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Statistical Solution for the Nonlocal Discrete Nonlinear Schrödinger Equation

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Abstract

In this article, we consider the nonlocal discrete nonlinear Schrödinger equation. We first prove that the associated process has a pullback-$\mathcal{D}_\delta$ attractor by overcoming the difficulties caused by the nonlocal operator. Then we establish the existence of a unique family of invariant Borel probability measures carried by the pullback attractor. Finally, we further construct statistical solutions for this nonlocal equation on infinite lattices.

Keywords Invariant measure · Statistical solution · Discrete nonlinear Schrödinger equation · Nonlocal

Mathematics Subject Classification 35B41 · 35D99 · 76F20

1 Introduction

In this article, we study the following discrete nonlocal Schrödinger equation:

$$i\dot{u}_n(t) + \sum_{m\in\mathbb{Z}} J(n-m)u_m(t) + f_n(u_n(t)) + i\gamma u_n(t) = g_n(t), \quad n \in \mathbb{Z}, \quad t > \tau, \tag{1.1}$$

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associated with the initial condition

$$u_n(\tau) = u_{n,\tau}, \qquad \tau \in \mathbb{R}, \tag{1.2}$$

where $J : \mathbb{Z} \to \mathbb{R}$ denotes the dispersive coupling operator, which is assumed to be even and $\sum_{m=1}^{\infty} |J(m)| < \infty$. In the system (1.1), the coupling parameters $J(m)$ include the long-range interactions, which have attracted much attention from many researchers mainly in the physic literature due to their wide applications. For example, a new form of (1.1) was proposed in [28] for the modelling of the nonlinear dynamics of the DNA molecule. Later, some specific form of (1.1) can be rigorously derived as the continuum limit of certain discrete physical systems with long-range lattice interactions; see [19]. Recently, the author in [30] proved the existence of global attractor for a nonlocal discrete Schrödinger equation. There are many works concerning the nonlocal discrete systems; see e.g., [1, 2, 14, 26, 30], etc. In this article, we are mainly interested in the invariant measures and statistical solutions for this nonlocal lattice system (1.1).

Note that if we choose the coupling parameters $J(m)$ in (1.1) as

$$J(m) = \sum_{j=0}^{2p} \binom{2p}{j} (-1)^j \delta_{m,j-p},$$

where p is any positive integer and $\delta_{m,k}$ is the Kronecker delta, then the nonlocal system (1.1) can be transformed into the following generalized discrete Schrödinger equation:

$$i\dot{u}_n(t) + \Delta_d^p u_n(t) + f_n(u_n(t)) + i\gamma u_n(t) = g_n(t), \quad n \in \mathbb{Z}, \quad t > \tau. \tag{1.3}$$

where $\Delta_d^p = \Delta_d \circ \cdots \circ \Delta_d$, p times, and Δ_d denotes the one-dimensional discrete Laplace operator given by $\Delta_d u_n = u_{n+1} + u_{n-1} - 2u_n$. It is well-known that the discrete Schrödinger equation is a very important model with a great variety of applications, ranging from physics to biology; see e.g., [27, 29] and the references therein.

In recent years, the discrete Schrödinger equation has been widely studied by mathematicians and physicists. There are various of works on global attractors [8, 11, 20, 30], pullback attractors [31], and bifurcations [17] for these equations under various boundary conditions. The interested reader is referred to [7, 15, 33, 34] for more results on the discrete Schrödinger equation. Particularly, Pereira [31] established the existence of pullback attractors for the nonautonomous discrete Schrödinger equation with delays. However, as far as we know, there are few papers studying the statistical solution of this nonlocal Schrödinger equation on infinite lattices.

In this article we mainly investigate invariant measures and statistical solutions for this nonlocal discrete system (1.1)–(1.2). We all know that these two concepts are very important in understanding the turbulence; see [10]. This is because measurements of many important aspects of turbulent flows are actually the measurements of some time-average quantities. Recently, there have been a series of works on invariant measures and statistical solutions of evolution systems; see [3–6, 18, 21, 24, 25, 35, 37, 39, 40, 42–44] for continuous systems. Especially, by using the notion of generalized Banach limits, Łukaszewicz, Real and Robinson [25] constructed invariant measures for general continuous dynamical systems on metric spaces. Later, Bronzi et al. [4] have developed an abstract framework for the theory of statistical solutions for general evolution systems. Based on these works, Zhao and Caraballo [42] used the natural translation semigroup and the trajectory attractor to construct trajectory statistical solutions for the globally modified Navier–Stokes equations.

Invariant measures and dynamics of lattice dynamical systems are widely studied by many researchers; see e.g., [12, 13, 22, 23, 36, 38, 41, 45]. Lattice dynamical systems are spatiotemporal systems with discretization in some variables, which have been widely used in many fields such as chemical reaction theory [9], biology [16], electrical engineering [32] and so on. Very recently, Zhao et al. [41] constructed the invariant Borel probability measures for the nonautonomous discrete Klein–Gordon–Schrödinger equations. Using some techniques in the above work, Wu and Huang [36] further construct the statistical solutions for discrete Klein–Gordon–Schrödinger type equations.

Our main purpose of this article is to construct the invariant Borel probability measures and statistical solutions for this discrete nonlocal Schrödinger equation. By using notions of generalized Banach limit and the theory given by Łukaszewicz and Robinson (see [24]), we establish the existence of invariant measures for (1.1)–(1.2). Then we further construct the statistical solutions of this nonlocal lattice system. We remark that the system (1.1)–(1.2) considered here consists of the nonlocal operator J, which can lead to some additional difficulties in giving the estimates of solutions. This requires us to utilize some more delicate analysis and techniques to overcome this term.

This work is organized as follows. Section 2 is devoted to the existence and boundedness of solutions of equations (1.1)–(1.2). In Sect. 3 we show that the process generated by (1.1)–(1.2) has a pullback-$\mathcal{D}_\delta$ attractor. In Sect. 4, we establish the existence of a unique family of invariant Borel probability measures carried by the pullback-$\mathcal{D}_\delta$ attractor. Finally, we further construct the statistical solution of the nonlocal system (1.1)–(1.2).

2 Existence and Boundedness of Solutions

In this section we study the existence and boundedness of solutions of equations (1.1)–(1.2). Let us first introduce some spaces. Set

$$\ell^2 = \{u = (u_n)_{n\in\mathbb{Z}} : \ u_n \in \mathbb{C} \text{ and } \sum_{n\in\mathbb{Z}} |u_n|^2 < +\infty\}$$

and equip it with the inner product and norm defined by

$$(u, v) = \sum_{n\in\mathbb{Z}} u_n \overline{v}_n, \quad \|u\|^2 = (u, u), \qquad \forall u = (u_n)_{n\in\mathbb{Z}}, \quad v = (v_n)_{n\in\mathbb{Z}} \in \ell^2.$$

Then $(\ell^2, (\cdot,\cdot))$ is a Hilbert space. Let $E = \ell^2$ and denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and norm, respectively.

For the sake of simplicity, we set

$$u = (u_n)_{n\in\mathbb{Z}}, \quad Au = ((Au)_n)_{n\in\mathbb{Z}} = \left(\sum_{m\in\mathbb{Z}} J(n-m)u_m\right)_{n\in\mathbb{Z}},$$
$$f(u(t)) = (f_n(u_n(t)))_{m\in\mathbb{Z}}, \quad u_\tau = (u_{\tau,n})_{n\in\mathbb{Z}}.$$

Then Eq. (1.1)–(1.2) can be written as a vector form

$$i\dot{u}(t) + Au(t) + f(u(t)) + i\gamma u(t) = g(t), \tag{2.1}$$
$$u(\tau) = u_\tau. \tag{2.2}$$

In order to establish our main results, in this article, we always assume that f in (2.1) satisfies the following conditions.

(F1) There exists $L_f > 0$ such that if $x_1, x_2 \in \mathbb{C}$, then

$$|f_n(x_1) - f_n(x_2)| \le L_f|x_1 - x_2|, \qquad n \in \mathbb{Z}.$$

(F2) There exist $k_1 = (k_{1,n})_{n\in\mathbb{Z}} \in \ell^\infty$, $k_2 = (k_{2,n})_{n\in\mathbb{Z}} \in \ell^2$ such that

$$|f_n(x)| \le k_{1,n}|x| + k_{2,n}, \qquad \forall x \in \mathbb{C}.$$

(F3) The number γ in (2.1) satisfies

$$\frac{\gamma}{2} > 2K_1 + \frac{L_f}{2},$$

where $K_1 = \|k_1\|_{\ell^\infty}$.

By $\mathcal{C}(\mathbb{R}, \ell^2)$ we denote the space of continuous functions from $\mathbb{R}$ to ℓ^2. Then if $h \in \mathcal{C}(\mathbb{R}, \ell^2)$,

$$\|h(t)\|^2 = \sum_{m\in\mathbb{Z}} |h_m(t)|^2 < +\infty.$$

Let us begin with some fundamental properties on the operator A in (2.1).

Lemma 2.1 *The operator $A : \ell^2 \to \ell^2$ is a bound linear operator and satisfies*

$$\|Au\| \le K_J\|u\|, \qquad \forall u \in \ell^2,$$

where $K_J = \left[2|J(0)|^2 + 8\left(\sum_{j=1}^{\infty} |J(j)|\right)^2\right]^{1/2}$.

Proof It is easy to see that A is a linear operator in E. Assume $u \in \ell^2$. Then by the definition of A, we have

$$
\begin{aligned}
\|Au\|^2 &= \sum_{n\in\mathbb{Z}} |\sum_{m\in\mathbb{Z}} J(n-m)u_m|^2 = \sum_{n\in\mathbb{Z}} |J(0)u_n + \sum_{m\neq n} J(n-m)u_m|^2 \\
&= \sum_{n\in\mathbb{Z}} \left| J(0)u_n + \sum_{m=1}^{\infty} J(m)u_{n-m} + \sum_{m=1}^{\infty} J(-m)u_{n+m} \right|^2 \\
&\leq \sum_{n\in\mathbb{Z}} \left(|J(0)u_n| + \sum_{m=1}^{\infty} |J(m)||u_{n-m} + u_{n+m}| \right)^2 \\
&\leq 2\sum_{n\in\mathbb{Z}} \left[|J(0)u_n|^2 + \left(\sum_{m=1}^{\infty} |J(m)||u_{n-m} + u_{n+m}| \right)^2 \right]. \qquad (2.3)
\end{aligned}
$$

In what follows we estimate the second term in (2.3). Indeed, by some basic computations, one has

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}} \left(\sum_{m=1}^{\infty} |J(m)||u_{n-m} + u_{n+m}| \right)^2 \\
&= \sum_{n\in\mathbb{Z}} \left[\sum_{m,i=1}^{\infty} |J(m)||J(i)|(|u_{n-m}| + |u_{n+m}|)(|u_{n-i}| + |u_{n+i}|) \right] \\
&\leq \sum_{m,i=1}^{\infty} |J(m)||J(i)| \left[\sum_{n\in\mathbb{Z}} (|u_{n-m}| + |u_{n+m}|)^2 \right]^{\frac{1}{2}} \left[\sum_{n\in\mathbb{Z}} (|u_{n-i}| + |u_{n+i}|)^2 \right]^{\frac{1}{2}} \\
&\leq 2\sum_{m,i=1}^{\infty} |J(m)||J(i)| \left[\sum_{n\in\mathbb{Z}} (|u_{n-m}|^2 + |u_{n+m}|^2) \right]^{\frac{1}{2}} \left[\sum_{n\in\mathbb{Z}} (|u_{n-i}|^2 + |u_{n+i}|^2) \right]^{\frac{1}{2}}. \qquad (2.4)
\end{aligned}
$$

Noticing that for $k = m, i$, we see that

$$
\sum_{n\in\mathbb{Z}} (|u_{n-k}|^2 + |u_{n+k}|^2) = \sum_{n\in\mathbb{Z}} |u_{n-k}|^2 + \sum_{n\in\mathbb{Z}} |u_{n+k}|^2 = 2\|u\|^2.
$$

Then it follows from (2.4) that

$$
\sum_{n\in\mathbb{Z}} \left(\sum_{m=1}^{\infty} |J(m)||u_{n-m} + u_{n+m}| \right)^2 \leq 4 \left[\sum_{j=1}^{\infty} |J(j)| \right]^2 \|u\|^2. \qquad (2.5)
$$

Combining (2.3) and (2.5), it yields that

$$\|Au\|^2 \leq 2|J(0)|^2\|u\|^2 + 8\left[\sum_{j=1}^{\infty}|J(j)|\right]^2\|u\|^2.$$

This completes the proof of this lemma. □

Lemma 2.2 *Let $g(t) = (g_n(t))_{n\in\mathbb{Z}} \in \mathcal{C}(\mathbb{R}, \ell^2)$. Then for each $u_\tau \in E$, there exists $T_0 > \tau$, such that Eq. (2.1)–(2.2) has a unique solution $u(t), t \geq \tau$ satisfying*

$$u \in \mathcal{C}([\tau, T_0), E) \cap \mathcal{C}^1((\tau, T_0), E).$$

Moreover, if $T_0 < +\infty$, then $\lim\limits_{t\to T_0^-}\|u(t)\|_E = +\infty$.

Proof Let $F(u,t) = iAu + if(u) - \gamma u - ig(t)$. Then we can rewrite the equation (2.1) as the following equivalent form

$$\dot{u} = F(u,t). \tag{2.6}$$

Assume $\mathcal{B} \subset E$ is a bounded subset and that $u^1, u^2 \in \mathcal{B}$. By assumption (F1), one can see that there exists $L_f > 0$ such that

$$\|f(u^1) - f(u^2)\| = \left(\sum_{n\in\mathbb{Z}}|f_n(u_n^1) - f_n(u_n^2)|^2\right)^{1/2} \leq L_f\|u^1 - u^2\|. \tag{2.7}$$

Thus we deduce from Lemmas 2.1 and (2.7) that

$$\begin{aligned}\|F(u^1,t) - F(u^2,t)\| &\leq \|A(u^1 - u^2)\| + \|f(u^1) - f(u^2)\| + \gamma\|u^1 - u^2\| \\ &\leq \left(K_J + L_f + \gamma\right)\|u^1 - u^2\|,\end{aligned}$$

which implies that $F(u,t)$ is locally Lipschitz from $E \times \mathbb{R}$ to E. By the classical theory of ODEs, the results of Lemma 2.1 hold. □

Lemma 2.3 *Assume $g(t) = (g_n(t))_{n\in\mathbb{Z}} \in \mathcal{C}(\mathbb{R}, \ell^2)$. Let $u(t)$ be a solution of (2.1)–(2.2) associated with the initial value $u_\tau \in E$. Then*

$$\|u(t)\|^2 \leq e^{-\delta(t-\tau)}\|u_\tau\|^2 + \frac{e^{-\delta t}}{\gamma}\int_\tau^t e^{\delta s}\|g(s)\|^2 ds + \frac{2K_2^2}{\delta\gamma}, \quad \forall t \geq \tau. \tag{2.8}$$

where $\delta = \frac{\gamma}{2} - 2K_1$ and $K_2 = \|k_2\|$.

Proof Taking the imaginary part of the inner product $(\cdot,\cdot)$ of the equation (2.1) with u in ℓ^2, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2+\mathbf{Im}(Au,u)+\mathbf{Im}(f(u),u)+\gamma\|u\|^2=\mathbf{Im}(g(t),u). \tag{2.9}$$

Some elementary computations give that

$$(Au,u)=J(0)\|u\|^2+2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}J(m)\mathbf{Re}(\overline{u_{n+m}}u_n). \tag{2.10}$$

By (F2), one can find

$$\mathbf{Im}(f(u),u)=\mathbf{Im}\sum_{n\in\mathbb{Z}}f_n(u_n)\bar{u}_n\le\sum_{n\in\mathbb{Z}}\left[k_{1,n}|u_n|+k_{2,n}\right]|u_n|\le K_1\|u\|^2+K_2\|u\|. \tag{2.11}$$

It is easy to see that

$$K_2\|u\|\le\frac{\gamma}{4}\|u\|^2+\frac{K_2^2}{\gamma}, \tag{2.12}$$

$$\mathbf{Im}(g(t),u)\le\frac{\gamma}{2}\|u\|^2+\frac{1}{2\gamma}\|g\|^2. \tag{2.13}$$

Thus we combine (2.9)–(2.13) to obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2+\delta\|u\|^2\le\frac{2K_2^2}{\gamma}+\frac{1}{\gamma}\|g\|^2. \tag{2.14}$$

Applying the Gronwall inequality to (2.14) on $[\tau,t]$ with $t\ge\tau$, one has

$$\|u(t)\|^2\le \mathrm{e}^{-\delta(t-\tau)}\|u_\tau\|^2+\frac{\mathrm{e}^{-\delta t}}{\gamma}\int_\tau^t \mathrm{e}^{\delta s}\|g(s)\|^2\mathrm{d}s+\frac{2K_2^2}{\delta\gamma},$$

which completes the proof of this lemma. □

In order to guarantee that the Eqs. (2.1)–(2.2) has a bounded pullback absorbing set, we further assume that the function g satisfies the following condition.
(F4) Assume $g(t)=(g_n(t))_{n\in\mathbb{Z}}\in\mathcal{C}(\mathbb{R},\ell^2)$ and that

$$\int_{-\infty}^t \mathrm{e}^{\delta s}\|g(s)\|^2\mathrm{d}s<M(t),\qquad t\in\mathbb{R},$$

where M is a continuous function on $\mathbb{R}$, which remains bounded on the interval $(-\infty,t)$ for each fixed t.

Therefore if the condition (F4) holds, then one can immediately conclude from Lemma 2.3 that the system (2.1)–(2.2) has a bounded pullback absorbing set.

Moreover, we infer from (F3) and Lemma 2.3 that for every $u_\tau \in E$, the corresponding solution $u(t)$ of (2.1)–(2.2) exists globally on $[\tau, +\infty)$. Furthermore, by Lemma 2.1 one can know that

$$u \in \mathcal{C}([\tau, +\infty), E) \cap \mathcal{C}^1((\tau, +\infty), E).$$

Thus the solution operators can generate a family of continuous processes $\{U(t, \tau)\}_{t \geq \tau}$ on E:

$$U(t, \tau) : u_\tau \mapsto u(t) \in E, \qquad t \geq \tau.$$

Let $\mathcal{P}(E)$ denote the family of all nonempty subsets of E and $\mathcal{D}_\delta$ denote the class of families of nonempty subsets $\mathcal{D} = \{D(s) : s \in \mathbb{R}\} \subset \mathcal{P}(E)$ satisfying

$$\lim_{s \to -\infty} \left(\mathrm{e}^{\frac{\delta s}{2}} \sup_{u \in D(s)} \|u\|^2 \right) = 0. \tag{2.15}$$

We usually call the class $\mathcal{D}_\delta$ a *universe* in $\mathcal{P}(E)$.

Remark 2.4 Clearly, the universe $\mathcal{D}_\delta$ contains all bounded subsets of E.

Lemma 2.5 *Let the assumptions* (F1)–(F4) *hold. Then the process* $\{U(t, \tau)\}_{t \geq \tau}$ *generated by* (2.1)–(2.2) *possesses a bounded pullback-*$\mathcal{D}_\delta$ *absorbing set*

$$\mathcal{B}_0 := \{B_0(s) : s \in \mathbb{R}\} \subset \mathcal{P}(E),$$

where $B_0(s) = B(0, r_\delta(s))$ *is a ball in* E *centered at* 0 *with radius* $r_\delta(s)$.

Proof Choose $r_\delta(t) = \rho_\delta^{1/2}(t)$, where

$$\rho_\delta(t) := 1 + \frac{\mathrm{e}^{-\delta t}}{\gamma} \int_{-\infty}^{t} \mathrm{e}^{\delta s} \|g(s)\|^2 \mathrm{d}s + \frac{2K_2^2}{\delta \gamma}.$$

Then one can easily deduce from Lemma 2.2 and the construction of $\mathcal{D}_\delta$ that the family $\mathcal{B}_0$ is the desired pullback-$\mathcal{D}_\delta$ bounded absorbing set for $\{U(t, \tau)\}_{t \geq \tau}$ in E.

3 Pullback Attractors

In this section we prove that the process $\{U(t, \tau)\}_{t \geq \tau}$ has a pullback-$\mathcal{D}_\delta$ attractor. To this end we first verify that the solutions of (2.1)–(2.2) have pullback-$\mathcal{D}_\delta$ asymptotic nullness.

Lemma 3.1 *Assume that the conditions* (F1)–(F4) *hold. Then for every* $t \in \mathbb{R}$, $\varepsilon > 0$ *and* $\mathcal{D} = \{D(s) : s \in \mathbb{R}\} \in \mathcal{D}_\delta$, *there exist two numbers* $N_* = N_*(t, \varepsilon, \mathcal{D}) \in \mathbb{N}$ *and* $\tau_* = \tau_*(t, \varepsilon, \mathcal{D}) \le t$ *such that*

$$\sup_{u_\tau \in D(\tau)} \sum_{|n| \ge N_*} |(U(t,\tau)u_\tau)_n|_E^2 \le \varepsilon^2, \qquad \forall \tau \le \tau_*.$$

Proof Define a smooth function $\chi(x) \in \mathcal{C}^1(\mathbb{R}_+, [0, 1])$ satisfying

$$\chi(x) = \begin{cases} 0, & 0 \le x \le 1; \\ 1, & x \ge 2, \end{cases} \quad \text{and} \quad |\chi'(x)| \le \chi_0 \,(\text{positive constant}), \qquad \forall x \ge 0.$$

Assume $\mathcal{D} = \{D(s) : s \in \mathbb{R}\} \in \mathcal{D}_\delta$ and that

$$u(t) = U(t,\tau)u_\tau = (u_n(t))_{n\in\mathbb{Z}} \in E$$

is a solution of (2.1)–(2.2) associated with the initial value $u_\tau \in D(\tau)$ for $t \ge \tau$. Let

$$v = (v_n)_{n\in\mathbb{Z}}, \qquad v_n = \chi(\frac{|n|}{N})u_n,$$

where N is a positive integer which will be decided later. For the sake of simplicity, we set

$$\chi_n = \chi(\frac{|n|}{N}), \quad \|w\|_\chi^2 = \sum_{n\in\mathbb{Z}} \chi_n |w_n|^2, \qquad n \in \mathbb{Z}.$$

Taking the imaginary part of the inner product $(\cdot,\cdot)$ of (2.1) with v in ℓ^2, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_\chi^2 + \gamma\|u\|_\chi^2 + \mathbf{Im}(Au, v) + \mathbf{Im}(f(u), v) = \mathbf{Im}(g, v). \tag{3.1}$$

By the definitions of A and χ, one has

$$\begin{aligned} \mathbf{Im}(Au, v) &= \mathbf{Im}\left[J(0)\|u\|_\chi^2 + \sum_{n\in\mathbb{Z}} \sum_{m=1}^{\infty} J(m)(\chi_{n+m} - \chi_n)\bar{u}_{n+m}u_n \right] \\ &\le \sum_{n\in\mathbb{Z}} \sum_{m=1}^{\infty} |J(m)||\chi_{n+m} - \chi_n||u_{n+m}||u_n|. \end{aligned} \tag{3.2}$$

It is easy to see that

$$|\chi_{n+m} - \chi_n| \le \frac{\chi_0}{N} m \quad \text{and} \quad |\chi_{n+m} - \chi_n| \le 2.$$

By Lemmas 2.3 and 2.5, we know that there exists $\tau_1 = \tau_1(t, \mathcal{D}) \leq t$ such that

$$\|u(t)\| \leq \rho_\delta(t), \qquad \tau \leq \tau_1. \tag{3.3}$$

Thus it follows from (3.2) and (3.3) that if $\tau \leq \tau_1$, then

$$\begin{aligned}
\mathbf{Im}(Au, v) &\leq \sum_{n\in\mathbb{Z}} \sum_{m=1}^{\infty} |J(m)||\chi_{n+m} - \chi_n||u_{n+m}||u_n| \\
&= \sum_{n\in\mathbb{Z}} \sum_{m=1}^{l} |J(m)||\chi_{n+m} - \chi_n||u_{n+m}||u_n| \\
&\quad + \sum_{n\in\mathbb{Z}} \sum_{m=l+1}^{\infty} |J(m)||\chi_{n+m} - \chi_n||u_{n+m}||u_n| \\
&\leq \frac{\chi_0}{N} \sum_{m=1}^{l} m|J(m)|\rho_\delta^2(t) + 2 \sum_{m=l+1}^{\infty} |J(m)|\rho_\delta^2(t),
\end{aligned} \tag{3.4}$$

where $l \geq 1$ is a positive integer. By virtue of assumption (F2), one finds that

$$\begin{aligned}
\mathbf{Im}(f(u), v) &\leq \sum_{n\in\mathbb{Z}} \chi_n \left(k_{1,n}|u_n| + k_{2,n}\right)|u_n| \leq K_1\|u\|_\chi^2 + K_{2,\chi}\|u\|_\chi \\
&\leq K_1\|u\|_\chi^2 + \frac{K_{2,\chi}^2}{\gamma} + \frac{\gamma}{4}\|u\|_\chi^2,
\end{aligned} \tag{3.5}$$

where $K_{2,\chi} = (\sum_{n\in\mathbb{Z}} \chi_n k_{2,n}^2)^{1/2}$. Note that

$$\mathbf{Im}(g, v) \leq \frac{1}{2\gamma}\|g(t)\|_\chi^2 + \frac{\gamma}{2}\|u\|_\chi^2. \tag{3.6}$$

Combining (3.4)–(3.6) and (3.1), we know that if $\tau \leq \tau_1$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_\chi^2 + \delta\|u\|_\chi^2 &\leq \frac{2\chi_0}{N} \sum_{m=1}^{l} m|J(m)|\rho_\delta^2(t) + 4 \sum_{m=l+1}^{\infty} |J(m)|\rho_\delta^2(t) \\
&\quad + \frac{2K_{2,\chi}^2}{\gamma} + \frac{1}{\gamma}\|g(t)\|_\chi^2,
\end{aligned} \tag{3.7}$$

where δ is the number given in Lemma 2.3. Now let $t \in \mathbb{R}$ and $\varepsilon > 0$ be arbitrary given numbers. Since $\sum_{m=1}^{\infty} |J(m)| < \infty$, we deduce from the definition of $\rho_\delta(t)$ that there exists $N_1 = N_1(\varepsilon, t)$ such that

$$4 \sum_{m=N_1+1}^{\infty} |J(m)|\rho_\delta^2(t) < \delta\varepsilon^2/6. \tag{3.8}$$

Moreover, using the fact that $\sum_{n\in\mathbb{Z}} k_{2,n}^2 < \infty$, one can pick $N_2 = N_2(t, \varepsilon)$ with $N_2 \geq N_1$ so that

$$\frac{2K_{2,\chi}^2}{\gamma} + \frac{2\chi_0}{N_2}\sum_{m=1}^{N_1} m|J(m)|\rho_\delta^2(t) < \delta\varepsilon^2/6. \tag{3.9}$$

Thus we conclude from (3.7) to (3.9) that for $N \geq N_2$ and $\tau \leq \tau_1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_\chi^2 + \delta\|u\|_\chi^2 \leq \frac{1}{\gamma}\|g(t)\|_\chi^2 + \delta\varepsilon^2/3. \tag{3.10}$$

Applying Gronwall inequality to (3.10), it yields

$$\|u\|_\chi^2 \leq \mathrm{e}^{-\delta(t-\tau)}\|u_\tau\|_\chi^2 + \frac{1}{\gamma}\int_\tau^t \mathrm{e}^{-\delta(t-s)}\sum_{|n|\geq N}|g_n(s)|^2\mathrm{d}s + \varepsilon^2/3, \qquad t \geq \tau, \tag{3.11}$$

provided $\tau \leq \tau_1$ and $N \geq N_2$. Noticing that for the given $t \in \mathbb{R}$, one can deduce from (F4) that there exists positive constant K_t (depending on t) satisfying

$$\int_{-\infty}^t \mathrm{e}^{\delta s}\|g(s)\|^2\mathrm{d}s < M_t,$$

from which it can be seen that there exists $N_3 = N_3(\varepsilon, t) \in \mathbb{N}$ such that

$$\frac{1}{\gamma}\int_\tau^t \mathrm{e}^{-\delta(t-s)}\sum_{|n|\geq N} g_n^2(s)\mathrm{d}s \leq \frac{1}{\gamma}\mathrm{e}^{-\delta t}\sum_{|n|\geq N}\int_{-\infty}^t \mathrm{e}^{\delta s}|g_n(s)|^2\mathrm{d}s < \varepsilon^2/3, \quad \forall N \geq N_3. \tag{3.12}$$

Because $u_\tau \in D(\tau)$ and $\mathcal{D} = \{D(s) : s \in \mathbb{R}\} \in \mathcal{D}_\delta$, we easily see from the construction of $\mathcal{D}_\delta$ that there is $\tau_* = \tau(\varepsilon, t, \mathcal{D})$ with $\tau_* \leq \tau_1$ so that

$$\mathrm{e}^{-\delta t}(\mathrm{e}^{\delta\tau}\sup_{u_\tau\in D(s)}\|u_\tau\|^2) < \varepsilon^2/3, \qquad \forall\tau \leq \tau_*. \tag{3.13}$$

Set $N_* = \max\{N_2, N_3\}$. Then we conclude from (3.11)-(3.13) that if $N \geq N_*$ and $\tau \leq \tau_*$,

$$\sum_{n\in\mathbb{Z}}\chi(\frac{|n|}{N})u_n^2 \leq \varepsilon^2.$$

The proof of Lemma 3.1 is complete. □

By virtue of Lemmas 2.5, 3.1 and [41, Theorem 2.1], we obtain the following main result.

Theorem 3.2 *Assume the conditions* (F1)–(F4) *hold. Then the family of continuous processes* $\{U(t,\tau)\}_{t\geq\tau}$ *corresponding to Eqs.* (2.1)–(2.2) *possesses a pullback-*$\mathcal{D}_\delta$ *attractor* $\mathcal{A}_{\mathcal{D}_\delta}=\{\mathcal{A}_{\mathcal{D}_\delta}(t):t\in\mathbb{R}\}$ *so that*

(i) *Compactness:* $\mathcal{A}_{\mathcal{D}_\delta}(t)$ *is a nonempty compact subset of* E *for every* $t\in\mathbb{R}$*;*
(ii) *Invariance:* $U(t,\tau)\mathcal{A}_{\mathcal{D}_\delta}(s)=\mathcal{A}_{\mathcal{D}_\delta}(t)$ *for* $t\geq s$*;*
(iii) *Pullback attracting:* $\mathcal{A}_{\mathcal{D}_\delta}$ *is pullback-*$\mathcal{D}_\delta$ *attracting in the following sense that*

$$\lim_{\tau\to-\infty}\operatorname{dist}_E(U(t,\tau)D(\tau),\mathcal{A}_{\mathcal{D}_\delta}(t))=0,\qquad\forall\mathcal{D}=\{D(s):s\in\mathbb{R}\}\in\mathcal{D}_\delta,\qquad t\in\mathbb{R}.$$

4 Invariant Measures on the Pullback Attractor

In this section we prove the existence of a unique family of invariant Borel probability measures for the process $\{U(t,\tau)\}_{t\geq\tau}$ in E generated by equations (2.1)–(2.2). We first recall two basic definitions.

Definition 4.1 [10] A generalized Banach limit is any functional, which we denoted by $\mathrm{LIM}_{T\to+\infty}$, defined on the space of all bounded real-valued functions on $[0,+\infty)$ that satisfies

(i) $\mathrm{LIM}_{T\to+\infty}\psi(T)\geq 0$ for nonnegative functions ψ;
(ii) $\mathrm{LIM}_{T\to+\infty}\psi(T)=\lim_{T\to+\infty}\psi(T)$ if the usual limit $\lim_{T\to+\infty}\psi(T)$ exists.

Definition 4.2 [24] A process $\{U(t,\tau)\}_{t\geq\tau}$ on a metric space X is called τ-continuous, if for each $x_0\in X$ and each $t\in\mathbb{R}$, the X-valued function $\tau\mapsto U(t,\tau)x_0$ is continuous and bounded on $(-\infty,t]$.

Remark 4.3 Note that we study the pullback asymptotic behavior of (2.1)–(2.2) and we require the generalized limit as $\tau\to-\infty$. Thus for a real-valued function ψ defined on the interval $(-\infty,0]$ and a Banach limit $\mathrm{LIM}_{T\to+\infty}$, we define

$$\mathop{\mathrm{LIM}}_{\tau\to-\infty}\psi(\tau)=\mathop{\mathrm{LIM}}_{\tau\to+\infty}\psi(-\tau).$$

In the following we establish the existence of a unique family of invariant Borel probability measures corresponding to the process $\{U(t,\tau)\}_{t\geq\tau}$ in E. By Łukaszewicz and Robinson [24, Theorem 3.1], we need to show the τ-continuous property of $\{U(t,\tau)\}_{t\geq\tau}$.

Lemma 4.4 *Assume that* $u^{(1)}(t)$ *and* $u^{(2)}(t)$ *are two solutions of the system* (2.1)–(2.2) *with initial values* $u_\tau^{(1)}$ *and* $u_\tau^{(2)}$*, respectively. Then*

$$\|u^{(1)}(t)-u^{(2)}(t)\|\leq\mathrm{e}^{(2\gamma-2L_f)(t-\tau)}\|u_\tau^{(1)}-u_\tau^{(2)}\|,\qquad t\geq\tau.$$

Proof Assume that $u^{(i)}(t)$ are two solutions of (2.1)–(2.2) with initial values $u_\tau^{(i)}\in E$ for $i=1,2$. Let

$$w(t)=u^{(1)}(t)-u^{(2)}(t).$$

Then

$$i\dot{w}(t) + Aw(t) + f(u^{(1)}(t)) - f(u^{(2)}(t)) + i\gamma w(t) = 0. \tag{4.1}$$

Taking the imaginary part of the inner product $(\cdot, \cdot)$ of (4.1) with w in ℓ^2 gives that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \mathbf{Im}(Aw, w) + \mathbf{Im}\left(f(u^{(1)}(t)) - f(u^{(2)}(t)), w\right) + \gamma\|w\|^2 = 0. \tag{4.2}$$

By (F1), we see that

$$\mathbf{Im}\left(f(u^{(1)}(t)) - f(u^{(2)}(t)), w\right) \leq L_f\|w\|^2. \tag{4.3}$$

Thus it follows from (2.10), (4.1) and (4.2) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + (2\gamma - 2L_f)\|w\|^2 \leq 0. \tag{4.4}$$

Applying Gronwall inequality to (4.4), we obtain

$$\|u^{(1)}(t) - u^{(2)}(t)\| \leq \mathrm{e}^{(2\gamma - 2L_f)(t-\tau)}\|u_\tau^{(1)} - u_\tau^{(2)}\|,$$

which completes the proof of this lemma. □

Lemma 4.5 *Assume that the assumptions* (F1)–(F4) *hold. Then for every fixed* $u_* \in E$ *and* $t \in \mathbb{R}$*, the* E*-valued function* $\tau \to U(t, \tau)u_*$ *is continuous and bounded on* $(-\infty, t]$.

Proof Given $u_* \in E$ and $t \in \mathbb{R}$. In what follows we shall show that for every $\varepsilon > 0$ and $s \leq t$, there is $\epsilon > 0$ so that if $r \in (-\infty, t]$ with $|r - s| < \epsilon$,

$$\|U(t, r)u_* - U(t, s)u_*\|_E < \varepsilon.$$

Without loss of generality we assume that $r < s$. Set

$$U(\cdot, s)U(s, r)u_* = u_*^{(1)}(\cdot), \qquad U(\cdot, s)U(r, r)u_* = u_*^{(2)}(\cdot).$$

By virtue of Lemma 4.4 and the property of the process $\{U(t, \tau)\}_{t\geq\tau}$, we obtain

$$\begin{aligned}\|U(t, r)u_* - U(t, s)u_*\| &= \|U(t, s)U(s, r)u_* - U(t, s)U(r, r)u_*\| \\ &\leq \mathrm{e}^{(2\gamma - 2L_f)(t-s)}\|U(s, r)u_* - U(r, r)u_*\|. \end{aligned} \tag{4.5}$$

Thus one can immediately conclude from (4.5) that if $|r - s|$ is sufficiently small, right hand side of (4.5) is as small as need. This shows that the E-valued function $\tau \to U(t, \tau)u_*$ is continuous with respect to $\tau \in (-\infty, t]$.

Now let us check that the E-valued function $\tau \to U(t,\tau)u_*$ is bounded on $(-\infty, t]$. Assume that u_* and $t \in \mathbb{R}$ are given as above. Observing that $u_\tau \in D(\tau)$, one can easily deduce from Lemma 2.3 and the assumption (F4) that

$$\begin{aligned}\lim_{\tau\to-\infty}\|U(t,\tau)u_*\|^2 &\le \lim_{\tau\to-\infty} \mathrm{e}^{-\delta(t-\tau)}\|u_\tau\|^2 + \frac{\mathrm{e}^{-\delta t}}{\gamma}\int_{-\infty}^{t}\mathrm{e}^{\delta s}\|g(s)\|^2\mathrm{d}s + \frac{2K_2^2}{\delta\gamma}\\ &= \frac{\mathrm{e}^{-\delta t}}{\gamma}\int_{-\infty}^{t}\mathrm{e}^{\delta s}\|g(s)\|^2\mathrm{d}s + \frac{2K_2^2}{\delta\gamma}. \qquad (4.6)\end{aligned}$$

Because the E-valued function $\tau \to U(t,\tau)u_*$ is continuous on $(-\infty, t]$ in E, we conclude from (4.6) that the E-valued function $\tau \to U(t,\tau)u_*$ is bounded on $(-\infty, t]$. The proof is complete. □

Thanks to Lemma 4.5, Theorem 3.2 and [24, Theorem 3.1], we have the following main result.

Theorem 4.6 *Let the assumptions* (F1)–(F4) *hold. Assume that* $\{U(t,\tau)\}_{t\ge\tau}$ *is the process generated by the system* (2.1)–(2.2) *and that* $\mathcal{A}_{\mathcal{D}_\delta} = \{\mathcal{A}_{\mathcal{D}_\delta}(t) : t \in \mathbb{R}\}$ *is the pullback-*$\mathcal{D}_\delta$ *attractor obtained by Theorem* 3.2. *Fix a generalised Banach limit* $\mathrm{LIM}_{T\to+\infty}$ *and let* $\varphi(\cdot) : \mathbb{R} \mapsto E$ *be a continuous map satisfying* $\varphi(\cdot) \in \mathcal{D}_\delta$. *Then there exists a unique family of Borel probability measures* $\{\mu_t\}_{t\in\mathbb{R}}$ *in* E *so that the support of the measure* μ_t *is contained in* $\mathcal{A}_{\mathcal{D}_\delta}(t)$ *and*

$$\mathop{\mathrm{LIM}}_{\tau\to-\infty}\frac{1}{t-\tau}\int_\tau^t \phi(U(t,s)\varphi(s))\mathrm{d}s = \int_{\mathcal{A}_{\mathcal{D}_\delta}(t)}\phi(z)\mathrm{d}\mu_t(z) = \int_E \phi(z)\mathrm{d}\mu_t(z)$$

for every real-value continuous functional ϕ *on* E. *Moreover,* μ_t *is invariant in the sense that*

$$\int_{\mathcal{A}_{\mathcal{D}_\delta}(t)}\phi(z)\mathrm{d}\mu_t(z) = \int_{\mathcal{A}_{\mathcal{D}_\delta}(\tau)}\phi(U(t,\tau)z)\mathrm{d}\mu_\tau(z), \qquad t \ge \tau.$$

Furthermore, if Φ *is a real-valued continuous and bounded functional on* E, *then*

$$\mathop{\mathrm{LIM}}_{\tau\to-\infty}\frac{1}{t-\tau}\int_\tau^t\int_E \Phi(U(t,s)z)\mathrm{d}\mu_s(z)\mathrm{d}s = \int_{\mathcal{A}_{\mathcal{D}_\delta}(t)}\Phi(z)\mathrm{d}\mu_t(z). \qquad (4.7)$$

5 Statistical Solutions

In this section we further verify that the invariant measure $\{\mu_t\}_{t\in\mathbb{R}}$ given in Theorem 4.6 is actually a statistical solution for (2.1)–(2.2). Let E be the Hilbert space introduced in Sect. 2 and E^* denote its dual. Let $\langle\cdot,\cdot\rangle$ denote the dual product between E^* and E.

For convenience, we also consider the following equivalent system:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F(u,t), \tag{5.1}$$

$$u(\tau) = u_\tau, \tag{5.2}$$

where $F(u,t) = iAu + if(u) - \gamma u - ig(t)$.

We begin with some basic definitions on statistical solutions; see [10] for details.

By $\mathcal{T}$ we denote the class of real-valued functionals Ψ on E that are bounded on any bounded subset of E and satisfy the following conditions.

(1) For every $u \in E$, the Fréchet derivative $\Psi'(u)$ exists. Specifically, for each $u \in E$, there is an element $\Psi'(u) \in E^*$ satisfying

$$\frac{\|\Psi(u+h) - \Psi(u) - \langle \Psi'(u), h\rangle\|}{\|h\|} \to 0 \qquad \text{as} \qquad \|h\| \to 0, \quad h \in E;$$

(2) The mapping $u \mapsto \Psi'(u)$ is continuous and bounded from E to E^*.
It is trivial to see that if $\Psi \in \mathcal{T}$ and $u(t)$ is a solution of equations (5.1)–(5.2),

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(u(t)) = \langle \Psi'(u(t)), F(u(t),t)\rangle. \tag{5.3}$$

Definition 5.1 Assume that $\mathcal{A}_{\mathcal{D}_\delta} = \{\mathcal{A}_{\mathcal{D}_\delta}(t) : t \in \mathbb{R}\}$ is the pullback-$\mathcal{D}_\delta$ attractor obtained by Theorem 3.2. A family of Borel probability measures ν_t is said to be a statistical solution for the system (5.1)–(5.2), if it satisfies

(i) the function

$$t \mapsto \int_{\mathcal{A}_{\mathcal{D}_\delta}(t)} \Psi(\varphi)\mathrm{d}\nu_t(\varphi)$$

is continuous on $[\tau, +\infty)$ for every $\Psi \in \mathcal{T}$;

(ii) for almost every $t \in [\tau, +\infty)$, the function $\varphi \mapsto \langle F(\varphi,t), v\rangle$ is ν_t-integral for each $v \in E$, and the mapping

$$t \mapsto \int_{\mathcal{A}_{\mathcal{D}_\delta}(t)} \langle F(\varphi,t), v\rangle \mathrm{d}\nu_t(\varphi)$$

belongs to $L^1_{\mathrm{loc}}([\tau, +\infty))$ for any $v \in E$;

(iii) for every test function Υ in $\mathcal{T}$, then it holds that

$$\int_{\mathcal{A}_{\mathcal{D}_\delta}(t)} \Upsilon(\varphi)\mathrm{d}\nu_t(\varphi) - \int_{\mathcal{A}_{\mathcal{D}_\delta}(\tau)} \Upsilon(\varphi)\mathrm{d}\nu_\tau(\varphi)$$
$$= \int_\tau^t \int_{\mathcal{A}_{\mathcal{D}_\delta}(\theta)} \langle \Upsilon'(\varphi), F(\varphi,\theta)\rangle \mathrm{d}\mu_\theta(\varphi)\mathrm{d}\theta$$

for all $t \geq \tau$.

Theorem 5.2 *Assume the conditions* (F1)–(F4) *hold. Then the family of invariant measures* $\{\mu_t\}_{t\in\mathbb{R}}$ *obtained in Theorem* 4.6 *are statistical solutions of the system* (5.1)–(5.2).

Proof We prove that the family of invariant measures $\{\mu_t\}_{t\in\mathbb{R}}$ obtained in Theorem 4.6 satisfy the conditions (i)–(iii) in Definition 5.1.

Let $\mathcal{A}_{\mathcal{D}_\delta} = \{\mathcal{A}_{\mathcal{D}_\delta}(t) : t \in \mathbb{R}\}$ be the pullback-$\mathcal{D}_\delta$ attractor given by Theorem 3.2 and $\Psi \in \mathcal{T}$. Let us first show that $t \mapsto \int_{\mathcal{A}_{\mathcal{D}_\delta}(t)} \Psi(\varphi)\mathrm{d}\nu_t(\varphi)$ is continuous on $[\tau, +\infty)$ for $\Psi \in \mathcal{T}$. Indeed, by the invariant property of $\{\mu_t\}_{t\in\mathbb{R}}$, one can find that

$$\int_{\mathcal{A}_{\mathcal{D}_\delta}(t)} \Psi(u)\mathrm{d}\mu_t(u) = \int_{\mathcal{A}_{\mathcal{D}_\delta}(\tau)} \Psi(U(t,\tau)u)\mathrm{d}\mu_\tau(u), \qquad t \geq \tau.$$

Thus we deduce from the continuity of the process $\{U(t,\tau)\}_{t\geq\tau}$ and the definition of $\mathcal{T}$ that the function

$$t \mapsto \int_{\mathcal{A}_{\mathcal{D}_\delta}(t)} \Psi(u)\mathrm{d}\mu_t(u)$$

is continuous.

Secondly, for each fixed $v \in E$, define

$$\Psi_1(u) = \langle F(u,t), v\rangle, \qquad \forall u \in E. \tag{5.4}$$

Then Ψ_1 maps E to $\mathbb{R}$. We claim that Ψ_1 is continuous. Assume that B is a bounded subset of E and that $u_1, u_2 \in B$. Then we see from the proof of Lemma 2.2 that

$$\begin{aligned}\|\Psi_1(u_1) - \Psi_1(u_2)\| &= \|\langle F(u_1,t) - F(u_2,t), v\rangle\| \leq \|F(u_1,t) - F(u_2,t)\|\|v\| \\ &\leq \left(K_J + L_f + \gamma\right)\|u_1 - u_2\|\|v\|,\end{aligned}$$

which implies that Ψ_1 is continuous on E. Thus the function $u \mapsto \langle F(u,t), v\rangle$ is μ_t-integral for each $v \in E$. Consequently, we deduce from the proof of the assertion (i) in Definition 5.1 that the mapping

$$t \mapsto \int_{\mathcal{A}_{\mathcal{D}_\delta}(t)} \langle F(u,t), v\rangle\mathrm{d}\mu_t(u)$$

belongs to $L^1_{\mathrm{loc}}([\tau,\infty))$ for every $v \in E$.

Finally, it remains to prove that $\{\mu_t\}_{t\in\mathbb{R}}$ satisfies (iii) in Definition 5.1. Let $\Psi \in \mathcal{T}$ and $t \geq \tau$. By (5.3) we see that

$$\Psi(u(t)) - \Psi(u(\tau)) = \int_\tau^t \langle \Psi'(u(\theta)), F(u(\theta),\theta)\rangle\mathrm{d}\theta. \tag{5.5}$$

Let $s < \tau$, $u_0 \in E$ and $u(\theta) = U(\theta, s)u_0$ for $\theta \geq s$. Then it follows by (5.5) that

$$\Psi(U(t,s)u_0) - \Psi(U(\tau,s)u_0) = \int_\tau^t \langle \Psi'(U(\theta,s)u_0), F(U(\theta,s)u_0,\theta)\rangle \mathrm{d}\theta. \quad (5.6)$$

By (4.7), we find that

$$\begin{aligned}
&\int_{\mathcal{A}_{\mathcal{D}_\delta}(t)} \Psi(u)\mathrm{d}\mu_t(u) - \int_{\mathcal{A}_{\mathcal{D}_\delta}(\tau)} \Psi(u)\mathrm{d}\mu_\tau(u) \\
&= \operatorname*{LIM}_{M\to-\infty} \frac{1}{t-M}\int_M^t \int_E \Psi(U(t,s)u_0)\mathrm{d}\mu_s(u_0)\mathrm{d}s \\
&\quad - \operatorname*{LIM}_{M\to-\infty} \frac{1}{\tau-M}\int_M^\tau \int_E \Psi(U(\tau,s)u_0)\mathrm{d}\mu_s(u_0)\mathrm{d}s \\
&= \operatorname*{LIM}_{M\to-\infty} \\
&\frac{1}{t-M}\left[\int_M^\tau \int_E \Psi(U(t,s)u_0)\mathrm{d}\mu_s(u_0)\mathrm{d}s + \int_\tau^t \int_E \Psi(U(t,s)u_0)\mathrm{d}\mu_s(u_0)\mathrm{d}s\right] \\
&\quad - \operatorname*{LIM}_{M\to-\infty} \frac{1}{\tau-M}\int_M^\tau \int_E \Psi(U(\tau,s)u_0)\mathrm{d}\mu_s(u_0)\mathrm{d}s. \quad (5.7)
\end{aligned}$$

Noticing that

$$\int_E \Psi(U(t,s)u_0)\mathrm{d}\mu_s(u_0) = \int_{\mathcal{A}_{\mathcal{D}_\delta}(s)} \Psi(U(t,s)u_0)\mathrm{d}\mu_s(u_0) = \int_{\mathcal{A}_{\mathcal{D}_\delta}(t)} \Psi(u_0)\mathrm{d}\mu_t(u_0)$$

is independent of s, one deduces that

$$\operatorname*{LIM}_{M\to-\infty} \frac{1}{t-M}\int_\tau^t \int_E \Psi(U(t,s)u_0)\mathrm{d}\mu_s(u_0)\mathrm{d}s = 0. \quad (5.8)$$

Thus we conclude from (5.7), (5.8) and the Fubini's Theorem that

$$\begin{aligned}
&\int_{\mathcal{A}_{\mathcal{D}_\delta}(t)} \Psi(u)\mathrm{d}\mu_t(u) - \int_{\mathcal{A}_{\mathcal{D}_\delta}(\tau)} \Psi(u)\mathrm{d}\mu_\tau(u) \\
&= \operatorname*{LIM}_{M\to-\infty} \frac{1}{\tau-M}\int_M^\tau \int_E \big(\Psi(U(t,s)u_0) - \Psi(U(\tau,s)u_0)\big)\mathrm{d}\mu_s(u_0)\mathrm{d}s \\
&= \operatorname*{LIM}_{M\to-\infty} \frac{1}{\tau-M}\int_M^\tau \int_E \int_\tau^t \langle \Psi'(U(\theta,s)u_0), F(U(\theta,s)u_0,\theta)\rangle \mathrm{d}\theta\mathrm{d}\mu_s(u_0)\mathrm{d}s \\
&= \operatorname*{LIM}_{M\to-\infty} \frac{1}{\tau-M}\int_M^\tau \int_\tau^t \int_E \langle \Psi'(U(\theta,s)u_0), F(U(\theta,s)u_0,\theta)\rangle \mathrm{d}\mu_s(u_0)\mathrm{d}\theta\mathrm{d}s.
\end{aligned} \quad (5.9)$$

Using the properties of the process and $\{\mu_t\}_{t\in\mathbb{R}}$ in Theorem 4.6, one can obtain

$$\begin{aligned}&\int_E \langle \Psi'(U(\theta,s)u_0), F(U(\theta,s)u_0,\theta)\rangle \mathrm{d}\mu_s(u_0)\\&=\int_E \langle \Psi'(U(\theta,\tau)U(\tau,s)u_0), F(U(\theta,\tau)U(\tau,s)u_0,\theta)\rangle \mathrm{d}\mu_s(u_0)\\&=\int_E \langle \Psi'(U(\theta,\tau)u_0), F(U(\theta,\tau)u_0,\theta)\rangle \mathrm{d}\mu_\tau(u_0).\end{aligned}$$

Because

$$\int_E \langle \Psi'(U(\theta,\tau)u_0), F(U(\theta,\tau)u_0,\theta)\rangle \mathrm{d}\mu_\tau(u_0)$$

is independent of s, one can see from (5.9) that

$$\begin{aligned}&\int_{\mathcal{A}_{\mathcal{D}_\delta}(t)} \Psi(u)\mathrm{d}\mu_t(u) - \int_{\mathcal{A}_{\mathcal{D}_\delta}(\tau)} \Phi(u)\mathrm{d}\mu_\tau(u)\\&=\int_\tau^t \int_E \langle \Psi'(U(\theta,\tau)u_0), F(U(\theta,\tau)u_0,\theta)\rangle \mathrm{d}\mu_\tau(u_0)\mathrm{d}\theta\\&=\int_\tau^t \int_E \langle \Psi'(u_0), F(u_0,\theta)\rangle \mathrm{d}\mu_\theta(u_0)\mathrm{d}\theta,\end{aligned}$$

which justifies the validity of (iii) in Definition 5.1. □

Remark 5.3 Although in this paper we study $m \in \mathbb{Z}$ (corresponding to the spatial domain $\mathbb{R}$) in equations (1.1)–(1.2), our main results on invariant measures and statistical solutions are still valid if $m \in \mathbb{Z}^k$ for some positive integer $k \geq 2$ (corresponding to the spatial domain $\mathbb{R}^k$). The interested reader is referred to Remark 4.1 in [41] for details.

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Declarations

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