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Exel-Pardo Algebras of Self-Similar k-Graphs

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Abstract

We introduce the Exel-Pardo $*$-algebra $\mathrm{EP}_R(G, \Lambda)$ associated to a self-similar k-graph (G, Λ, φ). We also prove the $\mathbb{Z}^k$-graded and Cuntz–Krieger uniqueness theorems for such algebras and investigate their ideal structure. In particular, we modify the graded uniqueness theorem for self-similar 1-graphs and then apply it to present $\mathrm{EP}_R(G, \Lambda)$ as a Steinberg algebra and to study the ideal structure.

Keywords Self-similar k-graph · Exel-Pardo algebra · Groupoid algebra · Ideal structure

Mathematics Subject Classification 16D70 · 16W50

1 Introduction

To give a unified framework like graph C^*-algebras for the Katsura's [11] and Nekrashevyche's algebras [17, 18], Exel and Pardo introduced self-similar graphs and their C^*-algebras in [7]. They then associated an inverse semigroup and groupoid model to this class of C^*-algebras and studied structural features by underlying self-similar graphs. Note that although only finite graphs are considered in [7], many of arguments and results may be easily generalized for countable row-finite graphs with no sources (see [8, 10] for example). Inspired from [7], Li and Yang in [15, 16] introduced self-similar action of a discrete countable group G on a row-finite k-graph Λ. They then associated a universal C^*-algebra $\mathcal{O}_{G,\Lambda}$ to (G, Λ) satisfying specific relations.

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The algebraic analogues of Exel-Pardo C^*-algebras, denoted by $\mathcal{O}^{\mathrm{alg}}_{(G,E)}$ in [6] and by $L_R(G, E)$ in [9], were introduced and studied in [6, 9]. In particular, Hazrat et al. proved a $\mathbb{Z}$-graded uniqueness theorem and gave a model of Steinberg algebras for $L_R(G, E)$ [9]. The initial aims to write the present paper are to give a much easier proof for [9, Theorem B] (a groupoid model for $L_R(G, E)$) using the $\mathbb{Z}$-graded uniqueness theorem and then to study the ideal structure. However, we do these here, among others, for a more general class of algebras associated to self-similar higher rank graphs (G, Λ), which is introduced in Sect. 2.

This article is organized as follows. Let R be a unital commutative $*$-ring. In Sect. 2, we introduce a universal $*$-algebra $\mathrm{EP}_R(G, \Lambda)$ of a self-similar k-graph (G, Λ) satisfying specific properties, which is called the *Exel-Pardo algebra of* (G, Λ). Our algebras are the higher rank generalization of those in [9, Theorem 1.6] and the algebraic analogue of $\mathcal{O}_{G,\Lambda}$ [15, 16]. Moreover, this class of algebras includes many important known algebras such as the algebraic Katsura algebras [9], Kumjian-Pask algebras [2], and the quotient boundary algebras $\mathcal{Q}^{\mathrm{alg}}_R(\Lambda \bowtie G)$ of a Zappa-Szép product $\Lambda \bowtie G$ introduced in Sect. 3. In Sect. 3, we give a specific example of Exel-Pardo algebras using boundary quotient algebras of semigroups. Indeed, for a single-vertex self-similar k-graph (G, Λ), the Zappa-Szép product $\Lambda \bowtie G$ is a cancellative semigroup. We prove that the quotient boundary algebra $\mathcal{Q}^{\mathrm{alg}}_R(\Lambda \bowtie G)$ (defined in Definition 3.1) is isomorphic to $\mathrm{EP}_R(G, \Lambda)$. Section 4 is devoted to proving a graded uniqueness theorem for the Exel-Pardo algebras. Note that using the description in Proposition 2.7, there is a natural $\mathbb{Z}^k$-grading on $\mathrm{EP}_R(G, \Lambda)$. Then, in Theorem 4.2, a $\mathbb{Z}^k$-graded uniqueness theorem is proved for $\mathrm{EP}_R(G, \Lambda)$ which generalizes and modifies [9, Theorem A]. In particular, we will see in Sects. 5 and 6 that this modification makes it more applicable.

In Sects. 5 and 6, we assume that our self-similar k-graphs are pseudo-free (Definition 5.1). In Sect. 5, we prove that every Exel-Pardo algebra $\mathrm{EP}_R(G, \Lambda)$ is isomorphic to the Steinberg algebra $A_R(\mathcal{G}_{G,\Lambda})$, where $\mathcal{G}_{G,\Lambda}$ is the groupoid introduced in [15]. We should note that the proof of this result is completely different from that of [9, Theorem B]. Indeed, the main difference between the proof of Theorem 5.5 and that of [9, Theorem B] is due to showing the injectivity of defined correspondence. In fact, in [9, Theorem B], the authors try to define a representation for $\mathcal{S}_{(G,E)}$ in $\mathrm{EP}_R(G, E)$ while we apply our graded uniqueness theorem, Theorem 4.2. This gives us an easier proof for Theorem 5.5, even in the 1-graph case. Finally, in Sect. 6, we investigate the ideal structure of $\mathrm{EP}_R(G, \Lambda)$. Using the Steinberg algebras, we can define a conditional expectation $\mathcal{E}$ on $\mathrm{EP}_R(G, \Lambda)$ and then characterize basic, $\mathbb{Z}^k$-graded, diagonal-invariant ideals of $\mathrm{EP}_R(G, \Lambda)$ by G-saturated G-hereditary subsets of Λ^0. These ideals are exactly basic, $Q(\mathbb{N}^k, G) \rtimes_{\mathcal{T}} \mathbb{Z}^k$-graded ideals of $\mathrm{EP}_R(G, \Lambda)$.

1.1 Notation and Terminology

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $k \geq 1$, we regard $\mathbb{N}^k$ as an additive semigroup with the generators $e_1, \ldots, e_k$. We use $\leq$ for the partial order on $\mathbb{N}^k$ given by $m \leq n$ if and only if $m_i \leq n_i$ for $1 \leq i \leq k$. We also write $m \vee n$ and $m \wedge n$ for the coordinate-wise maximum and minimum, respectively.

A *k-graph* is a countable small category $\Lambda = (\Lambda^0, \Lambda, r, s)$ equipped with a *degree functor* $d : \Lambda \to \mathbb{N}^k$ satisfying the unique factorization property: for $\mu \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\mu) = m + n$, then there exist unique $\alpha, \beta \in \Lambda$ such that $d(\alpha) = m$, $d(\beta) = n$, and $\mu = \alpha\beta$. We usually denote $\mu(0, m) := \alpha$ and $\mu(m, d(\mu)) := \beta$. We refer to Λ^0 as the vertex set and define $\Lambda^n := \{\mu \in \Lambda : d(\mu) = n\}$ for every $n \in \mathbb{N}^k$. For $A, B \subseteq \Lambda$, define $AB = \{\mu\nu : \mu \in A, \nu \in B, \text{ and } s(\mu) = r(\nu)\}$. Also, for $\mu, \nu \in \Lambda$, define $\Lambda^{min}(\mu, \nu) = \{(\alpha, \beta) \in \Lambda \times \Lambda : \mu\alpha = \nu\beta, d(\mu\alpha) = d(\mu) \vee d(\nu)\}$.

We say that Λ is *row-finite* if $v\Lambda^n$ is finite for all $n \in \mathbb{N}^k$ and $v \in \Lambda^0$. A *source* in Λ is a vertex $v \in \Lambda^0$ such that $v\Lambda^{e_i} = \emptyset$ for some $1 \leq i \leq k$.

Standing assumption. Throughout the article, we work only with row-finite k-graphs without sources.

Let $\Omega_k := \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}$. By defining $(m, n).(n, l) := (m, l)$, $r(m, n) := (m, m)$, and $s(m, n) := (n, n)$, then Ω_k is a row-finite k-graph without sources. A graph homomorphism $x : \Omega_k \to \Lambda$ is called an *infinite path of* Λ with the range $r(x) = x(0, 0)$, and we write Λ^∞ for the set of all infinite paths of Λ.

2 Exel-Pardo Algebras of Self-Similar *k*-Graphs

In this section, we associate a $*$-algebra to a self-similar k-graph as the algebraic analogue of [16, Definition 3.9]. Let us first review some definitions and notations.

Following [9], we consider $*$-algebras over $*$-rings. Let R be a unital commutative $*$-ring. Recall that a *$*$-algebra over R* is an algebra A equipped with an involution such that $(a^*)^* = a$, $(ab)^* = b^*a^*$, and $(ra + b)^* = r^*a^* + b^*$ for all $a, b \in A$ and $r \in R$. Then $p \in A$ is called a *projection* if $p^2 = p = p^*$, and $s \in A$ a *partial isometry* if $s = ss^*s$.

Definition 2.1 ([2, Definition 3.1]) Let Λ be a row-finite k-graph without sources. A *Kumjian-Pask Λ-family* is a collection $\{s_\mu : \mu \in \Lambda\}$ of partial isometries in a $*$-algebra A such that

(KP1) $\{s_v : v \in \Lambda^0\}$ is a family of pairwise orthogonal projections;
(KP2) $s_{\mu\nu} = s_\mu s_\nu$ for all $\mu, \nu \in \Lambda$ with $s(\mu) = r(\nu)$;
(KP3) $s_\mu^* s_\mu = s_{s(\mu)}$ for all $\mu \in \Lambda$; and
(KP4) $s_v = \sum_{\mu \in v\Lambda^n} s_\mu s_\mu^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

2.1 Self-Similar *k*-Graphs and Their Algebras

Let Λ be a row-finite k-graph without sources. An *automorphism of* Λ is a bijection $\psi : \Lambda \to \Lambda$ such that $\psi(\Lambda^n) \subseteq \Lambda^n$ for all $n \in \mathbb{N}^k$ with the properties $s \circ \psi = \psi \circ s$ and $r \circ \psi = \psi \circ r$. We denote by $\mathrm{Aut}(\Lambda)$ the group of automorphisms on Λ. Furthermore, if G is a countable discrete group, an *action* of G on Λ is a group homomorphism $g \mapsto \psi_g$ from G into $\mathrm{Aut}(\Lambda)$.

Definition 2.2 ([15]) Let Λ be a row-finite k-graph without sources and G a discrete group with identity e_G. We say that a triple (G, Λ, φ) is a *self-similar k-graph* whenever the following properties hold:

(1) G acts on Λ by a group homomorphism $g \mapsto \psi_g$. We prefer to write $g \cdot \mu$ for $\psi_g(\mu)$ to ease the notation.
(2) $\varphi : G \times \Lambda \to G$ is a 1-cocycle for the action $G \curvearrowright \Lambda$ such that for every $g \in G$, $\mu, \nu \in \Lambda$ and $v \in \Lambda^0$ we have

   (a) $\varphi(gh, \mu) = \varphi(g, h \cdot \mu)\varphi(h, \mu)$ (the 1-cocycle property),
   (b) $g \cdot (\mu\nu) = (g \cdot \mu)(\varphi(g, \mu) \cdot \nu)$ (the self-similar equation),
   (c) $\varphi(g, \mu\nu) = \varphi(\varphi(g, \mu), \nu)$, and
   (d) $\varphi(g, v) = g$.

For convenience, we usually write (G, Λ) instead of (G, Λ, φ).

Remark 2.3 In [15], the authors used the notation $g|_\mu$ for $\varphi(g, \mu)$. However, we would prefer to follow [7–9] for writing $\varphi(g, \mu)$.

Remark 2.4 If in equation (2)(a) of Definition 2.2, we set $g = h = e_G$, then we get $\varphi(e_G, \mu) = e_G$ for every $\mu \in \Lambda$. Moreover, [15, Lemma 3.5(ii)] shows that $\varphi(g, \mu) \cdot v = g \cdot v$ for all $g \in G$, $v \in \Lambda^0$, and $\mu \in \Lambda$.

Now we generalize the definition of Exel-Pardo $*$-algebras [9] to the k-graph case.

Definition 2.5 Let (G, Λ) be a self-similar k-graph. An *Exel-Pardo* (G, Λ)*-family* (or briefly (G, Λ)*-family*) is a set

$$\{s_\mu : \mu \in \Lambda\} \cup \{u_{v,g} : v \in \Lambda^0, g \in G\}$$

in a $*$-algebra satisfying

(1) $\{s_\mu : \mu \in \Lambda\}$ is a Kumjian-Pask Λ-family,
(2) $u_{v,e_G} = s_v$ for all $v \in \Lambda^0$,
(3) $u^*_{v,g} = u_{g^{-1} \cdot v, g^{-1}}$ for all $v \in \Lambda^0$ and $g \in G$,
(4) $u_{v,g} s_\mu = \delta_{v, g \cdot r(\mu)} s_{g \cdot \mu} u_{g \cdot s(\mu), \varphi(g,\mu)}$ for all $v \in \Lambda^0$, $\mu \in \Lambda$, and $g \in G$,
(5) $u_{v,g} u_{w,h} = \delta_{v, g \cdot w} u_{v,gh}$ for all $v, w \in \Lambda^0$ and $g, h \in G$.

Then the *Exel-Pardo algebra* $\mathrm{EP}_R(G, \Lambda)$ is the universal $*$-algebra over R generated by a (G, Λ)-family $\{s_\mu, u_{v,g}\}$.

Recall that the universality of $\mathrm{EP}_R(G, \Lambda)$ means that for every (G, Λ)-family $\{S_\mu, U_{v,g}\}$ in a $*$-algebra A, there exists a $*$-homomorphism $\phi : \mathrm{EP}_R(G, \Lambda) \to A$ such that $\phi(s_\mu) = S_\mu$ and $\phi(u_{v,g}) = U_{v,g}$ for all $v \in \Lambda^0$, $\mu \in \Lambda$, and $g \in G$. (See Sect. 2.2 for the construction of $\mathrm{EP}_R(G, \Lambda)$.) Throughout the paper, we will denote by $\{s_\mu, u_{v,g}\}$ the (G, Λ)-family generating $\mathrm{EP}_R(G, \Lambda)$.

2.2 The Construction of $\mathrm{EP}_R(G, \Lambda)$

Let (G, Λ) be a self-similar k-graph as in Definition 2.2. The following is a standard construction of a universal algebra $\mathrm{EP}_R(G, \Lambda)$ subject to desired relations. Consider the set of formal symbols

$$S = \left\{ S_\mu, S^*_\mu, U_{v,g}, U^*_{v,g} : \mu \in \Lambda, v \in \Lambda^0, g \in G \right\}.$$

Let $X = w(S)$ be the collection of finite words in S. We equip the free R-module

$$\mathbb{F}_R(X) := \left\{ \sum_{i=1}^{l} r_i x_i : l \geq 1, r_i \in R, x_i \in X \right\}$$

with the multiplication

$$\left(\sum_{i=1}^{l} r_i x_i \right) \left(\sum_{j=1}^{l'} s_j y_j \right) := \sum_{i,j} r_i s_j x_i y_j,$$

and the involution

$$\left(\sum r_i x_i \right)^* := \left(\sum r_i^* x_i^* \right)$$

where $x^* = s_l^* \dots s_1^*$ for each $x = s_1 \dots s_l$. Then $\mathbb{F}_R(X)$ is a $*$-algebra over R. If I is the (two-sided and self-adjoint) ideal of $\mathbb{F}_R(X)$ containing the roots of relations (1)-(5) in Definition 2.5, then the quotient $\mathbb{F}_R(X)/I$ is the Exel-Pardo algebra $\mathrm{EP}_R(G, \Lambda)$ with the desired universal property. Let us define $s_\mu := S_\mu + I$ and $u_{v,g} := U_{v,g} + I$ for every $\mu \in \Lambda$, $v \in \Lambda^0$, and $g \in G$. In case (G, Λ) is pseudo-free (Definition 5.1), Theorem 4.2 insures that all generators $\{s_\mu, u_{v,g}\}$ of $\mathrm{EP}_R(G, \Lambda)$ are nonzero.

Proposition 2.7 describes the elements of $\mathrm{EP}_R(G, \Lambda)$. First, see a simple lemma.

Lemma 2.6 *Let (G, Λ) be a self-similar graph (as in Definition 2.2) and $\{S, U\}$ a (G, Λ)-family. If $S_\mu U_{v,g} S_\nu^* \neq 0$ where $\mu, \nu \in \Lambda$, $v \in \Lambda^0$ and $g \in G$, then $s(\mu) = v = g \cdot s(\nu)$.*

Proof If $a = S_\mu U_{v,g} S_\nu^*$ is nonzero, then by Definition 2.5 we can write

$$\begin{aligned} S_\mu U_{v,g} S_\nu^* &= S_\mu (S_{s(\mu)} U_{v,g}) S_\nu^* \\ &= S_\mu (U_{s(\mu),e_G} U_{v,g}) S_\nu^* \\ &= S_\mu (\delta_{s(\mu),e_G \cdot v} U_{s(\mu),g}) S_\nu^*. \end{aligned}$$

Now, the hypothesis $a \neq 0$ forces $s(\mu) = v$. On the other hand, a similar computation gives

$$\begin{aligned} a &= S_\mu (U_{v,g} S_{s(\nu)}) S_\nu^* \\ &= S_\mu (U_{v,g} U_{s(\nu),e_G}) S_\nu^* \\ &= S_\mu (\delta_{v,g \cdot s(\nu)} U_{v,g}) S_\nu^*, \end{aligned}$$

and thus $v = g \cdot s(\nu)$. □

Proposition 2.7 *Let (G, Λ) be a self-similar graph. Then*

$$\mathrm{EP}_R(G, \Lambda) = \mathrm{span}_R\{s_\mu u_{s(\mu),g} s_\nu^* : g \in G,\ \mu, \nu \in \Lambda,\ \text{and } s(\mu) = g \cdot s(\nu)\}. \tag{2.1}$$

Proof Define $M := \text{span}_R\{s_\mu u_{s(\mu),g} s_\nu^* : g \in G,\ \mu, \nu \in \Lambda\}$. For every $g, h \in G$ and $\mu, \nu, \alpha, \beta \in \Lambda$ with $\alpha = \nu\alpha'$ for some $\alpha' \in \Lambda$, the relations of Definition 2.5 imply that

$$\begin{aligned}\left(s_\mu u_{s(\mu),g} s_\nu^*\right)\left(s_\alpha u_{s(\alpha),h} s_\beta^*\right) &= s_\mu u_{s(\mu),g}(s_\nu^* s_\alpha) u_{s(\alpha),h} s_\beta^* \\ &= s_\mu u_{s(\mu),g}(s_{\alpha'}) u_{s(\alpha),h} s_\beta^* \\ &= s_\mu \left(\delta_{s(\mu),g\cdot r(\alpha')} s_{g\cdot\alpha'} u_{g\cdot s(\alpha'),\varphi(g,\alpha')} u_{s(\alpha),h}\right) s_\beta^* \\ &= \delta_{s(\mu),g\cdot s(\nu)} s_{\mu(g\cdot\alpha')} \\ &\quad \left(\delta_{g\cdot s(\alpha'),\varphi(g,\alpha')\cdot s(\alpha)} u_{g\cdot s(\alpha'),\varphi(g,\alpha')h}\right) s_\beta^* \\ &\quad (\text{as } r(\alpha') = s(\nu)).\end{aligned}$$

In the case $\nu = \alpha\nu'$ for some $\nu' \in \Lambda$, the above multiplication may be computed similarly, and otherwise is zero. Hence, M is closed under multiplication. Also, we have

$$\left(s_\mu u_{s(\mu),g} s_\nu^*\right)^* = s_\nu u_{g^{-1}\cdot s(\mu),g^{-1}} s_\mu^*,$$

so $M^* \subseteq M$. Since

$$s_\mu = s_\mu u_{s(\mu),e_G} s_{s(\mu)}^* \qquad \text{and} \qquad u_{v,g} = s_v u_{v,g} s_{g\cdot v}^*$$

for all $g \in G$, $v \in \Lambda^0$, and $\mu \in \Lambda$, it follows that M is a $*$-subalgebra of $\text{EP}_R(G, \Lambda)$ containing the generators of $\text{EP}_R(G, \Lambda)$. In light of Lemma 2.6, this concludes the identification (2.1). □

2.3 The Unital Case

In case Λ is a k-graph with finite Λ^0, we may give a better description for Definition 2.5. Note that this case covers all unital Exel-Pardo algebras $\text{EP}_R(G, \Lambda)$.

Lemma 2.8 *Let (G, Λ) be a self-similar k-graph and let s_v be nonzero in $\text{EP}_R(G, \Lambda)$ for every $v \in \Lambda^0$. Then $\text{EP}_R(G, \Lambda)$ is a unital algebra if and only if the vertex set Λ^0 is finite.*

Proof If $\Lambda^0 = \{v_1, \dots, v_l\}$ is finite, then using identification (2.1), $P = \sum_{i=1}^{l} s_{v_i}$ is the unit of $\text{EP}_R(G, \Lambda)$. Conversely, if Λ^0 is infinite, then the set $\{s_v : v \in \Lambda^0\} \subseteq \text{EP}_R(G, \Lambda)$ contains infinitely many mutually orthogonal projections. Now again by (2.1), there is no element of $\text{EP}_R(G, \Lambda)$ which acts as an identity on each element of $\{s_v : v \in \Lambda^0\}$. □

Note that if $\{s, u\}$ is a (G, Λ)-family in a $*$-algebra A, then for each $g \in G$ we may define $u_g := \sum_{v \in \Lambda^0} u_{v,g}$ as an element of the multiplier algebra $\mathcal{M}(A)$ with the property $s_v u_g = u_{v,g}$ for all $v \in \Lambda^0$ (relations (2) and (5) of Definition 2.5 yield $s_v u_{w,g} = \delta_{v,w} u_{v,g}$). (See [1] for the definition of multiplier algebras.) Thus relations (3) and (5) of Definition 2.5 imply that $u : G \to \mathcal{M}(A)$, defined by $g \mapsto u_g$, is a unitary $*$-representation of G on $\mathcal{M}(A)$. In particular, in case Λ^0 is finite, u_g's lie all in A, and we may describe Definition 2.5 as the following:

Proposition 2.9 *Let (G, Λ) be a self-similar k-graph. Suppose also that Λ^0 is finite. Then $\mathrm{EP}_R(G, \Lambda)$ is the universal $*$-algebra generated by families $\{s_\mu : \mu \in \Lambda\}$ of partial isometries and $\{u_g : g \in G\}$ of unitaries satisfying*

(1) *$\{s_\mu : \mu \in \Lambda\}$ is a Kumjian-Pask Λ-family;*
(2) *$u : G \to \mathrm{EP}_R(G, \Lambda)$, by $g \mapsto u_g$, is a unitary $*$-representation of G on $\mathrm{EP}_R(G, \Lambda)$, in the sense that*
   (a) *$u_g u_h = u_{gh}$ for all $g, h \in G$, and*
   (b) *$u_g^* = u_g^{-1} = u_{g^{-1}}$ for all $g \in G$;*
(3) *$u_g s_\mu = s_{g \cdot \mu} u_{\varphi(g,\mu)}$ for all $g \in G$ and $\mu \in \Lambda$.*

3 An example: The Zappa-Szép Product $\Lambda \bowtie G$ and its $*$-Algebra

Let (G, Λ) be a self-similar k-graph such that $|\Lambda^0| = 1$. The C^*-algebra and quotient boundary C^*-algebra associated to the Zappa-Szép product $\Lambda \bowtie G$ as a semigroup were studied in [4, 14]. In this section, we first define $\mathcal{Q}_R^{\mathrm{alg}}(S)$ as the algebraic analogue of the quotient boundary C^*-algebra $\mathcal{Q}(S)$ of a cancellative semigroup S. Then we show that $\mathcal{Q}_R^{\mathrm{alg}}(\Lambda \bowtie G)$ is isomorphic to the Exel-Pardo algebra $\mathrm{EP}_R(G, \Lambda)$.

Let us recall some terminology from [4, 13]. Let S be a left-cancellative semigroup with an identity. Given $X \subseteq S$ and $s \in S$, define $sX := \{sx : x \in X\}$ and $s^{-1}X := \{r \in S : sr \in X\}$. Also, the set of *constructible right ideals in S* is defined as

$$\mathcal{J}(S) := \{s_1^{-1} r_1 \dots s_l^{-1} r_l S : l \geq 1,\ s_i, r_i \in S\} \cup \{\emptyset\}.$$

Then, a *foundation set in $\mathcal{J}(S)$* is a finite subset $F \subseteq \mathcal{J}(S)$ such that for each $Y \in \mathcal{J}(S)$, there exists $X \in F$ with $X \cap Y \neq \emptyset$.

The following is the algebraic analogue of [13, Definition 2.2].

Definition 3.1 Let S be a left-cancellative semigroup and R be a unital commutative $*$-ring. The *boundary quotient $*$-algebra of S* is the universal unital $*$-algebra $\mathcal{Q}_R^{\mathrm{alg}}(S)$ over R generated by a set of isometries $\{t_s : s \in S\}$ and a set of projections $\{q_X : X \in \mathcal{J}(S)\}$ satisfying

(1) $t_s t_r = t_{sr}$,
(2) $t_s q_X t_s^* = q_{sX}$,
(3) $q_S = 1$ and $q_\emptyset = 0$,
(4) $q_X q_Y = q_{X \cap Y}$, and moreover

(5) $\prod_{X\in F}(1-q_X)=0$
for all $s, r \in S$, $X, Y \in \mathcal{J}(S)$, and foundation sets $F \subseteq \mathcal{J}(S)$.

Let Λ be a k-graph such that $\Lambda^0 = \{v\}$. Then $\mu\nu$ is composable for all $\mu, \nu \in \Lambda$, and hence Λ may be considered as a semigroup with the identity v. Also, the unique factorization property implies that Λ is cancellative.

Definition 3.2 ([4, Definition 3.1]) Let (G, Λ) is a single-vertex self-similar k-graph. If we consider Λ as a semigroup, then the *Zappa-Szép product* $\Lambda \bowtie G$ is the semigroup $\Lambda \times G$ with the multiplication

$$(\mu, g)(\nu, h) := (\mu(g \cdot \nu), \varphi(g, \nu) \cdot h) \qquad (\mu, \nu \in \Lambda \text{ and } g, h \in G).$$

Remark 3.3 If Λ is a single-vertex k-graph, then [14, Lemma 3.2 (iv)] follows that

$$\mathcal{J}(\Lambda) = \left\{ \bigcup_{i=1}^{l} \mu_i \Lambda : l \geq 1, \mu_i \in \Lambda, d(\mu_1) = \cdots = d(\mu_l) \right\}.$$

In order to prove Theorem 3.6, the following lemmas are useful.

Lemma 3.4 *Let (G, Λ) be a self-similar k-graph with $\Lambda^0 = \{v\}$. Suppose that for each $\mu \in \Lambda$, the map $g \mapsto \varphi(g, \mu)$ is surjective. Then*

(1) $\mathcal{J}(\Lambda \bowtie G) = \mathcal{J}(\Lambda) \times \{G\}$, where $\emptyset \times G := \emptyset$.
(2) A finite subset $F \subseteq \mathcal{J}(\Lambda)$ is a foundation set if and only if $F' = F \times \{G\}$ is a foundation set in $\mathcal{J}(\Lambda \bowtie G)$.

Proof Statement (1) is just [14, Lemma 2.13]. For (2), suppose that $F \subseteq \mathcal{J}(\Lambda)$ is a foundation set, and let $Y \times G \in \mathcal{J}(\Lambda \bowtie G)$. Then there exists $X \in F$ such that $X \cap Y \neq \emptyset$. Thus $(X \times G) \cap (Y \times G) \neq \emptyset$, from which we conclude that $F \times \{G\}$ is a foundation set in $\mathcal{J}(\Lambda \bowtie G)$. The converse may be shown analogously. □

In the following, for $\mu \in \Lambda$ and $E \subseteq \Lambda$ we define

$$\mathrm{Ext}(\mu; E) := \{\alpha : (\alpha, \beta) \in \Lambda^{\min}(\mu, \nu) \text{ for some } \nu \in E\}.$$

Lemma 3.5 *Let (G, Λ) be a self-similar k-graph with $\Lambda^0 = \{v\}$. For every $X = \cup_{i=1}^{l} \mu_i \Lambda$ and $Y = \cup_{j=1}^{l'} \nu_j \Lambda$ in $\mathcal{J}(\Lambda)$, we have*

$$X \cap Y = \cup\{\mu_i \alpha \Lambda : 1 \leq i \leq l, \alpha \in \mathrm{Ext}(\mu_i; \{\nu_i, \ldots, \nu_{l'}\})\}.$$

Proof For any $\lambda \in X \cap Y$, there are $\alpha, \beta \in \Lambda$, $1 \leq i \leq l$, and $1 \leq j \leq l'$ such that $\lambda = \mu_i \alpha = \nu_j \beta$. Define

$$\alpha' := \alpha(0, d(\mu_i) \vee d(\nu_j) - d(\mu_i)) \quad \text{and} \quad \beta' := \beta(0, d(\mu_i) \vee d(\nu_j) - d(\nu_j)).$$

Then the factorization property implies that $\lambda = \mu_i \alpha' \lambda' = \nu_j \beta' \lambda'$ where $d(\mu_i \alpha') = d(\nu_j \beta') = d(\mu_i) \vee d(\nu_j)$ and $\lambda' = \lambda(d(\mu_i) \vee d(\nu_j), d(\lambda))$. It follows that $\lambda \in \mu_i \alpha' \Lambda$ with $\alpha' \in \mathrm{Ext}(\mu_i; \{\nu_j\})$ as desired. The reverse containment is trivial. □

The following result is inspired by [14, Theorem 3.3].

Theorem 3.6 *Let (G,Λ) be a self-similar k-graph with $\Lambda^0=\{v\}$ and let $\{s_\mu,u_g\}$ be the (G,Λ)-family generating $\mathrm{EP}_R(G,\Lambda)$ as in Proposition 2.9. Suppose that for every $\mu\in\Lambda$ the map $g\mapsto\varphi(g,\mu)$ is surjective. If the family $\{t_{(\mu,g)},q_X:(\mu,g)\in\Lambda\bowtie G, X\in\mathcal{J}(\Lambda\bowtie G)\}$ generates $\mathcal{Q}_R^{\mathrm{alg}}(\Lambda\bowtie G)$, then there exists an R-algebra $*$-isomorphism $\pi:\mathrm{EP}_R(G,\Lambda)\to\mathcal{Q}_R^{\mathrm{alg}}(\Lambda\bowtie G)$ such that $\pi(s_\mu)=t_{(\mu,e_G)}$ and $\pi(u_g)=q_{(v,g)}$ for all $\mu\in\Lambda$ and $g\in G$.*

Proof For every $\mu\in\Lambda$ and $g\in G$, define

$$S_\mu:=t_{(\mu,e_G)}\ \text{ and }\ U_g:=t_{(v,g)}.$$

We will show that $\{S,U\}$ is a (G,Λ)-family in $\mathcal{Q}_R^{\mathrm{alg}}(\Lambda\bowtie G)$, which is described in Proposition 2.9. First, for each $g\in G$ we have

$$U_gU_g^*=t_{(v,g)}\,q_{\Lambda\bowtie G}\,t_{(v,g)}^*=q_{(v,g)\Lambda\bowtie G}=q_{\Lambda\bowtie G}=1_{\mathcal{Q}_R^{\mathrm{alg}}(\Lambda\bowtie G)}.$$

So, $g\mapsto U_g$ is a unitary $*$-representation of G into $\mathcal{Q}_R^{\mathrm{alg}}(\Lambda\bowtie G)$. Moreover, (KP1)-(KP3) can be easily checked, so we verify (KP4) for $\{s_\mu:\mu\in\Lambda\}$. Fix some $n\in\mathbb{N}^k$. Then $\{\mu\Lambda:\mu\in\Lambda^n\}$ is a foundation set in $\mathcal{J}(\Lambda)$, and thus so is $F=\{\mu\Lambda\times G:\mu\in\Lambda^n\}$ in $\mathcal{J}(\Lambda\bowtie G)$ by Lemma 3.4. Hence we have

$$\begin{aligned}1-\sum_{\mu\in\Lambda^n}S_\mu S_\mu^*&=\prod_{\mu\in\Lambda^n}(1-S_\mu S_\mu^*) &&\text{(because } S_\mu S_\mu^*\text{s are pairwise orthogonal)}\\&=\prod_{\mu\in\Lambda^n}(1-t_{(\mu,e_G)}\,q_{\Lambda\bowtie G}\,t_{(\mu,e_G)}^*)\\&=\prod_{\mu\in\Lambda^n}(1-q_{\mu\Lambda\times G}) &&\text{(by eq. (2) of Definition 3.1)}\\&=\prod_{X\in F}(1-q_X)=0 &&\text{(by eq. (5) of Definition 3.1).}\end{aligned}$$

Because $S_v=U_{e_G}=1$, (KP4) is verified, and therefore $\{S_\mu:\mu\in\Lambda\}$ is a Kumjian-Pask Λ-family. Since for each $\mu\in\Lambda$ and $g\in G$,

$$U_gS_\mu=t_{(v,g)}t_{(\mu,e_G)}=t_{(v,g)(\mu,e_G)}=t_{(g\cdot\mu,\varphi(g,\mu))}=S_{g\cdot\mu}U_{\varphi(g,\mu)},$$

and so we have shown that $\{S,U\}$ is a (G,Λ)-family in $\mathcal{Q}_R^{\mathrm{alg}}(\Lambda\bowtie G)$. Now the universality implies that the desired $*$-homomorphism $\pi:\mathrm{EP}_R(G,\Lambda)\to\mathcal{Q}_R^{\mathrm{alg}}(\Lambda\bowtie G)$ exists.

Now we prove that π is an isomorphism. In order to do this, it suffices to find a homomorphism $\rho:\mathcal{Q}_R^{\mathrm{alg}}(\Lambda\bowtie G)\to\mathrm{EP}_R(G,\Lambda)$ such that $\rho\circ\pi=\mathrm{id}_{\mathrm{EP}_R(G,\Lambda)}$ and $\pi\circ\rho=\mathrm{id}_{\mathcal{Q}_R^{\mathrm{alg}}(\Lambda\bowtie G)}$. For any $(\mu,g)\in\Lambda\bowtie G$ and $X=(\cup_{i=1}^l\mu_i\Lambda)\times G\in\mathcal{J}(\Lambda\bowtie$

G), we define

$$T_{(\mu,g)} := s_\mu u_g \quad \text{and} \quad Q_X := \sum_{i=1}^{l} s_{\mu_i} s_{\mu_i}^*.$$

We will show that the family $\{T, Q\}$ satisfies the properties of Definitions 3.1. Relations (1)-(3) easily hold by the (G, Λ)-relations for $\{s, u\}$. Also, for every $X = (\cup_{i=1}^{l} \mu_i \Lambda) \times G$ and $Y = (\cup_{j=1}^{l'} \nu_j \Lambda) \times G$ in $\mathcal{J}(\Lambda \bowtie G)$, we have

$$\begin{aligned}
Q_X Q_Y &= \Big(\sum_{i=1}^{l} s_{\mu_i} s_{\mu_i}^*\Big)\Big(\sum_{j=1}^{l'} s_{\nu_j} s_{\nu_j}^*\Big) \\
&= \sum_{i,j} s_{\mu_i} (s_{\mu_i}^* s_{\nu_j}) s_{\nu_j}^* \\
&= \sum_{i,j} s_{\mu_i} \Big(\sum_{(\alpha,\beta)\in\Lambda^{\min}(\mu_i,\nu_j)} s_\alpha s_\beta^*\Big) s_{\nu_j}^* \qquad \text{(by [1, Lemma 3.3])} \\
&= \sum_{i,j} \sum_{\substack{\mu_i\alpha=\nu_j\beta \\ d(\mu_i\alpha)=d(\mu_i)\vee d(\nu_j)}} s_{\mu_i\alpha} s_{\nu_j\beta}^* \\
&= Q_{X\cap Y} \qquad \text{(by Lemma 3.5).}
\end{aligned}$$

For eq. (5) of Definition 3.1, let $F = \{X_i \times G := \cup_{j=1}^{t_i} \mu_{ij}\Lambda \times G : 1 \le i \le l\}$ be a foundation set in $\mathcal{J}(\Lambda \bowtie G)$. Then $F' = \{X_i = \cup_{j=1}^{t_i} \mu_{ij}\Lambda\}_{i=1}^{l}$ is a foundation set in $\mathcal{J}(\Lambda)$ by Lemma 3.4(2). Defining $n := \bigvee_{i,j} d(\mu_{ij})$, we claim that the set $M = \{\mu_{ij}\alpha : \alpha \in \Lambda^{n-d(\mu_{ij})}, 1 \le i \le l, 1 \le j \le t_i\}$ coincides with Λ^n. Indeed, if on the contrary there exists some $\lambda \in \Lambda^n \setminus M$, then $\Lambda^{\min}(\lambda, \mu_{ij}) = \emptyset$, and hence $\lambda\Lambda \cap \mu_{ij}\Lambda = \emptyset$ for all i and j. This yields that $\lambda\Lambda \cap X_i = \emptyset$ for every $X_i \in F'$, contradicting that F' is a foundation set in $\mathcal{J}(\Lambda)$.

Now one may compute

$$\begin{aligned}
\prod_{X_i\times G\in F} (1 - Q_{X_i\times G}) &= \prod_{i=1}^{l} (1 - \sum_{j=1}^{t_i} s_{\mu_{ij}} s_{\mu_{ij}}^*) \\
&= \prod_{i=1}^{l} \Big(1 - \sum_{j=1}^{t_i} s_{\mu_{ij}} (\sum_{\alpha\in\Lambda^{n-d(\mu_{ij})}} s_\alpha s_\alpha^*) s_{\mu_{ij}}^*\Big) \\
&= \prod_{i=1}^{l} \Big(1 - \sum_{j=1}^{t_i} \sum_{\alpha\in\Lambda^{n-d(\mu_{ij})}} s_{\mu_{ij}\alpha} s_{\mu_{ij}\alpha}^*\Big) \qquad (\star).
\end{aligned}$$

Observe that the projections $s_{\mu_{ij}\alpha} s_{\mu_{ij}\alpha}^*$ are pairwise orthogonal because $d(\mu_{ij}\alpha) = n$ for all i, j (see [2, Remark 3.2(c)]). Hence, using the above claim, expression $(\star)$

equals to

$$(\star) = 1 - \sum_{\lambda \in \Lambda^n} s_\lambda s_\lambda^* = 0 \qquad \text{(by (KP4))}.$$

Therefore, the family $\{T, Q\}$ satisfies the relations of Definition 3.1, and by the universality there exists an algebra $*$-homomorphism $\rho : \mathcal{Q}_R^{\mathrm{alg}}(\Lambda \bowtie G) \to \mathrm{EP}_R(G, \Lambda)$ such that $\rho(t_{(\mu,g)}) = T_{(\mu,g)}$ and $\rho(q_X) = Q_X$ for $(\mu, g) \in \Lambda \bowtie G$ and $X \in \mathcal{J}(\Lambda \bowtie G)$. It is clear that $\rho \circ \pi = \mathrm{id}_{\mathrm{EP}_R(G,\Lambda)}$ and $\pi \circ \rho = \mathrm{id}_{\mathcal{Q}_R^{\mathrm{alg}}(\Lambda \bowtie G)}$ because they fix the generators of $\mathrm{EP}_R(G, \Lambda)$ and $\mathcal{Q}_R^{\mathrm{alg}}(\Lambda \bowtie G)$, respectively. Consequently, π is an isomorphism, completing the proof. $\square$

4 A Graded Uniqueness Theorem

In this section, we prove a graded uniqueness theorem for $\mathrm{EP}_R(G, \Lambda)$ which generalizes and modifies [9, Theorem A] for self-similar k-graphs. This modification, in particular, helps us to prove Theorems 5.5 and 6.8.

Let us first recall some definitions. Let Γ be a group and A be an algebra over a ring R. A is called Γ-*graded* (or briefly, *graded* whenever the group is clear) if there is a family of R-submodules $\{A_\gamma : \gamma \in \Gamma\}$ of A such that $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ and $A_\gamma A_{\gamma'} \subseteq A_{\gamma\gamma'}$ for all $\gamma, \gamma' \in \Gamma$. Then each set A_γ is called a γ-*homogeneous component of* A. In this case, we say an ideal I of A is Γ-*graded* if $I = \bigoplus_{\gamma \in \Gamma}(I \cap A_\gamma)$. Note that an ideal I of A is Γ-graded if and only if it is generated by a subset of $\bigcup_{\gamma \in \Gamma} A_\gamma$, the homogeneous elements of A.

Furthermore, if A and B are two Γ-graded algebras over R, a homomorphism $\phi : A \to B$ is said to be a *graded homomorphism* if $\phi(A_\gamma) \subseteq B_\gamma$ for all $\gamma \in \Gamma$. Hence the kernel of a graded homomorphism is always a graded ideal. Also, if I is a graded ideal of A, then there is a natural Γ-grading $(A_\gamma + I)_{\gamma \in \Gamma}$ on the quotient algebra A/I, and thus the quotient map $A \to A/I$ is a graded homomorphism.

Lemma 4.1 *Let (G, Λ) be a self-similar k-graph. If for every $n \in \mathbb{Z}^k$, we define*

$$\mathrm{EP}_R(G, \Lambda)_n := \mathrm{span}_R\left\{ s_\mu u_{s(\mu),g} s_\nu^* : g \in G,\ \mu, \nu \in \Lambda,\ \text{and } d(\mu) - d(\nu) = n \right\},$$

then $(\mathrm{EP}_R(G, \Lambda)_n)_{n \in \mathbb{Z}^k}$ is a $\mathbb{Z}^k$-grading on $\mathrm{EP}_R(G, \Lambda)$.

Proof Consider the free $*$-algebra $\mathbb{F}_R(X)$ and its ideal I as in Sect. 2.2. If we define

$$\theta(S_\mu) := d(\mu),\ \theta(S_\mu^*) := -d(\mu),\ \text{and } \theta(U_{v,g}) := 0$$

for all $g \in G$, $v \in \Lambda^0$ and $\mu \in \Lambda$, then θ induces a $\mathbb{Z}^k$-grading on $\mathbb{F}_R(X)$. Also, since the generators of I are all homogenous, I is a graded ideal. Therefore, $\mathrm{EP}_R(G, \Lambda) \cong \mathbb{F}_R(X)/I$ is a $\mathbb{Z}^k$-graded algebra, and Proposition 2.7 concludes the result. $\square$

Theorem 4.2 (Graded Uniqueness) *Let* (G,Λ) *be a self-similar k-graph. Let* $\phi : \mathrm{EP}_R(G,\Lambda) \to B$ *be a* $\mathbb{Z}^k$*-graded R-algebra* $*$*-homomorphism into a* $\mathbb{Z}^k$*-graded* $*$*-algebra B. If* $\phi(a) \neq 0$ *for every nonzero element of the form* $a = \sum_{i=1}^{l} r_i u_{v,g_i}$ *with* $v \in \Lambda^0$ *and* $g_i^{-1} \cdot v = g_j^{-1} \cdot v$ *for* $1 \leq i, j \leq l$*, then* ϕ *is injective.*

Proof For convenience, we write $A = \mathrm{EP}_R(G,\Lambda)$. Since $A = \bigoplus_{n\in\mathbb{Z}^k} A_n$ and ϕ preserves the grading, it suffices to show that ϕ is injective on each A_n. So, fix some $b \in A_n$, and assume $\phi(b) = 0$. By equation (2.1), we can write

$$b = \sum_{i=1}^{l} r_i s_{\mu_i} u_{w_i,g_i} s_{\nu_i}^* \tag{4.1}$$

where $w_i = s(\mu_i) = g_i \cdot s(\nu_i)$ and $d(\mu_i) - d(\nu_i) = n$ for $1 \leq i \leq l$. Define $n' = \vee_{1\leq i\leq l} d(\mu_i)$. Then, for each $i \in \{1 \dots l\}$, (KP4) says that

$$s_{w_i} = \sum_{\lambda \in w_i \Lambda^{n'-d(\mu_i)}} s_\lambda s_\lambda^*,$$

and we can write

$$\begin{aligned}
s_{\mu_i} u_{w_i,g_i} s_{\nu_i}^* &= s_{\mu_i}(s_{s(\mu_i)}) u_{w_i,g_i} s_{\nu_i}^* \\
&= \sum s_{\mu_i}(s_\lambda s_\lambda^*) u_{w_i,g_i} s_{\nu_i}^* \\
&= \sum s_{\mu_i\lambda}\left(s_{\nu_i} u_{w_i,g_i}^* s_\lambda\right)^* \\
&= \sum s_{\mu_i\lambda}\left(s_{\nu_i} u_{g_i^{-1}\cdot w_i, g_i^{-1}} s_\lambda\right)^* \\
&= \sum s_{\mu_i\lambda}\left(s_{\nu_i} s_{g_i^{-1}\cdot\lambda} u_{g_i^{-1}\cdot s(\lambda), \varphi(g_i^{-1},\lambda)}\right)^* \\
&= \sum s_{\mu_i\lambda} u_{\varphi(g_i^{-1},\lambda)^{-1} g_i^{-1}\cdot s(\lambda), \varphi(g_i^{-1},\lambda)^{-1}} s_{\nu_i(g_i^{-1}\cdot\lambda)}^*
\end{aligned}$$

where the above summations are on $\lambda \in w_i \Lambda^{n'-d(\mu_i)}$. So, in each term of (4.1), we may assume $d(\mu_i) = n'$ and $d(\nu_i) = n' - n$. Now, for any $1 \leq j \leq l$, (KP3) yields that

$$s_{\mu_j}^* b s_{\nu_j} = s_{\mu_j}^* \left(\sum_{i=1}^{l} r_i s_{\mu_i} u_{w_i,g_i} s_{\nu_i}^*\right) s_{\nu_j} = \sum_{i\in[j]} r_i u_{w_i,g_i},$$

where $[j] := \{1 \leq i \leq l : (\mu_i, \nu_i) = (\mu_j, \nu_j)\}$. Thus

$$\phi\left(\sum_{i\in[j]} r_i u_{w_i,g_i}\right) = \phi(s_{\mu_j}^*)\phi(b)\phi(s_{\nu_j}) = 0$$

and hypothesis forces $\sum_{i\in[j]} r_i u_{w_i,g_i} = 0$. Therefore,

$$\sum_{i\in[j]} r_i s_{\mu_i} u_{w_i,g_i} s_{\nu_i}^* = s_{\mu_j}\Big(\sum_{i\in[j]} r_i u_{w_i,g_i}\Big) s_{\nu_j}^* = 0.$$

Since the index set $\{1,\ldots,l\}$ is a disjoint union of $[j]$'s, we obtain $b = 0$. It follows that ϕ is injective. □

5 $\mathrm{EP}_R(G,\Lambda)$ as a Steinberg Algebra

In this section, we want to prove an Steinberg algebra model for $\mathrm{EP}_R(G,\Lambda)$. Although our result will be the k-graph generalization of [9, Theorem B], note that our proof relies on the graded uniqueness theorem, Theorem 4.2, and is completely different from that of [9, Theorem B]. This gives us a much easier and shorter proof.

Let us first review some terminology about groupoids; see [19] for more details. A *groupoid* is a small category $\mathcal{G}$ with inverses. For each $\alpha \in \mathcal{G}$, we may define the range $r(\alpha) := \alpha\alpha^{-1}$ and the source $s(\alpha) := \alpha^{-1}\alpha$ satisfying $r(\alpha)\alpha = \alpha = \alpha s(\alpha)$. It follows that for every $\alpha, \beta \in \mathcal{G}$, the composition $\alpha\beta$ is well defined if and only if $s(\alpha) = r(\beta)$. The *unit space of* $\mathcal{G}$ is $\mathcal{G}^{(0)} := \{\alpha^{-1}\alpha : \alpha \in \mathcal{G}\}$. Throughout the paper we work with *topological groupoids*, which are ones equipped with a topology such that the maps r and s are continuous. Then a *bisection* is a subset $B \subseteq \mathcal{G}$ such that both restrictions $r|_B$ and $s|_B$ are homeomorphisms. In case $\mathcal{G}$ has a basis of compact open bisections, $\mathcal{G}$ is called an *ample groupoid*.

Let (G,Λ) be a self-similar k-graph. We also recall the groupoid $\mathcal{G}_{G,\Lambda}$ introduced in [15]. Let $C(\mathbb{N}^k, G)$ be the group of all maps form $\mathbb{N}^k$ to G with the pointwise multiplication. For $f, g \in C(\mathbb{N}^k, G)$, define the equivalence relation $f \sim g$ in case there exists $n_0 \in \mathbb{N}^k$ such that $f(n) = g(n)$ for all $n \geq n_0$. Write $Q(\mathbb{N}^k, G) := C(\mathbb{N}^k, G)/\sim$. Also, for each $z \in \mathbb{Z}^k$, let $\mathcal{T}_z : C(\mathbb{N}^k, G) \to C(\mathbb{N}^k, G)$ be the automorphism defined by

$$\mathcal{T}_z(f)(n) = \begin{cases} f(n-z) & n-z \geq 0 \\ e_G & \text{otherwise} \end{cases} \qquad (f \in C(\mathbb{N}^k, G),\ n \in \mathbb{N}^k).$$

Then $\mathcal{T}_z$ induces an automorphism, denoted again by $\mathcal{T}_z$, on $Q(\mathbb{N}^k, G)$, which is $\mathcal{T}_z([f]) = [\mathcal{T}_z(f)]$. So, $\mathcal{T} : \mathbb{Z}^k \to \mathrm{Aut}\, Q(\mathbb{N}^k, G)$ is a homomorphism and we consider the semidirect product group $Q(\mathbb{N}^k, G) \rtimes_{\mathcal{T}} \mathbb{Z}^k$.

Note that for every $g \in G$ and $x \in \Lambda^\infty$, one may define $\varphi(g,x) \in C(\mathbb{N}^k, G)$ by

$$\varphi(g,x)(n) := \varphi(g, x(0,n)) \qquad (n \in \mathbb{N}^k).$$

Moreover, [15, Lemma 3.7] says that there exists a unique action $G \curvearrowright \Lambda^\infty$ by defining

$$(g\cdot x)(m,n) := \varphi(g, x(0,m)) \cdot x(m,n) \qquad ((m,n) \in \Omega_k)$$

for every $g \in G$ and $x \in \Lambda^\infty$.

Definition 5.1 A self-similar k-graph (G, Λ) is said to be *pseudo-free* if for any $g \in G$ and $\mu \in \Lambda$, $g \cdot \mu = \mu$ and $\varphi(g, \mu) = e_G$ imply $g = e_G$.

According to [15, Lemma 5.6], in case (G, Λ) is pseudo-free, then we have

$$g \cdot \mu = h \cdot \mu \text{ and } \varphi(g, \mu) = \varphi(h, \mu) \implies g = h$$

for every $g, h \in G$ and $\mu \in \Lambda$.

Definition 5.2 Associated to (G, Λ) we define the subgroupoid

$$\mathcal{G}_{G,\Lambda} := \Big\{ \Big(\mu(g \cdot x); \mathcal{T}_{d(\mu)}([\varphi(g, x)]), d(\mu) - d(\nu); \nu x \Big) : g \in G, \\ \mu, \nu \in \Lambda, \ s(\mu) = g \cdot s(\nu) \}$$

of $\Lambda^\infty \times \Big(\mathcal{Q}(\mathbb{N}^k, G) \rtimes_{\mathcal{T}} \mathbb{Z}^k \Big) \times \Lambda^\infty$ with the range and source maps

$$r(x; [f], n - m; y) = x \quad \text{and} \quad s(x; [f], n - m; y) = y.$$

Note that if we set

$$Z(\mu, g, \nu) := \Big\{ \Big(\mu(g \cdot x); \mathcal{T}_{d(\mu)}([\varphi(g, x)]), d(\mu) - d(\nu); \nu x \Big) : x \in s(\nu)\Lambda^\infty \Big\},$$

then the basis

$$\mathcal{B}_{G,\Lambda} := \{ Z(\mu, g, \nu) : \mu, \nu \in \Lambda, g \in G, s(\mu) = g \cdot s(\nu) \}$$

induces a topology on $\mathcal{G}_{G,\Lambda}$. In case (G, Λ) is pseudo-free, [16, Proposition 3.11] shows that $\mathcal{G}_{G,\Lambda}$ is a Hausdorff groupoid with compact open base $\mathcal{B}_{G,\Lambda}$.

Definition 5.3 Let (G, Λ) be a pseudo-free self-similar k-graph and R a unital commutative $*$-ring. Then *the Steinberg algebra associated to* (G, Λ) is the R-algebra

$$A_R(\mathcal{G}_{G,\Lambda}) := \operatorname{span}_R\{1_B : B \text{ is a compact open bisection}\}$$

endowed with the pointwise addition, the multiplication $fg(\gamma) := \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$, and the involution $f^*(\gamma) := f(\gamma^{-1})^*$ for all $\gamma \in \mathcal{G}_{G,\Lambda}$.

To prove Theorem 5.5, we need the following lemma.

Lemma 5.4 *Let* (G, Λ) *be a pseudo-free self-similar* k*-graph. Let* $v, w \in \Lambda^0$ *and* $g, h \in G$ *with* $g \cdot v = h \cdot v = w$*. Then* $Z(v, g, w) \cap Z(v, h, w) = \emptyset$ *whenever* $g \neq h$*.*

Proof Suppose that $(g \cdot x; [\varphi(g, x)], 0; x) = (h \cdot y, [\varphi(h, y)], 0; y) \in Z(v, g, w) \cap Z(v, h, w)$ where $x, y \in Z(w)$. Then $y = x$, $g \cdot x = h \cdot x$ and $[\varphi(g, x)] = [\varphi(h, x)]$. Since (G, Λ) is pseudo-free, [15, Corollary 5.6] implies that $g = h$. □

Theorem 5.5 *Let (G, Λ) be a pseudo-free self-similar k-graph. Then there is a (unique) $*$-algebra isomorphism $\phi : \mathrm{EP}_R(G, \Lambda) \to A_R(\mathcal{G}_{G,\Lambda})$ such that*

$$\phi(s_\mu) = 1_{Z(\mu, e_G, s(\mu))} \quad \text{and} \quad \phi(u_{v,g}) = 1_{Z(v,g,g^{-1}\cdot v)}$$

for every $\mu \in \Lambda$, $v \in \Lambda^0$, and $g \in G$. In particular, the elements rs_μ and $ru_{v,g}$ with $r \in R\backslash\{0\}$ are all nonzero.

Proof For each $v \in \Lambda^0$, $\mu \in \Lambda$ and $g \in G$, define

$$S_\mu := 1_{Z(\mu, e_G, s(\mu))} \quad \text{and} \quad U_{v,g} := 1_{Z(v,g,g^{-1}\cdot v)}.$$

Since $S_\mu^* = 1_{Z(\mu, e_G, s(\mu))^{-1}} = 1_{Z(s(\mu), e_G, \mu)}$ and $U_{v,g}^* = 1_{Z(v,g,g^{-1}\cdot v)^{-1}} = 1_{Z(g^{-1}\cdot v, g^{-1}, v)}$, a long but straightforward computation shows that $\{S_\mu, U_{v,g}\}$ is a (G, Λ)-family in $A_R(\mathcal{G}_{G,\Lambda})$. Then, by the universal property, such $*$-homomorphism ϕ exists.

[16, Proposition 3.11] says that $\mathcal{G}_{G,\Lambda}$ is ample with compact open base $\mathcal{B}_{G,\Lambda}$. Since each element $Z(\mu, g, \nu)$ of $\mathcal{B}_{G,\Lambda}$ can be written as

$$Z(\mu, g, \nu) = Z(\mu, e_G, s(\mu)) Z(s(\mu), g, s(\nu)) Z(\nu, e_G, s(\nu))^{-1},$$

ϕ is surjective.

We will show the injectivity of ϕ by applying the graded uniqueness theorem. Note that the continuous 1-cocycle $c : \mathcal{G}_{G,\Lambda} \to \mathbb{Z}^k$, defined by $c(\mu(g \cdot x); [f], d(\mu) - d(\nu); \nu x) := d(\mu) - d(\nu)$, induces a $\mathbb{Z}^k$-grading on $A_R(\mathcal{G}_{G,\Lambda})$. Also, ϕ preserves the $\mathbb{Z}^k$-grading because it does on the generators. Now, to apply Theorem 4.2, we assume $\phi(a) = 0$ for an element of the form $a = \sum_{i=1}^{l} r_i u_{v,g_i}$ with $g_i^{-1} \cdot v = g_j^{-1} \cdot v$ for $1 \le i, j \le l$. We may also assume that the g_i's are distinct (otherwise, combine the terms with same g_i's). We then have

$$\phi(a) = \sum_{i=1}^{l} r_i 1_{Z(v, g_i, g_i^{-1}\cdot v)} = 0.$$

Lemma 5.4 says that the bisections $Z(v, g_i, g_i^{-1} \cdot v)$ are pairwise disjoint. Hence, for each i, if we pick some $\alpha \in Z(v, g_i, g_i^{-1} \cdot v)$, then $r_i = \phi(a)(\alpha) = 0$. Therefore $a = 0$, and Theorem 4.2 concludes that ϕ is injective. We are done. □

Combining [20, Theorem 6.7], [15, Theorem 5.9], and Theorem 5.5 gives the next corollary. (Although in [15] it is supposed $|\Lambda^0| < \infty$, but [15, Theorem 5.9] holds also for Λ with infinitely many vertices.)

Corollary 5.6 *Let (G, Λ) be a pseudo-free self-similar k-graph over an amenable group G. Then the complex algebra $\mathrm{EP}_{\mathbb{C}}(G, \Lambda)$ is a dense subalgebra of $\mathcal{O}_{G,\Lambda}$ introduced in [16].*

In the following, we see that the Kumjian-Pask algebra $\mathrm{KP}_R(\Lambda)$ from [2] can be embedded in $\mathrm{EP}_R(G, \Lambda)$.

Corollary 5.7 *Let (G, Λ) be a pseudo-free self-similar k-graph. Let the Kumjian-Pask algebra $\mathrm{KP}_R(\Lambda)$ be generated by a Kumjian-Pask Λ-family $\{t_\mu : \mu \in \Lambda\}$. Then the map $t_\mu \mapsto s_\mu$ embeds $\mathrm{KP}_R(\Lambda)$ into $\mathrm{EP}_R(G, \Lambda)$ as a $*$-subalgebra.*

Proof We know that $\mathrm{KP}_R(\Lambda)$ is $\mathbb{Z}^k$-graded by the homogenous components

$$\mathrm{KP}_R(\Lambda)_n := \mathrm{span}_R \left\{ t_\mu t_\nu^* : \mu, \nu \in \Lambda, d(\mu) - d(\nu) = n \right\}.$$

for all $n \in \mathbb{Z}^k$. Then, the universal property of Kumjian-Pask algebras gives a graded $*$-algebra homomorphism $\phi : \mathrm{KP}_R(\Lambda) \to \mathrm{EP}_R(G, \Lambda)$ such that $\phi(t_\mu) := s_\mu$ and $\phi(t_\mu^*) := s_\mu^*$ for every $\mu \in \Lambda$. Moreover, Theorem 5.5 shows that $\phi(rt_\mu) = rs_\mu \neq 0$ for all $r \in R \backslash \{0\}$ and $\mu \in \Lambda$. Therefore, the graded uniqueness theorem for Kumjian-Pask algebras [2, Theorem 4.1] implies that ϕ is injective. ◻

Definition 5.8 Let $\mathcal{G}$ be a topological groupoid. We say that $\mathcal{G}$ is *topologically principal* if the set of units with trivial isotropy group, that is $\{u \in \mathcal{G}^{(0)} : s^{-1}(u) \cap r^{-1}(u) = \{u\}\}$, is dense in $\mathcal{G}^{(0)}$.

The analogue of the topologically principal property for self-similar k-graphs is G-aperiodicity (see [15, Proposition 6.5]).

Definition 5.9 Let (G, Λ) be a self-similar k-graph. Then Λ is said to be *G-aperiodic* if for every $v \in \Lambda^0$, there exists $x \in v\Lambda^\infty$ with the property that

$$x(p, \infty) = g \cdot x(q, \infty) \implies g = e_G \text{ and } p = q \qquad (\forall g \in G, \ \forall p, q \in \mathbb{N}^k).$$

Theorem 5.10 (The Cuntz–Krieger uniqueness) *Let (G, Λ) be a pseudo-free self-similar k-graph. Let (G, Λ) be also G-aperiodic. Suppose that $\phi : \mathrm{EP}_R(G, \Lambda) \to A$ is a $*$-algebra homomorphism from $\mathrm{EP}_R(G, \Lambda)$ into a $*$-algebra A such that $\phi(rs_v) \neq 0$ for all $0 \neq r \in R$ and $v \in \Lambda^0$. Then ϕ is injective.*

Proof First note that $\mathcal{G}_{G,\Lambda}$ is a Hausdorff ample groupoid by [16, Proposition 3.11], and that $\mathcal{B}_{G,\Lambda}$ is a basis for $\mathcal{G}_{G,\Lambda}$ consisting compact open bisections. Also, [16, Lemma 3.12] says that $\mathcal{G}_{G,\Lambda}$ is topologically principal (so is effective in particular). So, we may apply [5, Theorem 3.2].

Denote by $\psi : \mathrm{EP}_R(G, \Lambda) \to A_R(\mathcal{G}_{G,\Lambda})$ the isomorphism of Theorem 5.5. If on the contrary ϕ is not injective, then neither is $\widetilde{\phi} := \phi \circ \psi^{-1} : A_R(\mathcal{G}_{G,\Lambda}) \to A$. Thus, by [5, Theorem 3.2], there exists a compact open subset $K \subseteq \mathcal{G}_{G,\Lambda}^{(0)}$ and $r \neq 0$ such that $\widetilde{\phi}(r1_K) = 0$. Since K is open, there is a unit $U = Z(\mu, e_G, \mu) \in \mathcal{B}_{G,\Lambda}$ such that $U \subseteq K$. So we get

$$\phi(rs_\mu s_\mu^*) = \widetilde{\phi}(r1_U) = \widetilde{\phi}(r1_{U \cap K}) = \widetilde{\phi}(r1_K)\widetilde{\phi}(1_U) = 0,$$

and hence

$$\phi(rs_{s(\mu)}) = \phi(s_\mu^*)\phi(rs_\mu s_\mu^*)\phi(s_\mu) = 0.$$

This contradicts the hypothesis, and therefore ϕ is injective. ◻

6 Ideal Structure

By an *ideal* we mean a two-sided and self-adjoint one. In this section, we characterize basic, $\mathbb{Z}^k$-graded and diagonal-invariant ideals of $\mathrm{EP}_R(G,\Lambda)$, which are exactly all basic $Q(\mathbb{N}^k,G)\rtimes_{\mathcal{T}}\mathbb{Z}^k$-graded ones.

Let (G,Λ) be a pseudo-free self-similar k-graph. Since $\mathcal{G}_{G,\Lambda}$ is a Hausdorff ample groupoid [15, Theorem 5.8], $\mathcal{G}_{G,\Lambda}^{(0)}$ is both open and closed, and for every $f\in A_R(\mathcal{G}_{G,\Lambda})$ the restricted function $f|_{\mathcal{G}_{G,\Lambda}^{(0)}}=f\chi_{\mathcal{G}_{G,\Lambda}^{(0)}}$ lies again in $A_R(\mathcal{G}_{G,\Lambda})$. Then $A_R(\mathcal{G}_{G,\Lambda}^{(0)})$ is a $*$-subalgebra of $A_R(\mathcal{G}_{G,\Lambda})$ and there is a conditional expectation $\mathcal{E}:A_R(\mathcal{G}_{G,\Lambda})\to A_R(\mathcal{G}_{G,\Lambda}^{(0)})$ defined by $\mathcal{E}(f)=f|_{\mathcal{G}_{G,\Lambda}^{(0)}}$ for $f\in A_R(\mathcal{G}_{G,\Lambda})$. Let $D:=\mathrm{span}_R\{s_\mu s_\mu^*:\mu\in\Lambda\}$ be the diagonal of $\mathrm{EP}_R(G,\Lambda)$. In light of Theorem 5.5, it is easy to check that the expectation is $\mathcal{E}:\mathrm{EP}_R(G,\Lambda)\to D$ defined by

$$\mathcal{E}\left(s_\mu u_{s(\mu),g}s_\nu^*\right)=\delta_{\mu,\nu}\delta_{g,e_G}s_\mu s_\mu^*\qquad(\mu,\nu\in\Lambda,\ s(\mu)=g\cdot s(\nu)).$$

Definition 6.1 An ideal I of $\mathrm{EP}_R(G,\Lambda)$ is called *diagonal-invariant* whenever $\mathcal{E}(I)\subseteq I$. Also, I is said to be *basic* if $rs_v\in I$ implies $s_v\in I$ for all $v\in\Lambda^0$ and $r\in R\backslash\{0\}$.

Definition 6.2 Let (G,Λ) be a self-similar k-graph. A subset $H\subseteq\Lambda^0$ is called

(1) *G-hereditary* if $r(\mu)\in H\implies g\cdot s(\mu)\in H$ for all $g\in G$ and $\mu\in\Lambda$;
(2) *G-saturated* if $v\in\Lambda^0$ and $s(v\Lambda^n)\subseteq H$ for some $n\in\mathbb{N}^k\implies v\in H$.

In the following, given any $H\subseteq\Lambda^0$, we denote by I_H the ideal of $\mathrm{EP}_R(G,\Lambda)$ generated by $\{s_v:v\in H\}$. Also, for each ideal I of $\mathrm{EP}_R(G,\Lambda)$, we define $H_I:=\{v\in\Lambda^0:s_v\in I\}$.

To prove Theorem 6.8 we need some structural lemmas about the ideals I_H and associated quotients $\mathrm{EP}_R(G,\Lambda)/I_H$.

Lemma 6.3 *If I is an ideal of* $\mathrm{EP}_R(G,\Lambda)$*, then $H_I:=\{v\in\Lambda^0:s_v\in I\}$ is a G-saturated G-hereditary subset of Λ^0.*

Proof The proof is straightforward. □

Lemma 6.4 *Let H be a G-saturated G-hereditary subset of Λ^0 and I_H the ideal of* $\mathrm{EP}_R(G,\Lambda)$ *generated by $\{s_v:v\in H\}$. Then we have*

$$I_H=\mathrm{span}_R\left\{s_\mu u_{s(\mu),g}s_\nu^*:g\in G,\ s(\mu)=g\cdot s(\nu)\in H\right\},\tag{6.1}$$

and I_H is a $\mathbb{Z}^k$-graded diagonal-invariant ideal.

Proof Denote by J the right-hand side of (6.1). The identity

$$s_\mu u_{s(\mu),g}s_\nu^*=s_\mu(s_{s(\mu)})u_{s(\mu),g}s_\nu^*$$

yields $J \subseteq I_H$. Also, using the description of $\mathrm{EP}_R(G,\Lambda)$ in Proposition 2.7, it is straightforward to check that J is an ideal of $\mathrm{EP}_R(G,\Lambda)$. So, by $s_v = s_v u_{v,e_G} s_v^*$, J contains all generators of I_H, and we have proved (6.1).

Now, (6.1) says that I_H is spanned by its homogenous elements, hence it is a graded ideal. Moreover, let $a = \sum_{i=1}^{l} s_{\mu_i} u_{s(\mu_i),g_i} s_{\nu_i}^* \in I_H$ such that $s(\mu_i) = g \cdot s(\nu_i) \in H$. Then, in particular, each term of a with $g_i = e_G$ belongs to I_H. Therefore, $\mathcal{E}(a) \in I_H$, and I_H is diagonal-invariant. □

Let H be a G-saturated G-hereditary subset of Λ^0 and consider the k-subgraph $\Lambda \setminus \Lambda H$. Then the restricted action $G \curvearrowright \Lambda \setminus \Lambda H$ is well defined, and hence $(G, \Lambda \setminus \Lambda H, \varphi|_{G\times\Lambda\setminus\Lambda H})$ is also a self-similar k-graph. So we have:

Lemma 6.5 *Let (G,Λ) be a pseudo-free self-similar k-graph. If H is a G-saturated G-hereditary subset of Λ^0, then $(G, \Lambda \setminus \Lambda H)$ is a pseudo-free self-similar k-graph.*

Proof The proof is straightforward. □

Lemma 6.6 *Let H be a G-saturated G-hereditary subset of Λ^0. For every $v \in \Lambda^0$ and $r \in R\setminus\{0\}$, $rs_v \in I_H$ implies $v \in H$.*

Proof Let $\{t_\mu, w_{v,g}\}$ be the generators of $\mathrm{EP}_R(G, \Lambda \setminus \Lambda H)$. If we define

$$S_\mu := \begin{cases} t_\mu & s(\mu) \notin H \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad U_{v,g} := \begin{cases} w_{v,g} & v \notin H \\ 0 & \text{otherwise}, \end{cases}$$

then $\{S_\mu, U_{v,g}\}$ is a (G,Λ)-family in $\mathrm{EP}_R(G, \Lambda\setminus\Lambda H)$, and by the universality, there is a $*$-homomorphism $\psi : \mathrm{EP}_R(G,\Lambda) \to \mathrm{EP}_R(G, \Lambda\setminus\Lambda H)$ such that $\psi(s_\mu) = S_\mu$ and $\psi(u_{v,g}) = U_{v,g}$ for all $\mu \in \Lambda$, $v \in \Lambda^0$ and $g \in G$. Since $\psi(s_v) = 0$ for every $v \in H$, we have $I_H \subseteq \ker\psi$. On the other hand, Theorem 5.5 implies that all $\psi(rs_v) = rt_v$ are nonzero for $v \in \Lambda^0\setminus H$ and $r \in R\setminus\{0\}$.

Now assume $rs_v \in I_H$ for some $v \in \Lambda^0$ and $r \in R\setminus\{0\}$. If $v \in \Lambda^0\setminus H$, then $\psi(rs_v) = rt_v \neq 0$, and we get $rs_v \notin \ker\psi \supseteq I_H$, a contradiction. □

In fact, Lemma 6.6 says that I_H is a basic ideal with $H_{I_H} = H$ for every G-saturated G-hereditary subset H of Λ^0.

Proposition 6.7 *Let H be a G-saturated G-hereditary subset of Λ^0. Let $\{t_\mu, w_{v,g}\}$ be the $(G, \Lambda \setminus \Lambda H)$-family generating $\mathrm{EP}_R(G, \Lambda \setminus \Lambda H)$. Then the map ψ : $\mathrm{EP}_R(G, \Lambda\setminus\Lambda H) \to \mathrm{EP}_R(G,\Lambda)/I_H$ defined by*

$$\psi(t_\mu w_{s(\mu),g} t_\nu^*) := s_\mu u_{s(\mu),g} s_\nu^* + I_H \qquad (\mu, \nu \in \Lambda \setminus \Lambda H,\ g \in G)$$

is an (R-algebra) $*$-isomorphism.

Proof If we set $T_\mu := s_\mu + I_H$ and $W_{v,g} := u_{v,g} + I_H$ for every $v \in \Lambda^0$, $\mu \in \Lambda$, and $g \in G$, then $\{T_\mu, W_{v,g}\}$ is a $(G, \Lambda \setminus \Lambda H)$-family in $\mathrm{EP}_R(G,\Lambda)/I_H$ (the relations of Definition 2.5 for $\{T_\mu, W_{v,g}\}$ immediately follow from those for $\{s_\mu, u_{v,g}\}$). So, the universality of $\mathrm{EP}_R(G, \Lambda \setminus \Lambda H)$ gives such $*$-homomorphism ψ. Note that $s_\mu \in I_H$ for each $\mu \in \Lambda H$ by (6.1), which gives the surjectivity of ψ.

To prove the injectivity, we apply the graded uniqueness theorem, Theorem 4.2. First, since I_H is a $\mathbb{Z}^k$-graded ideal, $\mathrm{EP}_R(G, \Lambda)/I_H$ has a natural $\mathbb{Z}^k$-grading and ψ is a graded homomorphism. Thus, we fix an element in $\mathrm{EP}_R(G, \Lambda \setminus \Lambda H)$ of the form $a = \sum_{i=1}^{l} r_i w_{v,g_i}$ such that $v \in \Lambda^0 \setminus H$ and $g_i^{-1} \cdot v = g_j^{-1} \cdot v$ for all $1 \le i, j \le l$. Without loss of generality, we may also suppose that the g_i's are distinct. If $\psi(a) = 0$, then $\psi(a) = \sum_{i=1}^{l} r_i u_{v,g_i} + I_H = I_H$ and $\sum_{i=1}^{l} r_i u_{v,g_i} \in I_H$. Thus, for each $1 \le j \le l$, we have

$$\left(\sum_{i=1}^{l} r_i u_{v,g_i}\right) u_{g_j^{-1}\cdot v, g_j^{-1}} = \sum_{i=1}^{l} r_i u_{v, g_i g_j^{-1}} \in I_H \qquad \text{(by eq. (5) in Definition 2.5)}$$

and since I_H is diagonal-invariant,

$$r_j s_v = r_j u_{v,e_G} = \mathcal{E}\left(\sum_{i=1}^{l} r_i u_{v, g_i g_j^{-1}}\right) \in I_H.$$

As $v \notin H$, Lemma 6.6 forces $r_j = 0$ for each $1 \le j \le l$, hence $a = 0$. Now Theorem 4.2 implies that ψ is an isomorphism. □

Theorem 6.8 *Let (G, Λ) be pseudo-free self-similar k-graph. Then $H \mapsto I_H$ is a one-to-one correspondence between G-saturated G-hereditary subsets of Λ^0 and basic, $\mathbb{Z}^k$-graded and diagonal-invariant ideals of* $\mathrm{EP}_R(G, \Lambda)$, *with inverse $I \mapsto H_I$.*

Proof The injectivity of $H \mapsto I_H$ follows from Lemma 6.6. Indeed, if $I_H = I_K$ for G-saturated G-hereditary subsets $H, K \subseteq \Lambda^0$, then Lemma 6.6 yields that $H = H_{I_H} = H_{I_K} = K$.

To see the surjectivity, we take a basic, $\mathbb{Z}^k$-graded and diagonal-invariant ideal I of $\mathrm{EP}_R(G, \Lambda)$, and then prove $I = I_{H_I}$. Write $J := I_{H_I}$ for convenience. By Proposition 6.7 we may consider $\mathrm{EP}_R(G, \Lambda \setminus \Lambda H_I) \cong \mathrm{EP}_R(G, \Lambda)/J$ as a $*$-R-algebra. Let $\{s_\mu, u_{v,g}\}$ and $\{t_\mu, w_{v,g}\}$ be the generators of $\mathrm{EP}_R(G, \Lambda)$ and $\mathrm{EP}_R(G, \Lambda)/J$, respectively. Since $J \subseteq I$, we may define the quotient map $q : \mathrm{EP}_R(G, \Lambda)/J \to \mathrm{EP}_R(G, \Lambda)/I$ such that

$$q(t_\mu) = s_\mu + I \quad \text{and} \quad q(w_{v,g}) = u_{v,g} + I$$

for all $\mu \in \Lambda$, $v \in \Lambda^0$ and $g \in G$. Notice that q preserves the grading because I is a $\mathbb{Z}^k$-graded ideal. So, we can apply Theorem 4.2 to show that q is an isomorphism. To do this, fix an element of the form $a = \sum_{i=1}^{l} r_i w_{v,g_i}$ with $v \in \Lambda^0 \setminus H_I$ such that $g_i^{-1} \cdot v = g_j^{-1} \cdot v$ for $1 \le i, j \le l$ and $q(a) = 0$. Then $\sum_{i=1}^{l} r_i u_{v,g_i} \in I$. As before, we may also assume that the g_i's are distinct. Thus, for each $1 \le j \le l$, we have

$$b_j := \left(\sum_{i=1}^{l} r_i u_{v,g_i}\right)\left(u_{g_j^{-1}\cdot v, g_j^{-1}}\right) = \sum_{i=1}^{l} r_i u_{v, g_i g_j^{-1}} \in I.$$

Since I is diagonal-invariant and basic, the case $r_j \neq 0$ yields $r_j s_v = r_j u_{v,e_G} = \mathcal{E}(b_j) \in I$, and thus $s_v \in I$ and $v \in H_I$, which contradicts the choice of v. It follows that r_j's are all zero, and hence $a = 0$. Now Theorem 4.2 implies that q is injective, or equivalently $I = J = I_{H_I}$ as desired. □

In the end, we remark the following about $Q(\mathbb{N}^k, G) \rtimes_{\mathcal{T}} \mathbb{Z}^k$-graded ideals of $\mathrm{EP}_R(G, \Lambda)$.

Remark 6.9 Let (G, Λ) be a pseudo-free self-similar k-graph and $\mathcal{G}_{G,\Lambda}$ be the associated groupoid. Denote by $\Gamma := Q(\mathbb{N}^k, G) \rtimes_{\mathcal{T}} \mathbb{Z}^k$ the group introduced in Sect. 5. If we define $c : \mathcal{G}_{G,\Lambda} \to \Gamma$ by $c(x; \gamma; y) := \gamma$, then c is a cocycle on $\mathcal{G}_{G,\Lambda}$ because $c(\alpha\beta) = c(\alpha) *_\Gamma c(\beta)$ for all $\alpha, \beta \in \mathcal{G}_{G,\Lambda}$ with $s(\alpha) = r(\beta)$. Hence, it induces a Γ-grading on $A_R(\mathcal{G}_{G,\Lambda}) = \mathrm{EP}_R(G, \Lambda)$ with the homogenous components

$$A_\gamma := \mathrm{span}_R\{1_V : V \subseteq c^{-1}(\gamma) \text{ is a compact open bisection }\}$$

(see [6, Proposition 5.1] for example). By a similar argument as in [6, §6.5] and combining Theorem 5.5 and [6, Theorem 5.3], we may obtain that the ideals of the form I_H, described in Theorem 6.8 above, are precisely the basic, Γ-graded ideals of $\mathrm{EP}_R(G, \Lambda)$.

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Declarations

Conflict of interest The author declares that he has no conflict of interest.

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