A Flexible Bounded Distribution: Information Measures and Lifetime Data Analysis

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Abstract
This article introduces the Fréchet log-logistic distribution (FLD) model, which offers a notable, bounded, and flexible distribution for modeling increasing and bathtub shapes failure rate phenomena. Compared to well-known distributions like Weibull and Fréchet distributions, the proposed FLD model provides a more adaptable solution to lifetime data modeling. Various mathematical and statistical characteristics of the model, including the hazard function, percentile function, moment generating function, entropy, and characterization, are considered. Real-life data applications of the FLD model are also explored using four data sets, and the results show that the proposed model is a viable alternative to existing lifetime probability models.

Keywords Odd link function · Hazard rate function · Reliability · Goodness-of-fit statistics · Mills ratio · Information criterion · Lifetime data analysis

Mathematics Subject Classification 60E05 · 62E10 · 62N05

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1 Introduction

In real-world lifetime phenomena, researchers often face a finite range of variations. Bounded domain distributions result from such finite variations. For a detailed discussion of such distributions, please refer to [1–4]. Bounded domain distributions are valuable in investigating lifetime phenomena such as yearly flood flow, yearly precipitation data, and reliability analysis. The best conventional bounded domain probability models in this regard are the beta, triangular, epsilon, PERT, uniform, arcsin, Bates, Topp–Leone, Kumaraswamy, triangular, von Mises distributions, epsilon-Lindley, and raised cosine. Applications of such distributions include wavelet, beamforming, and pattern synthesis, as well as storage volume of a reservoir of capacity $0 < z < z_{\max}$, modeling failure rate phenomena such as bathtub shape (BTS), inverted bathtub shape (IBTS), increasing failure rate (IFR), decreasing failure rate (DFR), working life expectancy modeling, and environmental data assessment.

Our motivational tactic starts with an overview of constrained function. The earliest phase on this path is to pick up the odd link function $\mathcal{D}(x; \Theta) = ln\left(\frac{1-\mathcal{G}(x)}{\mathcal{G}(x)}\right)$, which fulfills the provisions such as (i) $\mathcal{D}(x)$ is differentiable and monotonically non-decreasing (ii) $\mathcal{D}(x) \to a$ as $x \to 0$ and $\mathcal{D}(x) \to b$ as $x \to \infty$ with a baseline cumulative distribution function (CDF) $\mathcal{G}(x)$. Our second motivation deals with the selection of baseline distribution, i.e., the log-logistic distribution (LLD) studied by Verhulst [5], which has wide applications in modeling stochastic lifetime phenomena, such as in time-to-event analysis, especially in survival analysis, color, e.g., the modeling of mortality rate after cancer diagnosis or treatment [6], demography for modeling population, economics for the distribution of wealth or income inequality, engineering for reliability analysis, and hydrology for modeling stream flow rates and precipitation [6]. In this regard, we took the CDF of LLD, which is constructed on the interval $(0, \infty)$ as $\mathcal{G}(x|\alpha, \beta) = \frac{x^\beta}{x^\beta + \alpha^\beta}$. Incorporating the baseline CDF into $\mathcal{D}(x|\alpha, \beta, \theta)$, yields $\mathcal{D}(x|\alpha, \beta, \theta) = \beta ln(\frac{\alpha}{x})$. The next motivation is the extraction of the CDF of Fréchet distribution (FD), a model of extreme values (EVs). Extreme events are rare, but they may have a very high impact on the observed experiments [7]. This model has been used in modeling the data that appear in various spheres of life such as accelerated life monitoring, sea waves, horse racing, rainfall, environmental disasters, earthquakes, wind speeds, sea currents, track race records, relief periods and survival times data, and so on. Although the above model has multiple applications in various fields of study, it must first be defined as follows to extract it:

$$f(t; \theta) = \theta t^{-\theta-1} e^{-t^{-\theta}}, t > 0, \theta > 0,$$

with the corresponding CDF given by

$$e^{-(\mathcal{D}(x;\alpha,\beta))^{-\theta}} = \int_0^{\mathcal{D}(x;\alpha,\beta)} \theta y^{-\theta-1} e^{-y^{-\theta}} \mathrm{d}y, \alpha > 0, \beta > 0, \theta > 0,$$

However, it is important to note that the above is not a CDF according to the defined odd link function. To propose a bounded CDF under the name FLD, we must first

introduce a constraint $0 < x < \alpha$, then define the CDF as

$$\mathcal{F}(x; \alpha, \beta, \theta) = 1 - e^{-(\mathcal{D}(x, \alpha, \beta))^{-\theta}};$$

on incorporating the $\mathcal{D}(.)$ into the above expression, we get

$$\mathcal{F}_{\mathrm{FLD}}(x; \alpha, \beta, \theta) = 1 - e^{-(\beta \, ln(\frac{\alpha}{x}))^{-\theta}}, \beta > 0, \theta > 0, 0 < x < \alpha, \quad (1)$$

which after differentiation yields the PDF of FLD as

$$f_{\mathrm{FLD}}(x; \alpha, \theta, \beta) = \frac{\beta \theta}{x} \left(\beta \, ln \left(\frac{\alpha}{x} \right) \right)^{-\theta-1} e^{-(\beta \, ln(\frac{\alpha}{x}))^{-\theta}}. \quad (2)$$

An interesting feature appears when we set $\alpha = 1$, which in turn yields a two-parameter unit FLD defined as

$$f_{\mathrm{FLD}}(x; 1, \theta, \beta) = \frac{\beta \theta}{x} (\beta \, ln(\frac{1}{x}))^{-\theta-1} e^{-\left(\beta \, ln \left(\frac{1}{x} \right) \right)^{-\theta}}, \beta > 0, \theta > 0, 0 < x < 1. \quad (3)$$

Similarly, the survival function (SF) is

$$\mathcal{SF}_{\mathrm{FLD}}(x; \alpha, \theta, \beta) = e^{-(\beta \, ln(\frac{\alpha}{x}))^{-\theta}}. \quad (4)$$

The hazard rate function (HRF) describes the immediate future failure given that the unit has not failed at time x, and stated as

$$\mathcal{H}_{\mathrm{FLD}}(x; \alpha, \beta, \theta) = \frac{\beta \theta}{x} \left(\beta \, ln \left(\frac{\alpha}{x} \right) \right)^{-\theta-1}. \quad (5)$$

The reversed hazard rate function (RHRF) describes the probability of an immediate past failure, given that the unit has already failed at time x. For the FLD in Eq. (3), the RHRF is

$$\mathfrak{R}_{\mathrm{FLD}}(x; \alpha, \beta, \theta) = \frac{\beta \theta}{x \left(1 - e^{-(\beta \, ln(\frac{\alpha}{x}))^{-\theta}} \right)} \left(\beta \, ln \left(\frac{\alpha}{x} \right) \right)^{-\theta-1} e^{-(\beta \, ln(\frac{\alpha}{x}))^{-\theta}}. \quad (6)$$

Likewise, the concerned functions, for instance, the PDF and HRF, are highly adaptable in respective performance, will be depicted in the coming sections. More explicitly, the PDF takes on several forms, similar to symmetrical, right skewed, left skewed, L-shape and U-shape curves, see Figs. 1 and 2. Besides, we observe a positive, negative skewness, leptokurtic, mesokurtic, and platykurtic nature of the curvature, which undoubtedly suggests that it is conceived to establish the light and thick tailed prodigy, see Fig. 4. Such incidents are usually widespread in reliability presentations, queuing theory, and environmental phases. Our application segment will assist the readers for realistic predictions about the subsequent generation's future in a healthier sense. Unlike the Weibull distribution, the suggested model may show the BTS,
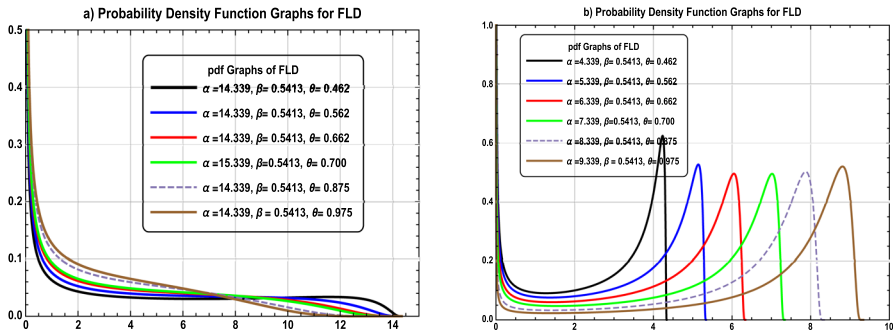
Fig. 1 FLD PDF graphs

which is uncommon in competing models such as LLD, gamma distribution (GD), inverse gamma distribution (IGD), three-parameter Weibull distribution (3P WD), three-parameter Kappa (3P Kappa) distribution, exponentiated exponential distribution (EED), FD and log-normal distribution (LND). One can visualize these eminent qualities from HRF plots, probability graphs and four real lifetime data applications, and it also has closed-form CDF and HRF functions. It can efficiently be used in lifetime data analysis, whose detail is given in the application section, which endorses the suitability of the proposed model both in IFR and BTS functions. Also, it portrays negatively, positively, and symmetric shapes of a density function, which is not seen in the corresponding models except MWD. Fourthly, due to its closed CDF or SF one can easily generates random numbers from it. A final motivation is its flexibility in exhibiting a unit domain distribution by fixing $\alpha = 1$. In addition, the notations and acronyms used in this study are listed in Appendix-A_0.

The rest of the manuscript is managed as follows. Section 2 is reserved for studying the curve behavior of PDF and HRF as well as characterizations based of five different lifetime gadgets. The percentile function, rth moment via moment generating function, various entropies, Lorenz, Bonferroni, scaled total time on test transform (TTT), conditional moments, mean deviation (MD), residual life (RL) functions, stress, and strength function are studied in Sect. 3. Section 4 studies the estimation of parameters. Section 5 deals with the competing models and applications. The conclusion is drawn in Sect. 6.

2 Curve Behavior of PDF and HRF Along with Characterizations

2.1 Shapes and Distribution of Curve

In this segment, we shall scrutinize the graphs of PDF and HRF of the FLD distribution as well as the characterization constructed on failure rate functions. For exploring the curve behavior of PDF and HRF, we shall first define the logarithmic form of Eq. (2), which after differentiation about x and equating to zero yields
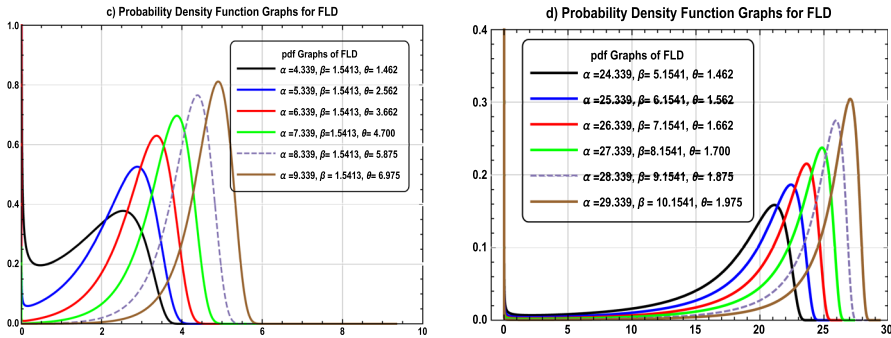
Fig. 2 FLD PDF graphs

$$\frac{d}{dx}(ln(f(x))) = (ln(f(x)))' = -\frac{1}{x} + \frac{1+\theta}{x \, ln\left(\frac{\alpha}{x}\right)} - \frac{\beta\theta \left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-1-\theta}}{x} = 0. \quad (7)$$

Since a closed-form solution is not possible, computational package can be used to find the mode numerically. Furthermore, we observed that $(ln(f(x)))'$, at extreme points exhibit a non-monotone behavior. For $\theta = 0$, the mode of the distribution lies at $x = \alpha e^{-1}$. It is also observed that

$$\lim_{x \to 0} e^{-(\beta ln \frac{\alpha}{x})^{-\theta}} \sim e^{-\left(\frac{\beta}{x}\right)^{-\theta}}, \ \lim_{x \to \alpha} e^{-(\beta ln \frac{\alpha}{x})^{-\theta}} \sim e^{-(\beta(x-a))^{-\theta}}.$$

So, as $x \to 0$, we have

$$\lim_{x \to 0} f(x) \sim \frac{\beta\theta}{x} \left(\frac{\beta}{x}\right)^{-\theta-1} e^{-\left(\frac{\beta}{x}\right)^{-\theta}}, \quad 0 < x < \infty,$$

and

$$\lim_{x \to \alpha} f(x) \sim \beta\theta \left(\beta(x-\alpha)\right)^{-\theta-1} e^{-(\beta(x-\alpha))^{-\theta}} \quad 0 < x < \alpha. \quad \text{as} \quad x \to \alpha.$$

Consequently, the HRF asymptotic form, as $x \to 0$, is

$$\mathcal{H}(x) \sim \frac{\beta\theta}{x} \left(\frac{\beta}{x}\right)^{-\theta-1}, 0 < x < \infty,$$

and as $x \to \alpha$

$$\mathcal{H}(x) \sim \beta\theta \left(\beta(x-\alpha)\right)^{-\theta-1}, 0 < x < \alpha.$$
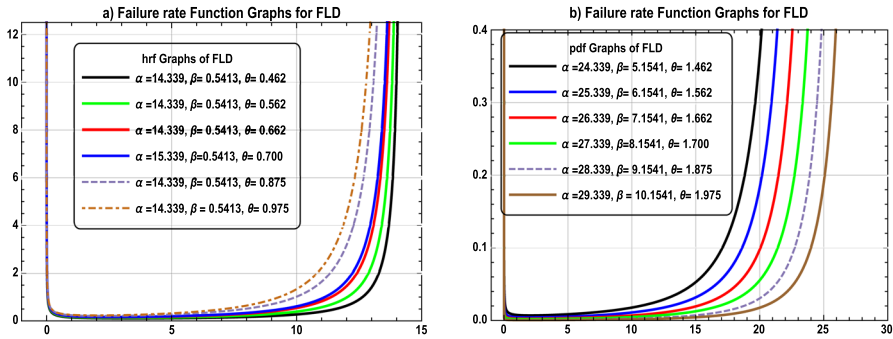
Fig. 3 FLD HRF graphs

After the differentiation of Eq. (5) with respect to x and setting the resultant derivative to zero, we get

$$\frac{d}{dx}(ln(\mathcal{H}(x))) = \frac{\theta\left(1+\theta-ln\left(\frac{\alpha}{x}\right)\right)\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}{x^2 ln\left(\frac{\alpha}{x}\right)^2} = 0. \qquad (8)$$

From the earlier equation, the mode of HRF is $x = \alpha e^{-\theta-1}$, which depends upon on α and θ only. Moreover, its second derivative is

$$\frac{d^2}{dx^2}(ln(\mathcal{H}(x))) = \frac{\theta\left((1+\theta)(2+\theta)-3(1+\theta)ln\left(\frac{\alpha}{x}\right)+2ln\left(\frac{\alpha}{x}\right)^2\right)}{x^3\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^\theta (ln\left(\frac{\alpha}{x}\right))^3} = 0, \qquad (9)$$

which implies that the change points (CP) are: $CP_1 = \alpha e^{\frac{1}{4}\left(-3-3\theta-\sqrt{-7-6\theta+\theta^2}\right)}$ and $CP_2 = \alpha e^{\frac{1}{4}\left(-3-3\theta+\sqrt{-7-6\theta+\theta^2}\right)}$ (Fig. 3).

2.2 Characterizations

As lifetime phenomena are usually related to reliability and HRF that should be as far as possible to be realistic, such a realistic mechanism can only be obtained if the model is characterizable. Characterization assists scholars in finding the genuine model by using any particular property. For a comprehensive study, readers are referred to [8–10]. In this regard, we characterize the proposed model by statistical lifetime gadgets like hazard rate, reverse hazard rate, Mills' ratio, and elasticity functions which are quite well known in physical sciences and criminological sciences as well as in lifetime data analysis. In this regard, respective characterizing conditions are defined in different propositions one by one as follows.

Definition 1 Let $X : \mathcal{S} \longrightarrow (0, \infty)$ be a continuous stochastic variate with PDF $f(x)$ iff the HRF fulfills the subsequent equation

$$\frac{f'(x)}{f(x)} = \frac{\mathcal{H}'(x)}{\mathcal{H}(x)} - \mathcal{H}(x),$$

and RHRF $\mathfrak{R}(x)$ fulfills the stated condition

$$\frac{f'(x)}{f(x)} = \frac{\mathfrak{R}'(\mathfrak{x})}{\mathfrak{R}(x)} + \mathfrak{R}(x),$$

respectively.

Proposition 1 *Let $X : \mathcal{S} \longrightarrow (0, \infty)$ be a continuous stochastic variate with the PDF* (2)*iff its HRF defined by* 5 *is twice differentiable and satisfies the expression*

$$\frac{\mathcal{H}'(x)}{(\mathcal{H}(x))^2} = \frac{\left(1 + \theta - ln\frac{\alpha}{x} - \theta(\beta ln(\frac{\alpha}{x}))^{-\theta}\right) x}{ln(\frac{\alpha}{x})\beta\theta(\beta ln(\frac{\alpha}{x}))^{-\theta-1} e^{-(\beta ln(\frac{\alpha}{x}))^{-\theta}}\beta\theta(\beta ln(\frac{\alpha}{x}))^{-\theta-1}} + 1. \quad (10)$$

Proof The proof is given in Appendix A1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Proposition 2 *Let $X : \mathcal{S} \longrightarrow (0, \infty)$ be a continuous stochastic variate with the PDF* 2 *iff Eq.* 6 *is twice differentiable and satisfies the given expression*

$$\frac{\mathfrak{R}'(x)}{(\mathfrak{R}(x))^2} = \frac{e^{-(\beta ln(\frac{\alpha}{x}))^{-\theta}}\theta\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-2\theta}\left(-\theta + (1 + \theta - ln\left(\frac{\alpha}{x}\right))\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{\theta}\right)\left(1 - e^{-(\beta ln(\frac{\alpha}{x}))^{-\theta}}\right)}{\left(ln\left(\frac{\alpha}{x}\right)\right)^2\left(\beta\theta\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-1-\theta}e^{-(\beta ln(\frac{\alpha}{x}))^{-\theta}}\right)^2} - 1. \quad (11)$$

Proof The proof is given in Appendix A1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Definition 2 Let $X : \mathcal{S} \longrightarrow (0, \infty)$ be a continuous stochastic variate having absolutely continuous CDF and PDF iff the Mills ratio defined as: $\mathfrak{M}(x) = \frac{1}{\mathcal{H}(x)}$ is twice differentiable function and satisfies the following equation:

$$\frac{f'(x)}{f(x)} + \frac{\mathfrak{M}'(x) + 1}{\mathfrak{M}(x)} = 0.$$

Proposition 3 *Let $X : \mathcal{S} \longrightarrow (0, \infty)$ be a continuous stochastic variate with PDF as defined in Eq.* 2 *iff its Mills ratio $\mathfrak{M}(x) = \frac{x}{\beta\theta(\beta ln(\frac{\alpha}{x}))^{-theta-1}}$ is twice differentiable and satisfies*

$$\mathfrak{M}'(x) = \frac{e^{-(\beta ln(\frac{\alpha}{x}))^{-\theta}}\theta\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-2\theta}\left(-\theta + (1 + \theta - ln\left(\frac{\alpha}{x}\right))\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{\theta}\right)\left(e^{-(\beta ln(\frac{\alpha}{x}))^{-\theta}}\right)}{\left(ln\left(\frac{\alpha}{x}\right)\right)^2\left(\beta\theta\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-1-\theta}e^{-(\beta ln(\frac{\alpha}{x}))^{-\theta}}\right)^2} + 1. \quad (12)$$

Proof The proof is given in Appendix A1.   □

Definition 3 Let $X : S \longrightarrow (0, \infty)$ be an absolutely continuous stochastic variate with CDF and PDF, the elasticity function, $\mathfrak{E}(x) = x\mathfrak{R}(x)$, of a twice differentiable distribution function satisfies the differential equation,

$$\frac{f'(x)}{f(x)} = \frac{\mathfrak{E}'(x)}{\mathfrak{E}(x)} + \frac{\mathfrak{E}(x)}{x} - \frac{1}{x}.$$

Proposition 4 *Let* $X : S \longrightarrow (0, \infty)$ *be an absolutely continuous random variate with PDF and CDF as defined in Eqs. 2 and 4 iff its elasticity function is defined* $\mathfrak{E}(x) = x\mathfrak{R}(x)$ *which is twice differentiable and satisfies the given expression*

$$\mathfrak{E}'(x) - \left(-\frac{1}{x} + \frac{1+\theta}{x \ln\left[\frac{\alpha}{x}\right]} - \frac{\beta\theta \left(\beta \ln\left[\frac{\alpha}{x}\right]\right)^{-1-\theta}}{x} - \frac{e^{-\left(\beta \ln\left[\frac{\alpha}{x}\right]\right)^{-\theta}} \beta\theta \left(\beta \ln\left[\frac{\alpha}{x}\right]\right)^{-1-\theta}}{\left(1 - e^{-\left(\beta \ln\left[\frac{\alpha}{x}\right]\right)^{-\theta}}\right) x} \right) \mathfrak{E}(x)$$

$$= \frac{\beta\theta (\beta \ln(\frac{\alpha}{x}))^{-\theta-1} e^{-\left(\beta \ln(\frac{\alpha}{x})\right)^{-\theta}}}{x(1 - e^{-\left(\beta \ln(\frac{\alpha}{x})\right)^{-\theta}})}. \tag{13}$$

Proof The proof is given in Appendix A1.   □

The following theorem was used in [11] to characterize different univariate continuous distributions. Here, we discuss characterizations of FLD distributions through Theorem [11] on the basis of simple relationship between two functions of X.

Theorem 1 *Let* $(S : \mathcal{F}; P)$ *be a given probability space, and let* $\mathbb{W} = [c, d]$ *be an interval for some* $c < d (c = -\infty; d = \infty$ *might as well be allowed). Let* $X : S \to \mathbb{W}$ *be a continuous random variable with distribution function* $\mathcal{F}$, *and let p and q be two real functions defined on* $\mathbb{W}$ *such that*

$$\mathbb{E}\left(p(X)|X \geq x\right) = \mathbb{E}\left(q(X)|X \geq x\right) \psi(x), x \in \mathbb{W},$$

is defined with some real function ψ. *Assume that* $p, q \in \mathcal{C}^1(\mathbb{W})$, $\lambda \in \mathcal{C}^2(\mathbb{W})$ *and* $\mathcal{F}$ *is a twice continuously differentiable and strictly monotone function on the set* $\mathbb{W}$. *Finally, assume that the equation* $q\psi = p$ *has no real solution in the interior of* $\mathbb{W}$. *Then* $\mathcal{F}$ *is uniquely determined by the functions* p, q *and* ψ, *particularly*

$$\mathcal{F} = \int_0^x \mathcal{C}\left|\frac{\psi'(u)}{\psi(u)q(u) - p(u)}\right| exp(-r(u)) du,$$

where the function r is a solution of the differential equation $r' = \frac{\psi'q}{\psi q - p}$ *and* $\mathcal{C}$ *is a constant to make* $\int_{\mathbb{W}} d\mathcal{F} = 1$.

Proposition 5 *Let* $X : S \longrightarrow (0, \infty)$ *be a continuous random variable, and let* $q(x) = \frac{3}{e^{2(\beta \ln \frac{\alpha}{x})-\theta}}$ *and* $p(x) = \frac{2}{e^{(\beta \ln \frac{\alpha}{x})-\theta}}$ *for* $x \in (0; \alpha)$. *The random variable X has*

PDF (2) *iff there exist functions ψ and p defined in Glanzel Theorem satisfying the differential equation*

$$\frac{\psi'(x)}{\psi(x)q(x) - p(x)} = \frac{e^{2(\beta ln(\frac{\alpha}{x}))^{-\theta}}\beta\theta(\beta ln(\frac{\alpha}{x}))^{-\theta-1}}{x}. \quad 0 < x < \alpha \qquad (14)$$

Proof The proof is given in Appendix A1. □

Corollary 1 *Let $X; \mathcal{S} \longrightarrow (0, \infty)$ be a continuous random variable, and let $q(x) = \frac{3}{e^{2(\beta ln\frac{\alpha}{x})^{-\theta}}}$ and $p(x) = \frac{2}{e^{(\beta ln\frac{\alpha}{x})^{-\theta}}}$ for $x \in (0; \alpha)$. The random variable X has the PDF* (2) *iff the ψ has the form $\psi(x) = e^{(\beta ln\frac{\alpha}{x})^{-\theta}}$.*

Remark 1 Since solution of differential Eq. 14 is

$$\psi(x) = \int \frac{e^{(\beta ln(\frac{\alpha}{u}))^{-\theta}}\beta\theta(\beta ln(\frac{\alpha}{u}))^{-\theta-1}}{u}du + D,$$

where D is an integral constant.

3 Lifetime Variate's Mathematical and Statistical Properties

This section is reserved only for the mathematical and statistical properties of the proposed model, which are quite helpful in the lifetime phenomenon. These properties include the quantile, moment generating function, moments, conditional moments, mean deviation, entropy, residual and reverse residual, Bonferroni, Lorenz curves, scaled TTT, and stress and strength probability.

3.1 Percentile Function

Let X be a continuous variate with CDF $\mathcal{F}_X : R \rightarrow [0, 1]$. Now, from this definition a percentile function $\mathcal{P}$ generally sends back a threshold measurement x. In this regard, inverse of the FLD percentile function yields $x = \mathcal{P}(p)$ as follows:

$$\mathcal{P}_p = \mathcal{F}^{-1}(p) = \alpha e^{-\frac{(-1)^{-1/\theta}ln[1-p]^{-1/\theta}}{\beta}}, \qquad (15)$$

where $p \in (0, 1)$. The median is $\alpha e^{-\frac{(-1)^{-1/\theta}ln[0.5]^{-1/\theta}}{\beta}}$.

3.2 Raw Moments from Moment Generating Function

In physical sciences, computing engineering, and environmental modeling, the term raw moment of any model is a computable measure related to the center, dispersion, skewness, and kurtosis of the model's graph. It not only addresses the curve behavior of a function but also assists in characterizing the probability functions. In order to
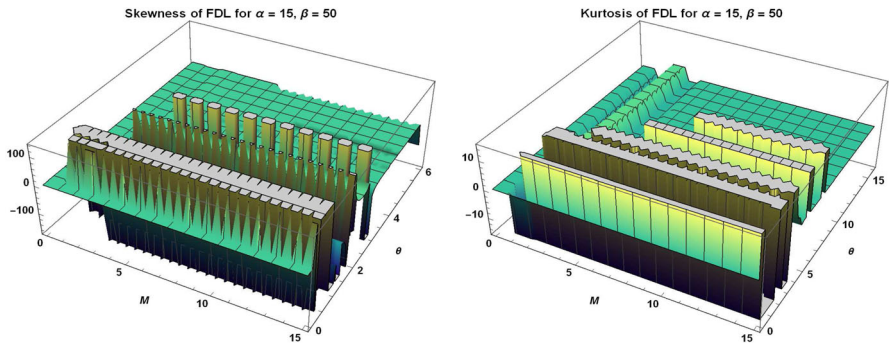
Fig. 4 Skewness and Kurtosis graphs

generate moments from the MGF, we shall first define the MGF of a continuous random variable X~ FLD(α, β, θ) as $\mathcal{M}_X(t) = \mathbb{E}(e^{tX}) = \int_0^\alpha e^{tx} f(x) dx$, where $|t| < 1$. Now, on incorporating Eq. (2) we get

$$\mathcal{M}_X(t) = \int_0^\alpha e^{tx} \frac{\beta\theta}{x} (\beta \ln(\frac{\alpha}{x}))^{-\theta-1} e^{-(\beta \ln(\frac{\alpha}{x}))^{-\theta}} dx.$$

Let $y = \left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}$, $dy = \frac{\theta\beta(\beta ln(\frac{\alpha}{x}))^{-\theta-1}dx}{x}$, if $x = 0, y = 0, x = \alpha, y = \infty$, so when $x = \alpha e^{-\frac{y^{-1/\theta}}{\beta}} \alpha$, on simplification we get

$$\mathcal{M}_X(t) = \sum_{r=0}^\infty \sum_{m=0}^\infty \frac{\alpha^r \left(\frac{-r}{\beta}\right)^m}{m!} \Gamma\left[1 - \frac{m}{\theta}\right] \frac{t^r}{r!}.$$

The rth raw moment for FLD distribution can be expressed as the coefficient of $\frac{t^r}{r!}$

$$\mu'_r = \sum_{m=0}^\infty \frac{\alpha^r \left(\frac{-r}{\beta}\right)^m}{m!} \Gamma\left[1 - \frac{m}{\theta}\right],$$

where $\Gamma[a] = \int_0^\infty t^{a-1} e^{-t} dt$ is a gamma function. On the basis of these, we are able to assess the skewness (departure from symmetry, $\frac{\mu_3}{\mu_2^{3/2}}$) and kurtosis (degree of peakedness $\frac{\mu_4}{\mu_2^2}$), where μ is the moment about mean. Since expressions for these ratios are not in closed form, the plots are portrayed in Fig. 4, indicating that the proposed model is negatively and positively skewed and symmetric. Moreover, the model is also leptokurtic, mesokurtic, and platykurtic in behavior.

3.3 Entropy

The measure of the degree of uncertainty is commonly known as entropy. It has many significant chattels that agree with our innate belief of haphazardness. For studying this, we like to mention some of its properties which are (1) It may be positive or negative. (2) It vanishes if and only if it is a particular event. (3) Entropy is augmented by the accumulation of an independent constituent and lessened by acclimatizing. However, adding this belief to continuous probability models postures some contests. The entropy is specially combatted in physics, computing engineering, and statistics, where it measures the number of ways a thermodynamic system is arranged, the total amount of information in each received message, and measures uncertainty and dispersion, respectively. A number of research papers and monographs have appeared over the past 60 years, discussing and extending the original work of Shannon. In this research work, we will discuss some of them under the following names.

3.3.1 Shannon's Entropy

The SE is a significant and famous notion in the field of physical sciences and computing engineering, and its applications are generally seen in the field of financial analysis, data compression, mathematics, statistics, and computing sciences. For continuous variable X, it can be expressed as

$$\mathcal{SE} = -\int_{-\infty}^{\infty} f(x) ln(f(x)) dx.$$

Incorporating the PDF in the above equation and simplifying it, we get

$$\mathcal{SE} = -ln(\beta) - ln(\theta) + \frac{(\theta + 1)\gamma}{\theta} + 1 + ln(\alpha) + \frac{1}{\beta}\Gamma\left(1 - \frac{1}{\theta}\right),$$
$$\mathcal{SE} = ln\left(\frac{\alpha}{\beta\theta}\right) + \frac{(\theta + 1)\gamma}{\theta} + 1 + \frac{1}{\beta}\Gamma\left(1 - \frac{1}{\theta}\right),$$

where γ is the Euler constant and $\frac{1}{\theta} < 1$.

3.3.2 Cumulative Residual Entropy

Another measure of randomness of a random variable X called the CRE, where the PDF of the random variable X is swapped by the CDF/SF in Shannon's definition. The CDF/SF is more consistent than the PDF, because the PDF is obtained as the derivative of the CDF/SF. If X is a random variable, then we can define the CRE as

$$\mathcal{CRE} = -\int_{-\infty}^{\infty} \mathcal{S}(x) ln(\mathcal{S}(x)) dx;$$

for FLD, it can be expressed as

$$\mathcal{CRE} = -\int_0^\alpha e^{-\left(\beta \, ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}} ln\left(e^{-\left(\beta \, ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\right) dx,$$

which on simplification yields

$$\mathcal{CRE} = \frac{\alpha}{\beta\theta} \sum_{m=0}^{\infty} \frac{\left(\frac{-1}{\beta}\right)^m}{m!} \Gamma\left(1 - \frac{m+1}{\theta}\right).$$

provided $\theta > m + 1$.

3.3.3 Tsallis Entropy

TE was presented by Tsallis [12] in 1988, and it is a generalization of Boltzmann–Gibbs statistics. For a nonnegative continuous random variable X with the PDF 2, Tsallis entropy of order λ is defined by $\mathcal{TE}_\lambda(X) = \frac{1}{\lambda-1}\left(1 - \int_0^\alpha (f(x))^\lambda \, dx\right); \lambda \neq 1, \lambda > 1$. For FLD, it can be expressed as

$$\mathcal{TE}_\lambda(X) = \frac{1}{\lambda - 1}\left(1 - \left(\frac{\theta\beta}{\alpha}\right)^{\lambda-1} \lambda^{\frac{1-(1+\theta)\lambda}{\theta}} \sum_{m=0}^{\infty} \frac{\left(\frac{(\lambda-1)\lambda^{\frac{1}{\theta}}}{\beta}\right)^m}{m!} \Gamma\left(\frac{-1-m+\lambda+\theta\lambda}{\theta}\right)\right),$$

provided $(\theta + 1)\lambda > m + 1$.

3.3.4 Cumulative Residual Tsallis Entropy

Based on the TE, Sati and Gupta [13] proposed a CRTE of order λ, which is given by $\mathcal{CRTE}_\lambda(X) = \frac{1}{\lambda-1}\left(1 - \int_0^\alpha (\mathcal{F}(x))^\lambda \, dx\right); \lambda \neq 1, \lambda > 1$

$$\mathcal{CRTE}_\lambda(X) = \frac{1}{\lambda - 1}\left(1 - \alpha \sum_{m=0}^{\infty} \frac{\left(-\frac{\lambda^{\frac{1}{\theta}}}{\beta}\right)^m}{m!} \Gamma\left(1 - \frac{m}{\theta}\right)\right),$$

provided $\theta > m$.

3.3.5 Rényi Entropy

The RE measures the assortment, uncertainty, or randomness of a system. The entropy is identified after Alfréd Rényi, who looked for the most general definition of information measures that maintain additivity for independent events. It is applied in ecology and statistics as the index of diversity [14], in quantum information as a measure of complex system [15] and in heart rate variability for detecting CAN [16]. The RE of
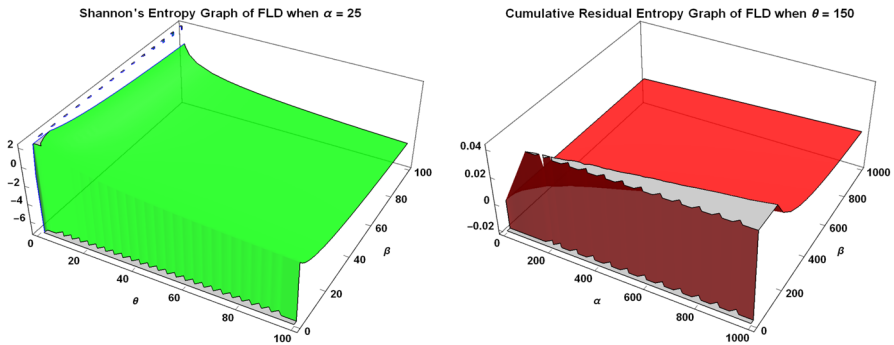
Fig. 5 FLD Shannon's and cumulative residual entropy graphs

order λ, where $\lambda \geq 0$, $\lambda \neq 0$ is defined as $\mathcal{RE} = \frac{1}{1-\lambda} ln \int_{\Re} f^{\lambda}(x) dx$. For the FLD, it can be expressed as

$$
\mathcal{RE}_{\lambda}(X) = \frac{1}{1-\lambda} \left(\left(\frac{1 - (1+\theta)\lambda}{\theta} \right) ln[\lambda] + (1-\lambda)ln \left[\frac{\alpha}{\theta\beta} \right] \right.
$$
$$
\left. ln \left[\sum_{m=0}^{\infty} \frac{\left(-\frac{(1-\lambda)\lambda^{\frac{1}{\theta}}}{\beta} \right)^m}{m!} \Gamma \left[\frac{-1 - m + \lambda + \theta\lambda}{\theta} \right] \right] \right),
$$

provided $(\theta + 1)\lambda > m + 1$.

3.3.6 Mathai–Haubold Entropy

An inaccuracy measure through disturbance or distortion of systems can be calculated via an entropy known as MHE proposed by [17] is defined as $\mathcal{MHE} = \frac{\int_{\Re} f^{2-\lambda}(x) dx - 1}{\lambda - 1}$. For the FLD it can be expressed as

$$
\mathcal{MHE}_{\lambda}(X)
$$
$$
= \frac{(2-\lambda)^{\frac{\lambda-1}{\theta} + \lambda - 2} \left(\frac{\theta\beta}{\alpha} \right)^{2-\lambda-1} \sum_{m=0}^{\infty} \frac{\left(-\frac{(2-\lambda-1)(2-\lambda)^{\frac{1}{\theta}}}{\beta} \right)^m}{m!} \Gamma \left[\frac{1-m-\theta(-2+\lambda)-\lambda}{\theta} \right] - 1}{\lambda - 1},
$$

provided $\frac{1}{\theta} > \frac{m+\theta(\lambda-2)+\lambda}{\theta}$.

Interpretations of Entropy From Figs. 5, 6, and 7, it is obvious that entropy occupies both negative and positive amount. Both cases are observed in all types of entropies, i.e., FLD is maximum entropy distribution and also has negative entropy, which indicates that the model also exhibits ordered pattern in entropy and it requires energy to achieve this state, i.e., negative entropy. However, the amount of entropy in positives cases is too low, which indicates that disorder exists very negligibly, small.
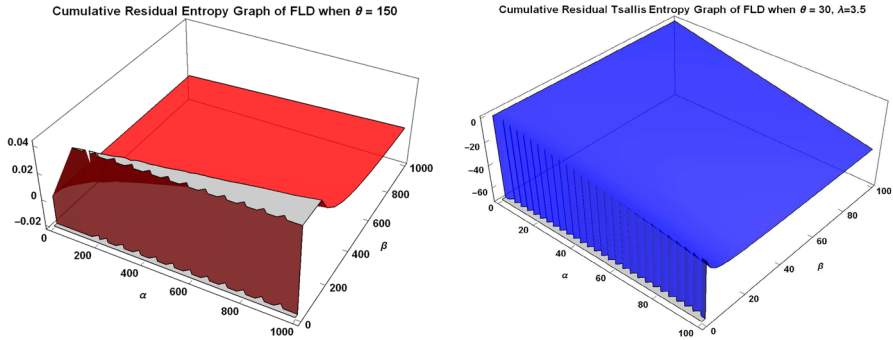
Fig. 6 Tsallis and cumulative residual Tsallis entropy graphs
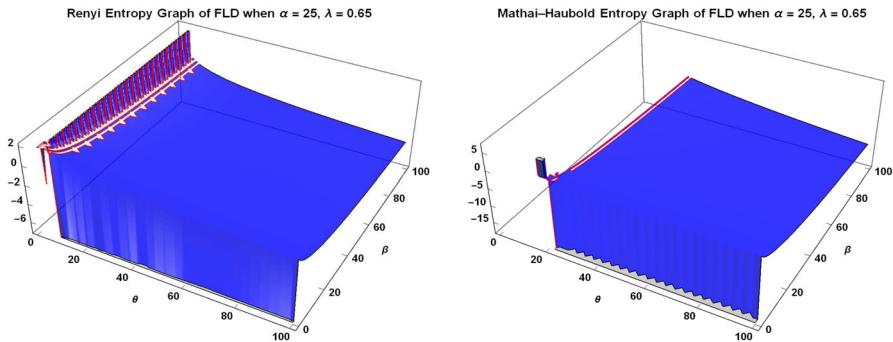
Fig. 7 Rényi and Mathai–Haubold entropy graphs

3.4 Conditional Moments and Mean Deviations

For lifetime probability models, the rth conditional moment is defined as $\mathbb{E}(X^r|X > t), r = 1, 2, ...,$ which has crucial and substantial concern in forecasting. So for this purpose the rth partial moments of the variate X defined as $\delta_r(t)$ for any real $r > 0$ is given as

$$\delta_r(t) = \int_{-\infty}^{t} x^r f(x) dx,$$

$$= \sum_{m=0}^{\infty} \frac{\alpha^r \left(-\frac{r}{\beta}\right)^m \left(\Gamma\left[1 - \frac{m}{\theta}\right] - \Gamma\left[1 - \frac{m}{\theta}, \left(\beta ln\left[\frac{\alpha}{t}\right]\right)^{-\theta}\right]\right)}{m!},$$

where $\Gamma(a, b) = \int_{b}^{\infty} t^{a-1} e^{-t}$ is an incomplete gamma function. Now, the rth conditional moment of the FLD distribution can be obtained as

$$\mathbb{E}(X^r | X > t) = \frac{1}{\mathcal{S}(t)} \left\{ \mathbb{E}(X^r) - \int_0^t x^r f(x) dx \right\},$$

$$= \frac{1}{\mathcal{S}(t)} \{ \mu_r' - \delta_r(t) \}.$$

The partial moments methodology is quite useful in finding the average deviance between the median and mean, where the median/mean aberration yields key evidence about typical of population. These partial moments can be used in many fields like economics and insurance. It delivers vital evidence about a population's attributes via the first partial moment. Now in this regard, the MD about the mean (μ') and median ($\tilde{\mu}$) are expressed as

$$\mathcal{MD}_{\mu'} = \mathbb{E} \mid X - \mu_1' \mid = 2\mu_1' \mathcal{F}(\mu_1') - 2\delta_1(\mu_1'), \tag{16}$$

where $\delta_r(t) = \int_{-\infty}^t x^r f(x) dx$ and

$$\mathcal{MD}_{\tilde{\mu}} = \mathbb{E} \mid X - \tilde{\mu} \mid = \mu_1' - 2\delta_1(\tilde{\mu}), \tag{17}$$

respectively, which in turn assists researchers to determine the volume of diffusion in a population, also $\mu_1' = \mathbb{E}(X)$, $\tilde{\mu} = \mathbb{M}(X) = \mathcal{P}(\frac{1}{2})$, and $\delta_1(t)$ is the first complete moment defined as above with $r = 1$.

3.5 Residual Lifetime Function and Its Characteristics

If $t > 0$ is the survival time, then RL and RRL stochastic variates from time t till the time of failure and the time passed from the failure of a section given that its life $\leq t$, are stated, respectively, as $\mathcal{RL}(t) := (X - t | X > t)$ and $\mathcal{R\bar{R}L}(t) := (t - X | X \leq t)$. The SF of the $\mathcal{RL}$ is $\mathcal{SF}_{\mathcal{RL}_{(t)}}, t \geq 0$; for the FLD distribution, it is given as

$$\mathcal{SF}_{\mathcal{RL}_{(t)}}(x) = \frac{\mathcal{SF}(x+t)}{\mathcal{SF}(t)} = \frac{e^{-(\beta \, ln(\frac{\alpha}{x+t}))^{-\theta}}}{e^{-(\beta \, ln(\frac{\alpha}{t}))^{-\theta}}},$$

$$\mathcal{SF}_{\mathcal{RL}_{(t)}}(x) = e^{-(\beta \, ln(\frac{\alpha}{x+t}))^{-\theta} - (\beta \, ln(\frac{\alpha}{t}))^{-\theta}},$$

and the corresponding PDF is

$$f_{\mathcal{RL}_{(t)}}(x) = \frac{\beta \theta}{x+t} \left(\beta \, ln \left(\frac{\alpha}{x+t} \right) \right)^{-\theta-1} e^{-(\beta \, ln(\frac{\alpha}{x+t}))^{-\theta} - (\beta \, ln(\frac{\alpha}{t}))^{-\theta}},$$

and HRF of $\mathcal{RL}_{(t)}$ is given by

$$\mathcal{H}_{\mathcal{RL}_{(t)}}(x) = \frac{\beta \theta}{x+t} \left(\beta \, ln \left(\frac{\alpha}{x+t} \right) \right)^{-\theta-1}.$$
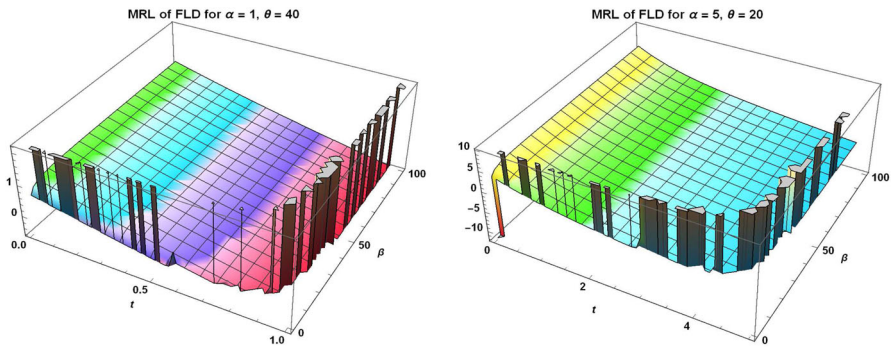
Fig. 8 FLD MRL function graphs

As MRL, suggests how much retentive apparatus will last for a specific point of time, is a vital gauge in reliability appraisal and modeling. It deals with condensed data for various decision-making malfunctions, such as fixing burn-in tests, planning enhanced life tests, determining warranty policy, and making maintenance decisions. Keeping in view the significance of the function, one can define it as

$$\mathcal{K}(t) = \mathbb{E}(\mathcal{RL}_{(t)}) = \frac{1}{\mathcal{SF}(t)} \int_t^\alpha x \mathrm{f}(x) \mathrm{d}x$$

$$= \frac{1}{\mathcal{SF}(t)} \{ \mathbb{E}(X) - \int_0^t x \mathrm{f}(x) \mathrm{d}x \} - t,$$

where

$$\delta_r(t) = \int_{-\infty}^t x^r \mathrm{f}(x) \mathrm{d}x.$$

The MRL function is muscularly linked to the failure rate function; the MRL classes are also related to the classes defined via the HRF. From Fig. 3, its is obvious that purported model belongs to BFR, which suggests that the linked MRL has an IBTS shape, see Fig. 8.

$$Mean = \mathcal{M}(t) = \frac{1}{\mathcal{S}(t)} \left\{ \mathbb{E}(X) - \alpha \sum_{m=0}^\infty \frac{\left(\frac{-1}{\beta}\right)^m}{m!} \right.$$

$$\left. \left(\Gamma\left(1 - \frac{m}{\theta}\right) - \Gamma\left(1 - \frac{m}{\theta}, \left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}\right)\right) \right\} - t,$$

Gupta and Kirmani [18] stated the substantial importance of the variance of RL function which compelled us to calculate it for FLD as (Fig. 9)
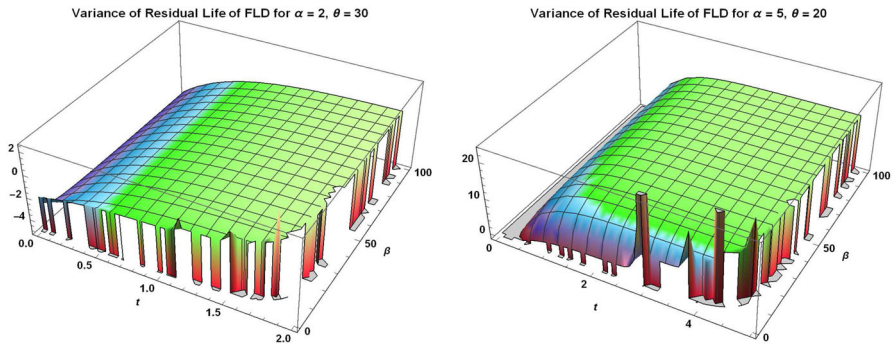
Fig. 9 FLD variance of residual life function graphs

$$
\mathcal{VAR}(t) = \mathbb{V}ariance(\mathcal{RL}_{(t)})
$$

$$
= \frac{2}{\mathcal{SF}(t)} \int_{t}^{\alpha} x\mathcal{SF}(x)\mathrm{d}x - 2t\mathcal{K}(t) - (\mathcal{K}(t))^2 ,
$$

$$
\mathcal{VAR}(t) = \frac{1}{\mathcal{SF}(t)} \left\{ \mathbb{E}(X^2) - \alpha^2 \sum_{m=0}^{\infty} \frac{\left(\frac{-1}{\beta}\right)^m}{m!} \right.
$$

$$
\left. \left(\Gamma\left(1 - \frac{m}{\theta}\right) - \Gamma\left(1 - \frac{m}{\theta}\right\} - t^2 - 2t\mathcal{K}(t) - (\mathcal{K}(t))^2 \right) \right\}.
$$

3.6 Reversed Residual Life Function and Its Features

Similarly, we have also calculated the SF of the RRL function $\mathcal{RV}_{(t)}$, $t \geq 0$ for the FLD as

$$
\mathcal{SF}_{\mathcal{RV}_{(t)}}(x) = \frac{\mathcal{F}(t-x)}{\mathcal{F}(t)} = \frac{1 - e^{-(\beta \, ln(\frac{\alpha}{t-x}))^{-\theta}}}{1 - e^{-(\beta \, ln(\frac{\alpha}{t}))^{-\theta}}}, \, 0 \leq x \leq t,
$$

and the corresponding PDF is

$$
f_{\mathcal{RV}_{(t)}}(x) = \frac{\beta\theta}{(t-x)(1 - e^{-(\beta \, ln(\frac{\alpha}{t}))^{-\theta}})} \left(\beta \, ln\left(\frac{\alpha}{t-x}\right) \right)^{-\theta-1} e^{-(\beta \, ln(\frac{\alpha}{t-x}))^{-\theta}}.
$$

The HRF of $\mathcal{RV}_{(t)}$ is given by

$$
\mathcal{H}_{\mathcal{RV}_{(t)}}(x) = \frac{\beta\theta}{(t-x)(1 - e^{-(\beta \, ln(\frac{\alpha}{t-x}))^{-\theta}})} \left(\beta \, ln\left(\frac{\alpha}{t-x}\right) \right)^{-\theta-1} e^{-(\beta \, ln(\frac{\alpha}{t-x}))^{-\theta}},
$$

with average and dispersion measures as

$$\mathbb{M}ean(\mathcal{RV}_{(t)}) = \mathcal{LW}(t) = \mathbb{E}(\mathcal{RV}_{(t)}) = t - \frac{1}{\mathcal{F}(t)} \int_0^t x \mathrm{f}(x) \mathrm{d}x = t - \frac{\delta_1(t)}{\mathcal{F}(t)},$$

$$\mathbb{V}ariance(\mathcal{RV}_{(t)}) = \mathcal{W}(t) = 2t\mathcal{LW}(t) - (\mathcal{LW}(t))^2 - \frac{2}{\mathcal{F}(t)} \int_0^t x \mathcal{F}(x) \mathrm{d}x,$$

$$\mathcal{W}(t) = 2t\mathcal{LW}(t) - (\mathcal{LW}(t))^2 - t^2 + \frac{\delta_2(t)}{\mathcal{F}(t)}.$$

3.7 Lorenz Curve

Suppose X is a nonnegative random variable, then the Lorenz curve for the FLD is expressed, for a given probability p, as

$$\mathcal{Z}(\mathrm{p}) = \frac{1}{\mu_1'} \delta_1(\mathrm{q}).$$

3.8 Bonferroni Curve

Suppose X be a nonnegative random variable, then the Bonferroni curve for FLD can be expressed as

$$\mathcal{B}(\mathrm{p}) = \frac{1}{\mathrm{p}\mu_1'} \delta_1(\mathrm{q}),$$

where $\mu_1' = \mathbb{E}(X)$, and $\mathrm{p} = \mathcal{P}(\mathrm{p})$ is the percentile function of X at percentile p and $\mathrm{q} = \mathcal{F}^{-1}(\mathrm{p})$.

3.9 Scaled Total Time on Test

The scaled TTT is important for the aging properties of the underlying distribution and can be applied to solve geometrically some stochastic maintenance problems. TTT for a CDF $\mathcal{F}$ can be defined as $\mathcal{SF}_{\mathcal{F}}(\mathcal{F}(t)) = \frac{1}{\mathbb{E}(X)} \int_0^t \mathcal{SF}(y)\mathrm{d}y$ see [19]. Therefore, for the FLD it can be expressed as

$$\mathcal{SF}_{\mathcal{F}}(\mathcal{F}(t)) = \frac{1}{\mathbb{E}(X)} \{ t e^{-(\beta \ln(\frac{\alpha}{t}))^{-\theta}} + \delta_1(t) \}.$$

3.10 Stress–Strength Reliability for the FLD Distribution

In this subsection, the mathematical form of stress–strength R for the FLD is calculated as $\mathcal{R} = \mathcal{P}(Y < X)$. It gauges the constituent reliability when it is exposed to random stress, Y, given that its strength is X. In this perspective, R can be believed as a measure of system execution and spontaneously develops electrical and electronic

systems. For comprehensive study about stress–strength reliability, readers may see [20]. Let $X \sim \text{FLD}(\alpha, \beta, \theta_1)$ and $Y \sim \text{FLD}(\alpha, \beta, \theta_2)$ denote the strength–stress random variable. Hence, the parameter R for the FLD is obtained as

$$\mathcal{R} = \mathcal{P}(Y < X) = \int_{x=0}^{\alpha} \int_{y=0}^{x} f(x; \alpha, \beta, \theta_1) f(y; \alpha, \beta, \theta_2) \mathrm{d}x \mathrm{d}y$$

$$= \int_{x=0}^{\alpha} f(x; \alpha, \beta, \theta_1) \mathcal{F}(x; \alpha, \beta, \theta_2) \mathrm{d}x,$$

which after simplification yields

$$\mathcal{R} = 1 - \int_{x=0}^{\alpha} \frac{\beta\theta}{x} \left(\beta \, ln(\frac{\alpha}{x})\right)^{-\theta_1 - 1} e^{-(\beta \, ln(\frac{\alpha}{x}))^{-\theta_1}} e^{-(\beta \, ln(\frac{\alpha}{x}))^{-\theta_2}} \mathrm{d}x,$$

$$= 1 - \int_{z=0}^{\infty} e^{-z} e^{-z^{\frac{\theta_2}{\theta_1}}} \mathrm{d}z, = 1 - \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \Gamma\left(1 + m\frac{\theta_2}{\theta_1}\right),$$

provided $m\frac{\theta_2}{\theta_1} > -1$.

4 Estimation of Model Parameters

4.1 Likelihood Method

Statistical implications are usually passed through three dissimilar methods like interval and point estimation as well as hypothesis testing. Even though abundant of approaches for the estimation of parameters were available in the statistical literature, the maximum likelihood method of estimation is the most excellently adaptable one. It owns projected chattels when fabricating the confidence regions and intervals as well as in test statistics. Asymptotic theories of these estimators convey simple calculations that toil well in the limited information contained in the samples. Statisticians frequently pursue to estimate quantities like the density of a test statistic that depends on the sample size so as to obtain better estimate distributions. The subsequent calculations for the MLEs in distribution theory can be handled either logically or mathematically. In this section, we are trying to cope with parameters estimation via the MLE method from the whole sample. Suppose $x_1, ..., x_n$ is a stochastic realization of size n from the FLD distribution given by Eq. (2). The logarithmic form of likelihood function of the FLD is expressed as

$$\ell_n(\Theta) = n ln(\beta) + n ln(\theta) - \sum_{i=1}^{n} ln(x_i) - (\theta + 1) \sum_{i=1}^{n} \beta ln\left(\frac{\alpha}{x_i}\right)$$

$$- \sum_{i=1}^{n} \left(\beta ln\left(\frac{\alpha}{x_i}\right)\right)^{-\theta}, \tag{18}$$

on partially differentiating Eq. (18) with respect to "θ" and "β," respectively, and equating it to zero, we get

$$\frac{\partial \ell_n(\Theta)}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} ln\left(\beta ln\left(\frac{\alpha}{x_i}\right)\right) - \sum_{i=1}^{n} -ln\left(\beta ln\left(\frac{\alpha}{x_i}\right)\right)\left(\beta ln\left(\frac{\alpha}{x_i}\right)\right)^{-\theta} = 0,$$

(19)

$$\frac{\partial \ell_n(\Theta)}{\partial \beta} = \frac{n}{\beta} - \frac{n(1+\theta)}{\beta} + \sum_{i=1}^{n} \theta ln\left(\frac{\alpha}{x_i}\right)\left(\beta ln\left(\frac{\alpha}{x_i}\right)\right)^{-1-\theta} = 0.$$

(20)

Because this expression cannot be worked out rationally, we desire to apply a computational package like Mathematica [12.0]. Therefore, we started estimation by utilizing the NMaximized commands. But, we have examined that we are unable to attain the fairly accurate estimate of α from Eq. (18). Since α is not based on x and it is the highest value in the domain of x, so we assume that the MLE for α as $\hat{\alpha} = max(x_1, x_2, ..., x_n) + \xi$ where $\xi > 0$ represents some arbitrary constants. However, when start estimation of parameter, α of the FLD, using limited sample information, we assume that all the components in the sample are the stochastic variate. Therefore, we should assume that the sample should be encompassed of independent observations on the random variable in demand. As the value of parameter α defines in the domain of attraction of the FLD random variable, during the process of estimation, it is a stipulation that α should be greater than the greatest element in the sample. That is why, we are looking for the vigorous value of $\hat{\alpha}$ which will perhaps be recognized under the condition that $\hat{\alpha}$ is greater than the greatest element in the sample. Likewise, the convergence of anticipated model's MLE is based on the massive value of $\hat{\alpha}$, which seems to be in accordance with the distribution theory of standard epsilon models; for comprehensive study, the reader is referred to [1]. Furthermore, the regularity conditions of MLEs, the FLD (α, β, θ) model fulfill the regularity conditions as declared by [21](pp. 419). Thus, the confidence interval belt MLE vector of $\hat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{\theta})$ is consistent and asymptotically normal family, i.e., $\sqrt{n}[\hat{\Theta}^T - \Theta^T] \sim \text{TVN}[0, I^{-1}]$, where I^{-1} is the inverse of the expected Fisher information matrix, which generates variance–covariance matrix based on the expectation of second-order log-likelihood derivatives.

5 Competing Models and Applications

In this section, we have studied four data sets ranging from lifetime perspective to participation aspects. The sources of data sets are mentioned in their respective section.

5.1 Models for Comparisons

For comparison purposes, we have studied ten competing models, including MWD [22], 3P Kappa distribution studied by [23], 3P WD [24], EED [25], WD and GD taken from Wikipedia, IGD [26], FD [7], LND [27], and LLD [6].

5.2 Detection of Hazard Rate Pattern

For failure rate identification pattern, the researchers usually prefer to use the TTT-transform, which helps to examine the shapes of the hazard rate of a given data. In this concern, Barlow and Campo [28] first applied it, for statistical inference problems under order restrictions. When modeling a data set, Aarset [29] used this test as a selection method [30]. As the developed model exhibits IBTS together with IFR, so we prefer to select various data sets exhibiting BTS or IFR.

5.3 Goodness-of-Fit and Discrimination Criterion

In this subsection, diverse standard discriminations are adopted, which are built on the log-likelihood $(-l)$ function. Let q denote the number of parameters to be fitted and $\hat{\Theta}$ be the MLEs of Θ, n is the size of data set, the number of classes is denoted by k, $z_j = \mathcal{F}_X(x_j)$ and the x_j, $j = 1, 2, 3, \ldots, n$ being the arranged observations. Then, the expressions for respective information criteria are given as

$$\text{AIC} = 2q - 2\ell(\hat{\Theta}), \quad \text{AICc} = \text{AIC} + \frac{2q(q+1)}{n-q-1}, \quad \text{BIC} = q\ln(n) - 2\ell(\hat{\Theta})$$

$$\text{HQIC} = -2\ell(\hat{\Theta}) + 2q\log(\log(n)), \quad \text{CAIC} = -2\ell(\hat{\Theta}) + \frac{2qn}{n-q-1}.$$

The finest model will be accepted if it possesses the least values of these gadgets. Additionally, brilliance of rival models is also confirmed through various goodness-of-fit test statistics which include both the parametric and nonparametric tests like KS, χ^2, W_0^* and A_0^*. Their corresponding expressions are given as follows:

$$\text{KS} = \max\left\{\frac{j}{k} - z_j, z_j - \frac{j-1}{k}\right\}, \chi^2 = \sum_{j=1}^{k} \frac{(o_j - e_j)^2}{e_j},$$

$$W_0^* = \sum_{j=1}^{k} \left(z_j - \frac{2j-1}{2k}\right)^2 + \frac{1}{12k},$$

$$A_0^* = \left(\frac{2.25}{k^2} + \frac{3}{4k} + 1\right)\left\{-k - \frac{1}{k}\sum_{j=1}^{k}(2j-1)\ln(z_j(1-z_{k-j+1}))\right\}.$$

Moreover, the VTS, suggested by [31], is also applied. For comprehensive procedural understanding, the readers are referred to [32]. The evaluations of the competing models are described in Table 14.

5.4 Application with Real Lifetime Data Sets

Example 1 The lengths of power failures, in minutes, are recorded in this data set, and the data are extracted from the book [33](pp. 51) with the following measurements:

Table 1 Data set I descriptive summary

SS	$\bar{X}$	$\tilde{X}$	$\hat{\sigma}$	$\hat{SK}$	$\hat{KU}$	$\hat{SE}$
45	74.022	78.00	39.2576	-0.0320	1.93677	3.5178

Table 2 Data set I theoretical measures

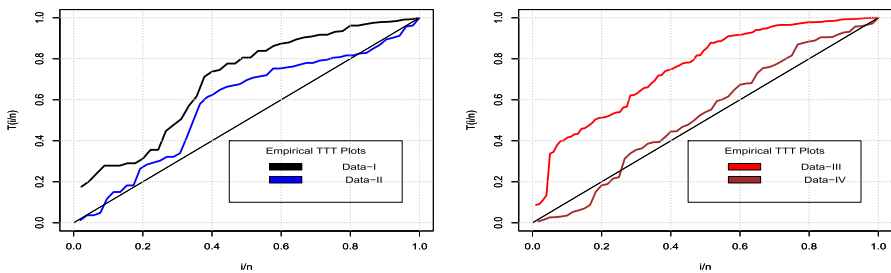| SS	$\bar{X}$	$\tilde{X}$	$\hat{\sigma}$	$\hat{SK}$	$\hat{KU}$	$\hat{SE}$
45	73.7476	74.9678	37.1584	-0.0164	2.3106	4.9937

Fig. 10 Empirical TTT plots

22, 18, 135, 15, 90, 78, 69, 98, 102, 83, 55, 28, 121, 120, 13, 22, 124, 112, 70, 66, 74, 89, 103, 24, 21, 112, 21, 40, 98, 87, 132, 115, 21, 28, 43, 37, 50, 96, 118, 158, 74, 78, 83, 93, 95.

Analytical Discussion About Data Set I Table 1 displays some descriptive features like SS, $\bar{X}$, $\tilde{X}$, $\hat{\sigma}$, $\hat{SK}$, $\hat{KU}$ and SE of the data, which reveals a close coordination between descriptive and theoretical results given by Table 2. The proposed model has a minimum value of χ^2 goodness-of-fit statistic as compared to rest of the models values. Also, FLD has minimum value of A_0^*, W_0^* and the KS statistic that supports the suitability of suggested model. Take note of that the parameter α is the highest value in the domain of Eq. 2, i.e., it is nonnegative only if $x \in (0, \alpha)$, which implies that the value of parameter α must fulfill the necessity that $\alpha > max_{i=1,2,3,\dots,n}(x_i)$, see [2]. In this regard, while finding the MLEs we observed that $\hat{\alpha}$ is larger than the $max_{i=1,2,3,\dots,n}(x_i)$ for any data set, which can be visualized in the related tables. Moreover, it is noteworthy that data portray an IFR behavior as portrayed in Fig. 10 along with maximum entropy model as exhibited in Table 13. So, such narrative demands a model that can not only model the positive skewness, leptokurtic behavior, IFR but also the maximum entropy issue in a pleasant way. In addition, for drawing valid conclusion, we have grouped the observation by using **R** as [13,22], (22,41.7], (41.7,73.4], (73.4,87.3], (87.3,98], (98,117], (117,158] and the frequencies are listed as 8, 5, 6, 7, 7, 5, 7, respectively. Moreover, Tables 3 and 15 (see Appendix A2) portray that the developed model is the most suitable one, with least values for all statistics and highest p value for χ^2 and with the smallest KS statistics. Furthermore, the VTS and histogram that are portrayed in Table 13 and Fig. 11, respectively, also support

Table 3 Data set I goodness-of-fit measure based on listed MLEs

Distribution	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	χ^2(d.f)	p value	A_0^*	W_0^*	KS	p value
FLD	500.6930	3.9049	0.5785	8.0047(4)	0.0914	0.5266	0.0923	0.1141	0.7358
MWD	0.0135	0.3826	0.0000	19.8421(2)	0.0000	4.0101	0.7574	0.2567	0.0174
3P Kappa	0.0793	12.9862	10.6009	62.6373(2)	0.0000	16.6664	3.4580	0.4704	0.0000
3P WD	1.9557	82.7211	0.571701	8.90066(4)	0.0636	1.1658	0.2242	0.1636	0.2901
EED	0.0238	2.8274	–	12.054(3)	0.0072	1.5958	0.3207	0.1845	0.1724
WD	1.9782	83.4093	–	8.89733(4)	0.0637	1.1501	0.2210	0.1628	0.2960
GD	2.7057	27.3580	–	11.1078(3)	0.0112	1.5259	0.3055	0.1824	0.1820
IGD	2.0184	93.9624	–	25.792(3)	0.0000	2.9095	0.5788	0.2561	0.0178
FD	1.3660	42.1148	–	29.0155(3)	0.00001	2.7979	0.5466	0.2486	0.0233
LN	4.1083	0.6924	–	12.9359(3)	0.0048	2.0985	0.4153	0.2134	0.0753
LLD	2.4680	66.2892	–	14.4229(3)	0.0024	1.5626	0.2680	0.1640	0.2878

Table 4 Data set II descriptive summary

SS	$\bar{X}$	$\tilde{X}$	$\hat{\sigma}$	$\hat{SK}$	$\hat{KU}$	$\hat{SE}$
52	80.9615	78.00	69.5092	2.17917	10.2266	3.7646

Table 5 Data set II theoretical measures

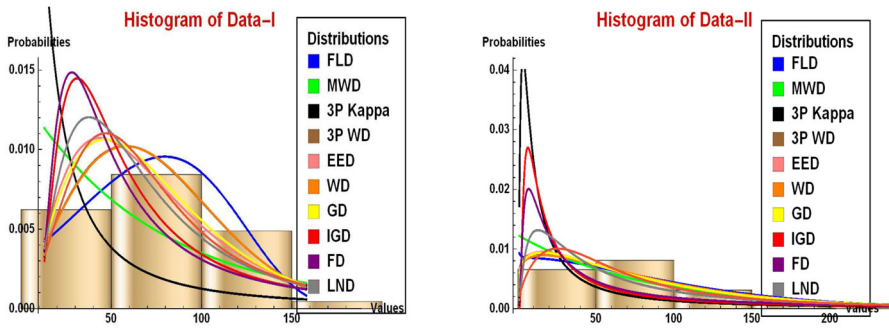| SS	$\bar{X}$	$\tilde{X}$	$\hat{\sigma}$	$\hat{SK}$	$\hat{KU}$	$\hat{SE}$
52	81.6837	78.9833	67.2136	1.14350	10.2601	− 8.4441

Fig. 11 Histogram of data set I and II

the above results. Hence, our proposed model seems to be a natural choice for such data sets.

Example 2 This data set is extracted from the book [34](pp. 12). This data set mentions the death times (in weeks) of patients with cancer (Aneuploid Tumor) of the tongue, with the measurements: 1, 3, 3, 4, 10, 13, 13, 16, 16, 24, 26, 27, 28, 30, 30, 32, 41, 51, 61, 65, 67, 70, 72, 73, 74, 77, 79, 80, 81, 87, 87, 88, 89, 91, 93, 93, 96, 97, 100, 101, 104, 104, 108, 109, 120, 131, 150, 157, 167, 231, 240, 400.

Discussion on the Fit of Data Set II It is evident from Table 4 that the data portray a maximum entropy, see Table 14, positive skewness and high kurtosis in such a way that skewness to kurtosis ratio is 0.2131. However, the theoretical aspect as given in Table 6 also portrays the same results. But Table 13 endorses both positive and negative entropies. Moreover, Table 5 also affirms the above statement. In addition, we have made frequency distribution of the above data set by **R** computational package. For this purpose, we have created different classes, such as [1,16], (16,31.1, (31.1,71.7], (71.7,87], (87,96.4],(96.4,117],(117,400] along with respective frequencies 9, 6, 7, 9, 6, 7, 8. In this regard, Tables 6 and 16 (see Appendix A2) display that the proposed model is highly recommended in moderate tailed behavior and BTS data set as portrayed in 10. Also, these tables provide enough evidence about the appropriateness of the proposed model, with reasonably small values of all the statistics with least loss of information attitude by depicting minimum value of all information criteria. Furthermore, Vuong test statistics and histogram as portrayed in Table 13

Table 6 Data set II goodness–of–fit measure based on listed MLEs

Distribution	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\chi^2(d.f.)$	$pvalue$	A_0^*	W_0^*	KS	p value
FLD	9.999910	13.8796	0.0859	6.47571 (3)	0.0909	1.0736	0.2258	0.1695	0.1506
MWD	0.0123	0.00000	0.0124	7.48314 (3)	0.0581	2.0036	0.4410	0.2186	0.0271
3P Kappa	12.9862	0.0673	10.6009	58.7827(3)	0.00001	12.3281	2.6777	0.4093	0.0000
3P WD	1.1407	83.9536	0.6065	7.6704 (3)	0.0533	1.3279	0.2841	0.1890	0.0802
EED	1.2432	0.0141	–	8.1960 (3)	0.0422	1.4384	0.3161	0.1974	0.0599
WD	1.1739	85.3614	–	7.74997(3)	0.0515	1.2114	0.2612	0.1829	0.0984
GD	1.2423	65.1697	–	8.1426 (3)	0.0432	1.3961	0.3060	0.1950	0.0653
IGD	0.5807	10.2521	–	37.1618(3)	0.0001	6.6602	1.3586	0.3140	0.0002
FD	0.6745	26.7895	–	24.1588(3)	0.000023	4.7311	0.9003	0.2521	0.0066
LN	3.9403	1.1736	–	12.7154 (3)	0.00529	2.8563	0.5827	0.2467	0.0083
LLD	1.6401	60.7004	–	13.424 (3)	0.00380	1.7472	0.3244	0.1947	0.0660

Table 7 Data set III descriptive summary

SS	X̄	X̃	$\hat{\sigma}$	SK̂	KÛ	SÊ
99	458.82	495.00	195.867	− 0.3250	1.9499	4.5391

Table 8 Data set III theoretical measures

SS	X̄	X̃	$\hat{\sigma}$	SK̂	KÛ	SÊ
99	464.78	495.68	192.0034	− 0.5695	2.5977	6.5431

and Fig. 11, respectively, indicate that the MWD and 3P WD are reasonably good competitors of the proposed model. Moreover, our claims are further consolidated by the above findings.

Example 3 The third data set is extracted from the book [34](pp. 12). This data set mentions the death times (in weeks) of patients with cancer (Aneuploid Tumor) of the tongue, with measurements: 159, 189, 191, 198, 200, 207, 220, 235, 245, 250, 256, 261, 265, 266, 280, 343, 356, 383, 403, 414, 428, 432, 317, 318, 399, 495, 525, 536, 549, 552, 554, 557, 558, 571, 586, 594, 596, 605, 612, 621, 628, 631, 636, 643, 647, 648, 649, 661, 663, 666, 670, 695, 697, 700, 705, 712, 713, 738, 748, 753, 40, 42, 51, 62, 163, 179, 206, 222, 228, 249, 252, 282, 324, 333, 341, 366, 385, 407, 420, 431, 441, 461, 462, 482, 517, 517, 524, 564, 567, 586, 619, 620, 621, 622, 647, 651, 686, 761, 763.

Discussion About the Third Data Set From Tables 7 and 8, the theoretical and observed descriptive statistics show a remarkable closeness to each other and it seems that third data set is being simulated by the proposed model. Furthermore, BFR can be observed from Fig. 10. For more authoritative advocacy of the proposed model, we have calculated the χ^2 by making classes of data set via the command **b**ins of **R** computational package as [40,238], (238,352], (352,495], (495,595], (595,650], (650,763] with the respective frequencies 17, 16, 17, 16, 17,17. Moreover, from Tables 9 and 17 (see Appendix A2), it is evident that our proposed model gives more viable results as compare to the competing models.

Example 4 The fourth data set measures the exact times of failure which is reported by [35] with the measurements:14, 34, 59, 61, 69, 80, 123, 142, 165, 210, 381, 464, 479, 556, 574, 839, 917, 969, 991, 1064, 1088, 1091, 1174, 1270, 1275, 1355, 1397, 1477, 1578, 1649, 1702, 1893, 1932, 2001, 2161, 2292, 2326, 2337, 2628, 2785, 2811, 2886, 2993, 3122, 3248, 3715, 3790, 3857, 3912, 4100, 4106, 4116, 4315, 4510, 4584, 5267, 5299, 5583, 6065, 9701.

Discussion and Analysis of the Fourth Data It is evident from Tables 10 and 11 that the theoretical and empirical measures are closely associated, indicating positive skewness and high kurtosis behaviors. Therefore, these data require a model that can work very well in positively skewed, moderate kurtosis and BFR distributions, as shown in Fig. 10. Furthermore, grouped data with the classes such as [14,184],

Table 9 Data set III goodness-of-fit measure based on listed MLEs

Distribution	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	χ^2(d.f)	p value	A_0^*	W_0^*	KS	p value
FLD	1071.7800	2.1739	1.54298	8.34898 (5)	0.138026	1.1351	0.2000	0.0966	0.3381
MWD	0.0021	0.1086	0.0000	96.3372 (4)	0.00001	12.4427	2.3584	0.2639	0.0000
3P Kappa	9911.6000	0.0000	11.1344	629.826 (2)	0.00001	57.6590	12.4599	0.6497	0.0000
3P WD	8.7941	1497.8400	−955.3790	16.0829 (5)	0.00001	1.6853	0.2835	0.1142	0.1680
EED	3.9015	0.0044	–	37.6862 (4)	0.00001	2.7551	0.3960	0.1290	0.0848
WD	2.5394	514.9810	–	22.2886 (5)	0.000462	2.0153	0.2987	0.1232	0.1119
GD	3.7405	122.6610	–	34.6771(5)	0.00001	2.4660	0.3596	0.1330	0.0694
IGD	1.9711	597.6550	–	91.3867 (4)	0.00001	6.8508	1.0736	0.1921	0.0018
FD	1.1901	281.3220	–	112.113 (4)	0.00001	8.3081	1.3304	0.2629	0.0000
LN	5.9891	0.6145	–	49.1192 (4)	0.00001	3.6468	0.5179	0.1457	0.0354
LLD	3.1426	434.8280	–	42.2235 (5)	0.00001	2.9423	0.3577	0.1459	0.0350

Table 10 Data set IV descriptive summary

SS	$\bar{X}$	$\tilde{X}$	$\hat{\sigma}$	$\hat{SK}$	$\hat{KU}$	$\hat{SE}$
60	2193.03	1675.50	1920.15	1.2613	5.2312	4.0943

Table 11 Data set IV theoretical measures

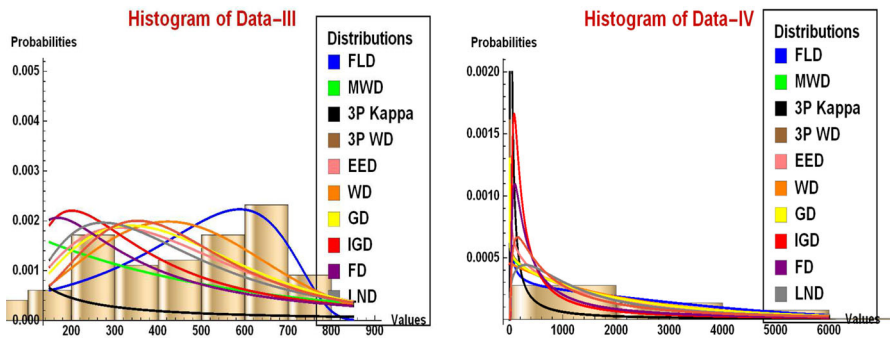| SS	$\bar{X}$	$\tilde{X}$	$\hat{\sigma}$	$\hat{SK}$	$\hat{KU}$	$\hat{SE}$
60	2197.04	1646.66	1881.03	1.2304	4.5710	8.0259

Fig. 12 Histogram of data set III and IV

(184,962], (962,1370], (1370,2120], (2120,3010],(3010,4110], (4110,9701] and the observed frequencies of each class, which are 9, 8, 9, 8, 9, 8, 9, respectively, also used for comparison purposes. So in this regard, Tables 12 and 18 (see Appendix A2) portray the comparison of the compared distributions. From these tables, it is evident that our proposed model is the suitable one, with least values for all statistics and highest p value for χ^2 statistics. However, encouraging aspects for the proposed model are its VTS values that ensure that our proposed model is better strategy than other competing models (Fig. 12, Table 13).

Comparison Via Vuong Test The brief summary of the VTS is portrayed in Table 14, where the possible paired values of the FLD with their competitor models are given. From these values, it is clear that the proposed model is suitable choice among these best competitors. Moreover, another encouraging aspect of the proposed model is having a least loss of information model which can be seen from Tables 15, 16, 17, and 18 of criterion, see Appendix A2.

6 Conclusion

The Fréchet log-logistic distribution introduced in this study is a notable bounded distribution. It can be used as a restricted substitute for the beta probability model because the unit support model is a limit case of the FLD, which is an uncommon form

Table 12 Data set IV goodness-of-fit measure based on listed MLEs

Distribution	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	χ^2(d.f)	p value	A_0^*	W_0^*	KS	p value
FLD	99,911	4.0820	0.2705	7.8705(6)	0.2477	0.2381	0.0239	0.0500	0.9983
MWD	0.0004	0.5407	0.0012	9.9673(6)	0.1260	0.7356	0.1137	0.0877	0.7451
3P Kappa	10.60	0.0277	12.98	198.61(2)	0.00001	30.4811	6.5073	0.5253	0.0000
3P WD	0.8629	2062.2	13.99	9.7767(6)	0.1343	1.2310	0.2027	0.1209	0.3439
EED	0.9159	0.0004	–	9.8884(6)	0.1294	0.7842	0.1240	0.0921	0.6885
WD	1.0009	2193.73	–	11.0755(6)	0.0860	0.8026	0.1022	0.0777	0.8618
GD	0.9307	2356.37	–	10.0636(6)	0.1220	0.7863	0.1208	0.0897	0.7198
IGD	0.4436	120.94	–	52.6808(3)	0.00001	6.7941	1.3897	0.2921	0.0001
FD	0.5796	531.95	–	37.9472(4)	0.00001	4.4481	0.8085	0.2156	0.0076
LN	7.0675	1.4392	–	19.0763(5)	0.00001	2.5337	0.4289	0.1654	0.0751
LLD	1.2920	1445.27	–	18.6902(6)	0.00001	1.8340	0.1959	0.1189	0.3649

Table 13 Theoretical entropy when $\lambda = 0.75$

Data sets	CRE	TCE	CRTE	RE	MHE
I	878.537	13.3444	2720.82	5.86789	3.55664
II	1.16732×10^6	2.52477	1.69468×10^6	1.95724	3.9183
III	-1958.92	9.47261	8569.69	4.57619	3.49023
IV	-3.58685×10^7	307.559	1.09047×10^8	17.3098	3.93821

Table 14 Comparison via VTS

Competing models	Statistics	Data I	Data II	Data III	Data IV
FLD-MWD	**Z**	4.7916	9.9705	7.3965	6.7851
	$Z_{0.05}$	1.645	1.645	1.645	1.645
FLD- 3P Kappa	**Z**	4.4715	4.8048	13.0030	7.9481
	$Z_{0.05}$	1.645	1.645	1.645	1.645
FLD-3P WD	**Z**	6.5014	6.4004	21.2992	2.8658
	$Z_{0.05}$	1.645	1.645	1.645	1.645
FLD-EED	**Z**	4.6408	18.5898	5.8529	5.8439
	$Z_{0.05}$	1.645	1.645	1.645	1.645
FLD-WD	**Z**	6.8798	13.5381	6.2153	6.8676
	$Z_{0.05}$	1.645	1.645	1.645	1.645
FLD-GD	**Z**	4.9265	18.7655	5.7082	6.1063
	$Z_{0.05}$	1.645	1.645	1.645	1.645
FLD-IGD	**Z**	3.7798	6.1086	5.4869	5.2836
	$Z_{0.05}$	1.645	1.645	1.645	1.645
FLD-FD	**Z**	3.5875	8.1484	6.5932	6.9552
	$Z_{0.05}$	1.645	1.645	1.645	1.645
FLD-LND	**Z**	4.4001	10.5250	5.6660	8.1816
	$Z_{0.05}$	1.645	1.645	1.645	1.645
FLD-LLD	**Z**	9.2496	27.2169	8.0684	13.5222
	$Z_{0.05}$	1.645	1.645	1.645	1.645

for a restricted domain model. Thus, this proposed model offers a flexible explanation to the dilemma of demonstrating constrained qualities. We have also shown that the PDF and HRF of the FLD are very accommodating in their curve shapes. The multiple skewed and kurtosis patterns shown in the PDF of the FLD indicate its versatility. We found that the FLD's HRF can approve a variety of shapes ranging from BTS to IFR with a left skewed J-shape, which normally comprises two phases: the lengthy usable period phase and the wear-out phase. These forms of the FLD's HRF make this model an appropriate choice for modeling determinations in a wide range of practical issues. The proposed model adopts a mathematically manipulated able form of its fundamental functions, including CDF, HRF, and SF, based on which we also characterize the FLD, indicating that all of the estimated parameters are reliable. Finally, in the applications

section, we explored the FLD model for modeling the failure time of a product, device, or human in respective units. Based on the empirical findings, we can infer that this suggested model outperforms the other competing models for all given data sets.

Appendix

A_0-Notation and Acronyms

BTS bathtub shape; IBTS inverted bathtub shape; CDF cumulative distribution function; EVs extreme values; FLD Fréchet log-logistic distribution; PDF probability density function; SF survival function; HRF hazard rate function; RHRF reversed hazard rate function; IFR increasing failure rate; DFR decreasing failure rate; TTT total time on test transform; CP change points; iff if and only if; $\mathcal{P}$ percentile function; $\tilde{\text{M}}$D Median; MGF moment generating function; SE Shannon entropy; CRE cumulative residual entropy; TE Tsallis entropy; CRTE cumulative residual Tsallis entropy; RE Rényi entropy; CAN cardiac autonomic neuropathy; MHE Mathai–Haubold entropy; RL residual life; RRL reversed residual life; MRL mean residual life; R reliability parameter; MD mean deviation; MLE maximum likelihood estimation; MWD modified Weibull distribution; 3P WD three-parameter Weibull distribution; LLD log-logistic distribution; GD gamma distribution; IGD inverse gamma distribution; WD Weibull distribution; 3P Kappa three-parameter Kappa; LND log-normal distribution; EED exponentiated exponential distribution; FD Fréchet distribution; SK Skewness; KU Kurtosis; KS Kolmogorov–Smirnov; W_0^* Cramer–Von Mises; A_0^* Anderson–Darling; χ^2 Chi-Square; AIC Akaike information criterion; AICc Corrected Akaike information criterion; BIC Bayesian information criterion; HQIC Hannan–Quinn information criterion; CAIC consistent Akaike information criterion; $\bar{X}$ mean; $\tilde{X}$ median; $\hat{\sigma}$ standard deviation; SS sample size; l log-likelihood; TVN tri-variate normal; IBTS inverted bathtub shape; VTS Vuong test statistics.

A1 Characterization

Proof of Proposition **1** ***Necessity*** If $X \sim \text{FLD}(\alpha, \beta, \theta)$, with the PDF 2, then logarithmic for its PDF can be expressed as

$$ln f(x) = ln(\beta) + ln(\theta) - ln(x) - (\theta + 1)ln\left(\beta ln\left(\frac{\alpha}{x}\right)\right) - \left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta},$$

on differentiating both sides with respect to x we get

$$\frac{f'(x)}{f(x)} = -\frac{1}{x} + \frac{\theta + 1}{x ln\left(\frac{\alpha}{x}\right)} - \frac{\beta\theta \left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta-1}}{x}, \tag{21}$$

where

$$\frac{\mathcal{H}'(x)}{\mathcal{H}(x)} = -\frac{1}{x} + \frac{\theta + 1}{x ln(\frac{\alpha}{x})},$$

and

$$\frac{f'(x)}{f(x)} = \frac{\left(1 + \theta - \ln\left(\frac{\alpha}{x}\right) - \theta\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}\right) x}{x \ln\left(\frac{\alpha}{x}\right) \beta\theta \left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{-1-\theta} e^{-\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}}. \tag{22}$$

On comparing Eq. 21 with Eq. 22, we get

$$\frac{\mathcal{H}'(x)}{\mathcal{H}(x)} = \frac{\left(1 + \theta - \ln\left(\frac{\alpha}{x}\right) - \theta\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}\right) x}{x \ln\left(\frac{\alpha}{x}\right) \beta\theta \left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{-1-\theta} e^{-\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}} + \frac{\beta\theta(\beta ln(\frac{\alpha}{x}))^{-\theta-1}}{x},$$

which after simplification yields Eq. 10.

Sufficiency Suppose Eq. 10 holds, then it may be rewritten as

$$\frac{\mathcal{H}'(x)}{(\mathcal{H}(x))^2} = \frac{\beta e^{\left(\beta\text{Log}\left(\frac{\alpha}{x}\right)\right)^{-\theta}} x \left(\beta\text{Log}\left(\frac{\alpha}{x}\right)\right)^{1+2\theta} \left(1 + \theta - \text{Log}\left(\frac{\alpha}{x}\right) - \theta\left(\beta\text{Log}\left(\frac{\alpha}{x}\right)\right)^{-\theta}\right) + \beta^2\theta^2\text{Log}\left(\frac{\alpha}{x}\right)}{\beta^2\theta^2},$$

the above differential equation yields

$$\mathcal{H}(u) = \frac{\beta\theta((\beta ln(\frac{\alpha}{u}))^{-\theta-1})}{u}. \tag{23}$$

After integrating the expression 23 from 0 to x we get

$$-ln(1 - \mathcal{F}(x)) = \left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta},$$

which implies that

$$\mathcal{F}(x) = 1 - e^{-((\beta ln(\frac{\alpha}{x}))^{-\theta})}, \beta > 0, \theta > 0, \alpha > 0, 0 \le x \le \alpha.$$

This completes the proof.   □

Proof of Proposition 2 Necessity If $X \sim$ FLD(α, β, θ), with PDF 2 then

$$lnf(x) = ln(\beta) + ln(\theta) - ln(x) - (\theta + 1)ln\left(\beta ln\left(\frac{\alpha}{x}\right)\right) - \left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta},$$

after differentiating both sides of the above expression with respect to x we get

$$\frac{f'(x)}{f(x)} = -\frac{1}{x} + \frac{\theta + 1}{x ln(\frac{\alpha}{x})} - \frac{\beta\theta(\beta ln(\frac{\alpha}{x}))^{-\theta-1}}{x}, \tag{24}$$

where

$$\frac{\mathfrak{R}'(x)}{\mathfrak{R}(x)} = -\frac{1}{x} + \frac{1+\theta}{x \ln\left(\frac{\alpha}{x}\right)} - \frac{\beta\theta\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{-1-\theta}}{x} - \frac{e^{-\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\beta\theta\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{-1-\theta}}{\left(1 - e^{-\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\right)x},$$

and

$$\frac{f'(x)}{f(x)} = \frac{\dfrac{e^{-\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\theta\left(\beta\ln\left(\frac{\alpha}{x}\right)\right)^{-2\theta}\left(-\theta+(1+\theta-\ln\left(\frac{\alpha}{x}\right))\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{\theta}\right)}{x^2\left(\ln\left(\frac{\alpha}{x}\right)\right)^2}}{\dfrac{\beta\theta}{x}\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{-\theta-1}e^{-\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}}. \tag{25}$$

On comparing Eq. 24 with Eq. 25, we get

$$\frac{\mathfrak{R}'(x)}{\mathfrak{R}(x)} = \frac{\dfrac{e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\theta\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-2\theta}\left(-\theta+(1+\theta-ln\left(\frac{\alpha}{x}\right))\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{\theta}\right)}{x^2\left(ln\left(\frac{\alpha}{x}\right)\right)^2}}{\dfrac{\beta\theta}{x}\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta-1}e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}}$$
$$-\frac{\beta\theta\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta-1}e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}}{x\left(1 - e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\right)},$$

which after simplification yields Eq. 11.

Sufficiency Suppose Eq. 11 holds, then we can rewrite it as

$$\frac{\mathfrak{R}'(x)}{(\mathfrak{R}(x))^2} = \frac{\dfrac{d\left(\frac{\beta\theta}{x}\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta-1}e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\right)}{dx}}{\left(\frac{\beta\theta}{x}\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta-1}e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\right)^2}\left(1 - e^{-\left(\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}\right)}\right) - 1;$$

after integrating the above expression from 0 to u, we get

$$\mathfrak{R}(u) = \frac{\beta\theta}{u\left(1 - e^{-\left(\beta ln\left(\frac{\alpha}{u}\right)\right)^{-\theta}}\right)}\left(\beta ln\left(\frac{\alpha}{u}\right)\right)^{-\theta-1}e^{-\left(\beta ln\left(\frac{\alpha}{u}\right)\right)^{-\theta}}, \tag{26}$$

which on integrating from 0 to x yields

$$ln(\mathcal{F}(x)) = ln(1 - (e^{-\beta ln\left(\frac{\alpha}{x}\right)})^{-\theta}),$$

which implies that

$$\mathcal{F}(x) = 1 - e^{-\left(\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}\right)}, \beta > 0, \theta >, \alpha > 0, 0 \le x \le \alpha.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Proposition 3 *Necessity* If $X \sim \text{FLD}(\alpha, \beta, \theta)$ has a CDF as defined in Eq. 1, then logarithmic form of its PDF can be expressed as

$$ln f(x) = ln(\beta) + ln(\theta) - ln(x) - (\theta + 1) ln\left(\beta ln\left(\frac{\alpha}{x}\right)\right) - \left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta},$$

on differentiating both sides with respect to x we get

$$\frac{f'(x)}{f(x)} = -\frac{1}{x} + \frac{\theta + 1}{x ln\left(\frac{\alpha}{x}\right)} - \frac{\beta \theta \left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta - 1}}{x}, \tag{27}$$

where

$$\frac{\mathfrak{M}'(x)}{\mathfrak{M}(x)} = -\frac{1}{x} + \frac{1 + \theta}{x \ln\left(\frac{\alpha}{x}\right)},$$

and

$$\frac{f'(x)}{f(x)} = \frac{\dfrac{e^{-\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}} \theta \left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-2\theta}\left(-\theta + \left(1+\theta-\ln\left(\frac{\alpha}{x}\right)\right)\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{\theta}\right)}{x^2\left(\ln\left(\frac{\alpha}{x}\right)\right)^2}}{\dfrac{\beta \theta}{x}\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta - 1} e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}}. \tag{28}$$

On comparing Eq. 27 with Eq. 28, we get

$$\mathfrak{M}'(x) = \frac{\dfrac{e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}} \theta \left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-2\theta}\left(-\theta + \left(1+\theta-ln\left(\frac{\alpha}{x}\right)\right)\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{\theta}\right)}{x^2\left(ln\left(\frac{\alpha}{x}\right)\right)^2}}{\dfrac{\beta \theta}{x}\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta - 1} e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}} + 1,$$

which after simplification yields Eq. 12.
Sufficiency Suppose Eq. 12 holds, then it may be rewritten as

$$\mathfrak{M}'(x) = \frac{\dfrac{d\left(\frac{\beta\theta}{x}\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta - 1} e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\right)}{dx} e^{-\left(\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}\right)}}{\left(\frac{\beta\theta}{x}\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta - 1} e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\right)^2} + 1,$$

On integrating the above equation from 0 to x, we get

$$\mathfrak{M}(x) = \frac{x}{\beta\theta\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta - 1}},$$

which is the Mills' ratio of FLD. This completes the proof. $\qquad\square$

Proof of Proposition 4 Necessity If $X \sim \mathrm{FLD}(\alpha, \beta, \theta)$, with a CDF defined by Eq. 1, then logarithmic form of its elasticity function can be expressed as

$$ln\mathfrak{E}(x) = ln(\theta) + ln(\beta) - (\theta + 1)ln\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)$$
$$- \left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta} - ln\left(1 - e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\right)$$

on differentiating both sides with respect to x we get

$$\frac{\mathfrak{E}'(x)}{\mathfrak{E}'(x)} = \frac{1 + \theta}{x ln\left(\frac{\alpha}{x}\right)} - \frac{\beta\theta\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-1-\theta}}{x} - \frac{e^{-\left(\beta\, ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\beta\theta\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{-1-\theta}}{\left(1 - e^{-\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\right)x},$$

on adding and subtracting $\frac{1}{x}$ in the above expression we get

$$\frac{\mathfrak{E}'(x)}{\mathfrak{E}'(x)} = -\frac{1}{x} + \frac{1 + \theta}{x ln\left(\frac{\alpha}{x}\right)} - \frac{\beta\theta\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-1-\theta}}{x}$$
$$- \frac{e^{-\left(\beta\, ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\beta\theta\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{-1-\theta}}{\left(1 - e^{-\left(\beta \ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\right)x} + \frac{1}{x},$$

which after simplification yields Eq. 13.
Sufficiency Suppose Eq. 13 holds, then it may be rewritten as

$$\mathfrak{E}'(x) = x\frac{\left(1 - e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\right)\frac{d}{dx}\left(\frac{\beta\theta\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta-1}e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}}{x}\right) - \left(\frac{\beta\theta\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta-1}e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}}{x}\right)^2}{\left(1 - e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\right)^2}$$
$$+ \frac{\left(\frac{\beta\theta\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta-1}e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}}{x}\right)}{\left(1 - e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\right)}.$$

On integrating the above equation from 0 to x, we get

$$\mathfrak{E}(x) = \frac{\beta\theta\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta-1}e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}}{\left(1 - e^{-\left(\beta ln\left(\frac{\alpha}{x}\right)\right)^{-\theta}}\right)},$$

which is the elasticity function of FLD. This completes the proof. □

Proof of Proposition 5 Necessity For a random variable X having the FLD with PDF 2 and CDF 1, we proceed as

$$(1 - \mathcal{F}(x))\mathbb{E}\left(q(x)|X \geq x\right) = e^{-3\left(\beta ln\frac{\alpha}{x}\right)^{-\theta}}, 0 < x < \alpha$$
$$(1 - \mathcal{F}(x))\mathbb{E}\left(p(x)|X \geq x\right) = e^{-2\left(\beta ln\frac{\alpha}{x}\right)^{-\theta}}, 0 < x < \alpha$$

Table 15 Model's information adaptability in data set I

Distribution	$-l$	AIC	AICC	BIC	HQIC	CAIC
FLD	225.897	457.795	458.380	463.215	454.468	458.380
MWD	238.696	483.393	483.978	488.813	480.066	483.978
3P Kappa	266.810	539.620	540.206	545.040	536.294	540.206
3P WD	227.255	460.511	461.096	465.931	457.184	461.096
EED	229.827	463.654	463.940	467.268	462.328	463.940
WD	227.256	458.511	458.797	462.125	457.185	458.797
GD	229.080	462.159	462.445	465.773	460.833	462.445
IGD	236.586	477.172	477.458	480.785	475.846	477.458
FD	238.457	480.914	481.200	484.528	479.588	481.200
LND	232.186	468.372	468.650	471.985	467.046	468.658
LLD	233.400	470.800	471.086	474.413	469.474	471.086

which implies that

$$\psi(x) = \frac{\mathbb{E}\left(p(x)|X \geq x\right)}{\mathbb{E}\left(q(x)|X \geq x\right)} = e^{(\beta ln \frac{\alpha}{x})^{-\theta}}, \ 0 < x < \alpha$$

$$\psi(x)q(x) - p(x) = e^{-(\beta ln \frac{\alpha}{x})^{-\theta}}, \ 0 < x < \alpha$$

and $\psi(x)q(x) - p(x) = e^{-(\beta ln \frac{\alpha}{x})^{-\theta}}$ for $0 < x < \alpha$. The differential equation (14) clearly holds.

Sufficiency If g and λ satisfy the differential Eq. 14, then

$$r'(x) = \frac{\psi'(x)q(x)}{\psi(x)q(x) - p(x)} = \frac{3\beta\theta(\beta ln(\frac{\alpha}{x}))^{-\theta-1}}{x}, \ \ 0 < x < \alpha$$

hence $r(x) = 3(\beta ln(\frac{\alpha}{x}))^{-\theta}$. Now from Theorem 1, X has PDF 2.   □

A2-Information Criteria

See Tables 15, 16, 17, and 18.

Table 16 Model's information adaptability in data set II

Distribution	$-l$	AIC	AICC	BIC	HQIC	CAIC
FLD	279.314	564.628	565.128	570.482	561.376	565.128
MWD	280.487	566.974	567.474	572.828	563.722	567.474
3P Kappa	305.663	617.326	617.826	623.180	614.074	617.826
3P WD	279.420	564.840	565.340	570.694	561.588	565.340
EED	279.830	563.660	563.905	567.562	562.408	563.905
WD	279.481	562.962	563.207	566.864	561.710	563.207
GD	279.769	563.538	563.783	567.440	562.286	563.783
IGD	306.082	616.164	616.409	620.066	614.912	616.409
FD	300.250	604.500	604.745	608.402	603.248	604.745
LND	287.003	578.006	578.251	581.908	576.754	578.251
LLD	284.557	573.114	573.359	577.016	571.862	573.359

Table 17 Model's information adaptability in data set III

Distribution	$-l$	AIC	AICC	BIC	HQIC	CAIC
FLD	654.595	1315.19	1315.44	1322.98	1312.24	1315.44
MWD	705.737	1417.47	1417.73	1425.26	1414.52	1417.73
3P Kappa	849.559	1705.12	1705.37	1712.9	1702.17	1705.37
3P WD	659.668	1325.34	1325.59	1333.12	1322.39	1325.59
EED	675.963	1355.93	1356.05	1361.12	1354.98	1356.05
WD	663.779	1331.56	1331.68	1336.75	1330.61	1331.68
GD	672.482	1348.96	1349.09	1354.15	1348.01	1349.09
IGD	708.044	1420.09	1420.21	1425.28	1419.14	1420.21
FD	715.874	1435.75	1435.87	1440.94	1434.8	1435.87
LND	685.179	1374.36	1374.48	1379.55	1373.41	1374.48
LLD	679.026	1362.05	1362.18	1367.24	1361.1	1362.18

Table 18 Model's information adaptability in data set IV

Distribution	$-l$	AIC	AICC	BIC	HQIC	CAIC
FLD	520.221	1046.44	1046.87	1052.73	1043.26	1046.87
MWD	521.459	1048.92	1049.35	1055.20	1045.74	1049.35
3P Kappa	595.067	1196.13	1196.56	1202.42	1192.95	1196.56
3P WD	520.472	1046.94	1047.37	1053.23	1043.76	1047.37
EED	521.436	1046.87	1047.08	1051.06	1045.69	1047.08
WD	521.582	1047.16	1047.37	1051.35	1045.98	1047.37
GD	521.479	1046.96	1047.17	1051.15	1045.78	1047.17
IGD	552.627	1109.25	1109.46	1113.44	1108.07	1109.46
FD	544.278	1092.56	1092.77	1096.74	1091.38	1092.77
LND	531.031	1066.06	1066.27	1070.25	1064.88	1066.27
LLD	529.666	1063.33	1063.54	1067.52	1062.15	1063.54

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