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Local Neighbor-Distinguishing Index of Graphs

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Abstract

Suppose that G is a graph and ϕ is a proper edge-coloring of G. For a vertex $v \in V(G)$, let $C_\phi(v)$ denote the set of colors assigned to the edges incident with v. The graph G is local neighbor-distinguishing with respect to the coloring ϕ if for any two adjacent vertices x and y of degree at least two, it holds that $C_\phi(x) \nsubseteq C_\phi(y)$ and $C_\phi(y) \nsubseteq C_\phi(x)$. The local neighbor-distinguishing index, denoted $\chi'_{\text{lnd}}(G)$, of G is defined as the minimum number of colors in a local neighbor-distinguishing edge-coloring of G. For $n \geq 2$, let H_n denote the graph obtained from the bipartite graph $K_{2,n}$ by inserting a 2-vertex into one edge. In this paper, we show the following results: (1) For any graph G, $\chi'_{\text{lnd}}(G) \leq 3\Delta - 1$; (2) suppose that G is a planar graph. Then $\chi'_{\text{lnd}}(G) \leq \lceil 2.8\Delta \rceil + 4$; and moreover $\chi'_{\text{lnd}}(G) \leq 2\Delta + 10$ if G contains no 4-cycles; $\chi'_{\text{lnd}}(G) \leq \Delta + 23$ if G is 3-connected; and $\chi'_{\text{lnd}}(G) \leq \Delta + 6$ if G is Hamiltonian.

Keywords Local neighbor-distinguishing index · Strict neighbor-distinguishing index · Edge-coloring · Planar graph · Factor

Mathematics Subject Classification 05C15

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1 Introduction

Only simple graphs are considered in this paper. Let G be a graph with vertex set $V(G)$, edge set $E(G)$, minimum degree $\delta(G)$ and maximum degree $\Delta(G)$ (for short, Δ). For a vertex $v \in V(G)$, let $d_G(v)$ denote the degree of v in G. Set $|G| = |V(G)|$ and $||G|| = |E(G)|$. A *k-vertex*, *k^--vertex*, and *k^+-vertex* of G are a vertex with degree k, at most k, and at least k, respectively. A graph G is *normal* if it contains no isolated edges, and *formal* if it contains no leaves. A graph G is called *planar* if it can be embedded in the plane such that all edges intersect in their end-vertices. A *plane graph* is a particular drawing of a planar graph in the plane. For two nonnegative integers p, q with $p < q$, we use $[p, q]$ to denote the set of all integers between p and q (including p and q).

An *edge-k-coloring* of a graph G is a mapping ϕ from the edge set $E(G)$ to the color set $\{1, 2, \ldots, k\}$ such that no two adjacent edges get same color. Here two edges are said to be *adjacent* if they share at least one common end vertex. The *chromatic index* $\chi'(G)$ of the graph G is defined as the smallest integer k such that G admits an edge-coloring using k colors. Given an edge-k-coloring ϕ of G and for a vertex $v \in V(G)$, we use $C_\phi(v)$ to denote the set of colors assigned to the edges incident with v. Suppose that x, y are any pair of adjacent vertices in G. We say that ϕ is *neighbor-distinguishing* if $C_\phi(x) \neq C_\phi(y)$, *strict neighbor-distinguishing* if $C_\phi(x) \nsubseteq C_\phi(y)$ and $C_\phi(y) \nsubseteq C_\phi(x)$, and *local neighbor-distinguishing* if $C_\phi(x) \nsubseteq C_\phi(y)$ and $C_\phi(y) \nsubseteq C_\phi(x)$ whenever $d_G(x), d_G(y) \geq 2$. The *neighbor-distinguishing index* $\chi'_{\mathrm{a}}(G)$ (*strict neighbor-distinguishing index* $\chi'_{\mathrm{snd}}(G)$, *local neighbor-distinguishing index* $\chi'_{\mathrm{lnd}}(G)$, respectively) of G is the smallest k such that G has a neighbor-distinguishing edge-k-coloring (a strict neighbor-distinguishing edge-k-coloring, a local neighbor-distinguishing edge-k-coloring, respectively).

As an easy observation, a graph G has a neighbor-distinguishing edge-coloring if and only if G is normal, and G has a strict neighbor-distinguishing edge-coloring if and only if G is formal. But the local neighbor-distinguishing edge-coloring is well defined for any graph G.

It is evident that $\chi'_{\mathrm{snd}}(G) \geq \chi'_{\mathrm{a}}(G) \geq \Delta$ for any formal graph G. Moreover, the following propositions hold obviously.

Proposition 1 *If G is a graph with $\delta(G) \geq 2$, then $\chi'_{\mathrm{lnd}}(G) = \chi'_{\mathrm{snd}}(G)$.*

Proposition 2 *If G is an r (≥ 2)-regular graph, then $\chi'_{\mathrm{lnd}}(G) = \chi'_{\mathrm{snd}}(G) = \chi'_{\mathrm{a}}(G)$.*

Zhang et al. [23] introduced the neighbor-distinguishing edge-coloring of graphs and proposed the following challenging conjecture.

Conjecture 1 *Every normal graph G, other than a 5-cycle, has $\chi'_{\mathrm{a}}(G) \leq \Delta + 2$.*

Akbari et al. [1] proved that every normal graph G satisfies $\chi'_{\mathrm{a}}(G) \leq 3\Delta$. This result was gradually improved to $\chi'_{\mathrm{a}}(G) \leq 2.5\Delta$ by Wang et al. [21], and to $\chi'_{\mathrm{a}}(G) \leq 2\Delta+2$ by Vučković [17]. In 2005, using probabilistic analysis, Hatami [10] showed that every normal graph G with $\Delta > 10^{20}$ has $\chi'_{\mathrm{a}}(G) \leq \Delta + 300$. Recently, this result was improved, by Joret and Lochet [13], to that $\chi'_{\mathrm{a}}(G) \leq \Delta + 19$ for a normal graph with sufficiently large Δ. Suppose that G is a normal planar graph. It was shown in [11] that if $\Delta \geq 12$ then $\chi'_a(G) \leq \Delta + 2$. Moreover, Wang and Huang [20] showed that if $\Delta \geq 16$, then $\Delta \leq \chi'_a(G) \leq \Delta + 1$, and $\chi'_a(G) = \Delta + 1$ if and only if G contains adjacent Δ-vertices. This result was improved in [19] to that if $\Delta \geq 14$, then $\Delta \leq \chi'_a(G) \leq \Delta + 1$, and $\chi'_a(G) = \Delta + 1$ if and only if G contains adjacent Δ-vertices.

The strict neighbor-distinguishing edge-coloring of graphs was studied in [24] (named there the Smarandachely adjacent vertex edge coloring). Let H_n ($n \geq 2$) denote the graph obtained from the bipartite graph $K_{2,n}$ by inserting a 2-vertex into one edge. It is easy to show that $\chi'_{\text{snd}}(H_n) = 2n + 1 = 2\Delta(H_n) + 1$. Based on this fact, Gu et al. [8] raised the following conjecture.

Conjecture 2 *Every connected formal graph G, different from H_Δ, has $\chi'_{\text{snd}}(G) \leq 2\Delta$.*

Because $\chi'_{\text{snd}}(K_{2,n}) = 2n = 2\Delta(K_{2,n})$, the upper bound 2Δ in Conjecture 2 is sharp. Conjecture 2 remains open, but it was confirmed for graphs with $\Delta \leq 3$ in [8] and K_4-minor-free graphs in [9].

In this paper, we continue to study the strict neighbor-distinguishing edge-coloring of graphs, in particular, for the class of planar graphs. As a helpful tool, we consider its relaxed form, i.e., local neighbor-distinguishing edge-coloring of graphs. Our main results in this paper are stated as follows:

- $\chi'_{\text{lnd}}(G) \leq 3\Delta - 1$ for any simple graph G;
- $\chi'_{\text{lnd}}(G) \leq \lceil 2.8\Delta \rceil + 4$ for a planar graph G;
- $\chi'_{\text{lnd}}(G) \leq 2\Delta + 10$ for a planar graph G without 4-cycles;
- $\chi'_{\text{lnd}}(G) \leq \Delta + 23$ for a 3-connected planar graph G;
- $\chi'_{\text{lnd}}(G) \leq \Delta + 6$ for a Hamiltonian planar graph G.

2 An Upper Bound

Let G be a graph and ϕ be a local neighbor-distinguishing edge-k-coloring of G. For the sake of briefness, ϕ is called a k-LNDE-coloring of G. Two adjacent vertices u and v are *exclusive* in ϕ if $C_\phi(u) \nsubseteq C_\phi(v)$ and $C_\phi(v) \nsubseteq C_\phi(u)$. To give an upper bound of the local neighbor-distinguishing index of a graph, we need to use the following result:

Lemma 2.1 ([23]) *For a cycle C_n with $n \geq 3$,*

$$\chi'_a(C_n) = \begin{cases} 3, & \text{if } n = 3; \\ 5, & \text{if } n = 5; \\ 4, & \text{if } n \neq 3, 5. \end{cases}$$

Theorem 2.2 *Every graph with $\Delta \geq 2$ has $\chi'_{\text{lnd}}(G) \leq 3\Delta - 1$.*

Proof The proof is by induction on the edge number $||G||$. If $||G|| \leq 3\Delta - 1$, then the result holds trivially since we can color the edges of G with distinct colors. Let G be a graph with $||G|| \geq 3\Delta \geq 6$. Without loss of generality, assume that G is connected. So, it follows that $\Delta \geq 2$ and $\delta(G) \geq 1$. In the following, we write simply $K = 3\Delta - 1$ and let $C = [1, K]$ denote the set of K colors.

First assume that $\delta(G) = 1$. Let v be a vertex adjacent to leaves $x_1, \ldots, x_l$ and 2^+-vertices $y_1, \ldots, y_k$, where $l \geq 1$ and $k \geq 0$. Let $H = G - x_1$. Then H is a graph with $||H|| < ||G||$ and $\Delta(H) \leq \Delta$. By the induction hypothesis, H admits a K-LNDE-coloring ϕ using the color set C. For $i \in [1, k]$, since v and y_i are exclusive in ϕ, there exists a color $r_i \in C_\phi(y_i) \backslash C_\phi(v)$. Set $R(v) = \{r_1, \ldots, r_k\}$, which is called the *second-level forbidden set* of vertex v. Obviously, $|R(v)| \leq k$. Based on ϕ, we color vx_1 with a color $a \in C \backslash (C_\phi(v) \cup R(v))$. Since $|C \backslash (C_\phi(v) \cup R(v))| \geq 3\Delta - 1 - |C_\phi(v)| - k \geq 3\Delta - 1 - (\Delta - 1) - (\Delta - 1) = \Delta + 1 \geq 3$, a exists and so the coloring is available. It is easy to check that the resultant coloring is a K-LNDE-coloring of G.

Next assume that $\delta(G) \geq 2$. If $\Delta = 2$, then G is a cycle. By Lemma 2.1 and Proposition 2, $\chi'_{\mathrm{lnd}}(G) = \chi'_{\mathrm{a}}(G) \leq 5 = 3\Delta - 1$. So assume that $\Delta \geq 3$. The proof is split into two cases as follows, depending on the size of $\delta(G)$.

Case I. $\delta(G) = 2$.

Let v be a 2-vertex with neighbors v_1, v_2 such that $d_G(v_1) \leq d_G(v_2)$. Without loss of generality, we may suppose that $d_G(v_2) \geq 3$ by the assumption that $\Delta \geq 3$. The proof is split into two subcases as follows.

- $d_G(v_1) = 2$. Let u_1 be the neighbor of v_1 other than v. If $u_1 = v_2$, then $H = G - vv_1$ is a graph with $||H|| < ||G||$ and $\Delta(H) = \Delta$. By the induction hypothesis, H admits a K-LNDE-coloring ϕ using the color set C. Based in ϕ, it suffices to color vv_1 with some color in $C \setminus C_\phi(v_2)$. If $u_1 \neq v_2$, then let $H = G - v$, which has a K-LNDE-coloring ϕ using the color set C by the induction hypothesis. We first color vv_2 with $a \in C \backslash (C_\phi(v_2) \cup R(v_2) \cup \{\phi(v_1u_1)\})$, where $R(v_2)$ is the second-level forbidden set of vertex v_2, as defined before. Then we color vv_1 with $b \in C \backslash (C_\phi(u_1) \cup C_\phi(v_2) \cup \{a\})$. For short, we write $C^+_\phi(v_2) = C_\phi(v_2) \cup R(v_2)$ in the following discussion. Since $|C \backslash (C^+_\phi(v_2) \cup \{\phi(v_1u_1)\})| \geq 3\Delta - 1 - 2(d_G(v_2) - 1) - 1 \geq \Delta \geq 2$ and $|C \backslash (C_\phi(u_1) \cup C_\phi(v_2) \cup \{a\})| \geq 3\Delta - 1 - 2\Delta \geq \Delta - 1 \geq 1$, both a and b exist and hence ϕ is extended to G.
- $d_G(v_1) \geq 3$. Let $H = G - v$, which admits a K-LNDE-coloring ϕ using C. Based on ϕ, we color vv_1 with $a \in C \backslash (C^+_\phi(v_1) \cup C_\phi(v_2))$, and vv_2 with $b \in C \backslash (C^+_\phi(v_2) \cup C_\phi(v_1) \cup \{a\})$. Since $|C \backslash (C^+_\phi(v_1) \cup C_\phi(v_2))| \geq 3\Delta - 1 - 2(\Delta - 1) - (\Delta - 1) \geq 2$ and $|C \backslash (C^+_\phi(v_2) \cup C_\phi(v_1) \cup \{a\})| \geq 3\Delta - 1 - 2(\Delta - 1) - (\Delta - 1) - 1 \geq 1$, a, b exist and ϕ is extended to G.

Case II. $\delta(G) \geq 3$.

Take a vertex $v \in V(G)$ with $d_G(v) = \delta(G) \geq 3$. Let $v_0, \ldots, v_{k-1}$ be the neighbors of v in G, where $k = d_G(v)$. Let $H = G - v$. Then H is a graph with $\delta(H) \geq 2$, $\Delta(H) \leq \Delta$, and $||H|| < ||G||$. By the induction hypothesis, H admits a K-LNDE-coloring ϕ using C. Let $x_1, \ldots, x_m$ be the neighbors of v_0 in H, where $m = d_G(v_0) - 1 \geq 2$. For $i \in [1, m]$, there exists a color $r_i \in C_\phi(x_i) \backslash C_\phi(v_0)$ since v_0 and x_i are exclusive in ϕ. Let $R(v_0) = \{r_1, \ldots, r_m\}$. Similarly, we can define $R(v_1), \ldots, R(v_{k-1})$. Let $U_i = C^+_\phi(v_i)$ for $i \in [0, k-1]$. Then $|U_i| = |C_\phi(v_i) \cup R(v_i)| \leq |C_\phi(v_i)| + |R(v_i)| \leq (\Delta - 1) + (\Delta - 1) = 2\Delta - 2$.

To extend ϕ to G, we design a coloring procedure as following.

Step 0. Color vv_0 with a color $c_0 \in C \backslash (U_0 \cup C_\phi(v_1))$, and then set $B_0 = \{c_0\}$.

Step 1. For $i \in [1, k-1]$, we do the following operation, where all indices are taken modulo k:

- If $B_{i-1} \subseteq C_\phi(v_{i+1})$, then we color vv_i with a color $c_i \in C \setminus (U_i \cup C_\phi(v_{i+1}))$; otherwise, we color vv_i with a color $c_i \in C \backslash (U_i \cup B_{i-1})$.
- Set $B_i = B_{i-1} \cup \{c_i\}$.

Step 2. If $i = k-1$, stop. Otherwise, set $i = i+1$, then go to Step 1.

Let π denote the resultant edge-coloring of G after the above iterative process is ended. Let $B = B_{k-1}$. Then $B = C_\pi(v)$. We will show that π is a K-LNDE-coloring of G.

Claim 1. π is a proper edge-K-coloring of G.

Proof We first prove the existence of the color c_i for $i \in [0, k-1]$. In fact, since $|C \backslash (U_0 \cup C_\phi(v_1))| \geq |C| - |U_0| - |C_\phi(v_1)| \geq (3\Delta - 1) - (2\Delta - 2) - (\Delta - 1) = 2$, c_0 exists. Assume that $1 \leq i \leq k-1$. If $B_{i-1} \subseteq C_\phi(v_{i+1})$, then $c_i \in C \setminus (U_i \cup C_\phi(v_{i+1}))$ by Step 1. Since $|C \backslash (U_i \cup C_\phi(v_{i+1}))| \geq (3\Delta - 1) - (2\Delta - 2) - (\Delta - 1) = 2$, c_i exists. Otherwise, $B_{i-1} \nsubseteq C_\phi(v_{i+1})$. By Step 1, $c_i \in C \backslash (U_i \cup B_{i-1})$. Since $|B_{i-1}| \leq i \leq k-1 = d_G(v) - 1 = \delta(G) - 1 \leq \Delta - 2$, it follows that $|C \backslash (U_i \cup B_{i-1})| \leq (3\Delta - 1) - (2\Delta - 2) - (\Delta - 2) = 3$; thus, c_i exists. Next, we need to show that $c_0, c_1, \ldots, c_{k-1}$ are mutually distinct. Actually, this is true from the definition of B_i's. Hence, π is a proper edge-K-coloring of G. □

Claim 2. π is local neighbor-distinguishing.

Proof It suffices to show that for any edge $xy \in E(G)$, we have

$$C_\pi(x) \nsubseteq C_\pi(y) \text{ and } C_\pi(y) \nsubseteq C_\pi(x)(*)$$

By symmetry, we consider the following three possibilities.

Case 1. $x, y \notin \{v, v_0, \ldots, v_{k-1}\}$.

Note that $\pi(e) = \phi(e)$ for each edge e incident with x or y in G. This implies that $C_\pi(x) = C_\phi(x)$ and $C_\pi(y) = C_\phi(y)$. Since $C_\phi(x) \nsubseteq C_\phi(y)$ and $C_\phi(y) \nsubseteq C_\phi(x)$, $(*)$ holds.

Case 2. $y \notin \{v, v_0, \ldots, v_{k-1}\}$ and $x = v_i$ for some $i \in [0, k-1]$.

By symmetry, suppose that $i = 0$, i.e., $x = v_0$. Then $C_\pi(y) = C_\phi(y)$ and $C_\pi(v_0) = C_\phi(v_0) \cup \{c_0\}$. Since $C_\phi(v_0) \nsubseteq C_\phi(y)$, it is immediate to derive that $C_\pi(v_0) = C_\phi(v_0) \cup \{c_0\} \nsubseteq C_\phi(y) = C_\pi(y)$. Conversely, there is a color $b \in R(v_0)$ such that $b \in C_\phi(y) \backslash C_\phi(v_0)$ and $c_0 \neq b$. This implies that $C_\pi(y) \nsubseteq C_\pi(v_0)$. Hence, $(*)$ holds.

Case 3. $x = v$ and $y = v_i$ for some $i \in [0, k-1]$.

Then $C_\pi(v) = B$ and $C_\pi(v_i) = C_\phi(v_i) \cup \{c_i\}$. Let $i \in [0, k-1]$. Since $d_G(v) = k = \delta(G) \leq d_G(v_i)$, it is easy to derive that $C_\pi(v_i) \not\subset C_\pi(v) = B$. Conversely, suppose that $B \subset C_\pi(v_i)$. We discuss three possibilities to get a contradiction.

- $i = 0$. Since $B_{k-2} = \{c_0, \ldots, c_{k-2}\} \subset B \subset C_\pi(v_0)$, it follows that $c_{k-1} \in C \backslash (U_{k-1} \cup C_\phi(v_0))$ by Step 1. This implies that $c_{k-1} \notin C_\phi(v_0)$, which contradicts the assumption that $c_{k-1} \in B_{k-1} = B \subset C_\pi(v_0)$.

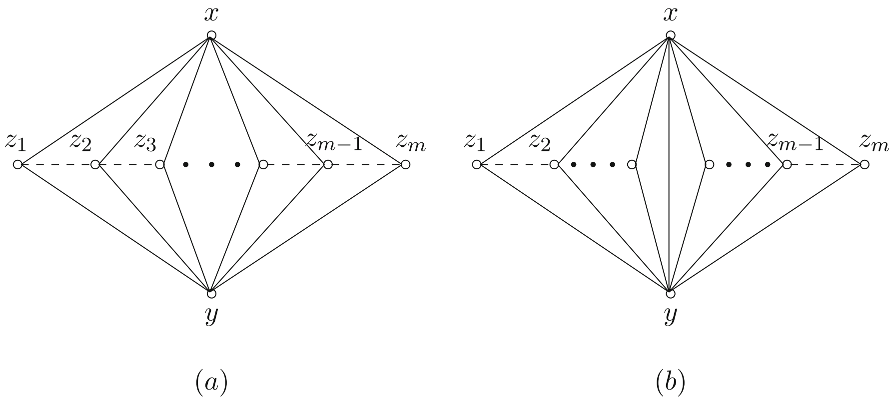

Fig. 1 **a** $B(x, y; m)$ with $xy \notin E(G)$; **b** $B(x, y; m)$ with $xy \in E(G)$

- $i = 1$. Step 0 implies that $c_0 \in C\backslash(U_0 \cup C_\phi(v_1))$. Thus, $c_0 \notin C_\pi(v_1)$. Since $c_0 \in B$, it follows that $B \not\subset C_\pi(v_1)$, which is a contradiction.
- $i \in [2, k-1]$. Note that $B_{i-2} = \{c_0, \ldots, c_{i-2}\} \subset B \subset C_\pi(v_i)$. By Step 1, $c_{i-1} \in C \setminus (U_{i-1} \cup C_\phi(v_i))$, implying $c_{i-1} \notin C_\phi(v_i)$, which contradicts the assumption that $c_{i-1} \in B_{i-1} \subset B \subset C_\pi(v_i)$.

By Claims 1 and 2, π is a K-LNDE-coloring of G. □

Theorem 2.3 follows immediately from Theorem 2.2 and Proposition 1:

Theorem 2.3 *Every formal graph G has $\chi'_{\text{snd}}(G) \leq 3\Delta - 1$.*

3 General Planar Graphs

Assume that G is a plane graph. A cycle C^* of G is called a *separating cycle* if there exist at least one vertex in the interior and exterior of C^*, respectively.

A *bunch* $B(x, y; m)$ of length $m \geq 3$ with x and y as *poles* is defined as m paths $Q_1, Q_2, \ldots, Q_m$ having the following properties, as shown in 1:

(a) Each Q_i has length 1 or 2 and joins x and y;
(b) For each $i \in [1, m-1]$, the cycle formed by Q_i and Q_{i+1} is not separating;
(c) This sequence of paths is maximal, that is, there does not exist a path Q_0 (or Q_{m+1}) that can be added to $B(x, y; m)$ with conditions (a) and (b) preserved.

If the length of Q_i is 2, say $P_i = xz_iy$, then z_i is called a *brother*. If $Q_i = xy$, then xy is called a *parental edge*. Assume that z_i exists. We further say that z_i is an *external brother* if $i \in \{1, m\}$, an *internal brother* if $i \in [2, m-1]$, and a *strictly internal brother* if $i \in [3, m-2]$. It is easy to see that each internal brother is of degree 2, 3, or 4 and adjacent only to the poses and possibly to one or two of brothers.

Borodin et al. [4] introduced the concept of the bunch in a plane graph and established a structural theorem on plane graphs. For our purposes, we only give the following simplified version of their theorem.

Lemma 3.1 *([4]) Every plane graph G with $\delta(G) \geq 2$ contains one of the following configurations:*

(A1) a k-vertex v, $k \in [2, 5]$, with neighbors $v_1, \ldots, v_k$ such that $d_G(v_i) \leq 25$ for all $i \in [1, k-1]$, and $d_G(v_1) + \cdots + d_G(v_{k-1}) \leq 38$;
(A2) a bunch $B(x, y; m)$ with $d_G(x) \geq 26$ and $m \geq 0.2 d_G(x)$.

Using Lemma 3.1, we can obtain an upper bound of the local neighbor-distinguishing index of planar graphs.

Theorem 3.2 *If G is a planar graph, then $\chi'_{\mathrm{lnd}}(G) \leq \lceil 2.8\Delta \rceil + 4$.*

Proof The proof proceeds by induction on $||G||$. If $||G|| \leq \lceil 2.8\Delta \rceil + 4$, then the result holds trivially. Let G be a planar graph with $||G|| \geq \lceil 2.8\Delta \rceil + 5 \geq 5$. Without loss of generality, suppose that G is connected and embedded in the plane. If $\Delta \leq 29$, then $\chi'_{\mathrm{lnd}}(G) \leq 3\Delta - 1 \leq \lceil 2.8\Delta \rceil + 4$ by Theorem 2.2. So suppose that $\Delta \geq 30$, which implies that $||G|| \geq \lceil 2.8\Delta \rceil + 4 \geq 88$. In what follows, let $C = [1, K]$, where $K = \lceil 2.8\Delta \rceil + 4$, be the set of K colors.

If G contains a 1-vertex v, then the graph $G - v$ admits a K-LNDE-coloring ϕ using the color set C by the induction hypothesis. Similarly to the proof of Theorem 2.2, ϕ can be extended to G.

So suppose that $\delta(G) \geq 2$. By Lemma 3.1, G contains the configurations (A1) or (A2). Our proof is split into the following cases.

Case 1. G contains a k-vertex v, $k \in [2, 5]$, with neighbors $v_1, \ldots, v_k$ such that $d_G(v_i) \leq 25$ for all $i \in [1, k-1]$, and $d_G(v_1) + \cdots + d_G(v_{k-1}) \leq 38$.

Without loss of generality, assume that $d_G(v_1) \leq \cdots \leq d_G(v_k)$. Depending on the size of k, we have some subcases to be considered below.

Case 1.1. $k = 2$.

Note that $d_G(v_1) \leq 25$. Let $H = G - v$, which admits a K-LNDE-coloring ϕ using C. Based on ϕ, we color vv_2 with a color $a \in C \backslash (C^+_\phi(v_2) \cup C_\phi(v_1))$, where $C^+_\phi(v_2) = C_\phi(v_2) \cup R(v_2)$ as defined in 2, and vv_1 with a color $b \in C \backslash (C^+_\phi(v_1) \cup C_\phi(v_2) \cup \{a\})$. Since $|C \backslash (C^+_\phi(v_2) \cup C_\phi(v_1))| \geq \lceil 2.8\Delta \rceil + 4 - 2(d_G(v_2) - 1) - (d_G(v_1) - 1) \geq 2.8\Delta + 7 - 2\Delta - 25 = 0.8\Delta - 18 \geq 6$ and $|C \backslash (C^+_\phi(v_1) \cup C_\phi(v_2) \cup \{a\})| \geq \lceil 2.8\Delta \rceil + 4 - 2(25 - 1) - (\Delta - 1) - 1 \geq 1.8\Delta - 44 \geq 10$, both a and b exist and hence ϕ is extended to G.

By Case 1.1, we may assume that $d_G(v_i) \geq 3$ for all $i \in [1, k]$ in the subsequent discussion.

Case 1.2. $k = 3$.

Note that $d_G(v_2) \leq 25$, and since $d_G(v_1) + d_G(v_2) \leq 38$, we derive that $d_G(v_1) \leq 19$. Let $H = G - \{vv_1, vv_2\}$, which has a K-LNDE-coloring ϕ using C. Suppose that $\phi(vv_3) = 1$. We color vv_2 with $a \in C \backslash (C^+_\phi(v_2) \cup C_\phi(v_1) \cup \{1\})$ and vv_1 with $b \in C \backslash (C^+_\phi(v_1) \cup C_\phi(v_2) \cup C_\phi(v_3) \cup \{a\})$. It is easy to calculate that $|C \backslash (C^+_\phi(v_2) \cup C_\phi(v_1) \cup \{1\})| \geq \lceil 2.8\Delta \rceil + 4 - 2(\Delta - 1) - (d_G(v_1) - 1) - 1 \geq 0.8\Delta + 6 - d_G(v_1) \geq 0.8\Delta + 6 - 19 \geq 11$ and $|C \backslash (C^+_\phi(v_1) \cup C_\phi(v_2) \cup C_\phi(v_3) \cup \{a\})| \geq \lceil 2.8\Delta \rceil + 4 - 2(d_G(v_1) - 1) - (d_G(v_2) - 1) - \Delta - 1 \geq 1.8\Delta + 6 - (d_G(v_1) + d_G(v_2)) - d_G(v_1) \geq 1.8\Delta + 6 - 38 - 19 \geq 3$. Hence, both a and b exist. Let ϕ' denote the resultant coloring after vv_1 and vv_2 are colored. Obviously, ϕ' is a proper edge-K-coloring of G. Note that $C_{\phi'}(v) = \{1, a, b\}$, $a \notin C_{\phi'}(v_1)$, $b \notin C_{\phi'}(v_2)$, and $a, b \notin C_{\phi'}(v_3)$. Since $d_G(v_i) \geq 3$ for $i \in [1, 3]$, v is exclusive with each of its neighbors. Consequently, ϕ' is a K-LNDE-coloring of G.

Case 1.3. $k = 4$.

Since $d_G(v_1) + d_G(v_2) + d_G(v_3) \leq 38$ and $d_G(v_i) \geq 3$ for $i \in [1, 4]$, it follows that $d_G(v_1) \leq 12$, $d_G(v_2) \leq 17$, and $d_G(v_3) \leq 25$. Let $H = G - \{vv_1, vv_2, vv_3\}$, which has a K-LNDE-coloring ϕ using C and with $\phi(vv_4) = 1$. We color vv_3 with $a \in C\backslash(C_\phi^+(v_3) \cup C_\phi(v_1) \cup C_\phi(v_2) \cup \{1\})$, vv_2 with $b \in C\backslash(C_\phi^+(v_2) \cup C_\phi(v_1) \cup C_\phi(v_3) \cup C_\phi(v_4) \cup \{a\})$, and vv_1 with $c \in C\backslash(C_\phi^+(v_1) \cup C_\phi(v_2) \cup C_\phi(v_3) \cup C_\phi(v_4) \cup \{a, b\})$. It is easy to check that $|C\backslash(C_\phi^+(v_3) \cup C_\phi(v_1) \cup C_\phi(v_2) \cup \{1\})| \geq 88 - 2(d_G(v_3) - 1) - (d_G(v_2) - 1) - (d_G(v_1) - 1) - 1 = 91 - d_G(v_3) - (d_G(v_1) + d_G(v_2) + d(v_3)) \geq 91 - 25 - 38 = 28$, $|C\backslash(C_\phi^+(v_2) \cup C_\phi(v_1) \cup C_\phi(v_3) \cup C_\phi(v_4) \cup \{a\})| \geq \lceil 2.8\Delta \rceil + 4 - 2(d_G(v_2) - 1) - (d_G(v_1) - 1) - (d_G(v_3) - 1) - \Delta - 1 \geq 1.8\Delta + 7 - d_G(v_2) - (d_G(v_1) + d_G(v_2) + d_G(v_3)) \geq 1.8\Delta + 7 - 17 - 38 \geq 6$, and $|C\backslash(C_\phi^+(v_1) \cup C_\phi(v_2) \cup C_\phi(v_3) \cup C_\phi(v_4) \cup \{a, b\})| \geq \lceil 2.8\Delta \rceil + 4 - 2(d_G(v_1) - 1) - (d_G(v_2) - 1) - (d_G(v_3) - 1) - \Delta - 2 \geq 1.8\Delta + 6 - d_G(v_1) - (d_G(v_1) + d_G(v_2) + d_G(v_3)) \geq 1.8\Delta + 6 - 12 - 38 \geq 10$. Hence, vv_1, vv_2, vv_3 can be colored properly. Let ϕ' denote the resultant coloring of G. It is easy to observe that $C_{\phi'}(v) = \{1, a, b, c\}$, and $b, c \notin C_{\phi'}(v_4)$, $b, c \notin C_{\phi'}(v_3)$, $a, c \notin C_{\phi'}(v_2)$, and $a, b \notin C_{\phi'}(v_1)$. Since $d_G(v_i) \geq 3$ for $i \in [1, 4]$, v is exclusive with each of its neighbors in ϕ'. Consequently, ϕ' is a K-LNDE-coloring of G.

Case 1.4. $k = 5$.

Since $d_G(v_1) + \cdots + d_G(v_4) \leq 38$ and $d_G(v_i) \geq 3$ for $i \in [1, 5]$, it is immediate to deduce that $d_G(v_1) \leq 9$, $d_G(v_2) \leq 11$, $d_G(v_3) \leq 16$, and $d_G(v_4) \leq 25$. Let $H = G - \{vv_1, vv_2, vv_3, vv_4\}$, which has a K-LNDE-coloring ϕ using C such that $\phi(vv_5) = 1$. Define the sets $M_4 = \bigcup_{i=1}^{4} C_\phi(v_i)$ and $M_5 = M_4 \cup C_\phi(v_5)$. We have to consider two possibilities as follows.

Case 1.4.1. $d_G(v_5) \geq 4$.

We color vv_4 with $a \in C\backslash(M_4 \cup R(v_4) \cup \{1\})$, vv_3 with $b \in C\backslash(M_4 \cup R(v_3) \cup \{1, a\})$, vv_2 with $c \in C \setminus (M_5 \cup R(v_2) \cup \{a, b\})$ and vv_1 with $d \in C \setminus (M_5 \cup R(v_1) \cup \{a, b, c\})$. It is easy to calculate that $|C\backslash(M_4 \cup R(v_4) \cup \{1\})| \geq 88 - 2(d_G(v_4) - 1) - (d_G(v_1) - 1) - (d_G(v_2) - 1) - (d_G(v_3) - 1) - 1 = 92 - d_G(v_4) - (d_G(v_1) + d_G(v_2) + d_G(v_3) + d_G(v_4)) \geq 92 - 25 - 38 = 29$, $|C\backslash(M_4 \cup R(v_3) \cup \{1, a\})| \geq 88 - 2(d_G(v_3) - 1) - (d_G(v_1) - 1) - (d_G(v_2) - 1) - (d_G(v_4) - 1) - 2 = 91 - d_G(v_3) - (d_G(v_1) + d_G(v_2) + d_G(v_3) + d_G(v_4)) \geq 91 - 16 - 38 = 37$, $|C\backslash(M_5 \cup R(v_2) \cup \{a, b\})| \geq \lceil 2.8\Delta \rceil + 4 - 2(d_G(v_2) - 1) - (d_G(v_1) - 1) - (d_G(v_3) - 1) - (d_G(v_4) - 1) - d_G(v_5) - 2 \geq 1.8\Delta + 7 - d_G(v_2) - (d_G(v_1) + d_G(v_2) + d_G(v_3) + d_G(v_4)) \geq 1.8\Delta + 7 - 11 - 38 \geq 12$, and $|C\backslash(M_5 \cup R(v_1) \cup \{a, b, c\})| \geq \lceil 2.8\Delta \rceil + 4 - 2(d_G(v_1) - 1) - (d_G(v_2) - 1) - (d_G(v_3) - 1) - (d_G(v_4) - 1) - d_G(v_5) - 3 \geq 1.8\Delta + 6 - d_G(v_1) - (d_G(v_1) + d_G(v_2) + d_G(v_3) + d_G(v_4)) \geq 1.8\Delta + 6 - 9 - 38 \geq 13$. Thus, the resultant coloring, denoted ϕ', is a proper edge-K-coloring of G. Observe that $C_{\phi'}(v) = \{1, a, b, c, d\}$, $c, d \notin C_{\phi'}(v_5)$, $c, b, d \notin C_{\phi'}(v_4)$, $a, c, d \notin C_{\phi'}(v_3)$, $a, b, d \notin C_{\phi'}(v_2)$, and $a, b, c \notin C_{\phi'}(v_1)$. Since $d_G(v_5) \geq 4$ and $d_G(v_i) \geq 3$ for $i \in [1, 4]$, v is exclusive with each of its neighbors in ϕ'. Hence, ϕ' is a K-LNDE-coloring of G.

Case 1.4.2. $d_G(v_5) = 3$.

It follows that $d_G(v_i) = 3$ for all $i \in [1, 4]$. It is evident that $|M_5| \le 2\times4+3 = 11$. We color vv_4 with $a \in C\backslash(M_5 \cup R(v_4))$, vv_3 with $b \in C\backslash(M_5 \cup R(v_3) \cup \{a\})$, vv_2 with $c \in C\backslash(M_5 \cup R(v_2) \cup \{a, b\})$, and vv_1 with $d \in C\backslash(M_5 \cup R(v_1) \cup \{a, b, c\})$. Then $|C\backslash(M_5 \cup R(v_4))| \ge 88-11-2 = 75$, $|C\backslash(M_5 \cup R(v_3) \cup \{a\})| \ge 88-11-2-1 = 74$, $|C\backslash(M_5 \cup R(v_2) \cup \{a, b\})| \ge 88 - 11 - 2 - 2 = 73$, $|C\backslash(M_5 \cup R(v_1) \cup \{a, b, c\})| \ge 88 - 11 - 2 - 3 = 72$. It is easy to check that the extended coloring is a K-LNDE-coloring of G.

Case 2. G contains a bunch $B(x, y; m)$ with $d_G(x) \ge 26$ and $m \ge 0.2d_G(x)$.

Here we use directly the notation in the definition of $B(x, y; m)$, as shown in 1. Since $d_G(x) \ge 26$, it follows that $m \ge 6$. We need to deal with the following two subcases.

Case 2.1. There exist two adjacent vertices u and w such that $3 \le d_G(u) \le d_G(w) \le 4$.

Case 2.1.1. $d_G(u) = 3$.

Let s, t be the neighbors of u other than w. In view of the proof of Case 1.1, we may assume that $d_G(s), d_G(t) \ge 3$. Let $H = G - \{uw, us\}$, which has a K-LNDE-coloring ϕ using C such that $\phi(ut) = 1$. We color us with $a \in C\backslash(C^+_\phi(s) \cup C_\phi(w) \cup \{1\})$ and uw with $b \in C\backslash(C^+_\phi(w) \cup C_\phi(s) \cup C_\phi(t) \cup \{a\})$. It is easy to check that $|C\backslash(C^+_\phi(s) \cup C_\phi(w) \cup \{1\})| \ge \lceil 2.8\Delta\rceil + 4 - 2(d_G(s) - 1) - (d_G(w) - 1) - 1 \ge 0.8\Delta + 6 - d_G(w) \ge 0.8\Delta + 6 - 4 = 26$ and $|C\backslash(C^+_\phi(w) \cup C_\phi(s) \cup C_\phi(t) \cup \{a\})| \ge \lceil 2.8\Delta\rceil + 4 - 2(d_G(w) - 1) - (d_G(s) - 1) - d_G(t) - 1 \ge 0.8\Delta + 6 - 2\times4 \ge 22$. Hence, the resultant coloring ϕ' is a proper edge-K-coloring of G. Since $C_{\phi'}(u) = \{1, a, b\}$, $a \notin C_{\phi'}(w)$, and $b \notin C_{\phi'}(s) \cup C_{\phi'}(t)$, u is exclusive with each of its neighbors in ϕ. Thus, ϕ is extended to G.

Case 2.1.2. $d_G(u) = d_G(w) = 4$.

Let s, t, z be the neighbors of u other than w. By Case 2.1.1, assume that $d_G(s), d_G(t), d_G(z) \ge 4$. Let $H = G - \{uw, us\}$, which has a K-LNDE-coloring ϕ using C such that $\phi(ut) = 1$ and $\phi(uz) = 2$. Since $d_H(u) = 2$, we see that $1 \notin C_\phi(z)$ and $2 \notin C_\phi(t)$. We color us with $a \in C\backslash(C^+_\phi(s) \cup C_\phi(w) \cup \{1, 2\})$ and uw with $b \in C\backslash(C^+_\phi(w) \cup C_\phi(s) \cup \{1, 2, a\})$. Since $|C\backslash(C^+_\phi(s) \cup C_\phi(w) \cup \{1, 2\})| \ge \lceil 2.8\Delta\rceil + 4 - 2(d_G(s) - 1) - (d_G(w) - 1) - 2 \ge 0.8\Delta + 5 - d_G(w) = 0.8\Delta + 5 - 4 \ge 25$ and $|C\backslash(C^+_\phi(w) \cup C_\phi(s) \cup \{1, 2, a\})| \ge \lceil 2.8\Delta\rceil + 4 - 2(d_G(w) - 1) - (d_G(s) - 1) - 3 \ge 1.8\Delta + 4 - 2 \times 4 \ge 50$, uw and us can be colored properly. Denote by ϕ' the resultant coloring. Noting that $C_{\phi'}(u) = \{1, 2, a, b\}$, $1 \notin C_{\phi'}(z)$, $2 \notin C_{\phi'}(t)$, $a \notin C_{\phi'}(w)$, and $b \notin C_{\phi'}(s)$, we obtain a K-LNDE-coloring of G.

Case 2.2. All strictly internal brothers are of degree 2 in G.

Let S denote the set of brothers z_i's with $d_G(z_i) = 2$ in $B(x, y; m)$. Obviously, S contains all strictly internal brothers of $B(x, y; m)$. Since $d_G(x) \ge 26$ and $m \ge 0.2d_G(x) > 5$, $B(x, y; m)$ has at least one strictly internal brother. Thus, $s := |S| \ge m - 5 \ge 1$. Let $H = G - S$, which has a K-LNDE-coloring ϕ using C. Let $E_x = \{wx \mid w \in S\}$ and $E_y = \{wy \mid w \in S\}$. For each edge $e_x \in E_x$ and each edge $e_y \in E_y$, we define a list assignment L as follows:

$L(e_x) = C \setminus (C^+_\phi(x) \cup C_\phi(y))$, $L(e_y) = C \setminus (C^+_\phi(y) \cup C_\phi(x))$.

First suppose that $xy \notin E(G)$. Then $s \geq m-4 \geq 2$. It is easy to compute that $|L(e_y)| \geq \lceil 2.8\Delta \rceil + 4 - 2(d_G(y)-s) - (d_G(x)-s) \geq 2.8\Delta + 4 + 3s - 2d_G(y) - d_G(x) \geq 0.8\Delta + 4 + s + 2s - d_G(x) \geq 0.8\Delta + 4 + s + 2(m-4) - d_G(x) \geq 0.8\Delta - 4 + s + 2 \times (0.2d_G(x)) - d_G(x) = 0.8\Delta - 4 + s - 0.6d_G(x) \geq 0.2\Delta - 4 + s \geq s+2$, and $|L(e_x)| \geq \lceil 2.8\Delta \rceil + 4 - 2(d_G(x)-s) - (d_G(y)-s) \geq 1.8\Delta + 4 + 2s + s - 2d_G(x) \geq 1.8\Delta + 4 + 2s + (m-4) - 2d_G(x) = 1.8\Delta + 2s + m - 2d_G(x) \geq 1.8\Delta + 2s + 0.2d_G(x) - 2d_G(x) \geq 2s$.

Next suppose that $xy \in E(G)$. In this case, $s \geq m-5 \geq 1$. Because $xy \in E(G)$, we have $\phi(xy) \in C_\phi(x) \cap C_\phi(y)$ and hence $|C_\phi(x) \cap C_\phi(y)| \geq 1$. So, $|L(e_y)| \geq \lceil 2.8\Delta \rceil + 4 - 2(d_G(y)-s) - (d_G(x)-s) + 1 \geq 2.8\Delta + 5 + 3s - 2d_G(y) - d_G(x) \geq 0.8\Delta + 5 + s + 2s - d_G(x) \geq 0.8\Delta + 5 + s + 2(m-5) - d_G(x) \geq 0.8\Delta - 5 + s + 2 \times (0.2d_G(x)) - d_G(x) = 0.8\Delta - 5 + s - 0.6d_G(x) \geq 0.2\Delta - 5 + s \geq s+1$, and $|L(e_x)| \geq \lceil 2.8\Delta \rceil + 4 - 2(d_G(x)-s) - (d_G(y)-s) + 1 \geq 1.8\Delta + 5 + 2s + s - 2d_G(x) \geq 1.8\Delta + 5 + 2s + (m-5) - 2d_G(x) = 1.8\Delta + 2s + m - 2d_G(x) \geq 1.8\Delta + 2s + 0.2d_G(x) - 2d_G(x) \geq 2s$.

In each of the above two cases, we first color the edges in E_y with distinct colors in $L(e_y)$ and then use C_y to denote the set of colors assigned to the edges in E_y. Then we color the edges in E_x with distinct colors in $L(e_x) \setminus C_y$. It is easy to check that the resultant coloring is a K-LNDE-coloring of G. □

By Proposition 1, we have the following:

Theorem 3.3 *If G is a formal planar graph, then $\chi'_{\mathrm{snd}}(G) \leq \lceil 2.8\Delta \rceil + 4$.*

4 Planar Graphs Without 4-Cycles

For the class of planar graphs without 4-cycles, we can show that Conjecture is almost true (away from a constant). To achieve this goal, we need to apply the following structural lemma.

Lemma 4.1 ([18]) *Let G be a planar graph with $\delta(G) \geq 2$ and without 4-cycles. Then G contains a k-vertex v, $k \in [2,4]$, whose neighbors $v_1, \ldots, v_k$ satisfy one of the following conditions, assuming $d_G(v_1) \leq \cdots \leq d_G(v_k)$:*

(1) $k = 2$ and $d_G(v_1) \leq 11$;
(2) $k = 3$ and $d_G(v_1) + d_G(v_2) \leq 14$;
(3) $k = 4$ and $d_G(v_1) + d_G(v_2) + d_G(v_3) \leq 15$.

Theorem 4.2 *If G is a planar graph without 4-cycles, then $\chi'_{\mathrm{lnd}}(G) \leq 2\Delta + 10$.*

Proof The proof proceeds by induction on $||G||$. If $||G|| \leq 2\Delta + 10$, then the result holds trivially. Let G be a connected planar graph with $||G|| \geq 2\Delta + 11 \geq 11$. If $\Delta \leq 11$, then $\chi'_{\mathrm{lnd}}(G) \leq 3\Delta - 1 \leq 2\Delta + 10$ by Theorem 2.2. So suppose that $\Delta \geq 12$. Again, let $K = 2\Delta + 10$ and $C = [1, K]$ denote a set of K colors. Hence, $|C| = K = 2\Delta + 10 \geq 34$.

First assume that $\delta(G) = 1$. Let u be a 1-vertex adjacent to a vertex v. Then $d_G(v) \geq 2$ by the assumption. Let $H = G - u$, which has a K-LNDE-coloring ϕ

using C. We color uv with a color $a \in C \backslash C_\phi^+(v)$. Since $|C \backslash C_\phi^+(v)| \geq 2\Delta + 10 - 2(d_G(v) - 1) \geq 2\Delta + 10 - 2(\Delta - 1) = 12$, ϕ is extended to G.

Next assume that $\delta(G) \geq 2$. By Lemma 4.1, G contains a k-vertex v, $k \in [2, 4]$, whose neighbors $v_1, \ldots, v_k$ satisfy one of the conditions (1) to (3), where $d_G(v_1) \leq \cdots \leq d_G(v_k)$. By the above proof, we may assume that $d_G(v_i) \geq 2$ for all $i \in [1, k]$.

Case 1. $k = 2$ and $d_G(v_1) \leq 11$.

Let $H = G - v$, which admits a K-LNDE-coloring ϕ using C. We have to discuss two possibilities.

- $d_G(v_1) = 2$. Let y be the neighbor of v_1 other than v. We color vv_2 with $a \in C \backslash (C_\phi^+(v_2) \cup \{\phi(yv_1)\})$ and vv_1 with $b \in C \backslash (C_\phi(y) \cup C_\phi(v_2) \cup \{a\})$. Since $|C \backslash (C_\phi^+(v_2) \cup \{\phi(v_1 y)\})| \geq 2\Delta + 10 - 2(d_G(v_2) - 1) - 1 \geq 11$ and $|C \backslash (C_\phi(y) \cup C_\phi(v_2) \cup \{a\})| \geq 2\Delta + 10 - \Delta - (\Delta - 1) - 1 = 10$, ϕ is extended to G.
- $d_G(v_1) \geq 3$. We color vv_2 with $a \in C \backslash (C_\phi^+(v_2) \cup C_\phi(v_1))$ and vv_1 with $b \in C \backslash (C_\phi^+(v_1) \cup C_\phi(v_2) \cup \{a\})$. Since $|C \backslash (C_\phi^+(v_2) \cup C_\phi(v_1))| \geq 2\Delta + 10 - 2(\Delta - 1) - 10 \geq 2$ and $|C \backslash (C_\phi^+(v_1) \cup C_\phi(v_2) \cup \{a\})| \geq 2\Delta + 10 - 2(d_G(v_1) - 1) - \Delta \geq \Delta + 10 - 2 \times 10 \geq 2$, ϕ is extended to G.

Now, by Case 1, we may assume that $d_G(v_i) \geq 3$ for all $i \in [1, k]$ in the following two situations.

Case 2. $k = 3$ and $d_G(v_1) + d_G(v_2) \leq 14$.

It follows that $d_G(v_1) \leq 7$ and $d_G(v_2) \leq 11$. Let $H = G - \{vv_1, vv_2\}$, which has a K-LNDE-coloring ϕ using C such that $\phi(vv_3) = 1$. We color vv_2 with $a \in C \backslash (C_\phi^+(v_2) \cup C_\phi(v_1) \cup \{1\})$ and vv_1 with $b \in C \backslash (C_\phi^+(v_1) \cup C_\phi(v_2) \cup C_\phi(v_3) \cup \{a\})$. Since $|C \backslash (C_\phi^+(v_2) \cup C_\phi(v_1) \cup \{1\})| \geq 2\Delta + 10 - 2(\Delta - 1) - (d_G(v_1) - 1) - 1 \geq 12 - (7 - 1) - 1 = 5$ and $|C \backslash (C_\phi^+(v_1) \cup C_\phi(v_2) \cup C_\phi(v_3) \cup \{a\})| \geq 2\Delta + 10 - 2(d_G(v_1) - 1) - (d_G(v_2) - 1) - \Delta - 1 \geq \Delta + 12 - d_G(v_1) - (d_G(v_1) + d_G(v_2)) \geq 24 - 7 - 14 = 3$, both a and b exist. Let ϕ' denote the resultant coloring. Then $C_{\phi'}(v) = \{1, a, b\}$, $a \notin C_{\phi'}(v_1)$, and $b \notin C_{\phi'}(v_2) \cup C_{\phi'}(v_3)$. Thus, v is exclusive with each of its neighbors and hence ϕ is extended to G.

Case 3. $k = 4$ and $d_G(v_1) + d_G(v_2) + d_G(v_3) \leq 15$.

Then $d_G(v_1) \leq 5$, $d_G(v_2) \leq 6$, and $d_G(v_3) \leq 9$. Let $H = G - \{vv_1, vv_2, vv_3\}$, which has a K-LNDE-coloring ϕ using C such that $\phi(vv_4) = 1$. We color vv_3 with $a \in C \backslash (C_\phi^+(v_3) \cup C_\phi(v_1) \cup C_\phi(v_2) \cup \{1\})$, vv_2 with $b \in C \backslash (C_\phi^+(v_2) \cup C_\phi(v_1) \cup C_\phi(v_3) \cup C_\phi(v_4) \cup \{a\})$, and vv_1 with $c \in C \backslash (C_\phi^+(v_1) \cup C_\phi(v_2) \cup C_\phi(v_3) \cup C_\phi(v_4) \cup \{a, b\})$. Noting that $|C \backslash (C_\phi^+(v_3) \cup C_\phi(v_1) \cup C_\phi(v_2) \cup \{a\})| \geq 34 - 2(d_G(v_3) - 1) - (d_G(v_2) - 1) - (d_G(v_1) - 1) - 1 = 37 - d_G(v_3) - (d_G(v_1) + d_G(v_2) + d_G(v_3)) \geq 37 - 9 - 15 = 13$, $|C \backslash (C_\phi^+(v_2) \cup C_\phi(v_1) \cup C_\phi(v_3) \cup C_\phi(v_4) \cup \{a\})| \geq 2\Delta + 10 - 2(d_G(v_2) - 1) - (d_G(v_1) - 1) - (d_G(v_3) - 1) - \Delta - 1 \geq \Delta + 13 - d_G(v_2) - (d_G(v_1) + d_G(v_2) + d_G(v_3)) \geq \Delta + 13 - 6 - 15 \geq 4$, and $|C \backslash (C_\phi^+(v_1) \cup C_\phi(v_2) \cup C_\phi(v_3) \cup C_\phi(v_4) \cup \{a, b\})| \geq 2\Delta + 10 - 2(d_G(v_1) - 1) - (d_G(v_2) - 1) - (d_G(v_3) - 1) - \Delta - 2 \geq \Delta + 12 - d_G(v_1) - (d_G(v_1) + d_G(v_2) + d_G(v_3)) \geq \Delta + 12 - 5 - 15 \geq 4$, vv_1, vv_2, vv_3 can be properly colored. Let ϕ' denote the resultant coloring. Since $C_{\phi'}(v) = \{1, a, b, c\}$, $a, b \notin C_{\phi'}(v_1)$, $a, c \notin C_{\phi'}(v_2)$, $c, b \notin C_{\phi'}(v_3)$, and $b, c \notin C_{\phi'}(v_4)$, ϕ' is a K-LNDE-coloring of G.

By Proposition 1, the following theorem holds automatically.

Theorem 4.3 *If G is a formal planar graph without 4-cycles, then $\chi'_{\mathrm{snd}}(G) \leq 2\Delta+10$.*

5 Planar Graphs with $[2, k]$-Factors

For two positive integers k_1, k_2 with $k_2 \geq k_1$, a spanning subgraph F of a graph G is called an $[k_1, k_2]$-*factor* if $k_1 \leq d_F(v) \leq k_2$ for all $v \in V(G)$. Tutte [15] showed that every 4-connected planar graph is Hamiltonian, i.e., it has a 2-connected $[2, 2]$-factor. By relaxing the 4-connected condition, Gao [7] showed that every 3-connected planar graph has a 2-connected $[2, 6]$-factor. Enomoto et al. [5] extended this result by showing that every 3-connected planar graph G with $\delta(G) \geq 4$ has a 2-connected $[2, 3]$-factor. Both numbers 6 and 3 in these two results are best possible with respect to the required conditions.

The *core* G_Δ of a graph G is the subgraph of G induced by Δ-vertices.

Lemma 5.1 *Let $k \geq 3$. If a connected graph G has a connected $[2, k]$-factor, then G contains a connected $[2, k]$-factor F whose core is acyclic.*

Proof Let F be a connected $[2, k]$-factor of G with the least number of edges. We claim that the core of F is acyclic. Suppose to the contrary that F_Δ contains a cycle C. Let $e = xy \in E(C)$ be an arbitrary edge, and set $F' = F - e$. Obviously, F' is a connected spanning subgraph of G. If $v \in V(G)\backslash\{x, y\}$, then $d_{F'}(v) = d_F(v)$. If $v \in \{x, y\}$, then $d_{F'}(v) = d_F(v) - 1 \geq k - 1 \geq 2$. It follows that F' is a connected $[2, k]$-factor of G with $||F'|| < ||F||$, which contradicts the choice of F. □

The following result can be derived from Lemma 5.1 and the results in [5] and [7].

Corollary 5.2 *Let G be a planar graph.*

(1) If G is 3-connected, then G contains a connected $[2, 6]$-factor F whose core is acyclic.
(2) If G is 3-connected and $\delta(G) \geq 4$, then G contains a connected $[2, 3]$-factor F whose core is acyclic.

The celebrated Vizing's Theorem gives a tight upper bound for the chromatic index of a simple graph.

Theorem 5.3 ([16]) *Every simple graph G has $\Delta \leq \chi'(G) \leq \Delta + 1$.*

A simple graph G is said to be *Class* 1 if $\chi'(G) = \Delta$ and *Class* 2 if $\chi'(G) = \Delta+1$.

Theorem 5.4 ([6]) *If the core of a simple graph G is a forest, then G is Class* 1.

Theorem 5.5 ([14]) *If G is a planar graph with $\Delta \geq 7$, then G is Class* 1.

An *edge-partition* of a graph G is a decomposition of G into subgraphs $G_1, \ldots, G_m$ such that $E(G) = E(G_1) \cup \cdots \cup E(G_m)$ and $E(G_i) \cap E(G_j) = \emptyset$ for $i \neq j$.

Suppose that H is a subgraph of a graph G. A *restricted-strong edge-k-coloring* of H on G is an edge-coloring $\phi : E(H) \to [1, k]$ such that any two edges $e_1, e_2 \in E(H)$ having distance at most two in G get distinct colors. The *restricted-strong chromatic* *index* of H on G, denoted $\chi'_{\mathrm{s}}(H|_G)$, is the smallest integer k such that H has a restricted-strong edge-k-coloring on G.

Since all planar graphs G considered in Theorems 5.6 to 5.11 contain a $[2, k]$-factor, it follows that $\delta(G) \geq 2$, which implies that $\chi'_{\mathrm{lnd}}(G) = \chi'_{\mathrm{snd}}(G)$ by Proposition 1.

Theorem 5.6 *Suppose that a connected graph G can be edge-partitioned into two graphs F and H such that F is a connected $[2, k]$-factor of G with $k \geq 2$. Then $\chi'_{\mathrm{snd}}(G) \leq \chi'(H) + \chi'_{\mathrm{s}}(F|_G)$.*

Proof Note that $2 \leq \delta(F) \leq \Delta(F) \leq k$ and $\Delta(H) \leq \Delta(G) - \delta(F) \leq \Delta(G) - 2$. Let $\chi'(H) = l$ and $\chi'_{\mathrm{s}}(F|_G) = m$. Let ϕ be an edge-l-coloring of H using the color set $C_1 = [1, l]$ and π be a restricted-strong-edge-coloring of F on G using the color set $C_2 = [l+1, l+m]$. We define an edge-coloring f of G as follows: $f(e) = \phi(e)$ for $e \in E(H)$, and $f(e) = \pi(e)$ for $e \in E(F)$. Obviously, f is a proper edge-$(l+m)$-coloring of G using the color set $C_1 \cup C_2$. If we can show that f is strict neighbor-distinguishing, then it holds naturally that $\chi'_{\mathrm{snd}}(G) \leq l+m = \chi'(H) + \chi'_{\mathrm{s}}(F|_G)$. In fact, for any edge $e = xy \in E(G)$, since $d_F(x), d_F(y) \geq 2$, there exist an edge $e_x \in E(F) \setminus \{e\}$ incident with x and an edge $e_y \in E(F) \setminus \{e\}$ incident with y. Since e_x and e_y have the distance 2 in G, it follows that $\pi(e_x) \notin C_\pi(y)$ and $\pi(e_y) \notin C_\pi(x)$. Because $C_1 \cap C_2 = \emptyset$, we have that $\pi(e_x) \notin C_f(y)$ and $\pi(e_y) \notin C_f(x)$, and so x and y are exclusive. $\square$

A family of graphs $\mathcal{G}$ is called *minor-closure* if it is closed under deleting vertices, deleting edges, or contracting edges. Let $\chi(G)$ denote the chromatic number of a graph G, which is defined as the smallest integer k for which the vertices of G can be colored using k colors such that no two adjacent vertices get same color. For a family of minor-closure graphs $\mathcal{G}$, we define $\chi(\mathcal{G}) = \max\{\chi(G) \mid G \in \mathcal{G}\}$. It is easily seen that the family of planar graphs, denoted $\mathcal{P}$, is minor-closure and $\chi(\mathcal{P}) \leq 4$ by the Four-Color Theorem [2].

Lemma 5.7 ([22]) *If F is a subgraph of a planar graph G, then $\chi'(F|_G) \leq 4\chi'(F)$.*

Combining Theorem 5.6 and Lemma 5.7, the following theorem holds automatically.

Theorem 5.8 *Let G be a connected planar graph with a connected $[2, k]$-factor F, and $H = G - E(F)$. Then $\chi'_{\mathrm{snd}}(G) \leq \chi'(H) + 4\chi'(F)$.*

Theorem 5.9 *Let G be a planar graph. Then the following statements (1) and (2) hold.*
(1) *If G is 3-connected, then $\chi'_{\mathrm{snd}}(G) \leq \Delta + 23$.*
(2) *If G is 3-connected and $\delta(G) \geq 4$, then $\chi'_{\mathrm{snd}}(G) \leq \Delta + 11$.*

Proof (1) Since G is 3-connected, it follows from Corollary 5.2(1) that G has a connected $[2, 6]$-factor F whose core is acyclic. Let $H = G - E(F)$. Then G is edge-partitioned into two subgraphs F and H. If $\Delta(F) \leq 5$, then $\chi'(F) \leq 5 + 1 = 6$ by Theorem 5.3. If $\Delta(F) = 6$, then $\chi'(F) = 6$ by Theorem 5.4. Hence, it always holds that $\chi'(F) \leq 6$. On the other hand, since $\Delta(H) \leq \Delta(G) - 2$, we have $\chi'(H) \leq \Delta(G) - 2 + 1 = \Delta - 1$ by Theorem 5.3. So, by Theorem 5.8, $\chi'_{\mathrm{snd}}(G) \leq \chi'(H) + 4\chi'(F) \leq \Delta(H) + 4 \times 6 \leq \Delta - 1 + 24 = \Delta + 23.$

(2) By Corollary 5.2(2), G has a connected $[2,3]$-factor F whose core is acyclic. Let $H = G - E(F)$. Then G is edge-partitioned into two subgraphs F and H. If $\Delta(F) = 2$, then $\chi'(F) \leq 2 + 1 = 3$ by Theorem 5.3. If $\Delta(F) = 3$, then $\chi'(F) = 3$ by Theorem 5.4. Hence, we always have that $\chi'(F) \leq 3$. By Theorem 5.3, $\chi'(H) \leq \Delta(H) + 1 \leq \Delta(G) - 2 + 1 = \Delta - 1$. By Theorem 5.8, $\chi'_{\text{snd}}(G) \leq \chi'(H) + 4\chi'(F) \leq \Delta - 1 + 4 \times 3 = \Delta + 11$. □

Theorem 5.10 *If a planar graph G is Hamiltonian, then $\chi'_{\text{snd}}(G) \leq \Delta + 6$.*

Proof Let $C = v_0v_1 \cdots v_{n-1}v_0$ be a Hamiltonian cycle of G, where $n = |V(G)|$. Let $H = G - E(C)$. Then $H \cup C$ is an edge-partition of G with $\Delta(H) \leq \Delta - 2$. Let $\chi'(H) = k$. First we give an edge-k-coloring of H using the color set $B_1 = [1, k]$. Then we define an edge-7-coloring π of C using the color set $B_2 = [k + 1, k + 7]$ in two ways below:

- Assume that n is even. Set $M = \{v_0v_1, v_2v_3, \ldots, v_{n-2}v_{n-1}\}$, and give a restricted-strong edge-4-coloring of M on G using the colors in $[k + 1, k + 4]$. Afterward, if $n \equiv 0 \pmod 4$, then we color alternatively $v_1v_2, v_3v_4, \ldots, v_{n-1}v_0$ with $k + 5$ and $k + 6$. If $n \equiv 2 \pmod 4$, then we color v_1v_2 with $k + 7$, and then color alternatively $v_3v_4, v_5v_6, \ldots, v_{n-1}v_0$ with $k + 5$ and $k + 6$.
- Assume that n is odd. Set $M = \{v_0v_1, v_2v_3, \ldots, v_{n-3}v_{n-2}\}$, and give a restricted-strong edge-4-coloring of M on G using $[k + 1, k + 4]$. Then we color $v_{n-1}v_0$ with $k + 7$ and then color alternatively $v_1v_2, v_3v_4, \ldots, v_{n-2}v_{n-1}$ with $k + 5$ and $k + 6$.

Let f denote the resultant edge-$(k + 7)$-coloring of G formed by combining ϕ and π, using the color set $B_1 \cup B_2$. It is easy to inspect that f is proper, i.e., any two adjacent edges having distinct colors. It remains to show that f is strict neighbor-distinguishing. Let $e = xy$ be an arbitrary edge of G. By the definition of M, at most one of x and y is not incident with any edge in M.

First assume that each of x and y is incident with an edge in M, respectively. Since M is a matching of G, there exist the unique $e_x \in M$ incident with x and the unique $e_y \in M$ incident with y. We have two possibilities as follows.

- $e_x = e_y$, that is, $e = e_x = e_y \in M$, say $e = v_iv_{i+1}$, where indices are taken modulo n. Then $v_{i-1}v_i, v_{i+1}v_{i+2} \in E(C) \setminus M$. By the definition of π, $\pi(v_{i-1}v_i), \pi(v_{i+1}v_{i+2}) \in [k + 5, k + 7]$ and $\pi(v_{i-1}v_i) \neq \pi(v_{i+1}v_{i+2})$. Noting that $\pi(v_{i-1}v_i) \notin C_f(v_{i+1})$ and $\pi(v_{i+1}v_{i+2}) \notin C_f(v_i)$, v_i and v_{i+1} are exclusive in f.
- $e_x \neq e_y$. Since e_x and e_y have distance 2 in G, the definition of π implies that $\pi(e_x) \neq \pi(e_y)$ and $\pi(e_x), \pi(e_y) \in [k + 1, k + 4]$. So, $\pi(e_x) \notin C_f(y)$ and $\pi(e_y) \notin C_f(x)$, and henceforth x and y are exclusive. Next assume that x is not incident with any edge in M. Then $x = v_{n-1}$. There are two subcases to be considered.
- $y \in \{v_0, v_{n-2}\}$, say $y = v_0$. Because $v_0v_1 \in M$ satisfies $\pi(v_0v_1) \in [k + 1, k + 4]$, and $v_{n-2}v_{n-1} \in E(C) \setminus M$ satisfies $\phi(v_{n-2}v_{n-1}) \in [k + 5, k + 7]$, it follows that $\pi(v_0v_1) \notin C_f(v_{n-1})$ and $\pi(v_{n-2}v_{n-1}) \notin C_f(v_0)$, and therefore, x and y are exclusive.

- $y \in \{v_1, \ldots, v_{n-2}\}$, say $y = v_i$ for some $i \in [1, n-2]$. Then exactly one of $v_{i-1}v_i$ and $v_i v_{i+1}$ belongs to M, whose color is $a \in [k+1, k+4]$. Moreover, assuming that $\pi(v_{n-1}v_0) = b$ and $\pi(v_{n-2}v_{n-1}) = c$, then $b, c \in [k+5, k+7]$. Since $a \notin C_f(v_{n-1})$ and at least one of b and c does not belong to $C_f(v_i)$, x and y are exclusive.

The above analysis and Theorem 5.3 show that $\chi'_{\text{snd}}(G) \leq k+4+3 \leq \Delta - 1 + 7 = \Delta + 6$. □

By Theorem 5.5 and the proof of Theorem 5.10, we can obtain the following better result.

Theorem 5.11 *If G is a Hamiltonian planar graph with $|G| \equiv 0 \pmod 4$ and $\Delta \geq 9$, then $\chi'_{\text{snd}}(G) \leq \Delta + 4$.*

A *Halin graph* is a plane graph $G = T \cup C$, where T is a plane tree with no vertex of degree two and at least one vertex of degree three or more, and C is a cycle connecting the pendant vertices of T in the cyclic order determined by the drawing of T.

Halin graphs are 3-connected, but any of their proper subgraphs is not. Bondy and Lovász [3] showed that Halin graphs are almost pancyclic with the possible exception of an even cycle. In particular, Halin graphs are Hamiltonian.

By Theorem 5.10, we have the following:

Corollary 5.12 *If G is a Halin graph, then $\chi'_{\text{snd}}(G) \leq \Delta + 6$.*

6 Concluding Remarks

In this section, we are going to provide some open problems on the local neighbor-distinguishing edge-coloring of graphs. In contrast to Conjecture 2, we first put forward the following conjecture:

Conjecture 3 *Every connected graph G, different from H_Δ, has $\chi'_{\text{lnd}}(G) \leq 2\Delta$. Observing the local neighbor-distinguishing index of $K_{2,n}$, we see that the upper bound 2Δ in Conjecture 3 is tight if it were true.*

Problem 1. *Does every planar graph satisfy Conjecture* 3 ?
Problem 2. *Is it true that there exists a constant c such that every planar graph G without* 4*-cycles has $\chi'_{\text{lnd}}(G) \leq \Delta + c$?*

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Conflict of interest The authors declared that they had no conflicts of interest with respect to their authorship or the publication of this paper.

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