



Two-Point Distortion Theorems for Harmonic Mappings

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Abstract

We establish two-point distortion theorems for sense-preserving planar harmonic mappings $f = h + \bar{g}$ in the unit disk \mathbb{D} which satisfy harmonic versions of the univalence criteria due to Becker and Nehari. In addition, we also find two-point distortion theorems for the cases when h is a normalized convex function and, more generally, when $h(\mathbb{D})$ is a c -linearly connected domain.

Keywords Two-point distortion · Harmonic mappings · Univalence criterion

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1 Introduction

To a large extent, the two-point distortion theorems provide us quantitative information on “how injective” a conformal function is, in the sense of estimating the distance (Euclidean or hyperbolic) between the images of any two points. That is, we seek lower and upper bounds for the distance between $f(a)$ and $f(b)$ for all a and b in the unit disk \mathbb{D} , and these bounds should involve, in some sense, $\rho(a, b) = |(a - b)/(1 - \bar{a}b)|$ and

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the derivative of the function f . The first two-point distortion theorem was introduced by Blatter in [5], who proved that if f is a conformal mapping, then

$$|f(a) - f(b)|^2 \geq \frac{\sinh^2(2d(a, b))}{8(\cosh(4d(a, b)))} (R(a)^2 + R(b)^2).$$

Here, $d(a, b) = \tanh^{-1}(\rho(a, b))$ and $R(z) = (1 - |z|^2)|f'(z)|$. Moreover, for different subclasses of univalent functions, such as convex, starlike, etc., there are two-point distortion theorems in which the bounds are more accurate. For instance, the authors in [16] show different two-point distortion theorems for bounded boundary rotation functions, among other related results.

In particular, for conformal maps onto convex domains, Kim and Minda in [17] (see the corollary of Theorem 3) proved the following result:

Theorem A. If $f : \mathbb{D} \rightarrow \mathbb{C}$ is a conformal convex mapping, then for all $p \geq 1$ and all a, b in \mathbb{D} we have that

$$|f(a) - f(b)| \geq \frac{\sinh(d(a, b))}{2(\cosh(pd(a, b)))^{1/p}} (R(a)^p + R(b)^p)^{1/p}. \tag{1.1}$$

On the other hand, univalence criteria involving both the pre-Schwarzian and Schwarzian derivatives of locally univalent functions defined in \mathbb{D} are well known and also provide their two-point distortion theorems. Let f be a locally univalent mapping, its Schwarzian derivative is defined by:

$$Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2} \left(\frac{f''}{f'}\right)^2 = Pf' - \frac{1}{2}Pf^2, \tag{1.2}$$

here, $Pf = f''/f'$ is named *pre-Schwarzian* derivative of f . Nehari in [22] showed that $\|Sf\| := \sup\{|Sf(z)|(1 - |z|^2)^2 : z \in \mathbb{D}\} \leq 2$ implies that f is univalent in \mathbb{D} . To see more univalence criteria that involve the Schwarzian derivative, we can see the p -criterion due to Nehari in [23]. In this context (every convex mappings satisfies that its Schwarzian norm is bounded by 2), Chuaqui and Pommerenke [7] gave a quantitative version of Nehari’s theorem by showing that the condition $\|Sf\| \leq 2$ implies that f has the two-point distortion property

$$|f(a) - f(b)| \geq d(a, b)\sqrt{R(a)R(b)}. \tag{1.3}$$

This result was extended to the mentioned p -criterion of univalence by Ma et al. in [21].

This paper aims to obtain two-point distortion theorems associated with different univalence criteria for complex-valued harmonic mappings defined in \mathbb{D} . These injectivity criteria are associated with both the pre-Schwarzian and Schwarzian derivatives and also with harmonic mappings whose analytic part is convex, see [6].

The manuscript is organized as follows: Sect. 2 deals with the preliminaries on complex-valued harmonic mappings and includes the above-mentioned univalence criteria, namely Theorems B, C, and D. Sections 3.1 and 3.2 are devoted to the two-point distortion theorems associated with the harmonic versions of the Becker and Nehari criteria, respectively. Special attention is paid in Sect. 3.3, since we not only deal with the case when the analytic part is convex, but also when it conformally maps the disk onto a *linearly connected* domain which means geometrically that it has no inward cusps. Indeed, we prove that these mappings form a linearly invariant family from which the two-point distortion theorem follows.

2 Preliminaries

Complex-valued harmonic functions f in a simply connected domain Ω are those who satisfy that $\Delta f = \partial^2 f / \partial z \partial \bar{z} = 0$, which is equivalent to $f = u + iv$ where u and v are real harmonic functions defined in Ω . These functions have a canonical representation $f = h + \bar{g}$, where h and g are analytic functions in Ω , that is unique up to an additive constant. When $\Omega = \mathbb{D}$, it is convenient to choose the additive constant so that $g(0) = 0$. The representation $f = h + \bar{g}$ is then unique and is called the canonical representation of f . A result of Lewy [18] states that f is locally univalent if and only if its Jacobian $J_f = |h'|^2 - |g'|^2$ does not vanish in Ω . Thus, harmonic mappings are either sense-preserving or sense-reversing depending on the conditions $J_f > 0$ or $J_f < 0$ throughout the domain Ω where f is locally univalent, respectively. Since $J_f > 0$ if and only if $J_{\bar{f}} < 0$, throughout this work we will consider sense-preserving mappings in \mathbb{D} . In this case, the analytic part h is locally univalent in \mathbb{D} since $h' \neq 0$, and the second complex dilatation of f , $\omega = g'/h'$, is an analytic function in \mathbb{D} with $|\omega| < 1$. The reader can find an elegant reference for this topic in [8].

Let S_H denote the family of sense-preserving univalent harmonic mappings $f = h + \bar{g}$ normalized by $h(0) = g(0) = 0$, and $h'(0) = 1$. A family $\mathcal{F} \in S_H$ is said to be linearly invariant if $f \in \mathcal{F}$ implies that, for all $a \in \mathbb{D}$,

$$\frac{f\left(\frac{z+a}{1+\bar{a}z}\right) - f(a)}{(1-|a|^2)h'(a)} \in \mathcal{F}.$$

The family \mathcal{F} is affine invariant if $f \in \mathcal{F}$ implies that, for all $\varepsilon \in \mathbb{D}$,

$$\frac{f + \varepsilon \bar{f}}{1 + \varepsilon g'(0)} \in \mathcal{F}.$$

The order of the family \mathcal{F} is define by $\alpha = \sup\{|h''(0)|/2 : f \in \mathcal{F}\}$. For more details on analytic linear invariant families, see [24], and [8] for affine and linear invariant families in the context of harmonic mappings.

Duren et al. in [10] showed, among other results, that if $f \in \mathcal{F}$ then

$$|f(a) - f(b)| \geq \frac{1}{2\alpha} (1 - \exp(-2\alpha d(a, b))) \max\{R(a), R(b)\},$$

where $R(z) = (1 - |z|^2)(|h'(z)| - |g'(z)|)$. Moreover, they also proved that

$$|f(a) - f(b)| \leq \frac{1}{2\alpha}(\exp(2\alpha d(a, b) - 1)) \min\{Q(a), Q(b)\},$$

where $Q(z) = (1 - |z|^2)(|h'(z)| + |g'(z)|)$. The above two inequalities represent the harmonic version of Blatter’s theorem. Note that in the analytic case, $\alpha = 2$, however, in the harmonic case, the size of α is unknown. In fact, the best known bound, when $\omega(0) = 0$, is $\alpha \leq 20.9197$. See [1, Thm. 1].

Hernández and Martín [12], defined the harmonic pre-Schwarzian and Schwarzian derivatives for sense-preserving harmonic mappings $f = h + \bar{g}$. Using those definitions, they generalized different results regarding analytic mappings to the harmonic case (see [12–15]). The reader can find in [11, 19, 20] relations of these operators with the order of an affine and linearly invariant family, and also necessary and sufficient conditions for which the norm of the pre-Schwarzian is bounded, among other results.

The pre-Schwarzian and Schwarzian derivatives of a sense-preserving harmonic mapping f are defined by

$$P_f = \frac{h''}{h'} - \frac{\bar{\omega}\omega'}{1 - |\omega|^2} = Ph - \frac{\bar{\omega}\omega'}{1 - |\omega|^2} = \frac{\partial}{\partial z} \log(J_f),$$

$$S_f = \frac{\partial}{\partial z} P_f - \frac{1}{2}(P_f)^2 = Sh + \frac{\bar{\omega}}{1 - |\omega|^2} \left(\omega' \frac{h''}{h'} - \omega'' \right) - \frac{3}{2} \left(\frac{\omega' \bar{\omega}}{1 - |\omega|^2} \right)^2,$$

where Ph and Sh are the standard pre-Schwarzian and Schwarzian derivatives, respectively, which are given by Eq. (1.2). It is easy to see that if f is analytic (ω is constant) then $P_f = h''/h'$ and $S_f = Sh$, recovering the classical definition of these operators. In [12], the authors proved that $S_f = 0$ if and only if f is a harmonic Möbius transformation, which is given by $f = h + \alpha \bar{h}$ with $\alpha \in \mathbb{D}$ and h is an analytic Möbius mapping. To motivate the two-point distortion theorems in the setting of harmonic mappings, notice that any harmonic Möbius mappings, $f = h + \alpha \bar{h}$, satisfy the relation

$$|f(a) - f(b)| = \sqrt{|h'(a)||h'(b)|} |a - b| |1 + \lambda\alpha|, \quad \lambda = \frac{\overline{h(a) - h(b)}}{h(a) - h(b)}, \quad a \neq b.$$

Thus, we can re-write this equation in terms of d to get

$$|f(a) - f(b)| = \sqrt{R_h(a)R_h(b)} \sinh(d(a, b)) |1 + \lambda\alpha|, \quad R_h(z) = |h'(z)|(1 - |z|^2).$$

In addition, the authors in [12] proved that $P_{f+\alpha\bar{f}} = P_f$ for all $a \in \mathbb{D}$, which is the key to prove the extension of Becker’s criterion of univalence for harmonic mappings, contained in the following theorem:

Theorem B. Let $f = h + \bar{g}$ be a locally univalent and sense-preserving harmonic function in the unit disk \mathbb{D} with dilatation ω . If for all $z \in \mathbb{D}$

$$(1 - |z|^2)|zP_f(z)| + \frac{|z\omega'(z)|(1 - |z|^2)}{1 - |\omega(z)|^2} \leq 1,$$

then f is univalent. The constant 1 is sharp.

The classical Becker's criterion can be found in [4] and is obtained by setting $\omega \equiv 0$ in theorem B.

In addition, in [15] the authors proved $S_{f+a\bar{f}} = S_f$ for all $a \in \mathbb{D}$ and the following generalization of Nehari's classical univalence criterion:

Theorem C. Let $f = h + \bar{g}$ be a locally univalent and sense-preserving harmonic function in the unit disk \mathbb{D} with dilatation ω . Then, there exists $\varepsilon > 0$ such that

$$(1 - |z|^2)^2 |S_f(z)| \leq \varepsilon$$

implies that f is univalent.

The classical result, due to Nehari, is obtained by taking $\omega = 0$, in which case $\varepsilon = 2$. Unfortunately, the value of ε is still unknown for the planar harmonic mappings setting. From the proof of this theorem (see [15]), we can see that this value ε is small enough to have $\|Sh\| \leq 2$, i.e., h satisfies Nehari's classical univalence criterion. Moreover, since S_f is invariant under the post-composition with affine harmonic mappings, we have that $h + ag$ satisfies Nehari's criterion for all $a \in \bar{\mathbb{D}}$.

The last criterion that we will consider in this manuscript appears in [6] and says the following:

Theorem D. Let $f = h + \bar{g}$ be a locally univalent and sense-preserving harmonic function in the unit disk \mathbb{D} with dilatation ω . If h is a convex mapping, then f is univalent. More generally, when $h(\mathbb{D})$ is c -linearly connected domain, if $\|\omega\|_\infty := \sup\{|\omega(z)| : z \in \mathbb{D}\} < 1/c$, then f is also univalent in the unit disk.

In the same spirit, Abu Muhana and Ponnusamy in [2] obtained that $f = h + \bar{g}$ is close to convex (in particular univalent) under other assumptions on the analytic part h .

3 Harmonic two-point distortion theorems

Throughout the section, we set φ as a locally univalent analytic mapping in \mathbb{D} , and let $f = h + \bar{g}$ be a locally univalent and sense-preserving harmonic mapping with dilatation ω defined in the unit disk. Recall that the fact that f is locally univalent and sense-preserving implies that $h' \neq 0$ and $\omega(\mathbb{D}) \subset \mathbb{D}$. Additionally, we define $R_\varphi(z) = (1 - |z|^2)|\varphi'(z)|$ and $R(z) = (1 - |z|^2)(|h'(z)| - |g'(z)|)$, and $Q(z) = (1 - |z|^2)(|h'(z)| + |g'(z)|)$.

3.1 Functions satisfying the Becker criterion of univalence

Let φ be a normalized locally univalent analytic mapping defined in \mathbb{D} . Its order is given by:

$$\text{ord}\langle\varphi\rangle = \sup_{z \in \mathbb{D}} |A_\varphi(z)| = \sup_{a \in \mathbb{D}} \left\{ \frac{1}{2} |\varphi_a''(0)| : \varphi_a(z) = \frac{\varphi\left(\frac{z+a}{1+\bar{a}z}\right) - \varphi(a)}{(1-|a|^2)\varphi'(a)} \right\},$$

where

$$A_\varphi(z) = \frac{1}{2}(1-|z|^2)\frac{\varphi''}{\varphi'}(z) - \bar{z}.$$

φ_a is named the *Koebe Transform* of φ . It is not difficult to see that $|A_{\varphi_a}(z)| = |A_\varphi((z+a)/(1+\bar{a}z))|$ (see [3, Eq. (4)]) which implies that the $\text{ord}\langle\varphi_a\rangle = \text{ord}\langle\varphi\rangle$. A well-known result (see [24]) asserts that if φ is a univalent mapping defined in the unit disk, normalized to $\varphi(0) = 0$ and $\varphi'(0) = 1$, and its order is α , then

$$\frac{1}{2\alpha} \left(\left(\frac{1+|z|}{1-|z|} \right)^\alpha - 1 \right) \geq |\varphi(z)| \geq \frac{1}{2\alpha} \left(1 - \left(\frac{1-|z|}{1+|z|} \right)^\alpha \right), \quad z \in \mathbb{D}. \quad (3.1)$$

In terms of hyperbolic metric d , this equation can be re-written as

$$\frac{1}{2\alpha} (\exp(2\alpha d(z, 0)) - 1) \geq |\varphi(z) - \varphi(0)| \geq \frac{1}{2\alpha} (1 - \exp(-2\alpha d(z, 0))).$$

Thus, by applying this inequality to φ_a and taking $b = (z+a)/(1+\bar{a}z)$, we have that

$$\frac{1}{2\alpha} (\exp(2\alpha d(a, b)) - 1) \geq \left| \frac{\varphi(b) - \varphi(a)}{(1-|a|^2)|\varphi'(a)|} \right| \geq \frac{1}{2\alpha} (1 - \exp(-2\alpha d(a, b))).$$

Since these inequalities hold for all a and b in \mathbb{D} , we can swap a with b to obtain that

$$\begin{aligned} |\varphi(a) - \varphi(b)| &\geq \frac{1}{2\alpha} (1 - \exp(-2\alpha d(a, b))) \max\{R_\varphi(a), R_\varphi(b)\}, \\ |\varphi(a) - \varphi(b)| &\leq \frac{1}{2\alpha} (\exp(2\alpha d(a, b)) - 1) \min\{R_\varphi(a), R_\varphi(b)\}. \end{aligned} \quad (3.2)$$

The following theorem is an application of (3.2) when φ satisfies the Becker’s criterion [4]. We haven’t been able to find any reference on this, so we include a proof.

Theorem 3.1 Let $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ be a locally univalent mapping with $\varphi(0) = 0$ and $\varphi'(0) = 1$ such that $|P\varphi(z)|(1 - |z|^2) \leq 1$. Then, for all $a, b \in \mathbb{D}$ we have that

$$\begin{aligned} |\varphi(a) - \varphi(b)| &\geq \frac{1 - \exp(-3d(a, b))}{3} \max\{R_\varphi(a), R_\varphi(b)\}, \\ |\varphi(a) - \varphi(b)| &\leq \frac{\exp(3d(a, b)) - 1}{3} \min\{R_\varphi(a), R_\varphi(b)\}. \end{aligned} \quad (3.3)$$

Proof We note that φ is univalent and its order satisfies that

$$\alpha = \text{ord}(\varphi) = \sup_{z \in \mathbb{D}} \left| \frac{1}{2}(1 - |z|^2) \frac{\varphi''}{\varphi'}(z) - \bar{z} \right| \leq \frac{1}{2} + |z| \leq \frac{3}{2}.$$

Since the real function $(\xi^\alpha - 1)/\alpha$ is increasing and $(\xi^{-\alpha} - 1)/\alpha$ is decreasing in α for any $\xi > 1$, using $\alpha \leq 3/2$ and inequality (3.2) again, the proof is complete. \square

Note that, under the hypothesis of Theorem B, the analytic part of $f = h + \bar{g}$ satisfies that $|Ph(z)|(1 - |z|^2) \leq 1$ and, therefore, it satisfies the Becker's criterion for analytic mappings in \mathbb{D} . Since $P_{f+\lambda\bar{f}} = P_f$, it follows that $\psi_\lambda = h + \lambda g$ satisfies Becker's univalence criterion for all $\lambda \in \mathbb{D}$; moreover, taking limits when $|\lambda| \rightarrow 1$, we can assert that Becker's criterion holds for $|\lambda| \leq 1$, which is the key for the following theorem:

Theorem 3.2 Let $f = h + \bar{g}$ be a sense-preserving harmonic mapping defined in \mathbb{D} satisfying the hypothesis of Theorem B. Then,

$$|f(a) - f(b)| \geq \frac{1}{3}(1 - \exp(-3d(a, b))) \max\{R(a), R(b)\},$$

and

$$|f(a) - f(b)| \leq \frac{1}{3}(\exp(3d(a, b)) - 1) \min\{Q(a), Q(b)\}.$$

Proof For a and b in \mathbb{D} , it follows that $f(a) - f(b) = h(a) - h(b) + \overline{(g(a) - g(b))}$, and thus, if $g(a) \neq g(b)$, then there exists a unimodular constant λ such that $f(a) - f(b) = \psi_\lambda(a) - \psi_\lambda(b)$. In the case, when $g(a) = g(b)$, we can consider $\lambda = 0$. In any case, ψ_λ satisfies Becker's criterion; thus, we can apply inequalities (3.3) and the fact that $|h'| + |g'| \geq |\psi'_\lambda| \geq |h'| - |g'|$ the proof is complete. \square

3.2 Functions satisfying Nehari's criterion

Let $f = h + \bar{g}$ be a sense-preserving harmonic mapping with dilatation ω . In [15], the authors show that there exists $\varepsilon > 0$ such that if $\|S_f\| = \sup\{(1 - |z|^2)^2 |S_f(z)| :$

$z \in \mathbb{D}\} \leq \varepsilon$, then f is univalent in \mathbb{D} . Moreover, they show that there exists a constant $k > 0$ such that

$$\|Sh\| \leq \varepsilon + kW_\varepsilon + \frac{3}{2}W_\varepsilon^2,$$

where

$$W_\varepsilon = \sup\{\|\omega^*\| := \sup_{z \in \mathbb{D}} \frac{|\omega'(z)|(1 - |z|^2)}{1 - |\omega(z)|^2} : \omega \in A_\varepsilon\}.$$

We say that an analytic mapping ω belongs to A_ε if and only there exists $f = h + \bar{g}$ with dilatation ω and $\|S_f\| \leq \varepsilon$. By Schwarz–Pick lemma, we have that $W_\varepsilon \leq 1$. Moreover, in [15] it has been proved that $W_\varepsilon \rightarrow 0$ when $\varepsilon \rightarrow 0$. We can assume that ε is given by

$$\varepsilon = \sup\{\delta : \delta + kW_\delta + \frac{3}{2}W_\delta^2 \leq 2\};$$

thus, Theorem C asserts that $\|S_f\| \leq \varepsilon$ is a sufficient condition for univalence for sense-preserving harmonic mappings. From the proof of Theorem C, it follows that for $\lambda \in \mathbb{D}$, the locally univalent analytic mapping $\psi_\lambda = h + \lambda g$ satisfies that $\|S\psi_\lambda\| \leq \varepsilon$ since $f_\lambda = f + \lambda \bar{f}$ has the same Schwarzian derivative. Thus, we can assert the following lemma:

Lemma 3.3 *Let $f = h + \bar{g}$ be a sense-preserving harmonic mapping defined in \mathbb{D} such that $\|S_f\| \leq \varepsilon$. Then, for any $\lambda \in \mathbb{D}$, the corresponding mapping ψ_λ satisfies that $\|S\psi_\lambda\| \leq 2$.*

In [21, Thm. 1.1 part a)], the authors proved that if $\|S\varphi\| \leq 2t$ with $t \in [0, 1]$, then

$$|\varphi(a) - \varphi(b)| \leq \sqrt{\frac{R_\varphi(a)R_\varphi(b)}{1+t}} \sinh\left(\sqrt{1+t}d(a,b)\right), \tag{3.4}$$

which together with the inequality (1.3), [7, Thm. 2], are the key points to prove the following theorem.

Theorem 3.4 *Let $f = h + \bar{g}$ be a sense-preserving harmonic mapping defined in \mathbb{D} . If $\|S_f\| \leq \varepsilon$, where ε is as in Theorem C, then*

$$\sqrt{\frac{Q(a)Q(b)}{2}} \sinh\left(\sqrt{2}d(a,b)\right) \geq |f(a) - f(b)| \geq d(a,b)\sqrt{R(a)R(b)}.$$

Proof For any a and b in the unit disk, it follows that $f(a) - f(b) = h(a) - h(b) + \frac{(g(a) - g(b))}{\dots}$, and thus, if $g(a) \neq g(b)$, then there exists an unimodular constant λ such that $f(a) - f(b) = \psi_\lambda(a) - \psi_\lambda(b)$. In the case, when $g(a) = g(b)$, we can

consider $\lambda = 0$. Using inequality (1.3) we get the lower bound and inequality (3.4) to the upper bound, we have that

$$\frac{R_{\psi_\lambda}(a)R_{\psi_\lambda}(b)}{2} \sinh^2 \left(\sqrt{2} d(a, b) \right) \geq |\psi_\lambda(a) - \psi_\lambda(b)|^2 \geq d(a, b)^2 R_{\psi_\lambda}(a)R_{\psi_\lambda}(b).$$

Since $|h'| + |g'| \geq |\psi'_\lambda| \geq |h'| - |g'|$, the proof is complete. \square

Note that if f is analytic, which is the case when $\omega = 0$, then $W_\varepsilon = 0$ and $\varepsilon = 2$, and thus, the last theorem is a generalization of [7, Thm. 2].

3.3 Functions with convex analytic part

We know that $f = h + \bar{g}$ is univalent (in fact close to convex) [6, Thm. 1] in the unit disk when h is a convex mapping. Moreover, this result can be extended to the case when h maps conformally \mathbb{D} onto a linearly connected domain, which is the subject of sub-section 3.1.1.

Theorem 3.5 *Let $f = h + \bar{g}$ be a sense-preserving normalized harmonic mapping defined in \mathbb{D} such that h is a convex function. Then, for all $a, b \in \mathbb{D}$*

$$|f(a) - f(b)| \geq (1 - \|\omega\|_\infty) \tanh(d(a, b)) \left(\frac{R_h(a) + R_h(b)}{2} \right),$$

and

$$|f(a) - f(b)| \leq (1 + \|\omega\|_\infty) (\exp(2d(a, b)) - 1) \frac{\min\{R_h(a), R_h(b)\}}{2}.$$

Proof We note that

$$|h(a) - h(b)| + |g(a) - g(b)| \geq |f(a) - f(b)| \geq |h(a) - h(b)| - |g(a) - g(b)|. \quad (3.5)$$

Considering the curve $\gamma = h^{-1}(R)$, where R is the segment joining $h(a)$ and $h(b)$, then for $\zeta = h^{-1}(w)$ we have that $h'(\zeta)d\zeta = dw$ and

$$|g(a) - g(b)| = \left| \int_\gamma g'(\zeta) d\zeta \right| = \left| \int_R \frac{g'(\zeta)}{h'(\zeta)} dw \right| \leq \|\omega\|_\infty \int_R |dw| = \|\omega\|_\infty |h(a) - h(b)|.$$

Hence, substituting this inequality in (3.5) we obtain

$$(1 + \|\omega\|_\infty) |h(a) - h(b)| \geq |f(a) - f(b)| \geq (1 - \|\omega\|_\infty) |h(a) - h(b)|.$$

Therefore, using inequality (1.1) in Theorem A (with $p = 1$), the lower bound follows. Now, since h is a normalized convex mapping, we can apply the upper bound in (3.2) with $\alpha = 1$. \square

3.3.1 When the analytic part h is c -linearly connected

We will pay special attention to the case where $h(\mathbb{D})$ is a linearly connected domain, which means that for any a and b in \mathbb{D} there exists a curve γ in $h(\mathbb{D})$ that joins $h(a)$ with $h(b)$, such that its length satisfies that

$$\ell_\gamma \leq c|h(a) - h(b)|. \tag{3.6}$$

In this case, we say that h is a c -linearly connected mapping. Observe that $c \geq 1$ and $c = 1$ if and only if $h(\mathbb{D})$ is a convex domain. Pommerenke said that the family of normalized c -linearly connected mappings is linearly invariant (see [25, p. 105]), but he did not provide any proof of this. Since we were not able to find a proof of this fact, here we will give one.

Theorem 3.6 *Let $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ be a conformal and c -linearly connected mapping defined in \mathbb{D} . Then,*

$$\varphi_a(z) = \frac{\varphi\left(\frac{z+a}{1+\bar{a}z}\right) - \varphi(a)}{(1-|a|^2)\varphi'(a)}, \quad a \in \mathbb{D},$$

is a c -linearly connected mapping.

Proof Let x and y be any two points in \mathbb{D} and $\sigma_a(z) = (z+a)/(1+\bar{a}z)$ an automorphism of the unit disk. Let Γ be a curve in $\varphi(\mathbb{D})$ whose length satisfies $\ell_\Gamma \leq c|\varphi(\sigma_a(x)) - \varphi(\sigma_a(y))|$. Let $\gamma = \sigma_a^{-1} \circ \varphi^{-1}(\Gamma)$ be a curve in \mathbb{D} which joins x with y . Thus, the curve $\Gamma_a = \varphi_a(\gamma)$ is completely contained in $\varphi_a(\mathbb{D})$ and joins $\varphi_a(x)$ with $\varphi_a(y)$. Moreover, the length of this curve satisfies that

$$\begin{aligned} \ell_{\Gamma_a} &= \int_\gamma |\varphi'_a(z)||dz| = \frac{1}{(1-|a|^2)|\varphi'(a)|} \int_{\sigma_a(\gamma)} |\varphi'(w)||dw| = \frac{\ell_\Gamma}{(1-|a|^2)|\varphi'(a)|} \\ &\leq \frac{c|\varphi(\sigma_a(x)) - \varphi(\sigma_a(y))|}{(1-|a|^2)|\varphi'(a)|} = c|\varphi_a(x) - \varphi_a(y)|. \end{aligned}$$

□

This theorem asserts that for any $c \geq 1$ the set $K_c = \{\varphi \in S : \varphi(\mathbb{D}) \text{ is } c\text{-linearly connected}\}$ is a linearly invariant family. Recall that S is the classical family of normalized conformal maps and K_1 the family of normalized convex mappings (see [9]). Let β be the order of K_c , which depends on c and tends to 2 when c goes to ∞ . Also, since $K_c \subset S$, we have that $\beta < 2$.

In [6], it was proved that if $h(\mathbb{D})$ is a linearly connected domain with constant c and $\|\omega\|_\infty < 1/c$, then f is univalent in the unit disk. The following theorem is the two-point distortion result for this criterion of univalence.

Theorem 3.7 *Let $f = h + \bar{g}$ be a sense-preserving harmonic mapping defined in \mathbb{D} such that h is in K_c and its dilatation satisfies that $\|\omega\|_\infty < 1/c$. Then, for any a and b in \mathbb{D}*

$$|f(a) - f(b)| \geq (1 - c\|\omega\|_\infty) \frac{1}{2\beta} (1 - \exp(-2\beta d(a, b))) \max\{R_h(a), R_h(b)\},$$

and

$$|f(a) - f(b)| \leq (1 + c\|\omega\|_\infty) \frac{1}{2\beta} (\exp(2\beta d(a, b)) - 1) \min\{R_h(a), R_h(b)\},$$

where β is the order of h .

Proof For any a and b in \mathbb{D} , there exists a curve $\gamma \subset h(\mathbb{D})$ such that its length ℓ_γ satisfies (3.6). Let $\Gamma = h^{-1}(\gamma) \subset \mathbb{D}$. Thus, considering $z = h^{-1}(\zeta)$ with $\zeta \in \gamma$, we have that

$$|g(a) - g(b)| = \left| \int_\Gamma g'(z) dz \right| \leq \int_\gamma \frac{|g'(z)|}{|h'(z)|} |d\zeta| \leq \|\omega\|_\infty \ell_\gamma \leq c\|\omega\|_\infty |h(a) - h(b)|.$$

As $|f(a) - f(b)| = |h(a) - h(b) + \overline{(g(a) - g(b))}|$, we have that

$$(1 + c\|\omega\|_\infty) |h(a) - h(b)| \geq |f(a) - f(b)| \geq (1 - c\|\omega\|_\infty) |h(a) - h(b)|. \quad (3.7)$$

Since $h \in K_c$ with order β , applying (3.2), the proof follows. \square

By applying the same arguments as in [6, p. 1191], it can be seen that if $h \in K_c$ and the dilatation ω satisfies $\|\omega\|_\infty \leq m < 1/c$, then, for any λ with modulus 1, the corresponding analytic function $\psi_\lambda = h + \lambda g$ is also a k -linearly connected mapping with constant k given by $k = c(1 + m)/(1 - mc)$. Since

$$\frac{\psi''_\lambda}{\psi'_\lambda} = \frac{h''}{h'} + \frac{\lambda \omega'}{1 + \lambda \omega},$$

the order of ψ_λ , namely β_λ , satisfies that $\beta_\lambda \leq \min\{2, \beta + \|\omega^*\|(1 + \|\omega\|_\infty)\}$. Thus, we can assert the following corollary.

Corollary 3.8 *Let $f = h + \bar{g}$ be a locally univalent and sense-preserving harmonic mapping defined in \mathbb{D} such that $\omega(0) = 0$, h is in K_c and its dilatation satisfies that $\|\omega\|_\infty \leq m < 1/c$. Then, for any a and b in \mathbb{D}*

$$|f(a) - f(b)| \geq \frac{1}{2\beta_\lambda} (1 - \exp(-2\beta_\lambda d(a, b))) \max\{R_{\psi_\lambda}(a), R_{\psi_\lambda}(b)\},$$

and

$$|f(a) - f(b)| \leq \frac{1}{2\beta_\lambda} (\exp(2\beta_\lambda d(a, b)) - 1) \min\{R_{\psi_\lambda}(a), R_{\psi_\lambda}(b)\},$$

where $\lambda = (g(a) - g(b))/\overline{(g(a) - g(b))}$ when $g(a) - g(b) \neq 0$, and otherwise, $\lambda = 0$.

Proof We first assume that $g(a) - g(b) \neq 0$; hence, $|f(a) - f(b)| = |\psi_\lambda(a) - \psi_\lambda(b)|$. Since ψ_λ is a k -linearly connected mapping, we can apply the inequality (3.2) and the corollary follows. On the other hand, if $g(a) - g(b) = 0$, then $|f(a) - f(b)| = |h(a) - h(b)|$ and $\psi_0 = h$. Again, we can apply inequality (3.2) to get the statement. \square

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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