (3.22), we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} f(x,u_n)(u_n-v)\phi \mathrm{d}x + \int_{\mathbb{R}^N} h(u_n-v)\phi \mathrm{d}x \\
&\geq \int_{\mathbb{R}^N} \phi\, |\nabla v|^{p(u_n)-2} \nabla v \nabla(u_n-v)\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)-2} \nabla u_n \nabla \phi(u_n-v)\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N} |v|^{p(u_n)-2}\, v\phi(u_n-v)\mathrm{d}x+\frac{1}{n}\int_{\mathbb{R}^N} \phi\, |\nabla v|^{p^+-2}\, \nabla v\nabla(u_n-v)\mathrm{d}x\\
&+\frac{1}{n}\int_{\mathbb{R}^N} |\nabla u_n|^{p^+-2}\, \nabla u_n\nabla\phi(u_n-v)\mathrm{d}x+\frac{1}{n}\int_{\mathbb{R}^N} |v|^{p^+-2}\, v\phi(u_n-v)\mathrm{d}x\\
&+\frac{1}{n}\int_{\mathbb{R}^N} \phi\, |\nabla v|^{p^--2}\, \nabla v\nabla(u_n-v)\mathrm{d}x+\frac{1}{n}\int_{\mathbb{R}^N} |\nabla u_n|^{p^--2}\, \nabla u_n\nabla\phi(u_n-v)\mathrm{d}x\\
&+\frac{1}{n}\int_{\mathbb{R}^N} |v|^{p^--2}\, v\phi(u_n-v)\mathrm{d}x.
\end{aligned}
\tag{3.23}
$$

We have

$$
\begin{aligned}
&\left|\frac{1}{n}\int_{\mathbb{R}^N} \phi\, |\nabla v|^{p^+-2}\, \nabla v\nabla(u_n-v)\mathrm{d}x\right|\\
&\leq \frac{|\phi|_\infty}{n}\int_{\mathbb{R}^N} |\nabla v|^{p^+-1}\, |\nabla(u_n-v)|\, \mathrm{d}x\\
&\leq \frac{|\phi|_\infty}{n}\left(\int_{\mathbb{R}^N} |\nabla v|^{p^+}\, \mathrm{d}x\right)^{\frac{p^+-1}{p^+}}\left(\int_{\mathbb{R}^N} |\nabla(u_n-v)|^{p^+}\, \mathrm{d}x\right)^{\frac{1}{p^+}}\\
&= |\phi|_\infty\left(\frac{1}{n}\right)^{\frac{p^+-1}{p^+}}\left(\frac{1}{n}\right)^{\frac{1}{p^+}}\left(\int_{\mathbb{R}^N} |\nabla v|^{p^+}\, \mathrm{d}x\right)^{\frac{p^+-1}{p^+}}\left(\int_{\mathbb{R}^N} |\nabla(u_n-v)|^{p^+}\, \mathrm{d}x\right)^{\frac{1}{p^+}}\\
&\leq |\phi|_\infty\left(\frac{1}{n}\right)^{\frac{p^+-1}{p^+}}\left(\frac{1}{n}\, \|u_n-v\|^{p^+}_{W^{1,p^+}(\mathbb{R}^N)}\right)^{\frac{1}{p^+}}\|v\|^{p^+-1}_{W^{1,p^+}(\mathbb{R}^N)}.
\end{aligned}
\tag{3.24}
$$

By (3.20), we know that

$$
\sup_{n\geq 1}\left(\frac{1}{n}\, \|u_n\|^{p^+}_{W^{1,p^+}(\mathbb{R}^N)}\right)<+\infty.
$$

Then, from (3.24), we obtain

$$
\frac{1}{n}\int_{\mathbb{R}^N} \phi\, |\nabla v|^{p^+-2}\, \nabla v\nabla(u_n-v)\mathrm{d}x\to 0,\ n\to+\infty. \tag{3.25}
$$

Similarly,

$$
\frac{1}{n}\int_{\mathbb{R}^N} \phi\, |v|^{p^+-2}\, v(u_n-v)\mathrm{d}x\to 0,\ n\to+\infty, \tag{3.26}
$$

$$
\frac{1}{n}\int_{\mathbb{R}^N} \phi\, |\nabla v|^{p^--2}\, \nabla v\nabla(u_n-v)\mathrm{d}x\to 0,\ n\to+\infty, \tag{3.27}
$$

and,

$$
\frac{1}{n}\int_{\mathbb{R}^N} \phi\, |v|^{p^--2}\, v(u_n-v)\mathrm{d}x\to 0,\ n\to+\infty. \tag{3.28}
$$

Now, note that $f(x, u_n(x))(u_n(x) - v(x))\phi(x) \to f(x, u(x))(u(x) - v(x))\phi(x)$, a.e. $x \in \mathbb{R}^N$. Using the boundedness of $(u_n)_n$ in $L^\infty(\mathbb{R}^N)$ and taking into account that $g \in L^1(\mathbb{R}^N)$, one can easily apply the Lebesgue's dominated convergence theorem to immediately deduce that

$$\int_{\mathbb{R}^N} f(x, u_n)(u_n - v)\phi \mathrm{d}x \to \int_{\mathbb{R}^N} f(x, u)(u - v)\phi \mathrm{d}x, \ n \to +\infty. \tag{3.29}$$

Similarly,

$$\int_{\mathbb{R}^N} h(u_n - v)\phi \mathrm{d}x \to \int_{\mathbb{R}^N} h(u - v)\phi \mathrm{d}x, \ n \to +\infty. \tag{3.30}$$

In view of (3.30), (3.29), (3.28), (3.27), (3.26) and (3.25), one can pass to the limit in (3.23) as n tends to $+\infty$, and finally obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} f(x, u)(u - v)\phi \mathrm{d}x + \int_{\mathbb{R}^N} h(u - v)\phi \mathrm{d}x \\ &\geq \lim_{n\to+\infty} \int_{\mathbb{R}^N} \phi \, |\nabla v|^{p(u_n)-2} \, \nabla v \nabla(u_n - v)\mathrm{d}x \\ &\quad + \lim_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)-2} \, \nabla u_n \nabla\phi (u_n - v)\mathrm{d}x \\ &\quad + \lim_{n\to+\infty} \int_{\mathbb{R}^N} \phi \, |v|^{p(u_n)-2} \, v(u_n - v)\mathrm{d}x. \end{aligned} \tag{3.31}$$

Let $k_0 > 0$ be such that $\mathrm{supp}(\phi) \subset \Omega_{k_0} = \left\{x \in \mathbb{R}^N, \ |x| < k_0\right\}$. Assume that $v \in X \cap W^{1,s}(\Omega_{k_0})$ where $s = \frac{p^+-1}{p^--1} p^- > p^+$. We have

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} \phi \left(|\nabla v|^{p(u_n)-2} \, \nabla v - |\nabla v|^{p(u)-2} \, \nabla v \right) \nabla(u_n - v)\mathrm{d}x \right| \\ &\leq \left(\int_{\mathbb{R}^N} \phi \left| \, |\nabla v|^{p(u_n)-2} \, \nabla v - |\nabla v|^{p(u)-2} \, \nabla v \right|^{\frac{p^-}{p^--1}} \mathrm{d}x \right)^{\frac{p^--1}{p^-}} \\ &\quad \times \left(\int_{\mathbb{R}^N} \phi \, |\nabla(u_n - v)|^{p^-} \, \mathrm{d}x \right)^{\frac{1}{p^-}}. \end{aligned} \tag{3.32}$$

Observe that

$$\begin{aligned} &\phi \left| \, |\nabla v|^{p(u_n)-2} \, \nabla v - |\nabla v|^{p(u)-2} \, \nabla v \right|^{\frac{p^-}{p^--1}} \\ &\leq \phi 2^{\frac{p^-}{p^--1}} \, |\nabla v|^{\frac{p^-(p^+-1)}{p^--1}} \, \mathbb{1}_{\{|\nabla v| \geq 1\}} + \phi 2^{\frac{p^-}{p^--1}} \mathbb{1}_{\{|\nabla v| \leq 1\}} \\ &\leq \phi 2^{\frac{p^-}{p^--1}} \left(1 + |\nabla v|^{\frac{p^-(p^+-1)}{p^--1}} \right). \end{aligned}$$

Taking into account that, for a.e. $x \in \mathbb{R}^N$, $p(u_n(x)) \to p(u(x))$ as $n \to +\infty$, then we can apply the Lebesgue dominated convergence theorem to get

$$\int_{\mathbb{R}^N} \phi \left| |\nabla v|^{p(u_n)-2} \nabla v - |\nabla v|^{p(u)-2} \nabla v \right|^{\frac{p^-}{p^--1}} \mathrm{d}x \to 0, \ n \to +\infty.$$

Having in mind that the sequence $(u_n)_n$ is bounded in $W^{1,p^-}(\Omega_k)$, $\forall\, k \geq 1$, we infer

$$\sup_{n \geq 1} \int_{\mathbb{R}^N} \phi\, |\nabla (u_n - v)|^{p^-} \mathrm{d}x < +\infty.$$

By (3.32), it follows

$$\int_{\mathbb{R}^N} \phi \left(|\nabla v|^{p(u_n)-2} \nabla v - |\nabla v|^{p(u)-2} \nabla v \right) \nabla (u_n - v) \mathrm{d}x \to 0, \\ n \to +\infty. \tag{3.33}$$

In a similar way, we get

$$\int_{\mathbb{R}^N} \phi \left(|v|^{p(u_n)-2} v - |v|^{p(u)-2} v \right) (u_n - v) \mathrm{d}x \to 0, \ n \to +\infty. \tag{3.34}$$

In view of (3.34) and (3.33), from (3.31), it comes

$$\begin{aligned} &\int_{\mathbb{R}^N} f(x,u)(u-v)\phi \mathrm{d}x + \int_{\mathbb{R}^N} h(u-v)\phi \mathrm{d}x \\ &\geq \lim_{n \to +\infty} \int_{\mathbb{R}^N} \phi\, |\nabla v|^{p(u)-2} \nabla v \nabla (u_n - v) \mathrm{d}x \\ &\quad + \lim_{n \to +\infty} \int_{\mathbb{R}^N} \phi\, |v|^{p(u)-2} v (u_n - v) \mathrm{d}x \\ &\quad + \lim_{n \to +\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)-2} \nabla u_n \nabla \phi (u_n - v) \mathrm{d}x. \end{aligned} \tag{3.35}$$

It is easy to see that the linear mapping

$$\xi \longmapsto \int_{\mathbb{R}^N} \phi\, |\nabla v|^{p(u)-2} \nabla v \nabla \xi \mathrm{d}x,$$

is in the topological dual of $W^{1,p^-}(\Omega_{k_0})$. Indeed, for $v \in X \cap W^{1,s}(\Omega_{k_0})$ and $\xi \in W^{1,p^-}(\Omega_{k_0})$, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \phi\, |\nabla v|^{p(u)-2} \nabla v \nabla \xi \mathrm{d}x \right| &\leq \int_{\Omega_{k_0}} \phi\, |\nabla v|^{p(u)-1} |\nabla \xi|\, \mathrm{d}x \\ &\leq \left(\int_{\Omega_{k_0}} |\nabla \xi|^{p^-} \mathrm{d}x \right)^{\frac{1}{p^-}} \left(\int_{\Omega_{k_0}} \phi\, |\nabla v|^{\frac{(p(u)-1)p^-}{p^--1}} \mathrm{d}x \right)^{\frac{p^--1}{p^-}} \end{aligned}$$

$$\leq \left(\int_{\Omega_{k_0}} |\nabla \xi|^{p^-} \, \mathrm{d}x \right)^{\frac{1}{p^-}} \left(\int_{\Omega_{k_0}} \phi \left(1 + |\nabla v|^{\frac{(p^+-1)p^-}{p^--1}} \right) \mathrm{d}x \right)^{\frac{p^--1}{p^-}}.$$

Since $(u_n)_n$ is weakly convergent to u in $W^{1,p^-}(\Omega_{k_0})$, then

$$\begin{aligned}
&\lim_{n \to +\infty} \int_{\mathbb{R}^N} \phi \, |\nabla v|^{p(u)-2} \, \nabla v \nabla (u_n - v) \mathrm{d}x \\
&= \lim_{n \to +\infty} \int_{\Omega_{k_0}} \phi \, |\nabla v|^{p(u)-2} \, \nabla v \nabla (u_n - v) \mathrm{d}x \\
&= \int_{\Omega_{k_0}} \phi \, |\nabla v|^{p(u)-2} \, \nabla v \nabla (u - v) \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \phi \, |\nabla v|^{p(u)-2} \, \nabla v \nabla (u - v) \mathrm{d}x. \qquad (3.36)
\end{aligned}$$

Similarly,

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} \phi \, |v|^{p(u)-2} \, v (u_n - v) \mathrm{d}x = \int_{\mathbb{R}^N} \phi \, |v|^{p(u)-2} \, v (u - v) \mathrm{d}x. \qquad (3.37)$$

Inserting (3.37) and (3.36) in (3.35), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} (f(x,u) + h)(u - v) \phi \mathrm{d}x \geq & \int_{\mathbb{R}^N} \phi \, |\nabla v|^{p(u)-2} \, \nabla v \nabla (u - v) \mathrm{d}x \\
& + \int_{\mathbb{R}^N} \phi \, |v|^{p(u)-2} \, v (u - v) \mathrm{d}x \\
& + \lim_{n \to +\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)-2} \, \nabla u_n \nabla \phi (u_n - v) \mathrm{d}x.
\end{aligned} \qquad (3.38)$$

In particular,

$$\begin{aligned}
&\int_{\mathbb{R}^N} (f(x,u) + h)(u - v) \phi \mathrm{d}x \\
&\geq \int_{\mathbb{R}^N} \phi \, |\nabla v|^{p(u)-2} \, \nabla v \nabla (u - v) \mathrm{d}x + \int_{\mathbb{R}^N} \phi \, |v|^{p(u)-2} \, v (u - v) \mathrm{d}x \\
&\quad + \lim_{n \to +\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)-2} \, \nabla u_n \nabla \phi (u_n - v) \mathrm{d}x, \ \forall \, v \in D(\mathbb{R}^N).
\end{aligned} \qquad (3.39)$$

Next, we claim that the inequality (3.39) can be extended to $W^{1,p(u)}(\Omega_{k_0})$ in the sense that

$$\int_{\mathbb{R}^N} (f(x,u) + h)(u - v) \phi \mathrm{d}x$$

$$\geq \int_{\mathbb{R}^N} \phi\, |\nabla v|^{p(u)-2}\, \nabla v \nabla (u-v) \mathrm{d}x + \int_{\mathbb{R}^N} \phi\, |v|^{p(u)-2}\, v(u-v) \mathrm{d}x$$
$$+ \lim_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)-2}\, \nabla u_n \nabla \phi (u_n - v) \mathrm{d}x,\ \forall\, v \in W^{1,p(u)}(\Omega_{k_0}). \tag{3.40}$$

To see that, let $v \in W^{1,p(u)}(\Omega_{k_0})$. By Proposition 1, there exists a sequence $(v_j)_j \subset D(\mathbb{R}^N)$ such that $v_j|_{\Omega_{k_0}} \to v$ strongly in $W^{1,p(u)}(\Omega_{k_0})$. Clearly, up to a subsequence, $v_j(x) \to v(x)$ a.e. $x \in \mathbb{R}^N$ and $(v_j)_j$ is bounded in $L^\infty(\Omega_{k_0})$. By (2.1), we have

$$\left| \int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)-2}\, \nabla u_n \nabla \phi (v_j - v) \mathrm{d}x \right|$$
$$\leq 2 \left| \, |\nabla u_n|^{p(u_n)-1} \right|_{L^{\frac{p(u_n)}{p(u_n)-1}}(\mathbb{R}^N)} \left| \, |\nabla \phi|\, (v_j - v) \right|_{L^{p(u_n)}(\mathbb{R}^N)}. \tag{3.41}$$

We have,

$$\int_{\mathbb{R}^N} |\nabla \phi|^{p(u_n)} \left| v_j - v \right|^{p(u_n)} \mathrm{d}x = \int_{\Omega_{k_0}} |\nabla \phi|^{p(u_n)} \left| v_j - v \right|^{p(u_n)} \mathrm{d}x$$
$$\leq \int_{\Omega_{k_0}} \left(1 + |\nabla \phi|^{p^+}\right) \left(\left| v_j - v \right|^{p^+} + \left| v_j - v \right|^{p^-} \right) \mathrm{d}x.$$

By (2.2), it yields

$$\lim_{j\to+\infty} \sup_{n\geq 1} \left| \, |\nabla \phi|\, (v_j - v) \right|_{L^{p(u_n)}(\mathbb{R}^N)} = 0. \tag{3.42}$$

Since

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)} \mathrm{d}x \leq c_4,\ \forall\, n \geq 1,$$

then by (2.2)

$$\sup_{n\geq 1} \left| \, |\nabla u_n|^{p(u_n)-1} \right|_{L^{\frac{p(u_n)}{p(u_n)-1}}(\mathbb{R}^N)} < +\infty. \tag{3.43}$$

We deduce from (3.41), (3.42) and (3.43) that

$$\lim_{j\to+\infty} \lim_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)-2}\, \nabla u_n \nabla \phi (v_j - v) \mathrm{d}x = 0,$$

which implies that

$$\lim_{j\to+\infty} \lim_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)-2}\, \nabla u_n \nabla \phi (u_n - v_j) \mathrm{d}x$$
$$= \lim_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)-2}\, \nabla u_n \nabla \phi (u_n - v) \mathrm{d}x.$$

The extensions of the other terms in (3.39) are immediate.

For $s>0$ and $w\in W^{1,p(u)}(\mathbb{R}^N)\subset W^{1,p(u)}(\Omega_{k_0})$, choosing $v=u-sw$ as test function in (3.40), it yields

$$\begin{aligned} s\int_{\mathbb{R}^N}(f(x,u)+h)\phi w\mathrm{d}x \geq\; & s\int_{\mathbb{R}^N}\phi\left|\nabla u-s\nabla w\right|^{p(u)-2}(\nabla u-s\nabla w)\nabla w\mathrm{d}x\\ &+s\int_{\mathbb{R}^N}\phi\left|u-sw\right|^{p(u)-2}(u-sw)w\mathrm{d}x\\ &+\lim_{n\to+\infty}\int_{\mathbb{R}^N}\left|\nabla u_n\right|^{p(u_n)-2}\nabla u_n\nabla\phi(u_n-u)\mathrm{d}x\\ &+s\lim_{n\to+\infty}\int_{\mathbb{R}^N}\left|\nabla u_n\right|^{p(u_n)-2}\nabla u_n\nabla\phi w\mathrm{d}x. \end{aligned}\tag{3.44}$$

By (2.1), we have

$$\begin{aligned} &\left|\int_{\mathbb{R}^N}\left|\nabla u_n\right|^{p(u_n)-2}\nabla u_n\nabla\phi(u_n-u)\mathrm{d}x\right|\\ &\leq\int_{\mathbb{R}^N}\left|\nabla u_n\right|^{p(u_n)-1}\left|(u_n-u)\nabla\phi\right|\mathrm{d}x\\ &\leq 2\left|\,\left|\nabla u_n\right|^{p(u_n)-1}\right|_{L^{\frac{p(u_n)}{p(u_n)-1}}(\mathbb{R}^N)}\left|\,\left|(u_n-u)\nabla\phi\right|\,\right|_{L^{p(u_n)}(\mathbb{R}^N)}. \end{aligned}\tag{3.45}$$

On the other hand, by the Lebesgue dominated convergence theorem, we can easily see that

$$\int_{\mathbb{R}^N}\left|(u_n-u)\nabla\phi\right|^{p(u_n)}\mathrm{d}x\to 0,\ n\to+\infty.$$

Hence, from (3.45) we infer

$$\int_{\mathbb{R}^N}\left|\nabla u_n\right|^{p(u_n)-2}\nabla u_n\nabla\phi(u_n-u)\mathrm{d}x\to 0,\ n\to+\infty.\tag{3.46}$$

Taking (3.46) into account, dividing by $s>0$ and tending s to 0^+ in (3.44), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N}(f(x,u)+h)\phi w\mathrm{d}x \geq & \int_{\mathbb{R}^N}\phi\left|\nabla u\right|^{p(u)-2}\nabla u\nabla w\mathrm{d}x+\int_{\mathbb{R}^N}\phi u^{p(u)-1}w\mathrm{d}x\\ &+\lim_{n\to+\infty}\int_{\mathbb{R}^N}\left|\nabla u_n\right|^{p(u_n)-2}\nabla u_n\nabla\phi w\mathrm{d}x. \end{aligned}$$

Clearly, that last inequality holds also with $(-w)$ instead of w. Therefore,

$$\begin{aligned} \int_{\mathbb{R}^N}(f(x,u)+h)\phi w\mathrm{d}x = & \int_{\mathbb{R}^N}\phi\left|\nabla u\right|^{p(u)-2}\nabla u\nabla w\mathrm{d}x+\int_{\mathbb{R}^N}\phi u^{p(u)-1}w\mathrm{d}x\\ &+\lim_{n\to+\infty}\int_{\mathbb{R}^N}\left|\nabla u_n\right|^{p(u_n)-2}\nabla u_n\nabla\phi w\mathrm{d}x. \end{aligned}\tag{3.47}$$

At this step, we established the inequality (3.47) for all $\phi \in X$ such that $\phi \geq 0$ and $\text{supp}(\phi)$ is compact. But, it is obvious that the same identity holds also for all $\phi \in X$ such that $\text{supp}(\phi)$ is compact. In particular, it holds for all $\phi \in D(\mathbb{R}^N)$. Let $\eta \in D(\mathbb{R}^N)$ be a cut-off function such that $0 \leq \eta \leq 1,\ \eta(x) = 0$, if $|x| \geq 2,\ \eta(x) = 1$, if $|x| \leq 1$. For an integer $m \geq 1$ and $x \in \mathbb{R}^N$, set $\eta_m(x) = \eta\left(\frac{x}{m}\right)$. Plainly, there exists a positive constant c_5 such that

$$|\nabla \eta_m(x)| = \frac{1}{m}\left|\nabla \eta\left(\frac{x}{m}\right)\right| \leq \frac{c_5}{m},\ \forall\, m \geq 1,\ \forall\, x \in \mathbb{R}^N.$$

Taking $\phi = \eta_m$ as test function in (3.47), it yields

$$\int_{\mathbb{R}^N} (f(x,u) + h)\eta_m w \mathrm{d}x = \int_{\mathbb{R}^N} \eta_m |\nabla u|^{p(u)-2} \nabla u \nabla w \mathrm{d}x + \int_{\mathbb{R}^N} \eta_m u^{p(u)-1} w \mathrm{d}x \\ + \lim_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)-2} \nabla u_n \nabla \eta_m w \mathrm{d}x. \tag{3.48}$$

We have

$$\left|\int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)-2} \nabla u_n \nabla \eta_m w \mathrm{d}x\right| \\ \leq 2\left|\, |\nabla u_n|^{p(u_n)-1}\right|_{L^{\frac{p(u_n)}{p(u_n)-1}}(\mathbb{R}^N)} \left|\, |\nabla \eta_m| \cdot |w|\, \right|_{L^{p(u_n)}(\mathbb{R}^N)}. \tag{3.49}$$

By (3.43), we know that the sequence $\left(\left|\, |\nabla u_n|^{p(u_n)-1}\right|_{L^{\frac{p(u_n)}{p(u_n)-1}}(\mathbb{R}^N)}\right)_n$ is bounded. On the other hand, by (2.2) we have

$$\left|\, |\nabla \eta_m| \cdot |w|\, \right|_{L^{p(u_n)}(\mathbb{R}^N)} \leq \left(\int_{\mathbb{R}^N} |\nabla \eta_m|^{p(u_n)} |w|^{p(u_n)} \mathrm{d}x\right)^{\frac{1}{p^+}} \\ + \left(\int_{\mathbb{R}^N} |\nabla \eta_m|^{p(u_n)} |w|^{p(u_n)} \mathrm{d}x\right)^{\frac{1}{p^-}}. \tag{3.50}$$

For m large enough, it yields

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla \eta_m|^{p(u_n)} |w|^{p(u_n)} \mathrm{d}x &= \int_{m\leq|x|\leq 2m} |\nabla \eta_m|^{p(u_n)} |w|^{p(u_n)} \mathrm{d}x \\ &\leq \left(\frac{c_5}{m}\right)^{p^-} \int_{m\leq|x|\leq 2m} |w|_\infty^{p(u_n)} \mathrm{d}x \\ &\leq \frac{c_6}{m^{p^-}} \left|\left\{x \in \mathbb{R}^N,\ m \leq |x| \leq 2m\right\}\right| \\ &= \frac{c_7 m^N}{m^{p^-}},\ \forall\, n \geq 1. \end{aligned} \tag{3.51}$$

Combining (3.51) with (3.50), from (3.49) we get

$$\left|\lim_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p(u_n)-2} \nabla u_n \nabla \eta_m w \mathrm{d}x\right| \leq c_8 m^{\frac{N-p^-}{p^+}},$$

which, since $p^- > N$, implies

$$\lim_{m\to+\infty}\left(\lim_{n\to+\infty}\int_{\mathbb{R}^N}|\nabla u_n|^{p(u_n)-2}\nabla u_n\nabla\eta_m w\mathrm{d}x\right)=0. \tag{3.52}$$

Since $w \in W^{1,p(u)}(\mathbb{R}^N)$, then the functions $(f(x,u)+h)w$, $|\nabla u|^{p(u)-2}\nabla u\nabla w$ and $u^{p(u)-1}w$ belong to $L^1(\mathbb{R}^N)$. By consequence, one can apply the Lebesgue dominated convergence theorem to obtain that

$$\lim_{m\to+\infty}\int_{\mathbb{R}^N}(f(x,u)+h)\eta_m w\mathrm{d}x=\int_{\mathbb{R}^N}(f(x,u)+h)w\mathrm{d}x, \tag{3.53}$$

$$\lim_{m\to+\infty}\int_{\mathbb{R}^N}\eta_m|\nabla u|^{p(u)-2}\nabla u\nabla w\mathrm{d}x=\int_{\mathbb{R}^N}|\nabla u|^{p(u)-2}\nabla u\nabla w\mathrm{d}x, \tag{3.54}$$

and

$$\lim_{m\to+\infty}\int_{\mathbb{R}^N}\eta_m u^{p(u)-1}w\mathrm{d}x=\int_{\mathbb{R}^N}u^{p(u)-1}w\mathrm{d}x. \tag{3.55}$$

In view of (3.52),(3.53),(3.54) and (3.55), from (3.48) we conclude that

$$\int_{\mathbb{R}^N}(f(x,u)+h)w\mathrm{d}x=\int_{\mathbb{R}^N}|\nabla u|^{p(u)-2}\nabla u\nabla w\mathrm{d}x$$
$$+\int_{\mathbb{R}^N}u^{p(u)-1}w\mathrm{d}x,\ \forall\, w\in W^{1,p(u)}(\mathbb{R}^N).$$

Since $h\neq 0$, then $u\neq 0$. This ends the proof of Theorem 1.2.

4 Proof of Theorem 1.4

Using the same arguments as in the first part of the proof of Theorem 1.2, we can easily show that, for each $n\geq 1$, there exists $u_n\in X=W^{1,p^+}(\mathbb{R}^N)\cap W^{1,p^-}(\mathbb{R}^N)$ such that $u_n\geq 0$ and

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla u_n|^{p(\alpha(u_n))-2}\nabla u_n\nabla v\mathrm{d}x+\int_{\mathbb{R}^N}|u_n|^{p(\alpha(u_n))-2}u_n v\mathrm{d}x\\&\quad+\frac{1}{n}\left(\int_{\mathbb{R}^N}|\nabla u_n|^{p^+-2}\nabla u_n\nabla v\mathrm{d}x+\int_{\mathbb{R}^N}|u_n|^{p^+-2}u_n v\mathrm{d}x\right)\\&\quad+\frac{1}{n}\left(\int_{\mathbb{R}^N}|\nabla u_n|^{p^--2}\nabla u_n\nabla v\mathrm{d}x+\int_{\mathbb{R}^N}|u_n|^{p^--2}u_n v\mathrm{d}x\right)\\&=\int_{\mathbb{R}^N}f(x,u_n)v\mathrm{d}x+\int_{\mathbb{R}^N}hv\mathrm{d}x,\ \forall\, v\in X.\end{aligned} \tag{4.1}$$

Moreover, we have $u_n \in L^\infty(\mathbb{R}^N)$ and the sequence $(u_n)_n$ is bounded in $L^\infty(\mathbb{R}^N)$. Furthermore, there exists a positive constant $c_9 > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p(\alpha(u_n))}\,\mathrm{d}x + \int_{\mathbb{R}^N} u_n^{p(\alpha(u_n))}\mathrm{d}x + \frac{1}{n}\left(\|u_n\|^{p^+}_{W^{1,p^+}(\mathbb{R}^N)} + \|u_n\|^{p^-}_{W^{1,p^-}(\mathbb{R}^N)}\right) \le c_9,\ \forall\, n \ge 1.$$

Proceeding as for the local case treated in Theorem 1.2, we can also prove that there exists $u \in L^{p^+}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap W^{1,p^-}_{loc}(\mathbb{R}^N)$ such that, up to subsequence, $u_n \rightharpoonup u$ weakly in $L^{p^+}(\mathbb{R}^N)$, $u_n \rightharpoonup u$ weakly in $W^{1,p^-}(\Omega_k)$, $\forall\, k \ge 1$, and $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$. Since $p^+ < +\infty$, then the sequence $(p(\alpha(u_n)))_n$ is bounded in $\mathbb{R}$. By the Bolzano-Weierstrass theorem, there is $p_0 \in \mathbb{R}$ such that, up to a subsequence, $p(\alpha(u_n)) \to p_0$ strongly in $\mathbb{R}$. Arguing as for the claim (3.21), we can prove that

$$\int_{\mathbb{R}^N} |\nabla u|^{p_0}\,\mathrm{d}x + \int_{\mathbb{R}^N} u^{p_0}\mathrm{d}x < +\infty,\ \text{ i.e. } u \in W^{1,p_0}(\mathbb{R}^N).$$

Finally, proceeding exactly as at the end of the proof of Theorem 1.2 (i.e., arguing by approximation with the classical Sobolev space $W^{1,p_0}(\mathbb{R}^N)$ playing the role of the Sobolev space of variable exponent $W^{1,p(u)}(\mathbb{R}^N)$), we can see that

$$\int_{\mathbb{R}^N} |\nabla u|^{p_0-2}\,\nabla u \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} |u|^{p_0-2}\,uv\mathrm{d}x = \int_{\mathbb{R}^N} (f(x,u)+h)v\mathrm{d}x,\ \forall\, v \in W^{1,p_0}(\mathbb{R}^N). \tag{4.2}$$

In order to conclude the proof of Theorem 1.4, it remains to prove that $p_0 = p(\alpha(u))$. For $n \ge 1$, set $p_n = p(\alpha(u_n))$. Without loss of generality, we can split the set $\{p_n,\ n \ge 1\}$ into $\left\{p_{\xi(n)},\ n \ge 1\right\} \cup \left\{p_{\psi(n)},\ n \ge 1\right\}$, where $(p_{\xi(n)})_n$ and $(p_{\psi(n)})_n$ are two subsequences of $(p_n)_n$ such that

$$p_{\xi(n)} \ge p_0,\ \text{ and } p_{\psi(n)} < p_0,\ \forall\, n \ge 1.$$

We claim that, up to a subsequence, $(u_{\xi(n)})_n$ and $(u_{\psi(n)})_n$ are both converging to u in $W^{1,p^-}_{loc}(\mathbb{R}^N)$. Let $\phi \in X$ be such that $\phi \ge 0$ and supp(ϕ) is compact. First, observe that, as for the identity (3.47), we can easily see that, for all $w \in X$, we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p_n-2}\,\nabla u_n \nabla \phi w \mathrm{d}x = \int_{\mathbb{R}^N} (f(x,u)+h)\phi w \mathrm{d}x - \int_{\mathbb{R}^N} |\nabla u|^{p_0-2}\,\nabla u \nabla w \phi \mathrm{d}x - \int_{\mathbb{R}^N} u^{p_0-1}\phi w \mathrm{d}x. \tag{4.3}$$

Taking $v = \phi u$ as test function in (4.2), it yields

$$\int_{\mathbb{R}^N} (f(x, u+h)u\phi \mathrm{d}x = \int_{\mathbb{R}^N} \phi \left|\nabla u\right|^{p_0} \mathrm{d}x + \int_{\mathbb{R}^N} \phi u^{p_0} \mathrm{d}x \\ + \int_{\mathbb{R}^N} \left|\nabla u\right|^{p_0-2} \nabla u \nabla \phi u \mathrm{d}x. \tag{4.4}$$

Combining (4.3) (where we take $w = u$) and (4.4), we get

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \left|\nabla u_n\right|^{p_n-2} \nabla u_n \nabla \phi u \mathrm{d}x = \int_{\mathbb{R}^N} \left|\nabla u\right|^{p_0-2} \nabla u \nabla \phi u \mathrm{d}x. \tag{4.5}$$

Choosing $v = \phi u_n$ as test function in (4.1), it yields

$$\begin{aligned} &\int_{\mathbb{R}^N} \phi \left|\nabla u_n\right|^{p_n} \mathrm{d}x + \int_{\mathbb{R}^N} \left|\nabla u_n\right|^{p_n-2} \nabla u_n \nabla \phi u_n \mathrm{d}x + \int_{\mathbb{R}^N} u_n^{p_n} \phi \mathrm{d}x \\ &\quad + \frac{1}{n} \left(\int_{\mathbb{R}^N} \left|\nabla u_n\right|^{p^+-2} \nabla u_n \nabla \phi u_n \mathrm{d}x + \int_{\mathbb{R}^N} \phi \left|\nabla u_n\right|^{p^+} \mathrm{d}x + \int_{\mathbb{R}^N} u_n^{p^+} \phi \mathrm{d}x \right) \\ &\quad + \frac{1}{n} \left(\int_{\mathbb{R}^N} \left|\nabla u_n\right|^{p^--2} \nabla u_n \nabla \phi u_n \mathrm{d}x + \int_{\mathbb{R}^N} \phi \left|\nabla u_n\right|^{p^-} \mathrm{d}x + \int_{\mathbb{R}^N} u_n^{p^-} \phi \mathrm{d}x \right) \\ &= \int_{\mathbb{R}^N} (f(x, u_n)u_n + hu_n)\phi \mathrm{d}x. \end{aligned} \tag{4.6}$$

By the boundedness of the sequence $(u_n)_n$ in $L^\infty(\mathbb{R}^N)$, we have

$$\frac{1}{n} \left| \int_{\mathbb{R}^N} u_n^{p^+} \phi \mathrm{d}x + \int_{\mathbb{R}^N} u_n^{p^-} \phi \mathrm{d}x \right| \to 0, \; n \to +\infty. \tag{4.7}$$

By the Lebesgue's dominated convergence theorem, we easily get

$$\int_{\mathbb{R}^N} (f(x, u_n)u_n + hu_n)\phi \mathrm{d}x \to \int_{\mathbb{R}^N} (f(x, u)u + hu)\phi \mathrm{d}x, \tag{4.8}$$

$$\int_{\mathbb{R}^N} u_n^{p_n} \phi \mathrm{d}x \to \int_{\mathbb{R}^N} u^{p_0} \phi \mathrm{d}x. \tag{4.9}$$

Moreover, using again the boundedness of $(u_n)_n$ in $L^\infty(\mathbb{R}^N)$, it yields

$$\begin{aligned} \frac{1}{n} \left| \int_{\mathbb{R}^N} \left|\nabla u_n\right|^{p^+-2} \nabla u_n \nabla \phi u_n \mathrm{d}x \right| &\le \frac{c_{10}}{n} \int_{\mathbb{R}^N} \left|\nabla u_n\right|^{p^+-1} \left|\nabla \phi\right| \mathrm{d}x \\ &\le \frac{c_{11}}{n} n^{\frac{p^+-1}{p^+}} \left(\frac{1}{n} \left|\nabla u_n\right|^{p^+}_{L^{p^+}(\mathbb{R}^N)} \right)^{\frac{p^+-1}{p^+}} \\ &\le \frac{c_{12}}{n^{\frac{1}{p^+}}}, \; \forall\, n \ge 1. \end{aligned}$$

Hence,

$$\frac{1}{n}\int_{\mathbb{R}^N} |\nabla u_n|^{p^+-2}\,\nabla u_n \nabla\phi u_n \mathrm{d}x \to 0,\ n \to +\infty. \tag{4.10}$$

Similarly,

$$\frac{1}{n}\int_{\mathbb{R}^N} |\nabla u_n|^{p^--2}\,\nabla u_n \nabla\phi u_n \mathrm{d}x \to 0,\ n \to +\infty. \tag{4.11}$$

On the other hand, by Hölder's inequality we have

$$\left|\int_{\mathbb{R}^N} |\nabla u_n|^{p_n-2}\,\nabla u_n \nabla\phi (u_n-u)\mathrm{d}x\right| \le \left(\int_{\mathbb{R}^N} |\nabla u_n|^{p_n}\,\mathrm{d}x\right)^{\frac{p_n}{p_n-1}} \times \left(\int_{\mathbb{R}^N} |\nabla \phi|^{p_n}\,|u_n-u|^{p_n}\,\mathrm{d}x\right)^{\frac{1}{p_n}},\ \forall\, n \ge 1.$$

By the virtue of the Lebesgue's dominated convergence theorem, it comes

$$\int_{\mathbb{R}^N} |\nabla \phi|^{p_n}\,|u_n-u|^{p_n}\,\mathrm{d}x \to 0,\ n \to +\infty.$$

That fact together with the boundedness of the sequence $\left(\int_{\mathbb{R}^N} |\nabla u_n|^{p_n}\,\mathrm{d}x\right)_n$ gives

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p_n-2}\,\nabla u_n \nabla\phi (u_n-u)\mathrm{d}x \to 0,\ n \to +\infty.$$

But,

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p_n-2}\,\nabla u_n \nabla\phi u_n \mathrm{d}x = \int_{\mathbb{R}^N} |\nabla u_n|^{p_n-2}\,\nabla u_n \nabla\phi u \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u_n|^{p_n-2}\,\nabla u_n \nabla\phi (u_n-u)\mathrm{d}x,$$

in view (4.5), we deduce that

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p_n-2}\,\nabla u_n \nabla\phi u_n \mathrm{d}x \to \int_{\mathbb{R}^N} |\nabla u|^{p_0-2}\,\nabla u \nabla\phi u \mathrm{d}x. \tag{4.12}$$

Having in mind that $\phi \ge 0$, taking (4.12), (4.11), (4.10), (4.9), (4.8) and (4.7) into account, we can pass to the upper limit in (4.6) as n tends to $+\infty$:

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^N} \phi\,|\nabla u_n|^{p_n}\,\mathrm{d}x \le \int_{\mathbb{R}^N} \phi\,|\nabla u|^{p_0}\,\mathrm{d}x. \tag{4.13}$$

Inequality (4.13) is valid for all nonnegative function $\phi \in X$ having a compact support. That fact immediately implies that

$$\limsup_{n\to+\infty}\int_{|x|<\rho} |\nabla u_n|^{p_n}\,\mathrm{d}x \le \int_{|x|<\rho} |\nabla u|^{p_0}\,\mathrm{d}x,\ \forall\,\rho > 0. \tag{4.14}$$

Since $p_{\xi(n)} \geq p_0$, then one can apply Hölder's inequality to obtain

$$\int_{|x|<\rho} \left|\nabla u_{\xi(n)}\right|^{p_0} \mathrm{d}x \leq |B(0,\rho)|^{1-\frac{p_0}{p_{\xi(n)}}} \left(\int_{|x|<\rho} \left|\nabla u_{\xi(n)}\right|^{p_{\xi(n)}} \mathrm{d}x\right)^{\frac{p_0}{p_{\xi(n)}}}, \ \forall\, n \geq 1,\ \forall\, \rho > 0, \tag{4.15}$$

where $B(0,\rho) = \left\{x \in \mathbb{R}^N,\ |x| < \rho\right\}$. Having in mind that $p_{\xi(n)} \to p_0$ and using (4.14), passing to the upper limit in (4.15), we infer

$$\limsup_{n\to+\infty} \int_{|x|<\rho} \left|\nabla u_{\xi(n)}\right|^{p_0} \mathrm{d}x \leq \int_{|x|<\rho} |\nabla u|^{p_0} \mathrm{d}x. \tag{4.16}$$

Now, observing that $p_0 \geq p^-$, it follows that $W^{1,p_0}(B(0,\rho))$ is continuously embedded into $W^{1,p^-}(B(0,\rho))$. Since $u_{\xi(n)} \rightharpoonup u$ weakly in $W^{1,p^-}(B(0,\rho))$, then $u_{\xi(n)} \rightharpoonup u$ weakly in $W^{1,p_0}(B(0,\rho))$, which implies that

$$\liminf_{n\to+\infty} \int_{|x|<\rho} \left|\nabla u_{\xi(n)}\right|^{p_0} \mathrm{d}x \geq \int_{|x|<\rho} |\nabla u|^{p_0} \mathrm{d}x. \tag{4.17}$$

Combining (4.17) and (4.16), we get

$$\int_{|x|<\rho} \left|\nabla u_{\xi(n)}\right|^{p_0} \mathrm{d}x \to \int_{|x|<\rho} |\nabla u|^{p_0} \mathrm{d}x,\ n \to +\infty.$$

Having in mind that

$$\int_{|x|<\rho} u_{\xi(n)}^{p_0} \mathrm{d}x \to \int_{|x|<\rho} u^{p_0} \mathrm{d}x,$$

we deduce that $u_{\xi(n)} \to 0$ strongly in $W^{1,p_0}(B(0,\rho))$. Since $p_0 \geq p^-$, then $u_{\xi(n)} \to u$ strongly in $W^{1,p^-}(B(0,\rho))$. Since ρ is arbitrary, then we can conclude that $u_{\xi(n)} \to u$ strongly in $W^{1,p^-}_{loc}(\mathbb{R}^N)$.

Let, as usual, $\phi \in X$ be such that $\phi \geq 0$ and $\mathrm{supp}(\phi)$ is compact. Now, taking $v = \phi(u_n - u)$ as test function in (4.1), it yields

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n|^{p_n-2} \nabla u_n \nabla\phi(u_n-u)\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u_n|^{p_n-2} \nabla u_n \nabla(u_n-u)\phi \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} u_n^{p_n-1}(u_n-u)\phi \mathrm{d}x \\
&\quad + \frac{1}{n}\left(\int_{\mathbb{R}^N} |\nabla u_n|^{p^+-2} \nabla u_n \nabla((u_n-u)\phi)\mathrm{d}x + \int_{\mathbb{R}^N} u_n^{p^+-1}(u_n-u)\phi \mathrm{d}x\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{n}\left(\int_{\mathbb{R}^N}|\nabla u_n|^{p^--2}\nabla u_n\nabla((u_n-u)\phi)\mathrm{d}x+\int_{\mathbb{R}^N}u_n^{p^--1}(u_n-u)\phi\mathrm{d}x\right)\\
&=\int_{\mathbb{R}^N}(f(x,u_n)+h)(u_n-u)\phi\mathrm{d}x.
\end{aligned}
\tag{4.18}
$$

Using the same arguments as previously (i.e., using the boundedness of the sequence $(u_n)_n$ in $L^\infty(\mathbb{R}^N)$ and the Lebesgue's dominated convergence theorem), one can easily see that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}|\nabla u_n|^{p_n-2}\nabla u_n\nabla\phi(u_n-u)\mathrm{d}x\to 0,\ n\to+\infty,\\
&\int_{\mathbb{R}^N}u_n^{p_n-1}(u_n-u)\phi\mathrm{d}x\to 0,\ n\to+\infty,\\
&\frac{1}{n}\left(\int_{\mathbb{R}^N}|\nabla u_n|^{p^+-2}\nabla u_n\nabla((u_n-u)\phi)\mathrm{d}x+\int_{\mathbb{R}^N}u_n^{p^+-1}(u_n-u)\phi\mathrm{d}x\right)\\
&\quad\to 0,\ n\to+\infty,\\
&\frac{1}{n}\left(\int_{\mathbb{R}^N}|\nabla u_n|^{p^--2}\nabla u_n\nabla((u_n-u)\phi)\mathrm{d}x+\int_{\mathbb{R}^N}u_n^{p^--1}(u_n-u)\phi\mathrm{d}x\right)\\
&\quad\to 0,\ n\to+\infty,
\end{aligned}
$$

and

$$
\int_{\mathbb{R}^N}(f(x,u_n)+h)(u_n-u)\phi\mathrm{d}x\to 0,\ n\to+\infty.
$$

From (4.18), we infer

$$
\int_{\mathbb{R}^N}|\nabla u_n|^{p_n-2}\nabla u_n\nabla(u_n-u)\phi\mathrm{d}x\to 0,\ n\to+\infty.
$$

In particular,

$$
\int_{\mathbb{R}^N}\left|\nabla u_{\psi(n)}\right|^{p_{\psi(n)}-2}\nabla u_{\psi(n)}\nabla(u_{\psi(n)}-u)\phi\mathrm{d}x\to 0,\ n\to+\infty. \tag{4.19}
$$

Next, we recall the following classical monotonicity inequalities: for all $\eta_1,\eta_2\in\mathbb{R}^N$, we have

$$
2^{1-q}|\eta_1-\eta_2|^q\le\left(|\eta_1|^{q-2}\eta_1-|\eta_2|^{q-2}\eta_2\right)(\eta_1-\eta_2),\ \forall\, q\ge 2, \tag{4.20}
$$

$$
\begin{aligned}
(q-1)|\eta_1-\eta_2|^q\le&\Big(\Big(|\eta_1|^{q-2}\eta_1\\
&-|\eta_2|^{q-2}\eta_2\Big)(\eta_1-\eta_2)\Big)^{\frac{q}{2}}\left(|\eta_1|^q+|\eta_2|^q\right)^{\frac{2-q}{2}},\ \forall\, 1<q<2.
\end{aligned}
\tag{4.21}
$$

From (4.20) and (4.21), we can also establish the following useful inequalities: for all $\eta_1, \eta_2 \in \mathbb{R}^N$, we have

$$|\eta_1|^q - |\eta_2|^q \geq q\,|\eta_2|^{q-2}\,\eta_2(\eta_1 - \eta_2) + c_q\,|\eta_1 - \eta_2|^q\,,\ \forall\, q \geq 2, \tag{4.22}$$

$$\begin{aligned}|\eta_1|^q - |\eta_2|^q &\geq q\,|\eta_2|^{q-2}\,\eta_2(\eta_1-\eta_2)\\ &+c'_q\,|\eta_1-\eta_2|^2\left(|\eta_1|^q + |\eta_2|^q\right)^{\frac{q-2}{q}}\,,\ \forall\, 1 < q < 2,\end{aligned} \tag{4.23}$$

where c_q and c'_q are two positive constants depending (continuously) only in q. See, for example, [16].

Case 1: $p_0 > 2$. Since $p_{\psi(n)} \to p_0$, then there exists $n_0 \geq 1$ large enough such that $p_{\psi(n)} > 2,\ \forall\, n \geq n_0$. Applying inequality (4.22) with $\eta_1 = \nabla u$ and $\eta_2 = \nabla u_{\psi(n)}$, it yields

$$\begin{aligned}&\int_{\mathbb{R}^N} \phi\,|\nabla u|^{p_{\psi(n)}}\,\mathrm{d}x - \int_{\mathbb{R}^N} \phi\,\left|\nabla u_{\psi(n)}\right|^{p_{\psi(n)}}\,\mathrm{d}x\\ &\geq p_{\psi(n)}\int_{\mathbb{R}^N} \left|\nabla u_{\psi(n)}\right|^{p_{\psi(n)}-2}\,\nabla u_{\psi(n)}\nabla(u - u_{\psi(n)})\mathrm{d}x\\ &+c_{p_{\psi(n)}}\int_{\mathbb{R}^N} \phi\,\left|\nabla(u_{\psi(n)} - u)\right|^{p_{\psi(n)}}\,\mathrm{d}x.\end{aligned} \tag{4.24}$$

Since $\phi\,|\nabla u|^{p_{\psi(n)}} \leq \phi\left(1 + |\nabla u|^{p_0}\right),\ \forall\, n$, then one can use the Lebesgue's dominated convergence theorem to obtain

$$\int_{\mathbb{R}^N} \phi\,|\nabla u|^{p_{\psi(n)}}\,\mathrm{d}x \to \int_{\mathbb{R}^N} \phi\,|\nabla u|^{p_0}\,\mathrm{d}x,\ n \to +\infty. \tag{4.25}$$

Moreover, proceeding as for the sequence $(p_{\xi(n)})_n$ (i.e., by taking $v = \phi u_{\psi(n)}$ as test function in (4.1)), we can easily show that

$$\int_{\mathbb{R}^N} \phi\,\left|\nabla u_{\psi(n)}\right|^{p_{\psi(n)}}\,\mathrm{d}x \to \int_{\mathbb{R}^N} \phi\,|\nabla u|^{p_0}\,\mathrm{d}x. \tag{4.26}$$

Combining (4.19), (4.25) and (4.26), after passing to the limit as n tends to $+\infty$ in (4.24), we deduce that

$$\int_{\mathbb{R}^N} \phi\,\left|\nabla(u_{\psi(n)} - u)\right|^{p_{\psi(n)}}\,\mathrm{d}x \to 0,\ n \to +\infty. \tag{4.27}$$

We have,

$$\int_{\mathbb{R}^N} \phi\,\left|\nabla(u_{\psi(n)}-u)\right|^{p^-}\,\mathrm{d}x \leq \left(\int_{\mathbb{R}^N} \phi\mathrm{d}x\right)^{\frac{p_{\psi(n)}-p^-}{p^-}}\left(\int_{\mathbb{R}^N} \phi\,\left|\nabla(u_{\psi(n)}-u)\right|^{p_{\psi(n)}}\,\mathrm{d}x\right)^{\frac{p^-}{p_{\psi(n)}}}.$$

From (4.27), it follows that

$$\int_{\mathbb{R}^N} \phi \left|\nabla(u_{\psi(n)} - u)\right|^{p^-} \mathrm{d}x \to 0,\ n \to +\infty.$$

By the virtue of the Lebesgue's dominated convergence Theorem, it comes

$$\int_{\mathbb{R}^N} \phi \left|u_{\psi(n)} - u\right|^{p^-} \mathrm{d}x \to 0.$$

Therefore, $u_{\psi(n)} \to u$ strongly in $W^{1,p^-}_{loc}(\mathbb{R}^N)$.
Case 2: $p_0 \le 2$. In this case, $p_{\psi(n)} < 2,\ \forall\, n \ge 1$. Applying inequality (4.23) with $\eta_1 = \nabla u$ and $\eta_2 = \nabla u_{\psi(n)}$, it yields

$$\begin{aligned} &\int_{\mathbb{R}^N} \phi\, |\nabla u|^{p_{\psi(n)}} \mathrm{d}x - \int_{\mathbb{R}^N} \phi \left|\nabla u_{\psi(n)}\right|^{p_{\psi(n)}} \mathrm{d}x \\ &\ge p_{\psi(n)} \int_{\mathbb{R}^N} \left|\nabla u_{\psi(n)}\right|^{p_{\psi(n)}-2} \nabla u_{\psi(n)} \nabla(u - u_{\psi(n)})\mathrm{d}x \\ &+ c'_{p_{\psi(n)}} \int_{\mathbb{R}^N} \phi \left|\nabla(u_{\psi(n)} - u)\right|^2 \left(\left|\nabla u_{\psi(n)}\right|^{p_{\psi(n)}} + |\nabla u|^{p_{\psi(n)}}\right)^{\frac{p_{\psi(n)}-2}{p_{\psi(n)}}} \mathrm{d}x. \end{aligned} \tag{4.28}$$

Using (4.19), (4.25) and (4.26), we deduce from (4.28) that

$$\begin{aligned} &\int_{\mathbb{R}^N} \phi \left|\nabla(u_{\psi(n)} - u)\right|^2 \left(\left|\nabla u_{\psi(n)}\right|^{p_{\psi(n)}} + |\nabla u|^{p_{\psi(n)}}\right)^{\frac{p_{\psi(n)}-2}{p_{\psi(n)}}} \mathrm{d}x \to 0, \\ &\quad n \to +\infty. \end{aligned} \tag{4.29}$$

We have

$$\begin{aligned} &\int_{\mathbb{R}^N} \phi \left|\nabla(u_{\psi(n)} - u)\right|^{p_{\psi(n)}} \mathrm{d}x \\ &= \int_{\mathbb{R}^N} \phi \left|\nabla(u_{\psi(n)} - u)\right|^{p_{\psi(n)}} \left(\left|\nabla u_{\psi(n)}\right|^{p_{\psi(n)}} + |\nabla u|^{p_{\psi(n)}}\right)^{\frac{p_{\psi(n)}-2}{2}} \left(\left|\nabla u_{\psi(n)}\right|^{p_{\psi(n)}} + |\nabla u|^{p_{\psi(n)}}\right)^{\frac{2-p_{\psi(n)}}{2}} \mathrm{d}x \\ &\le \left(\int_{\mathbb{R}^N} \phi \left|\nabla(u_{\psi(n)} - u)\right|^2 \left(\left|\nabla u_{\psi(n)}\right|^{p_{\psi(n)}} + |\nabla u|^{p_{\psi(n)}}\right)^{\frac{p_{\psi(n)}-2}{p_{\psi(n)}}}\right)^{\frac{p_{\psi(n)}}{2}} \\ &\quad \left(\int_{\mathbb{R}^N} \left(\left|\nabla u_{\psi(n)}\right|^{p_{\psi(n)}} + |\nabla u|^{p_{\psi(n)}}\right) \mathrm{d}x\right)^{\frac{2-p_{\psi(n)}}{2}}. \end{aligned} \tag{4.30}$$

Clearly,

$$\sup_{n \ge 1} \int_{\mathbb{R}^N} \left(\left|\nabla u_{\psi(n)}\right|^{p_{\psi(n)}} + |\nabla u|^{p_{\psi(n)}}\right) \mathrm{d}x < +\infty.$$

By (4.29), inequality (4.30) leads to

$$\int_{\mathbb{R}^N} \phi \left|\nabla(u_{\psi(n)} - u)\right|^{p_{\psi(n)}} \mathrm{d}x \to 0, \ n \to +\infty.$$

As in the previous case, we deduce that $(u_{\psi(n)})_n$ is strongly convergent to u in $W^{1,p^-}_{loc}(\mathbb{R}^N)$. Hence, $u_n \to u$ strongly in $W^{1,p^-}_{loc}(\mathbb{R}^N)$. Consequently, $\alpha(u_n) \to \alpha(u)$ in $\mathbb{R}$ and by the continuity of the function p, we conclude that $p(\alpha(u_n)) \to p(\alpha(u)) = p_0$. This ends the proof of Theorem 1.4.

Declarations

Conflict of interest This work does not have any conflict of interest.

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