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The Mihlin Multiplier Theorem on Anisotropic Mixed-Norm Hardy Spaces

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Abstract
Let $\vec{p} \in (0, 1]^n$ and $H_A^{\vec{p}}(\mathbb{R}^n)$ be the anisotropic mixed-norm Hardy spaces associated with a dilation matrix A. In this paper, we obtain a Mihlin multiplier theorem on anisotropic Hardy spaces $H_A^{\vec{p}}(\mathbb{R}^n)$, when $\vec{p}$ depends on eccentricities of A and the level of regularity of a multiplier symbol. This extends both the multiplier theorems in classical Hardy spaces and anisotropic Hardy spaces.

Keywords Multiplier · Anisotropic mixed-norm Hardy space · Fourier transform

Mathematics Subject Classification Primary 42B35 · Secondary 42B30, 42B20

1 Introduction and the Main Result

Let A be an $n \times n$ matrix, and $|\det A| = b$. We say that A is a *dilation* matrix if all the eigenvalues λ of A satisfy $|\lambda| > 1$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A, ordered by their norm from smallest to largest. Define λ_- and λ_+, such that $1 < \lambda_- < |\lambda_1|$ and $|\lambda_n| < \lambda_+$. Then $\ln \lambda_\pm / \ln b$ are called the *eccentricities* of dilation A. We point out that, if A is diagonalizable, we may let $\lambda_- := |\lambda_1|$ and $\lambda_+ := |\lambda_n|$. Otherwise, we may choose them sufficiently close to these equalities in accordance with what we

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need in our arguments. In addition, there is a sequence of *nested ellipsoids* $\{B_j\}_{j\in\mathbb{Z}}$ associated with A such that

$$B_{j+1} = A(B_j) \quad \text{and} \quad |B_0| = 1.$$

If A^* is the adjoint of A, then A^* is also a dilation matrix with the same determinant b and eigenvalues as well as eccentricities, with its own nested ellipsoids $\{B_j^*\}_{j\in\mathbb{Z}}$. We refer the reader to [6] for more properties about the dilation.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space, and let $\mathcal{S}'(\mathbb{R}^n)$ be the space of tempered distributions. Given a multi-index $\vec{p} := (p_1, \ldots, p_n)$ with $p_i \in (0, \infty)$ for any $1 \le i \le n$, the *mixed-norm Lebesgue space* $L^{\vec{p}}(\mathbb{R}^n)$ consists of all measurable functions f, for which

$$\begin{aligned}\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} &:= \Big\| \ldots \|f\|_{L^{p_1}_{x_1}} \ldots \Big\|_{L^{p_n}_{x_n}} \\ &:= \left\{ \int_{\mathbb{R}} \ldots \left[\int_{\mathbb{R}} |f(x_1, \ldots, x_n)|^{p_1}\, dx_1 \right]^{\frac{p_2}{p_1}} \ldots dx_n \right\}^{\frac{1}{p_n}} < \infty.\end{aligned}$$

If $p_1 = \ldots = p_n = p$, then the space $L^{\vec{p}}(\mathbb{R}^n)$ reduces to the classical Lebesgue space $L^p(\mathbb{R}^n)$. The *anisotropic mixed-norm Hardy space* $H_A^{\vec{p}}(\mathbb{R}^n)$ associated with dilation matrix A is defined as

$$H_A^{\vec{p}}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \ \|f\|_{H_A^{\vec{p}}(\mathbb{R}^n)} := \left\| \sup_{k\in\mathbb{Z}} |f * \varphi_k| \right\|_{L^{\vec{p}}(\mathbb{R}^n)} < \infty \right\},$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n} \varphi(x)\, dx \neq 0$ and $\varphi_k(x) := b^k \varphi(A^k x)$ for any $k \in \mathbb{Z}$. If $p_1 = \ldots = p_n = p$ and the dilation matrix

$$A := \begin{pmatrix} 2 & 0 & \ldots & 0 \\ 0 & 2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 2 \end{pmatrix}, \tag{1.1}$$

then the anisotropic mixed-norm Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$ coincides with the classical Hardy space $H^p(\mathbb{R}^n)$ of Fefferman-Stein [20].

The mixed-norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ was systematically studied by Benedek-Panzone in [5], which goes back to Hörmander [28]. After that, many works on these spaces have been done due to the importance of $L^{\vec{p}}(\mathbb{R}^n)$, not only in harmonic analysis but also in partial differential equations and geometric inequalities. For instance, in a series of recent papers of Chen-Sun [9–11], they studied the Hardy-Littlewood-Sobolev inequalities on $L^{\vec{p}}(\mathbb{R}^n)$ and characterized the boundedness of multilinear fractional integral operators on $L^{\vec{p}}(\mathbb{R}^n)$. When A is an anisotropic diagonal matrix, precisely,

$$A := \begin{pmatrix} 2^{a_1} & 0 & \dots & 0 \\ 0 & 2^{a_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 2^{a_n} \end{pmatrix} \tag{1.2}$$

with $1 \le a_i < \infty$ for $1 \le i \le n$, the anisotropic mixed-norm Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$ was first introduced and studied by Cleanthous-Georgiadis-Nielsen [15], and further developed by the author and his collaborators in [29–31, 33, 35]. This anisotropic mixed-norm Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$ associated with the diagonal matrix (1.2) was later extended to the general dilation matrix A (no need to be diagonal matrix or even no need to be diagonalizable) by the author and his collaborators in [32]. Here we refer to [13, 14, 17, 22–24, 26, 27, 34, 36, 37, 43, 44] for more detials on (anisotropic) mixed-norm function spaces and their applications.

This paper is devoted to studying the Mihlin multiplier theorem on the Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$. To state the multiplier theorem, let $\hat{f}$ and $\check{f}$ denote the *Fourier transform* and *inverse Fourier transform* of f, respectively. To be exact, when $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi \imath x\cdot\xi}\,dx \quad\text{and}\quad \check{f}(\xi) := \hat{f}(-\xi) = \int_{\mathbb{R}^n} f(x)e^{2\pi \imath x\cdot\xi}\,dx, \quad \forall \xi \in \mathbb{R}^n,$$

where $\imath := \sqrt{-1}$; when $f \in \mathcal{S}'(\mathbb{R}^n)$, $\langle \hat{f}, \phi\rangle := \langle f, \hat{\phi}\rangle$ for any $\phi \in \mathcal{S}(\mathbb{R}^n)$. Let $m \in L^\infty(\mathbb{R}^n)$. We say the measurable function m is a *Fourier multiplier* on $H_A^{\vec{p}}(\mathbb{R}^n)$ if its associated Fourier multiplier operator T_m, initially defined by

$$T_m f(x) := (m\hat{f})^\vee(x) = \int_{\mathbb{R}^n} m(\xi)\hat{f}(\xi)e^{2\pi \imath x\cdot\xi}\,d\xi, \quad \forall x \in \mathbb{R}^n,$$

for $f \in L^2(\mathbb{R}^n) \cap H_A^{\vec{p}}(\mathbb{R}^n)$, is bounded on $H_A^{\vec{p}}(\mathbb{R}^n)$. For a dilation matrix A, define the *dilation operator* D_A by

$$D_A f(x) = f(Ax), \quad \forall x \in \mathbb{R}^n.$$

For any $\Omega \in \mathbb{R}^n$ and $N \in \mathbb{N}\cup\{0\}$, denote by $C^N(\Omega)$ the set of all functions on Ω whose derivatives with order no greater than N exist and are continuous. Then the following anisotropic Mihlin condition was introduced in [4, 47]. Let A be a dilation matrix and $m \in C^N(\mathbb{R}^n\backslash\{0\})$ with $N \in \mathbb{N}\cup\{0\} =: \mathbb{Z}_+$. We say m satisfies the *anisotropic Mihlin condition of order* N if there exists a constant $C := C_N$ such that for any multi-indices α with $|\alpha| \le N$,

$$\left| D_{A^*}^{-j}\partial_\xi^\alpha D_{A^*}^j m(\xi)\right| \le C, \quad \forall \xi \in B_{j+1}^* \setminus B_j^*,\ j \in \mathbb{Z}, \tag{1.3}$$

where, for any $j \in \mathbb{Z}$, $D^j_{A^*} m(\xi) := m((A^*)^j \xi)$. Henceforth, we always use C to denote a *positive constant* which may depend on the dilation matrix A and scalar parameters such as n and $\vec{p}$, and may vary from line to line, but independent of the main parameters such as $f \in H^{\vec{p}}_A(\mathbb{R}^n)$. Given a vector $\vec{p} := (p_1, \ldots, p_n)$, let $p_- := \min\{p_1, \ldots, p_n\}$ and $p_+ := \max\{p_1, \ldots, p_n\}$. For any $s \in \mathbb{R}$, we always use $\lfloor s \rfloor$ to denote the largest integer no greater than s.

Now we can state the Mihlin multiplier theorem on the Hardy space $H^{\vec{p}}_A(\mathbb{R}^n)$ as follows.

Theorem 1.1 *Let A be a dilation matrix, $\vec{p} \in (0, 1]^n$, $N \in \mathbb{N}$ and*

$$M := \left(N \frac{\ln \lambda_-}{\ln b} - 1 \right) \frac{\ln b}{\ln \lambda_+}.$$

If m satisfies the anisotropic Mihlin condition of order N and T_m is the Fourier multiplier operator, then $T_m : H^{\vec{p}}_A(\mathbb{R}^n) \to H^{\vec{p}}_A(\mathbb{R}^n)$ is bounded, provided $\vec{p}$ satisfies

$$0 \le \frac{1}{p_-} - 1 < \lfloor M \rfloor \frac{(\ln \lambda_-)^2}{\ln b \ln \lambda_+}.$$

Recall that the study of the Fourier multiplier theory was initiated by Mihlin [42] and Hörmander [28] in the late 1950 s. Then the multiplier theory for Triebel-Lizorkin spaces and Besov-Lipschitz spaces was considered by Peetre [45] in 1975; for classical Hardy spaces was studied by Taibleson-Weiss [46] and Baernstein-Sawyer [3]; for Hardy spaces in the parabolic setting was inversitaged by Calderón-Torchinsky [7, 8]; for anisotropic Hardy spaces was obtained by Wang [47]. Additionally, Fourier multipliers (or more general operators) on the anisotropic mixed-norm setting were well studied by Cleanthous et al. in [16] as well as by Georgiadis et al. in [21, 23, 25], and the extensions on manifolds, Lie groups or discrete settings were considered in [1, 2, 12, 18, 19, 21, 38, 39].

Next we give some remarks on Theorem 1.1.

Remark 1.2 Let $p_1 = \ldots = p_n = p$ and the dilation matrix A be as in (1.1). Then $p_- = p$, $\lambda_- = \lambda_+ = 2$, $b = |\det A| = 2^n$ and the Hardy space $H^{\vec{p}}_A(\mathbb{R}^n)$ goes back to the classical Hardy space $H^p(\mathbb{R}^n)$, and hence $M = N - n$ and $\frac{n}{N} < p \le 1$. This theorem, in this case, recovers the classical case.

Remark 1.3 Let $p_1 = \ldots = p_n = p$. Then the anisotropic mixed-norm Hardy space $H^{\vec{p}}_A(\mathbb{R}^n)$ reduces to the anisotropic Hardy space $H^p_A(\mathbb{R}^n)$, and hence Theorem 1.1 coincides with the result in anisotropic Hardy space setting.

Remark 1.4 Let the dilation matrix A be as in (1.1). Then $\lambda_- = \lambda_+ = 2, b = |\det A| = 2^n$ and the Hardy space $H^{\vec{p}}_A(\mathbb{R}^n)$ goes back to the isotropic mixed-norm Hardy space $H^{\vec{p}}(\mathbb{R}^n)$, and hence $M = N - n$ and $\frac{n}{N} < p_- \le 1$. We point out that, even in this case, Theorem 1.1 is also new.

Finally, we make some conventions on notation. The notation $f \lesssim g$ means $f \le Cg$ and, if $f \lesssim g \lesssim f$, then we write $f \sim g$. We also use the following convention: If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, *rather than* $f \lesssim g = h$ or $f \lesssim g \le h$.

2 Proof of the Main Theorem

To prove Theorem 1.1, the main ingredients are the atoms of $H_A^{\vec{p}}(\mathbb{R}^n)$ introduced in [32] and the criterion on the boundedness of sublinear operators on $H_A^{\vec{p}}(\mathbb{R}^n)$ established in [32]. Moreover, the Calderón-Zygmund operator theory on anisotropic mixed-norm Hardy spaces $H_A^{\vec{p}}(\mathbb{R}^n)$ also plays an important role in our proof.

We begin with giving the following notion of the homogeneous quasi-norm.

Definition 2.1 For any given dilation A, a *homogeneous quasi-norm*, with respect to A, is a measurable mapping $\rho : \mathbb{R}^n \to [0, \infty)$ satisfying

(i) if $x \ne 0$, then $\rho(x) \in (0, \infty)$;
(ii) for any $x \in \mathbb{R}^n$, $\rho(Ax) = b\rho(x)$;
(iii) there exists some $R \in [1, \infty)$ such that

$$\rho(x + y) \le R[\rho(x) + \rho(y)], \quad \forall x, y \in \mathbb{R}^n.$$

For a fixed dilation A, the associated homogeneous quasi-norms are non-unique. But they are equivalent to each other (see [6, p. 6 Lemma 2.4]). Thus, in what follows, we may use the following *step homogeneous quasi-norm* ρ defined by setting

$$\rho(x) := \sum_{j\in\mathbb{Z}} b^j \mathbf{1}_{B_{j+1}\setminus B_j}(x) \quad \text{when } x \in \mathbb{R}^n \setminus \{0\}, \quad \text{or else} \quad \rho(0) := 0$$

for both simplicity and convenience. In addition, if A^* is the adjoint of a given dilation matrix A, then A^* is also a dilation matrix with the same determinant and eigenvalues, with its own nested ellipsoids $\{B_j^*\}_{j\in\mathbb{Z}}$ and step homogeneous quasi-norms ρ_*. Given a dilation A, we say that $(\vec{p}, r, s)$ is an *admissible triplet* if $\vec{p} \in (0, 1]^n$, $r \in (1, \infty]$ and

$$s \in \left[\left\lfloor \left(\frac{1}{p_-} - 1\right) \frac{\ln b}{\ln \lambda_-} \right\rfloor, \infty\right) \cap \mathbb{Z}_+.$$

We now present the definition of $(\vec{p}, r, s)$-atom from [32, Definition 4.1] as follows.

Definition 2.2 Let $(\vec{p}, r, s)$ be admissible. A measurable function a on $\mathbb{R}^n$ is called a $(\vec{p}, r, s)$-*atom* if

(i) supp $a \subset x + B_k$ for some $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$;
(ii) $\|a\|_{L^r(\mathbb{R}^n)} \le \frac{|B_k|^{1/r}}{\|\mathbf{1}_{x+B_k}\|_{L^{\vec{p}}(\mathbb{R}^n)}}$;
(iii) for any $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \le s$, $\int_{\mathbb{R}^n} a(x)x^\gamma \, dx = 0$.

We now recall the following notion from [47, Definition 3.1].

Definition 2.3 Let $R \in \mathbb{Z}_+$ and $K \in C^R(\mathbb{R}^n \setminus \{0\})$. We say that K is a *Calderón-Zygmund convolution kernel of order R* if there exists a constant C such that for all multi-indices α with $|\alpha| \le R$, and for any $j \in \mathbb{Z}$ and $x \in B_{j+1} \backslash B_j$,

$$\left| D_A^{-j} \partial_\xi^\alpha D_A^j K(x) \right| \le \frac{C}{\rho(x)}.$$

If K is such a kernel, we say K satisfies *CZC-R* and its associated singular integral operator T is defined by $Tf := K * f$, which is called a *Calderón-Zygmund operator of order R*.

For the *CZC-R* kernel and the Mihlin condition (1.3), we have the following key lemma (see [47, Lemma 3.2]).

Lemma 2.4 *Let $N \in \mathbb{Z}_+$ and $m \in L^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$. Suppose m satisfies the Mihlin condition of order N as in (1.3), and define K by $K := \check{m}$. Then K is a Calderón-Zygmund convolution kernel of order R provided $R \in \mathbb{N}$ and*

$$0 \le R < \left(N \frac{\ln \lambda_-}{\ln b} - 1 \right) \frac{\ln b}{\ln \lambda_+}.$$

To prove our main theorem, the following lemma plays an important role.

Lemma 2.5 *Let $(\vec{p}, \infty, s)$ be an admissible triplet, $m \in L^\infty(\mathbb{R}^n)$ and T_m the associated Fourier multiplier operator initially defined on $L^2(\mathbb{R}^n) \cap H_A^{\vec{p}}(\mathbb{R}^n)$. If there exists a positive constant C such that, for any $(\vec{p}, \infty, s)$-atom a, $\|T_m a\|_{H_A^{\vec{p}}(\mathbb{R}^n)} \le C$, then T_m has a unique bounded extension $\widetilde{T}_m : H_A^{\vec{p}}(\mathbb{R}^n) \to H_A^{\vec{p}}(\mathbb{R}^n)$.*

To prove Lemma 2.5, we need the following Lemma 2.6 from [32, Corollary 4], which gives the boundedness criterion about sublinear operators on $H_A^{\vec{p}}(\mathbb{R}^n)$. To state it, we first recall that a complete vector space $\mathcal{B}$, equipped with a quasi-norm $\|\cdot\|_{\mathcal{B}}$, is called a *quasi-Banach space* if

(i) $\|\varphi\|_{\mathcal{B}} = 0$ if and only if φ is the zero element of $\mathcal{B}$;
(ii) there exists a positive constant $C \in [1, \infty)$ such that, for any $\varphi,\ \phi \in \mathcal{B}$, $\|\varphi+\phi\|_{\mathcal{B}} \le C(\|\varphi\|_{\mathcal{B}} + \|\phi\|_{\mathcal{B}})$.

In addition, for any given $\gamma \in (0, 1]$, a *γ-quasi-Banach space* $\mathcal{B}_\gamma$ is a quasi-Banach space equipped with a quasi-norm $\|\cdot\|_{\mathcal{B}_\gamma}$ satisfying that there exists a constant $C \in [1, \infty)$ such that, for any $K \in \mathbb{N}$ and $\{\varphi_i\}_{i=1}^K \subset \mathcal{B}_\gamma$,

$$\left\| \sum_{i=1}^K \varphi_i \right\|_{\mathcal{B}_\gamma}^\gamma \le C \sum_{i=1}^K \|\varphi_i\|_{\mathcal{B}_\gamma}^\gamma .$$

Let $\mathcal{B}_\gamma$ be a γ-quasi-Banach space with $\gamma \in (0, 1]$ and $\mathcal{Y}$ a linear space. An operator T from $\mathcal{Y}$ to $\mathcal{B}_\gamma$ is said to be $\mathcal{B}_\gamma$-*sublinear* if there exists a positive constant $\widetilde{C}$ such that, for any $K \in \mathbb{N}$, $\{\mu_i\}_{i=1}^K \subset \mathbb{C}$ and $\{\varphi_i\}_{i=1}^K \subset \mathcal{Y}$,

$$\left\| T\left(\sum_{i=1}^K \mu_i \varphi_i\right)\right\|_{\mathcal{B}_\gamma}^\gamma \leq \widetilde{C} \sum_{i=1}^K |\mu_i|^\gamma \, \|T(\varphi_i)\|_{\mathcal{B}_\gamma}^\gamma$$

and, for any $\varphi,\ \phi \in \mathcal{Y}$, $\|T(\varphi) - T(\phi)\|_{\mathcal{B}_\gamma} \leq \widetilde{C}\|T(\varphi - \phi)\|_{\mathcal{B}_\gamma}$ (see [30, 48]).

Lemma 2.6 *Let $(\vec{p}, \infty, s)$ be an admissible triplet, $\gamma \in (0, 1]$ and $\mathcal{B}_\gamma$ a γ-quasi-Banach space. If T is a $\mathcal{B}_\gamma$-sublinear operator defined on all continuous $(\vec{p}, \infty, s)$-atoms satisfying*

$$\sup\left\{\|T(a)\|_{\mathcal{B}_\gamma} : \, a \text{ is any continuous } (\vec{p}, \infty, s)\text{-atom}\right\} < \infty,$$

then T uniquely extends to a bounded $\mathcal{B}_\gamma$-sublinear operator from $H_A^{\vec{p}}(\mathbb{R}^n)$ into $\mathcal{B}_\gamma$.

Additionally, the following property of Fourier transform of elements in Hardy spaces $H_A^{\vec{p}}(\mathbb{R}^n)$ is required in the proof of Lemma 2.5 (see [41, Theorem 3.1]).

Lemma 2.7 *Let $\vec{p} \in (0, 1]^n$. Then, for any $f \in H_A^{\vec{p}}(\mathbb{R}^n)$, there exists a continuous function g on $\mathbb{R}^n$ such that $\widehat{f} = g$ in $\mathcal{S}'(\mathbb{R}^n)$, and there exists a positive constant C such that*

$$|g(x)| \leq C\|f\|_{H_A^{\vec{p}}(\mathbb{R}^n)} \max\left\{[\rho_*(x)]^{\frac{1}{p_+}-1},\ [\rho_*(x)]^{\frac{1}{p_-}-1}\right\}, \quad \forall x \in \mathbb{R}^n.$$

With the help of Lemmas 2.6 and 2.7, we next show Lemma 2.5.

Proof of Lemma 2.5 From [32, Lemma 3.4], we infer that, for any $\{f_i\}_{i\in\mathbb{N}} \subset H_A^{\vec{p}}(\mathbb{R}^n)$,

$$\left\|\sum_{i=1}^K f_i\right\|_{H_A^{\vec{p}}(\mathbb{R}^n)}^{p_-} \leq C \sum_{i=1}^K \|f_i\|_{H_A^{\vec{p}}(\mathbb{R}^n)}^{p_-},$$

which implies that $(H_A^{\vec{p}}(\mathbb{R}^n), \|\cdot\|_{H_A^{\vec{p}}(\mathbb{R}^n)})$ is a p_--quasi-Banach space. By Lemma 2.7, we find that $\hat{f}$ agrees with a continuous function in the sense of distribution for any $f \in H_A^{\vec{p}}(\mathbb{R}^n)$. Moreover, applying [32, Lemma 3.4] again, we know that there exists a positive constant C such that, for any $K \in \mathbb{N}$, $\{\mu_i\}_{i=1}^K \subset \mathbb{C}$ and $\{\varphi_i\}_{i=1}^K \subset L^2(\mathbb{R}^n) \cap H_A^{\vec{p}}(\mathbb{R}^n)$,

$$\left\|T_m\left(\sum_{i=1}^K \mu_i\varphi_i\right)\right\|_{H_A^{\vec{p}}(\mathbb{R}^n)}^{p_-} = \left\|\sum_{i=1}^K \mu_i T_m\varphi_i\right\|_{H_A^{\vec{p}}(\mathbb{R}^n)}^{p_-} \leq C\sum_{i=1}^K |\mu_i|^{p_-}\, \|T_m\varphi_i\|_{H_A^{\vec{p}}(\mathbb{R}^n)}^{p_-},$$

which implies that the Fourier multiplier T_m is a $H_A^{\vec{p}}(\mathbb{R}^n)$-*sublinear* operator. Then combining the assumption $\|T_m a\|_{H_A^{\vec{p}}(\mathbb{R}^n)} \le C$ for all $(\vec{p}, \infty, s)$-atom a and the criterion on the boundedness of sublinear operators on $H_A^{\vec{p}}(\mathbb{R}^n)$ (see Lemma 2.6), we conclude that the operator T_m has a unique bounded extension $\widetilde{T}_m : H_A^{\vec{p}}(\mathbb{R}^n) \to H_A^{\vec{p}}(\mathbb{R}^n)$. This hence completes the proof. □

When applying Lemma 2.5, we must first show that the condition in Lemma 2.5 is satisfied. Thus, we need the following result, which is just a consequence of [40, Theorem 3]. In what follows, for any $s \in \mathbb{N}$, an operator T is said to have the *vanishing moments up to order s* if, for any $f \in L^2(\mathbb{R}^n)$ with compact support and satisfying that, for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le s$, $\int_{\mathbb{R}^n} x^\alpha f(x)\,dx = 0$, it holds true that $\int_{\mathbb{R}^n} x^\alpha Tf(x)\,dx = 0$.

Lemma 2.8 *Let $(\vec{p}, \infty, s)$ be an admissible triplet and $\ell \in \mathbb{N}$ with*

$$0 \le \frac{1}{p_-} - 1 < \ell \frac{(\ln \lambda_-)^2}{\ln b \ln \lambda_+}.$$

Assume that T is a Calderón-Zygmund operator of order ℓ and has the vanishing moment conditions up to order $s_0 := \lfloor (1/p_- - 1) \ln b / \ln \lambda_- \rfloor$. Then there exists a positive constant C such that for any $(\vec{p}, \infty, s)$-atom a,

$$\|Ta\|_{H_A^{\vec{p}}(\mathbb{R}^n)} \le C.$$

Using Lemmas 2.5 and 2.8, we now show Theorem 1.1.

Proof of Theorem 1.1 Let m satisfy the anisotropic Mihlin condition of order N as in (1.3). Without loss of generality, we may assume $M \notin \mathbb{N}$. Otherwise, if $M \in \mathbb{N}$, then let $\widetilde{\lambda}_-$ and $\widetilde{\lambda}_+$ be defined as

$$1 < \lambda_- < \widetilde{\lambda}_- < |\lambda_1| \le \ldots \le |\lambda_n| < \widetilde{\lambda}_+ < \lambda_+$$

such that the new

$$\widetilde{M} := \left(N \frac{\ln \widetilde{\lambda}_-}{\ln b} - 1 \right) \frac{\ln b}{\ln \widetilde{\lambda}_+},$$

defined in terms of the new eccentricities $\ln \widetilde{\lambda}_\pm / \ln b$, is slightly larger and no longer an integer satisfying $\lfloor \widetilde{M} \rfloor = \lfloor M \rfloor$. Notice that, applying Lemma 2.4, we conclude that $K := \check{m}$ is a Calderón-Zygmund convolution kernel of order R provided $R \in \mathbb{Z}_+$ and

$$0 \le R < \left(N \frac{\ln \lambda_-}{\ln b} - 1 \right) \frac{\ln b}{\ln \lambda_+}.$$

Thus, we may take

$$R := \left\lfloor \left(N \frac{\ln \lambda_-}{\ln b} - 1 \right) \frac{\ln b}{\ln \lambda_+} \right\rfloor$$

and then, from the assumption of Theorem 1.1, it follows that $R = \lfloor M \rfloor$.

We now show that there exists a positive constant C such that, for all $(\vec{p}, \infty, s)$-atom a, the singular integral operator T associated with kernel K defined by $Tf := K * f$ satisfying

$$\|Ta\|_{H_A^{\vec{p}}(\mathbb{R}^n)} \le C$$

when

$$0 \le \frac{1}{p_-} - 1 < \lfloor M \rfloor \frac{(\ln \lambda_-)^2}{\ln b \ln \lambda_+}.$$

Indeed, we first note that T is a Calderón-Zygmund operator of order $\lfloor M \rfloor$. Moreover, by the definition of operator T and the vanishing moments condition of $(\vec{p}, \infty, s)$-atom a, we know that, for any $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \le s$,

$$\begin{aligned}\int_{\mathbb{R}^n} Ta(x) x^\gamma \, dx &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x-y) K(y) \, dy \, x^\gamma \, dx \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x-y) x^\gamma \, dx \, K(y) \, dy = 0,\end{aligned}$$

which implies that T has the vanishing moment conditions up to order s. Therefore, the operator T satisfies all assumptions of Lemma 2.8 and hence, from Lemma 2.8, we infer that $\|Ta\|_{H_A^{\vec{p}}(\mathbb{R}^n)} \le C$. By this and the fact that $T = T_m$, we find that $\|T_m a\|_{H_A^{\vec{p}}(\mathbb{R}^n)} \le C$. Combining this and Lemma 2.5, we finally conclude that T_m has a unique bounded extension $\widetilde{T}_m : H_A^{\vec{p}}(\mathbb{R}^n) \to H_A^{\vec{p}}(\mathbb{R}^n)$ and hence Theorem 1.1 is proved. □

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Declarations

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