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Reconstruction and Error Analysis Based on Multiple Sampling Functionals over Mixed Lebesgue Spaces

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Abstract

Signal sampling issue has been extensively studied based on a single sampling functional over classical Lebesgue spaces. This paper focuses on discussing the signal reconstruction and error analysis based on multiple sampling functionals over mixed Lebesgue spaces. We firstly explore the stabilities for two kinds of sampling functionals, respectively. Then the corresponding iterative reconstruction algorithms are established. Finally, the error between the reconstruction signal in the presence of noisy and the original signal f is analyzed.

Keywords Average sampling · Mixed Lebesgue space · Stability · Reconstruction algorithm · Random noise · Error analysis

Mathematics Subject Classification 94A20 · 46E30 · 94A12

1 Introduction

The mixed Lebesgue space is a natural generalization of the classical Lebesgue space, which was firstly in depth introduced by Benedek and Panzone [7]. In fact, it arises from considering a function containing several independent variables of different properties. For instance, a multivariate function depending on both spatial and time variables may belong to a mixed Lebesgue space. Moreover, the flexibility of the separate integrability for each variable is of interests and potentially useful in the study of time-based partial

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differential equations [10]. The definition of mixed Lebesgue spaces $L^{p,q}(\mathbb{R}^{d+1})$ [14, 16, 17, 23] is given as follows.

Definition 1.1 Let $1 \le p, q < \infty$, then $L^{p,q}(\mathbb{R}^{d+1})$ consists of all measurable functions f on $\mathbb{R}^{d+1}$ such that

$$\|f\|_{L^{p,q}} := \left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} |f(x,y)|^q \mathrm{d}y\right)^{\frac{p}{q}} \mathrm{d}x\right]^{\frac{1}{p}} < \infty.$$

The corresponding sequence spaces are

$$l^{p,q}(\mathbb{Z}^{d+1}) = \left\{ c,\ \|c\|_{l^{p,q}} := \left(\sum_{k_1 \in \mathbb{Z}} \left(\sum_{k_2 \in \mathbb{Z}^d} |c(k_1,k_2)|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} < \infty \right\}.$$

It is easy to check $L^{p,p}(\mathbb{R}^{d+1}) = L^p(\mathbb{R}^{d+1})$ and $l^{p,p}(\mathbb{Z}^{d+1}) = l^p(\mathbb{Z}^{d+1})$, respectively.

Signal sampling and its reconstruction theories are ubiquitous tools in a wide range of applications. The most well-known result is Shannon sampling theorem which gives an explicit reconstruction formula and states that every band-limited function can be reconstructed from its uniform samples. However, due to the slow decay and infinite support of the *sinc* function, it is often less efficient for numerical implementation. Moreover, there are many data just can be observed on non-uniform sampling set, such as in communication theory, medical imaging, astronomical measurement and among many others [1–4, 6, 21, 25]. In general, sampling problems have been studied in the following shift-invariant space [3–6, 11, 20]

$$V_p(\Phi) := \left\{ \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}^d} c_i(k)\phi_i(\cdot - k),\ c_i = \left\{c_i(k)\right\} \in l^p(\mathbb{Z}^d) \right\},$$

where $1 \le p < \infty$. The vector function $\Phi := (\phi_1, \cdots, \phi_r)^T$ is usually called the generator of the space $V_p(\Phi)$. If $r = d = 1$, $p = 2$ and $\phi(\cdot) = sinc(\cdot)$, then $V_2(\phi)$ reduces to the classical space of band-limited functions.

Although the samples are usually supposed to be the exact values of a signal f in classical sampling theory, in fact, only the local average values can be derived. More precisely, the samples of f can be taken near the points which belong to a countable index set. In the last decades, the average sampling theory has drawn considerable attentions including in band-limited signals [8, 13], shift-invariant signals [1, 3–6, 15, 22], non-decaying signals [18] and multi-channel sampling problems [9, 12].

In addition, the multiple sampling functionals can be traced back to the multi-channel sampling problem [19]. Unser and Zerubia [24] showed that the multi-channel sampling can achieve higher stability and be more suitable for analyzing large bandwidth signals. Recently, Zhang [27] studied the non-uniform average sampling problem in multiply generated shift-invariant subspaces of mixed Lebesgue spaces and provided two fast reconstruction algorithms for two types of average sampled values. Wang and Zhang [26] considered the average sampling problem for signals in shift-invariant subspaces of weighted mixed Lebesgue spaces. More precisely, the sampling stability and iterative reconstruction algorithms are established for two kinds of average sampling functionals. However, the work in Refs. [26, 27] only discussed the average sampling problems by using single sampling functional. Motivated by above literature, this paper investigates the stabilities and reconstruction algorithms of average sampling based on multiple sampling functionals over mixed Lebesgue spaces.

The rest of the paper is organized as follows. In Sect. 2, the definitions and preliminaries are introduced briefly. In order to recover the signal exactly from the average sampling, Sect. 3 provides the sampling stabilities for two kinds of average sampling functionals, respectively. In Sect. 4, the iterative algorithms for the reconstruction are presented. Finally, since the samples are usually contaminated by random noises, the error analysis is discussed in Sect. 5.

2 Definitions and Preliminaries

This section collects some definitions, notations and preliminary results for future convenience. We begin with the mixed Wiener amalgam spaces $W(L^{p,q})(\mathbb{R}^{d+1})$.

Definition 2.1 [16] Let $1 \le p, q < \infty$, then a measurable function f belongs to $W(L^{p,q})(\mathbb{R}^{d+1})$ if it satisfies

$$\|f\|^p_{W(L^{p,q})} := \sum_{n\in\mathbb{Z}} \operatorname*{ess\,sup}_{x\in[0,1]} \left[\sum_{l\in\mathbb{Z}^d} \operatorname*{ess\,sup}_{y\in[0,1]^d} |f(x+n, y+l)|^q \right]^{\frac{p}{q}} < \infty.$$

Moreover, $W_0(L^{p,q})(\mathbb{R}^{d+1})$ denotes the space of all continuous functions in $W(L^{p,q})(\mathbb{R}^{d+1})$.

For the simpler case, a function f belongs to $W(L^p)(\mathbb{R}^{d+1})$ $(1 \le p < \infty)$ if

$$\|f\|^p_{W(L^p)} := \sum_{k\in\mathbb{Z}^{d+1}} \operatorname*{ess\,sup}_{x\in[0,1]^{d+1}} |f(x+k)|^p < \infty$$

holds. Furthermore, it is easy to check that $W(L^p) \subset W(L^{p,p})$ and $W(L^p) \subset W(L^q) \subset L^q$ $(1 \le p \le q < \infty)$ (the details please see Ref. [3]).

With $\Phi := (\phi_1, \cdots, \phi_r)^T \in W(L^{1,1})^{(r)} := \underbrace{W(L^{1,1}) \times \cdots \times W(L^{1,1})}_{r \text{ times}}$, the underlying shift-invariant space is given by

$$V_{p,q}(\Phi) := \left\{ \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}^d} C(k_1, k_2)^T \Phi(\cdot - k_1, \cdot - k_2), \quad \{C(k_1, k_2)\} \in (l^{p,q})^{(r)}(\mathbb{Z}^{d+1}) \right\},$$

where $C(k_1,k_2) := (c_1(k_1,k_2),\cdots,c_r(k_1,k_2))^T$. On the other hand, the corresponding norm $\|\Phi\|$ of a vector function $\Phi := (\phi_1,\cdots,\phi_r)^T$ stands for $\|\Phi\| := \sum_{i=1}^r \|\phi_i\|$.

Moreover, we assume that for any $\xi \in \mathbb{R}^{d+1}$ and all $k \in \mathbb{Z}^{d+1}$, the sequences

$$\left\{\widehat{\phi}_1(\xi+2\pi k), \widehat{\phi}_2(\xi+2\pi k), \cdots, \widehat{\phi}_r(\xi+2\pi k)\right\}$$

are linearly independent in this paper, where $\widehat{\phi_i}$ stands for the Fourier transform of ϕ_i. For instance, let ϕ_i $(i=1,\cdots,r)$ be the $d+1$-dimensional tensor product orthonormal wavelets. Then for fixed $\xi \in \mathbb{R}^{d+1}$ and $k \in \mathbb{Z}^{d+1}$, $\{\widehat{\phi}_1(\xi+2\pi k),\cdots,\widehat{\phi}_r(\xi+2\pi k)\}$ are linearly independent thanks to Theorem 1.4 in Ref. [17]. It is well-known that there exists the dual functions $\tilde{\phi}_1,\cdots,\tilde{\phi}_r \in W(L^{1,1})(\mathbb{R}^{d+1})$ such that for any $f \in V_{p,q}(\Phi)$ $(1 < p,q < \infty)$,

$$f(x,y) = \sum_{i=1}^{r}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^d} \langle f,\ \tilde{\phi}_i(\cdot-k_1,\cdot-k_2)\rangle \phi_i(x-k_1,y-k_2).$$

All these claims can be found in Ref. [17].

The following lemma is a natural extension of Theorem 3.1 in Ref. [16].

Lemma 2.1 *Let* $\Phi \in W(L^{1,1})^{(r)}(\mathbb{R}^{d+1})$ *and* $\{C(k_1,k_2)\} \in (l^{p,q})^{(r)}(\mathbb{Z}^{d+1})$ $(1 \le p,q < \infty)$. *Then for any* $f = \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^d} C(k_1,k_2)^T\Phi(\cdot-k_1,\cdot-k_2)$,

$$\|f\|_{L^{p,q}} \le \sum_{i=1}^{r} \|c_i\|_{l^{p,q}}\|\phi_i\|_{W(L^{1,1})}.$$

To prove the sampling stability, we need the definition of relatively separated set.

Definition 2.2 [14] A sampling set $\Gamma = \{\gamma_{j,k} = (x_j, y_k),\ \ (j,k) \in \mathbb{J} := \mathbb{J}_1 \times \mathbb{J}_2\}$ is called to be (δ_1,δ_2)-relatively separated, if

$$B_{\Gamma,x}(\delta_1) = \sup_{x\in\mathbb{R}} \sum_{j\in\mathbb{J}_1} \chi_{B(x_j,\delta_1)}(x) < \infty,$$
$$B_{\Gamma,y}(\delta_2) = \sup_{y\in\mathbb{R}^d} \sum_{k\in\mathbb{J}_2} \chi_{B(y_k,\delta_2)}(y) < \infty$$

and

$$A_{\Gamma,x}(\delta_1) = \inf_{x\in\mathbb{R}} \sum_{j\in\mathbb{J}_1} \chi_{B(x_j,\delta_1)}(x) \ge 1,$$
$$A_{\Gamma,y}(\delta_2) = \inf_{y\in\mathbb{R}^d} \sum_{k\in\mathbb{J}_2} \chi_{B(y_k,\delta_2)}(y) \ge 1$$

for some $\delta_1 > 0$ and $\delta_2 > 0$. Here, $\mathbb{J} = \mathbb{J}_1 \times \mathbb{J}_2$ is a countable index set, $\chi_{B(x,\delta)}(\cdot)$ denotes the characteristic function on a ball with the center point x and the radius δ.

Given a relatively separated sampling set Γ, two kinds of average sampling schemes based on Γ are considered in the present paper.

- The first average sampling scheme is given by

$$\langle f,\ \psi_a^l(\cdot-\gamma_{j,k})\rangle = f*\widetilde{\psi_a^l}(\gamma_{j,k}),\quad (j,k)\in\mathbb{J},$$

where $\psi^l\in L^1(\mathbb{R}^{d+1})$ $(l=1,\cdots,s)$ satisfy that for $a>0$,

$$\sum_{l=1}^{s}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\psi^l(x,y)\mathrm{d}y\mathrm{d}x=1,\quad \psi_a^l(\cdot):=\frac{1}{a^{d+1}}\psi^l\Big(\frac{\cdot}{a}\Big),\quad \widetilde{\psi_a^l}(\cdot)=\overline{\psi_a^l(-\cdot)}.$$

- The second average sampling scheme is defined by

$$\langle f,\ \psi_{j,k}^l\rangle=\int_{\mathbb{R}}\int_{\mathbb{R}^d}f(x,y)\psi_{j,k}^l(x,y)\mathrm{d}y\mathrm{d}x,\quad (j,k)\in\mathbb{J},$$

where the average sampling functionals $\{\psi_{j,k}^l,\ (j,k)\in\mathbb{J}\}$ $(l=1,\cdots,s)$ satisfy that

(i). supp $\psi_{j,k}^l\subset B(\gamma_{j,k},\widetilde{a})$ for some $\widetilde{a}>0$;

(ii). There exists a constant $M>0$ such that $\int_{\mathbb{R}}\int_{\mathbb{R}^d}|\psi_{j,k}^l(x,y)|\mathrm{d}y\mathrm{d}x\le M$ for all $(j,k)\in\mathbb{J}$;

(iii). $\sum_{l=1}^{s}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\psi_{j,k}^l(x,y)\mathrm{d}y\mathrm{d}x=1$ for all $(j,k)\in\mathbb{J}$.

3 Sampling Stability

Before illustrating the main results, the oscillation (or modulus of continuity) of a continuous function $f\in L^1(\mathbb{R}^{d+1})$ is given by

$$osc_\delta(f)(x):=\sup_{|y|\le\delta}|f(x+y)-f(x)|.$$

Lemma 3.1 [16] *If $\phi\in W_0(L^1)(\mathbb{R}^{d+1})$, then the following two statements hold:*

$$\|osc_\delta(\phi)\|_{W(L^1)}\le 2^{2d+3}\|\phi\|_{W(L^1)},\tag{3.1}$$

and

$$\lim_{\delta\to 0}\|osc_\delta(\phi)\|_{W(L^1)}=0.\tag{3.2}$$

Denote

$$\Phi^a:=\Phi-\Phi*\Big(\sum_{l=1}^{s}\widetilde{\psi_a^l}\Big)=\Big(\phi_1-\phi_1*\Big(\sum_{l=1}^{s}\widetilde{\psi_a^l}\Big),\cdots,\phi_r-\phi_r*\Big(\sum_{l=1}^{s}\widetilde{\psi_a^l}\Big)\Big)^T.$$

Then the next two lemmas are introduced.

Lemma 3.2 *Let* $\psi^l \in L^1(\mathbb{R}^{d+1})$ $(l = 1, \cdots, s)$ *satisfy* $\sum_{l=1}^s \int_{\mathbb{R}^{d+1}} \psi^l(t)\mathrm{d}t = 1$. *Then for any* $\Phi = (\phi_1, \cdots, \phi_r)^T \in W_0(L^1)^{(r)}(\mathbb{R}^{d+1})$,

$$\lim_{a\to 0^+} \|\Phi^a\|_{W(L^1)} = 0.$$

Proof According to the definition of Φ^a and the fact

$$\|\Phi^a\|_{W(L^1)} = \Big\|\Phi - \Phi * \Big(\sum_{l=1}^s \widetilde{\psi_a^l}\Big)\Big\|_{W(L^1)} = \sum_{i=1}^r \Big\|\phi_i - \phi_i * \Big(\sum_{l=1}^s \widetilde{\psi_a^l}\Big)\Big\|_{W(L^1)},$$

one only needs to show $\|\phi_i - \phi_i * (\sum_{l=1}^s \widetilde{\psi_a^l})\|_{W(L^1)} \to 0$ as $a \to 0^+$ for each $i = 1, \cdots, r$.

It follows from $\sum_{l=1}^s \int_{\mathbb{R}^{d+1}} \psi^l(t)\mathrm{d}t = 1$ that

$$\begin{aligned}\left|\phi_i(x) - \phi_i * \Big(\sum_{l=1}^s \widetilde{\psi_a^l}\Big)(x)\right| &= \left|\sum_{l=1}^s \int_{\mathbb{R}^{d+1}} \phi_i(x)\overline{\psi_a^l(t)}\mathrm{d}t - \sum_{l=1}^s \int_{\mathbb{R}^{d+1}} \phi_i(x+t)\overline{\psi_a^l(t)}\mathrm{d}t\right| \\ &\le \sum_{l=1}^s \int_{\mathbb{R}^{d+1}} |\phi_i(x) - \phi_i(x+t)||\overline{\psi_a^l(t)}|\mathrm{d}t \\ &= \sum_{l=1}^s \Big(\int_{|t|\ge\sqrt{a}} + \int_{|t|<\sqrt{a}}\Big)|\phi_i(x) - \phi_i(x+t)||\overline{\psi_a^l(t)}|\mathrm{d}t \\ &:= I_1(x) + I_2(x).\end{aligned} \tag{3.3}$$

For $I_1(x)$, it is clear to see that

$$\begin{aligned}\|I_1\|_{W(L^1)} &= \sum_{k\in\mathbb{Z}^{d+1}} \operatorname*{ess\,sup}_{x\in[0,1]^{d+1}} \sum_{l=1}^s \int_{|t|\ge\sqrt{a}} |\phi_i(x+k) - \phi_i(x+t+k)||\overline{\psi_a^l(t)}|\mathrm{d}t \\ &\le \sum_{k\in\mathbb{Z}^{d+1}} \sum_{l=1}^s \int_{|t|\ge\sqrt{a}} \left(\operatorname*{ess\,sup}_{x\in[0,1]^{d+1}} |\phi_i(x+k)| + \operatorname*{ess\,sup}_{x+t\in[0,1]^{d+1}} |\phi_i(x+t+k)|\right) |\overline{\psi_a^l(t)}|\mathrm{d}t.\end{aligned}$$

Obviously, the above inequality reduces to

$$\|I_1\|_{W(L^1)} \le 2\|\phi_i\|_{W(L^1)} \sum_{l=1}^s \int_{|t|\ge\frac{1}{\sqrt{a}}} |\overline{\psi^l(t)}|\mathrm{d}t.$$

Therefore, $\|I_1\|_{W(L^1)} \longrightarrow 0$ as $a \to 0^+$ follows from $\psi^l \in L^1(\mathbb{R}^{d+1})$.

For $I_2(x)$, by the definition of oscillation,

$$\|I_2\|_{W(L^1)} \le \sum_{l=1}^s \int_{|t|\le\sqrt{a}} \sum_{k\in\mathbb{Z}^{d+1}} \operatorname*{ess\,sup}_{x\in[0,1]^{d+1}} |osc_{\sqrt{a}}(\phi_i)(x+k)||\overline{\psi_a^l(t)}|\mathrm{d}t$$

$$= \|osc_{\sqrt{a}}(\phi_i)\|_{W(L^1)} \sum_{l=1}^{s} \int_{|t|\le\sqrt{a}} |\overline{\psi_a^l(t)}| \mathrm{d}t.$$

This with $\phi_i \in W_0(L^1)$ and (3.2) shows $\|osc_{\sqrt{a}}(\phi_i)\|_{W(L^1)} \longrightarrow 0$ as $a \to 0^+$. Then $\|I_2\|_{W(L^1)} \longrightarrow 0$.

Hence, the above arguments tell that for each $i = 1, \cdots, r$,

$$\left\| \phi_i - \phi_i * \left(\sum_{l=1}^{s} \widetilde{\psi_a^l} \right) \right\|_{W(L^1)} \le \|I_1\|_{W(L^1)} + \|I_2\|_{W(L^1)} \longrightarrow 0, \quad \text{as } a \to 0^+$$

thanks to (3.3). The proof is done. □

Lemma 3.3 *Under the assumptions of Lemma 3.2, the oscillation of* Φ^a *satisfies*

$$\lim_{\delta\to 0} \|osc_\delta(\Phi^a)\|_{W(L^1)} = 0,$$

where $osc_\delta(\Phi^a) = (osc_\delta(\phi_1^a), \cdots, osc_\delta(\phi_r^a))^T$ *and* $a \in \mathbb{R}$.

Proof Similar to the discussion of Lemma 3.2, one should prove that for each $i = 1, \cdots, r$,

$$\lim_{\delta\to 0} \|osc_\delta(\phi_i^a)\|_{W(L^1)} = 0.$$

Recall that $\phi_i^a := \phi_i - \phi_i * \left(\sum_{l=1}^{s} \widetilde{\psi_a^l} \right)$. Then

$$osc_\delta(\phi_i^a)(x) \le osc_\delta(\phi_i)(x) + \int_{\mathbb{R}^{d+1}} osc_\delta(\phi_i)(x-z) \left| \sum_{l=1}^{s} \widetilde{\psi_a^l}(z) \right| \mathrm{d}z. \tag{3.4}$$

For the second term of the right-hand side of (3.4), it is easy to find that

$$\begin{aligned} &\left\| \int_{\mathbb{R}^{d+1}} osc_\delta(\phi_i)(\cdot - z) \left| \sum_{l=1}^{s} \widetilde{\psi_a^l}(z) \right| \mathrm{d}z \right\|_{W(L^1)} \\ &= \sum_{k\in\mathbb{Z}^{d+1}} \operatorname*{ess\,sup}_{x\in[0,1]^{d+1}} \int_{\mathbb{R}^{d+1}} osc_\delta(\phi_i)(x+k-z) \left| \sum_{l=1}^{s} \widetilde{\psi_a^l}(z) \right| \mathrm{d}z \\ &\le \|osc_\delta(\phi_i)\|_{W(L^1)} \sum_{l=1}^{s} \int_{\mathbb{R}^{d+1}} \left| \widetilde{\psi_a^l}(z) \right| \mathrm{d}z. \end{aligned}$$

Combining this with (3.2) and $\psi^l \in L^1(\mathbb{R}^{d+1})$, one obtains that for each $a \in \mathbb{R}$,

$$\lim_{\delta\to 0} \left\| \int_{\mathbb{R}^{d+1}} osc_\delta(\phi_i)(\cdot - z) \left| \sum_{l=1}^{s} \widetilde{\psi_a^l}(z) \right| \mathrm{d}z \right\|_{W(L^1)} = 0. \tag{3.5}$$

Furthermore, by (3.4)–(3.5),

$$\|osc_\delta(\phi_i^a)\|_{W(L^1)} \le \|osc_\delta(\phi_i)\|_{W(L^1)} + \left\| \int_{\mathbb{R}^{d+1}} osc_\delta(\phi_i)(\cdot - z) \Big| \sum_{l=1}^{s} \widetilde{\psi_a^l}(z) \Big| \mathrm{d}z \right\|_{W(L^1)} \longrightarrow 0$$

as $\delta \to 0$, which completes the proof. □

Lemma 3.4 is necessary for later discussions.

Lemma 3.4 *If $\Phi \in W(L^{1,1})^{(r)}(\mathbb{R}^{d+1})$, and $\{\widehat{\phi_i}(\xi + 2k\pi),\ k \in \mathbb{Z}^{d+1}\}$ are linearly independent, then for any $f \in V_{p,q}(\Phi)(\mathbb{R}^{d+1})$ $(1 < p, q < \infty)$,*

$$\sum_{i=1}^{r} \|c_i\|_{l^{p,q}} \le \|\widetilde{\Phi}\| \cdot \|f\|_{L^{p,q}},$$

where $\|\widetilde{\Phi}\|$ is a positive constant only depending on Φ.

Proof For each fixed $i = 1, \cdots, r$, there exists a dual function $\tilde{\phi}_i$ such that

$$c_i(k_1, k_2) = \langle f,\ \tilde{\phi}_i(\cdot - k_1, \cdot - k_2)\rangle \tag{3.6}$$

thanks to $f \in V_{p,q}(\Phi)$ (see Ref. [17]). Take $b = \{b(k_1, k_2),\ k_1 \in \mathbb{Z}, k_2 \in \mathbb{Z}^d\} \in l^{p',q'}(\mathbb{Z}^{d+1})$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then by (3.6), one knows that

$$\begin{aligned}\langle c_i,\ b\rangle &= \sum_{k_1 \in \mathbb{Z}, k_2 \in \mathbb{Z}^d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} f(x, y)\overline{\tilde{\phi}_i(x - k_1, y - k_2)}\mathrm{d}y\mathrm{d}x \cdot \overline{b(k_1, k_2)} \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} f(x, y) \sum_{k_1 \in \mathbb{Z}, k_2 \in \mathbb{Z}^d} \overline{b(k_1, k_2)\tilde{\phi}_i(x - k_1, y - k_2)}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

This with the Hölder inequality and Lemma 2.1 shows that

$$\begin{aligned}|\langle c_i,\ b\rangle| &\le \|f\|_{L^{p,q}} \left\| \sum_{k_1 \in \mathbb{Z}, k_2 \in \mathbb{Z}^d} b(k_1, k_2)\tilde{\phi}_i(\cdot - k_1, \cdot - k_2) \right\|_{L^{p',q'}} \\ &\le \|f\|_{L^{p,q}} \|b\|_{l^{p',q'}} \|\tilde{\phi}_i\|_{W(L^{1,1})}.\end{aligned}$$

Thus, $\|c_i\|_{l^{p,q}} \le \|f\|_{L^{p,q}} \|\tilde{\phi}_i\|_{W(L^{1,1})}$ which leads to

$$\sum_{i=1}^{r} \|c_i\|_{l^{p,q}} \le \|\widetilde{\Phi}\| \cdot \|f\|_{L^{p,q}}$$

with $\|\widetilde{\Phi}\| := \sum_{i=1}^{r} \|\tilde{\phi}_i\|_{W(L^{1,1})}$. The proof is finished. □

We are in a position to state the first stability theorem.

Theorem 3.1 *Suppose that $\Phi \in W_0(L^1)^{(r)}(\mathbb{R}^{d+1})$ and Γ is a (δ_1, δ_2)-relatively separated set. If δ_1, δ_2 and a are chosen such that*

$$r_1 := \|\widetilde{\Phi}\| \left(\left\| osc_{\sqrt{\delta_1^2+\delta_2^2}}(\Phi) \right\|_{W(L^1)} + \left\| osc_{\sqrt{\delta_1^2+\delta_2^2}}(\Phi^a) \right\|_{W(L^1)} + \|\Phi^a\|_{W(L^1)} \right) < 1, \tag{3.7}$$

then for any signal $f \in V_{p,q}(\Phi)$ $(1 < p, q < \infty)$,

$$\begin{aligned} \Big(\frac{2\delta_1}{A_{\Gamma,x}(\delta_1)}\Big)^{-\frac{1}{p}} \Big(\frac{V_d \delta_2^d}{A_{\Gamma,y}(\delta_2)}\Big)^{-\frac{1}{q}} (1-r_1)\|f\|_{L^{p,q}} &\le \left\| \left\{ \sum_{l=1}^{s} \langle f,\ \psi_a^l(\cdot - \gamma_{j,k})\rangle \right\}_{(j,k)\in\mathbb{J}} \right\|_{l^{p,q}} \\ &\le \Big(\frac{2\delta_1}{B_{\Gamma,x}(\delta_1)}\Big)^{-\frac{1}{p}} \Big(\frac{V_d \delta_2^d}{B_{\Gamma,y}(\delta_2)}\Big)^{-\frac{1}{q}} (1+r_1)\|f\|_{L^{p,q}}, \end{aligned} \tag{3.8}$$

where $V_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$ is the volume of d-dimensional unit ball, and $\|\widetilde{\Phi}\|$ is the positive constant given in Lemma 3.4.

Proof For any $x \in B(x_j, \delta_1)$ and $y \in B(y_k, \delta_2)$, it follows from $f \in V_{p,q}(\Phi)$ that

$$\begin{aligned} &\Big| \sum_{l=1}^{s} \langle f,\ \psi_a^l(\cdot - \gamma_{j,k})\rangle - f(x,y) \Big| \\ &\quad \le \sum_{i=1}^{r} \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}^d} |c_i(k_1,k_2)| \Big| \phi_i(x-k_1, y-k_2) - \phi_i * \Big(\sum_{l=1}^{s} \widetilde{\psi_a^l}\Big)(x_j - k_1, y_k - k_2) \Big|. \end{aligned}$$

Moreover, by $\phi_i^a := \phi_i - \phi_i * \Big(\sum_{l=1}^{s} \widetilde{\psi_a^l}\Big)$, the above inequality reduces to

$$\begin{aligned} &\Big| \sum_{l=1}^{s} \langle f,\ \psi_a^l(\cdot - \gamma_{j,k})\rangle - f(x,y) \Big| \\ &\quad \le \sum_{i=1}^{r} \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}^d} |c_i(k_1,k_2)| \Big(osc_{\sqrt{\delta_1^2+\delta_2^2}}(\phi_i)(x-k_1, y-k_2) \\ &\qquad + osc_{\sqrt{\delta_1^2+\delta_2^2}}(\phi_i^a)(x-k_1, y-k_2) + (\phi_i^a)(x-k_1, y-k_2) \Big) \\ &\quad := F_1(x,y). \end{aligned} \tag{3.9}$$

Furthermore, due to Lemma 2.1 and $W(L^1) \subset W(L^{1,1})$,

$$\begin{aligned} \|F_1\|_{L^{p,q}} &\le \sum_{i=1}^{r} \|c_i\|_{l^{p,q}} \Big(\left\| osc_{\sqrt{\delta_1^2+\delta_2^2}}(\phi_i) \right\|_{W(L^1)} + \left\| osc_{\sqrt{\delta_1^2+\delta_2^2}}(\phi_i^a) \right\|_{W(L^1)} + \|\phi_i^a\|_{W(L^1)} \Big) \\ &\le \|\widetilde{\Phi}\| \Big(\left\| osc_{\sqrt{\delta_1^2+\delta_2^2}}(\Phi) \right\|_{W(L^1)} + \left\| osc_{\sqrt{\delta_1^2+\delta_2^2}}(\Phi^a) \right\|_{W(L^1)} + \|\Phi^a\|_{W(L^1)} \Big) \|f\|_{L^{p,q}} \\ &= r_1 \|f\|_{L^{p,q}} \end{aligned} \tag{3.10}$$

thanks to Lemma 3.4 and (3.7).

Define

$$\alpha_j(x) := \frac{\chi_{B(x_j,\delta_1)}(x)}{\sum_{j'\in\mathbb{J}_1}\chi_{B(x_{j'},\delta_1)}(x)}, \quad \beta_k(y) := \frac{\chi_{B(y_k,\delta_2)}(y)}{\sum_{k'\in\mathbb{J}_2}\chi_{B(y_{k'},\delta_2)}(y)}, \quad (j,k)\in\mathbb{J}.$$

Then with Γ being (δ_1,δ_2)-relatively separated, it is clear to see that

$$2\delta_1 B^{-1}_{\Gamma,x}(\delta_1) \le \|\alpha_j\|_{L^1} \le 2\delta_1 A^{-1}_{\Gamma,x}(\delta_1), \quad \text{for each } j\in\mathbb{J}_1 \tag{3.11}$$

and

$$V_d\delta_2^d B^{-1}_{\Gamma,y}(\delta_2) \le \|\beta_k\|_{L^1} \le V_d\delta_2^d A^{-1}_{\Gamma,y}(\delta_2), \quad \text{for each } k\in\mathbb{J}_2. \tag{3.12}$$

On the other hand, it follows from (3.9) that

$$\Big|\sum_{l=1}^{s}\langle f,\ \psi_a^l(\cdot-\gamma_{j,k})\rangle\Big|\alpha_j^{\frac{1}{p}}(x)\beta_k^{\frac{1}{q}}(y) \le |f(x,y)|\alpha_j^{\frac{1}{p}}(x)\beta_k^{\frac{1}{q}}(y)+|F_1(x,y)|\alpha_j^{\frac{1}{p}}(x)\beta_k^{\frac{1}{q}}(y). \tag{3.13}$$

Taking l^q-norm and L^q-norm about the variable $k\in\mathbb{J}_2$, $y\in\mathbb{R}^d$, respectively, on the both sides of (3.13), one knows that

$$\left\|\left\{\sum_{l=1}^{s}\langle f,\ \psi_a^l(\cdot-\gamma_{j,k})\rangle\|\beta_k\|_{L^1}^{\frac{1}{q}}\right\}_{k\in\mathbb{J}_2}\right\|_{l^q}\alpha_j^{\frac{1}{p}}(x) \le \alpha_j^{\frac{1}{p}}(x)\|f(x,\cdot)\|_{L^q}+\alpha_j^{\frac{1}{p}}(x)\|F_1(x,\cdot)\|_{L^q} \tag{3.14}$$

thanks to $\|\{\beta_k^{\frac{1}{q}}\}_{k\in\mathbb{J}_2}\|_{l^q}=1$. Moreover, taking l^p-norm and L^p-norm about the variables $j\in\mathbb{J}_1$, $x\in\mathbb{R}$, respectively, on the both sides of (3.14), one obtains that

$$\left\|\left\{\sum_{l=1}^{s}\langle f,\ \psi_a^l(\cdot-\gamma_{j,k})\rangle\|\beta_k\|_{L^1}^{\frac{1}{q}}\|\alpha_j\|_{L^1}^{\frac{1}{p}}\right\}_{(j,k)\in\mathbb{J}}\right\|_{l^{p,q}} \le \|f\|_{L^{p,q}}+\|F_1\|_{L^{p,q}} \le (1+r_1)\|f\|_{L^{p,q}} \tag{3.15}$$

because of $\|\{\alpha_j^{\frac{1}{p}}\}_{j\in\mathbb{J}_1}\|_{l^p}=1$ and (3.10).

Combining (3.15) with the left-hand sides of (3.11)–(3.12), the right-hand side of (3.8) is established.

In addition, due to (3.9),

$$|f(x,y)|\alpha_j^{\frac{1}{p}}(x)\beta_k^{\frac{1}{q}}(y) \le \Big|\sum_{l=1}^{s}\langle f,\ \psi_a^l(\cdot-\gamma_{j,k})\rangle\Big|\alpha_j^{\frac{1}{p}}(x)\beta_k^{\frac{1}{q}}(y)+|F_1(x,y)|\alpha_j^{\frac{1}{p}}(x)\beta_k^{\frac{1}{q}}(y).$$

Moreover, similar to the discussions of (3.13)–(3.15), one knows

$$\begin{aligned}\left\|\left\{\sum_{l=1}^{s}\langle f,\ \psi_a^l(\cdot-\gamma_{j,k})\rangle\|\beta_k\|_{L^1}^{\frac{1}{q}}\|\alpha_j\|_{L^1}^{\frac{1}{p}}\right\}_{(j,k)\in\mathbb{J}}\right\|_{l^{p,q}} &\ge \|f\|_{L^{p,q}}-\|F_1\|_{L^{p,q}}\\ &\ge (1-r_1)\|f\|_{L^{p,q}}.\end{aligned} \tag{3.16}$$

Clearly, the assumption of $r_1 < 1$ is necessary in (3.16). In fact, $r_1 < 1$ could follow from Lemmas 3.2–3.3. Therefore, the inequality (3.16) with (3.11)–(3.12) implies that the left-hand side of (3.8) holds, which concludes the desired conclusion. □

The next lemma is useful to prove Theorem 3.2.

Lemma 3.5 *If $\Phi \in W_0(L^1)^{(r)}(\mathbb{R}^{d+1})$, then for any $f \in V_{p,q}(\Phi)$ $(1 \leq p, q < \infty)$,*

$$\|osc_\delta(f)\|_{L^{p,q}} \leq \sum_{i=1}^{r} \|c_i\|_{l^{p,q}} \|osc_\delta(\phi_i)\|_{W(L^1)}.$$

Proof According to the definition of oscillation and $f \in V_{p,q}(\Phi)$,

$$\begin{aligned} osc_\delta(f)(x) &\leq \sup_{|y|\leq\delta} \sum_{i=1}^{r} \sum_{k\in\mathbb{Z}^{d+1}} |c_i(k)||\phi_i(x+y-k) - \phi_i(x-k)| \\ &\leq \sum_{i=1}^{r} \sum_{k\in\mathbb{Z}^{d+1}} |c_i(k)| osc_\delta(\phi_i)(x-k). \end{aligned}$$

Moreover, by Lemma 2.1,

$$\begin{aligned} \|osc_\delta(f)\|_{L^{p,q}} &\leq \left\| \sum_{i=1}^{r} \sum_{k\in\mathbb{Z}^{d+1}} |c_i(k)| osc_\delta(\phi_i)(\cdot - k) \right\|_{L^{p,q}} \\ &\leq \sum_{i=1}^{r} \|c_i\|_{l^{p,q}} \|osc_\delta(\phi_i)\|_{W(L^1)} \end{aligned}$$

thanks to $W(L^1) \subset W(L^{1,1})$, which is the desired conclusion. □

The sampling stability for the second kind of average sampling functional is explored by Theorem 3.2.

Theorem 3.2 *Suppose that $\Phi \in W_0(L^1)^{(r)}(\mathbb{R}^{d+1})$ and Γ is a (δ_3, δ_4)-relatively separated set. If δ_3, δ_4 and $\widetilde{a}$ are chosen such that*

$$r_2 := sM\|\widetilde{\Phi}\| \left\| osc_{\widetilde{a}+\sqrt{\delta_3^2+\delta_4^2}}(\Phi) \right\|_{W(L^1)} < 1, \tag{3.17}$$

then for any signal $f \in V_{p,q}(\Phi)$ $(1 < p, q < \infty)$,

$$\begin{aligned} &\left(\frac{2\delta_3}{A_{\Gamma,x}(\delta_3)}\right)^{-\frac{1}{p}} \left(\frac{V_d \delta_4^d}{A_{\Gamma,y}(\delta_4)}\right)^{-\frac{1}{q}} (1-r_2)\|f\|_{L^{p,q}} \leq \left\| \left\{ \sum_{l=1}^{s} \langle f, \psi_{j,k}^l \rangle \right\}_{(j,k)\in\mathbb{J}} \right\|_{l^{p,q}} \\ &\leq \left(\frac{2\delta_3}{B_{\Gamma,x}(\delta_3)}\right)^{-\frac{1}{p}} \left(\frac{V_d \delta_4^d}{B_{\Gamma,y}(\delta_4)}\right)^{-\frac{1}{q}} (1+r_2)\|f\|_{L^{p,q}}, \end{aligned} \tag{3.18}$$

where $V_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$ is the volume of d-dimensional unit ball, and $\|\widetilde{\Phi}\|$ is the positive constant given in Lemma 3.4.

Proof For any $f \in V_{p,q}(\Phi)$, it follows from supp $\psi^l_{j,k} \subset B(\gamma_{j,k}, \widetilde{a})$ and $\sum_{l=1}^s \int_{\mathbb{R}} \int_{\mathbb{R}^d} \psi^l_{j,k}(x, y) \mathrm{d}y\mathrm{d}x = 1$ that

$$\left|\sum_{l=1}^{s} \langle f,\ \psi^l_{j,k}\rangle - f(x,y)\right| \le \sum_{l=1}^{s} \iint_{B(\gamma_{j,k},\widetilde{a})} |f(u,v) - f(x,y)||\psi^l_{j,k}(u,v)|dvdu$$
$$\le sMosc_{\widetilde{a}+\sqrt{\delta_3^2+\delta_4^2}}(f)(x,y) := F_2(x,y) \tag{3.19}$$

due to $\int_{\mathbb{R}} \int_{\mathbb{R}^d} |\psi^l_{j,k}(x, y)| \mathrm{d}y\mathrm{d}x \le M$. This with Lemma 3.5 and Lemma 3.4 shows

$$\|F_2\|_{L^{p,q}} = sM \left\|osc_{\widetilde{a}+\sqrt{\delta_3^2+\delta_4^2}}(f)\right\|_{L^{p,q}} \le sM \sum_{i=1}^{r} \|c_i\|_{l^{p,q}} \left\|osc_{\widetilde{a}+\sqrt{\delta_3^2+\delta_4^2}}(\phi_i)\right\|_{W(L^1)}$$
$$\le sM\|\widetilde{\Phi}\|\|f\|_{L^{p,q}} \left\|osc_{\widetilde{a}+\sqrt{\delta_3^2+\delta_4^2}}(\Phi)\right\|_{W(L^1)} = r_2\|f\|_{L^{p,q}} \tag{3.20}$$

thanks to (3.17).

Define

$$\alpha_j(x) := \frac{\chi_{B(x_j,\delta_3)}(x)}{\sum_{j'\in\mathbb{J}_1} \chi_{B(x_{j'},\delta_3)}(x)}, \quad \beta_k(y) := \frac{\chi_{B(y_k,\delta_4)}(y)}{\sum_{k'\in\mathbb{J}_2} \chi_{B(y_{k'},\delta_4)}(y)}, \quad (j,k) \in \mathbb{J}.$$

It follows from (3.19) that

$$\left|\sum_{l=1}^{s} \langle f,\ \psi^l_{j,k}\rangle\right| \alpha_j^{\frac{1}{p}}(x)\beta_k^{\frac{1}{q}}(y) \le |f(x,y)|\alpha_j^{\frac{1}{p}}(x)\beta_k^{\frac{1}{q}}(y) + |F_2(x,y)|\alpha_j^{\frac{1}{p}}(x)\beta_k^{\frac{1}{q}}(y).$$

The reminding proof is similar to the arguments in (3.14)–(3.15), one firstly takes the l^q-norm and L^q-norm for the variable k and y, respectively, then takes the l^p-norm and L^p-norm for the variable j and x, respectively. By (3.20),

$$\left\|\left\{\sum_{l=1}^{s} \langle f,\ \psi^l_{j,k}\rangle \|\beta_k\|_{L^1}^{\frac{1}{q}} \|\alpha_j\|_{L^1}^{\frac{1}{p}}\right\}_{(j,k)\in\mathbb{J}}\right\|_{l^{p,q}} \le \|f\|_{L^{p,q}} + \|F_2\|_{L^{p,q}} \le (1+r_2)\|f\|_{L^{p,q}}.$$

Therefore, the right-hand side of (3.18) can be concluded from the above inequality, the left-hand side of (3.18) can be obtained by the similar method as (3.16). The proof is done. □

4 Iterative Reconstruction Algorithms

In this section, the iterative reconstruction algorithms for recovering original signals are provided. Before introducing the main results, we demonstrate the following important concept.

Definition 4.1 [14] Let $\mathbb{J} := \mathbb{J}_1 \times \mathbb{J}_2$ be a countable index set, and Γ be a (δ_1, δ_2)-relatively separated set. Then $\{u_{j,k}(x, y),\ (j, k) \in \mathbb{J}\}$ is called a **BUPU** (Bounded Uniform Partition of Unity) associated with Γ, if

(i). $0 \le u_{j,k}(x, y) \le 1$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}^d$ and $(j, k) \in \mathbb{J}$;

(ii). supp $u_{j,k} \subset B(\gamma_{j,k}, \sqrt{\delta_1^2 + \delta_2^2})$ for each $(j, k) \in \mathbb{J}$;

(iii). $\sum_{j\in\mathbb{J}_1} \sum_{k\in\mathbb{J}_2} u_{j,k}(x, y) = 1$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}^d$.

Lemma 4.1 *Let $\Phi \in W_0(L^1)^{(r)}(\mathbb{R}^{d+1})$, Γ be a (δ'_5, δ'_6)-relatively separated set and $\{u_{j,k}(x, y),\ (j, k) \in \mathbb{J}\}$ be a* **BUPU** *associated with Γ. Then for any $f \in V_{p,q}(\Phi)$ $(1 \le p, q < \infty)$,*

$$\|Q_\Gamma f\|_{L^{p,q}} \le (1 + 2^{2d+3}) \sum_{i=1}^{r} \|c_i\|_{l^{p,q}} \|\phi_i\|_{W(L^1)},$$

where $Q_\Gamma f := \sum_{j\in\mathbb{J}_1} \sum_{k\in\mathbb{J}_2} f(x_j, y_k) u_{j,k}$.

Proof Clearly,

$$\|Q_\Gamma f\|_{L^{p,q}} \le \|f - Q_\Gamma f\|_{L^{p,q}} + \|f\|_{L^{p,q}}. \tag{4.1}$$

Then the main work of Lemma 4.1 is to estimate $\|f - Q_\Gamma f\|_{L^{p,q}}$. According to the definition of $Q_\Gamma f$, supp $u_{j,k} \subset B(\gamma_{j,k}, \sqrt{\delta_5'^2 + \delta_6'^2})$ and $\sum_{j\in\mathbb{J}_1} \sum_{k\in\mathbb{J}_2} u_{j,k}(x, y) = 1$,

$$\begin{aligned} |f(x, y) - (Q_\Gamma f)(x, y)| &= \left| \sum_{j\in\mathbb{J}_1} \sum_{k\in\mathbb{J}_2} f(x, y) u_{j,k}(x, y) - \sum_{j\in\mathbb{J}_1} \sum_{k\in\mathbb{J}_2} f(x_j, y_k) u_{j,k}(x, y) \right| \\ &\le \sum_{j\in\mathbb{J}_1} \sum_{k\in\mathbb{J}_2} osc_{\sqrt{\delta_5'^2+\delta_6'^2}}(f)(x, y) u_{j,k}(x, y) \\ &= osc_{\sqrt{\delta_5'^2+\delta_6'^2}}(f)(x, y). \end{aligned}$$

Then with Lemma 3.5, one concludes

$$\begin{aligned} \|f - Q_\Gamma f\|_{L^{p,q}} &\le \sum_{i=1}^{r} \|c_i\|_{l^{p,q}} \left\| osc_{\sqrt{\delta_5'^2+\delta_6'^2}}(\phi_i) \right\|_{W(L^1)} \\ &\le 2^{2d+3} \sum_{i=1}^{r} \|c_i\|_{l^{p,q}} \|\phi_i\|_{W(L^1)} \end{aligned} \tag{4.2}$$

thanks to (3.1).

On the other hand, Lemma 2.1 tells that $\|f\|_{L^{p,q}} \le \sum_{i=1}^{r} \|c_i\|_{l^{p,q}} \|\phi_i\|_{W(L^1)}$. Combining this with (4.1)–(4.2), the proof is completed. □

Let P be the projection operator from $L^{p,q}(\mathbb{R}^{d+1})$ onto $V_{p,q}(\Phi)$,

$$Pf(x, y) := \sum_{i=1}^{r} \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}^d} \langle f,\ \tilde{\phi}_i(\cdot - k_1, \cdot - k_2)\rangle \phi_i(x - k_1, y - k_2),$$

where $\tilde{\phi}_1, \cdots, \tilde{\phi}_r \in W(L^{1,1})$ are the dual functions of $\phi_1, \cdots, \phi_r \in W(L^{1,1})$, respectively. Then we provide the following lemma.

Lemma 4.2 *Define $A_{\Gamma,a} f := \sum_{l=1}^{s} \sum_{j\in\mathbb{J}_1} \sum_{k\in\mathbb{J}_2} (f * \widetilde{\psi_a^l})(\gamma_{j,k}) u_{j,k}$. Then there exist a (δ_5, δ_6)-relatively separated set Γ and $a_0 > 0$ such that the operator $I - PA_{\Gamma,a}$ is contractive on $V_{p,q}(\Phi)$ $(1 < p, q < \infty)$ for $a \le a_0$.*

Proof For any $f \in V_{p,q}(\Phi)$, one knows that

$$\begin{aligned} \|f - PA_{\Gamma,a} f\|_{L^{p,q}} &= \|f - PQ_\Gamma f + PQ_\Gamma f - PA_{\Gamma,a} f\|_{L^{p,q}} \\ &\le \|P\|_{\text{op}} \Big(\|f - Q_\Gamma f\|_{L^{p,q}} + \|Q_\Gamma f - A_{\Gamma,a} f\|_{L^{p,q}} \Big). \end{aligned} \tag{4.3}$$

By (4.2) and Lemma 3.4,

$$\begin{aligned} \|f - Q_\Gamma f\|_{L^{p,q}} &\le \sum_{i=1}^{r} \|c_i\|_{l^{p,q}} \left\| osc_{\sqrt{\delta_5'^2+\delta_6'^2}}(\phi_i) \right\|_{W(L^{1,1})} \\ &\le \|\widetilde{\Phi}\| \left\| osc_{\sqrt{\delta_5'^2+\delta_6'^2}}(\Phi) \right\|_{W(L^1)} \|f\|_{L^{p,q}}. \end{aligned} \tag{4.4}$$

On the other hand, it follows from Lemmas 4.1 and 3.4 that

$$\begin{aligned} \|Q_\Gamma f - A_{\Gamma,a} f\|_{L^{p,q}} &= \left\| Q_\Gamma \left(f - f * \Big(\sum_{l=1}^{s} \widetilde{\psi_a^l} \Big) \right) \right\|_{L^{p,q}} \\ &\le (1 + 2^{2d+3}) \sum_{i=1}^{r} \|c_i\|_{l^{p,q}} \left\| \phi_i - \phi_i * \Big(\sum_{l=1}^{s} \widetilde{\psi_a^l} \Big) \right\|_{W(L^1)} \\ &\le (1 + 2^{2d+3}) \|\widetilde{\Phi}\| \left\| \Phi - \Phi * \Big(\sum_{l=1}^{s} \widetilde{\psi_a^l} \Big) \right\|_{W(L^1)} \|f\|_{L^{p,q}}. \end{aligned} \tag{4.5}$$

Combining (4.3) with (4.4) and (4.5), one obtains that

$$\|f - PA_{\Gamma,a} f\|_{L^{p,q}}$$

$$\leq \|\widetilde{\Phi}\|\|P\|_{\text{op}} \left(\left\| osc_{\sqrt{\delta_5'^2+\delta_6'^2}}(\Phi) \right\|_{W(L^1)} + (1+2^{2d+3})\|\Phi^a\|_{W(L^1)} \right) \|f\|_{L^{p,q}}. \tag{4.6}$$

By (3.2),

$$\left\| osc_{\sqrt{\delta_5'^2+\delta_6'^2}}(\Phi) \right\|_{W(L^1)} = \sum_{i=1}^{r} \left\| osc_{\sqrt{\delta_5'^2+\delta_6'^2}}(\phi_i) \right\|_{W(L^1)} \longrightarrow 0$$

when $\sqrt{\delta_5'^2 + \delta_6'^2} \to 0$. Moreover, Lemma 3.2 shows

$$\lim_{a \to 0^+} \|\Phi^a\|_{W(L^1)} = 0.$$

Therefore, there exist δ_5, δ_6 and $a_0 > 0$ such that

$$\alpha_1 := \|\widetilde{\Phi}\|\|P\|_{\text{op}} \left(\left\| osc_{\sqrt{\delta_5^2+\delta_6^2}}(\Phi) \right\|_{W(L^1)} + (1+2^{2d+3})\|\Phi^{a_0}\|_{W(L^1)} \right) < 1.$$

Hence, (4.6) reduces to

$$\|f - PA_{\Gamma,a} f\|_{L^{p,q}} \leq \alpha_1 \|f\|_{L^{p,q}} < \|f\|_{L^{p,q}},$$

which implies that $I - PA_{\Gamma,a}$ is a contractive operator. □

Theorem 4.1 *If $\Phi \in W_0(L^1)^{(r)}(\mathbb{R}^{d+1})$, then there exist a (δ_5, δ_6)-relatively separated set Γ and $a_0 > 0$ such that each signal $f \in V_{p,q}(\Phi)$ $(1 < p, q < \infty)$ can be recovered from $\left\{\langle f, \ \psi_a^l(\cdot - \gamma_{j,k})\rangle\right\}_{(j,k)\in\mathbb{J}}$ by the following iterative algorithm:*

$$\begin{cases} f_1 = PA_{\Gamma,a} f, \\ f_n = PA_{\Gamma,a}(f - f_{n-1}) + f_{n-1}, \quad n \geq 2. \end{cases} \tag{4.7}$$

Furthermore,

$$\|f - f_n\|_{L^{p,q}} \leq \alpha_1^n \|f\|_{L^{p,q}}$$

for some $\alpha_1(\delta_5, \delta_6, a_0, \Phi) < 1$.

Proof Let $e_n = f - f_n$. Then by (4.7),

$$e_n = f - f_n = f - f_{n-1} - PA_{\Gamma,a}(f - f_{n-1}) = (I - PA_{\Gamma,a})e_{n-1}.$$

Using Lemma 4.2 and choosing δ_5, δ_6 and $a_0 > 0$ such that $\alpha_1 = \|I - PA_{\Gamma,a}\|_{\text{op}} < 1$, then

$$\|e_n\|_{L^{p,q}} \leq \alpha_1 \|e_{n-1}\|_{L^{p,q}} \leq \alpha_1^2 \|e_{n-2}\|_{L^{p,q}} \leq \cdots \leq \alpha_1^n \|f\|_{L^{p,q}}.$$

Furthermore, $\|e_n\|_{L^{p,q}} \longrightarrow 0$ as $n \to \infty$, which completes the proof. □

Similar to Lemma 4.2, we establish the next lemma for the second kind of sampling functional.

Lemma 4.3 *Define* $A_\Gamma f := \sum_{l=1}^{s}\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}\langle f,\ \psi_{j,k}^l\rangle u_{j,k}$. *Then there exist a* (δ_7, δ_8)*-relatively separated set* Γ *and* $\widetilde{a}_0 > 0$ *such that the operator* $I - PA_\Gamma$ *is contractive on* $V_{p,q}(\Phi)$ $(1 < p, q < \infty)$.

Proof For any $f \in V_{p,q}(\Phi)$, one obtains

$$\begin{aligned}\|f - PA_\Gamma f\|_{L^{p,q}} &= \|f - PQ_\Gamma f + PQ_\Gamma f - PA_\Gamma f\|_{L^{p,q}}\\ &\le \|P\|_{\text{op}}\Big(\|f - Q_\Gamma f\|_{L^{p,q}} + \|Q_\Gamma f - A_\Gamma f\|_{L^{p,q}}\Big). \end{aligned} \tag{4.8}$$

According to the definitions of $Q_\Gamma f,\ A_\Gamma f$ and $\sum_{l=1}^{s}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\psi_{j,k}^l(x,y)\mathrm{d}y\mathrm{d}x = 1$,

$$\begin{aligned}&|Q_\Gamma f(x,y) - A_\Gamma f(x,y)|\\ &= \left|\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2} f(x_j, y_k)u_{j,k}(x,y) - \sum_{l=1}^{s}\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}\langle f,\ \psi_{j,k}^l\rangle u_{j,k}(x,y)\right|\\ &\le \sum_{l=1}^{s}\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}^d}|f(x_j,y_k) - f(v,t)||\psi_{j,k}^l(v,t)|\mathrm{d}t\mathrm{d}v\right)u_{j,k}(x,y).\end{aligned}$$

Moreover, by supp $\psi_{j,k}^l \subset B(\gamma_{j,k}, \widetilde{a})$ $(l = 1, \cdots, s)$, the above inequality yields

$$\begin{aligned}|Q_\Gamma f(x,y) - A_\Gamma f(x,y)| &\le \sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2} osc_{\widetilde{a}}(f)(x_j,y_k)u_{j,k}(x,y)\sum_{l=1}^{s}\int_{\mathbb{R}}\int_{\mathbb{R}^d}|\psi_{j,k}^l(v,t)|\mathrm{d}t\mathrm{d}v\\ &\le sMQ_\Gamma[osc_{\widetilde{a}}(f)](x,y)\end{aligned}$$

thanks to $\int_{\mathbb{R}}\int_{\mathbb{R}^d}|\psi_{j,k}^l(s,t)|\mathrm{d}t\mathrm{d}s \le M$ and the definition of Q_Γ. This with Lemma 4.1 and Lemma 3.4 shows that

$$\begin{aligned}\|Q_\Gamma f - A_\Gamma f\|_{L^{p,q}} &\le sM(1 + 2^{2d+3})\sum_{i=1}^{r}\|c_i\|_{l^{p,q}}\|osc_{\widetilde{a}}(\phi_i)\|_{W(L^1)}\\ &\le sM(1 + 2^{2d+3})\|\widetilde{\Phi}\|\|osc_{\widetilde{a}}(\Phi)\|_{W(L^1)}\|f\|_{L^{p,q}}.\end{aligned} \tag{4.9}$$

Combining (4.8) with (4.4) and (4.9), one concludes

$$\begin{aligned}\|f - PA_\Gamma f\|_{L^{p,q}} \le \|\widetilde{\Phi}\|\|P\|_{\text{op}}\Big(&\left\|osc_{\sqrt{\delta_7'^2+\delta_8'^2}}(\Phi)\right\|_{W(L^1)}\\ &+ sM(1 + 2^{2d+3})\|osc_{\widetilde{a}}(\Phi)\|_{W(L^1)}\Big)\|f\|_{L^{p,q}}.\end{aligned}$$

Furthermore, (3.2) implies that

$$\lim_{\sqrt{\delta_7'^2+\delta_8'^2}\to 0}\left\|osc_{\sqrt{\delta_7'^2+\delta_8'^2}}(\Phi)\right\|_{W(L^1)}=0 \quad \text{and} \quad \lim_{\widetilde{a}\to 0}\|osc_{\widetilde{a}}(\Phi)\|_{W(L^1)}=0.$$

Therefore, there exist δ_7, δ_8 and $\widetilde{a}_0 > 0$ such that

$$\alpha_2 := \|\widetilde{\Phi}\|\|P\|_{\text{op}}\left(\left\|osc_{\sqrt{\delta_7^2+\delta_8^2}}(\Phi)\right\|_{W(L^1)} + sM(1+2^{2d+3})\|osc_{\widetilde{a}_0}(\Phi)\|_{W(L^1)}\right) < 1.$$

Hence, it leads to the conclusion, i.e.,

$$\|f - PA_\Gamma f\|_{L^{p,q}} \le \alpha_2\|f\|_{L^{p,q}} < \|f\|_{L^{p,q}}.$$

The proof is finished. □

Similar to the proof of Theorem 4.1, we can derive the second iterative algorithm immediately.

Theorem 4.2 *If $\Phi \in W_0(L^1)(\mathbb{R}^{d+1})^{(r)}$, then there exist a (δ_7, δ_8)-relatively separated set Γ and $\widetilde{a}_0 > 0$ such that each signal $f \in V_{p,q}(\Phi)$ $(1 < p, q < \infty)$ can be recovered from $\{\langle f, \psi^l_{j,k}\rangle\}_{(j,k)\in\mathbb{J}}$ by the following iterative algorithm:*

$$\begin{cases} f_1 = PA_\Gamma f, \\ f_n = PA_\Gamma(f - f_{n-1}) + f_{n-1}, \quad n \ge 2. \end{cases} \tag{4.10}$$

Furthermore,

$$\|f - f_n\|_{L^{p,q}} \le \alpha_2^n\|f\|_{L^{p,q}}$$

for some $\alpha_2(\delta_7, \delta_8, \widetilde{a}_0, \Phi) < 1$.

5 Error Analysis

In many applications, the samples are often contaminated by random noises. Motivated by the work of Aldroubi et al. [5] and Jiang [14], we investigate the error analysis if the samples are destroyed by random noises in this section.

Firstly, we propose the following inequality under mixed norm.

Lemma 5.1 *If $\phi \in W(L^1)(\mathbb{R}^{d+1})$ and $f \in L^{p,q}(\mathbb{R}^{d+1})$ $(1 \le p, q < \infty)$, then*

$$\|f * \phi\|_{l^{p,q}} \le \|f\|_{L^{p,q}}\|\phi\|_{W(L^{1,1})}.$$

Proof Obviously, $\phi \in W(L^1)(\mathbb{R}^{d+1}) \subset W(L^p)(\mathbb{R}^{d+1}) \subset L^p(\mathbb{R}^{d+1})$ holds for each $p \geq 1$ and

$$\|f*\phi\|_{l^{p,q}} = \left[\sum_{k_1\in\mathbb{Z}}\Big(\sum_{k_2\in\mathbb{Z}^d}\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d} f(x,y)\phi(k_1-x,k_2-y)\mathrm{d}y\mathrm{d}x\Big|^q\Big)^{\frac{p}{q}}\right]^{\frac{1}{p}} < \infty.$$

For fixed x and k_1, denote $f_x(y) := f(x,y)$ and $\phi_{k_1-x}(k_2-y) := \phi(k_1-x,k_2-y)$. Then

$$\begin{aligned}\|f*\phi\|_{l^{p,q}} &= \left[\sum_{k_1\in\mathbb{Z}}\Big(\sum_{k_2\in\mathbb{Z}^d}\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d} f_x(y)\phi_{k_1-x}(k_2-y)\mathrm{d}y\mathrm{d}x\Big|^q\Big)^{\frac{p}{q}}\right]^{\frac{1}{p}}\\ &= \left[\sum_{k_1\in\mathbb{Z}}\Big(\sum_{k_2\in\mathbb{Z}^d}\Big|\int_{\mathbb{R}}\sum_{l\in\mathbb{Z}^d}\int_{[0,1]^d} f_x(y+l)\phi_{k_1-x}(k_2-y-l)\mathrm{d}y\mathrm{d}x\Big|^q\Big)^{\frac{p}{q}}\right]^{\frac{1}{p}}\\ &= \left[\sum_{k_1\in\mathbb{Z}}\Big\|\int_{\mathbb{R}}\int_{[0,1]^d}\sum_{l\in\mathbb{Z}^d} f_x(y+l)\phi_{k_1-x}(\cdot-y-l)\mathrm{d}y\mathrm{d}x\Big\|_{l^q}^p\right]^{\frac{1}{p}}.\end{aligned}$$

This with the generalized Minkowski inequality shows

$$\|f*\phi\|_{l^{p,q}} \leq \left[\sum_{k_1\in\mathbb{Z}}\Big(\int_{\mathbb{R}}\int_{[0,1]^d}\Big(\sum_{k_2\in\mathbb{Z}^d}\Big|\sum_{l\in\mathbb{Z}^d} f_x(y+l)\phi_{k_1-x}(k_2-y-l)\Big|^q\Big)^{\frac{1}{q}}\mathrm{d}y\mathrm{d}x\Big)^p\right]^{\frac{1}{p}}. \tag{5.1}$$

Let $f_{x,y}(\cdot) := f_x(\cdot+y)$ and $\phi_{k_1-x,y}(\cdot) := \phi_{k_1-x}(\cdot-y)$. Then it follows from Young's inequality that

$$\begin{aligned}&\int_{[0,1]^d}\Big(\sum_{k_2\in\mathbb{Z}^d}\Big|\sum_{l\in\mathbb{Z}^d} f_x(y+l)\phi_{k_1-x}(k_2-y-l)\Big|^q\Big)^{\frac{1}{q}}\mathrm{d}y\\ &= \int_{[0,1]^d}\|f_{x,y}*\phi_{k_1-x,y}\|_{l^q}\mathrm{d}y \leq \int_{[0,1]^d}\|f_{x,y}\|_{l^q}\|\phi_{k_1-x,y}\|_{l^1}\mathrm{d}y.\end{aligned} \tag{5.2}$$

Using the Hölder inequality with $\frac{1}{q}+\frac{1}{q'}=1$, (5.2) reduces to

$$\begin{aligned}&\int_{[0,1]^d}\Big(\sum_{k_2\in\mathbb{Z}^d}\Big|\sum_{l\in\mathbb{Z}^d} f_x(y+l)\phi_{k_1-x}(k_2-y-l)\Big|^q\Big)^{\frac{1}{q}}\mathrm{d}y\\ &\leq \Big(\int_{[0,1]^d}\|f_{x,y}\|_{l^q}^q\mathrm{d}y\Big)^{\frac{1}{q}}\Big(\int_{[0,1]^d}\|\phi_{k_1-x,y}\|_{l^1}^{q'}\mathrm{d}y\Big)^{\frac{1}{q'}}\\ &= \Big(\int_{[0,1]^d}\sum_{l\in\mathbb{Z}^d}|f_x(y+l)|^q\mathrm{d}y\Big)^{\frac{1}{q}}\Big(\int_{[0,1]^d}\Big[\sum_{l\in\mathbb{Z}^d}|\phi_{k_1-x}(l-y)|\Big]^{q'}\mathrm{d}y\Big)^{\frac{1}{q'}}.\end{aligned}$$

Because the function $\sum_{l\in\mathbb{Z}^d}|\phi_{k_1-x}(l-\cdot)|$ is **1**-periodic, the above inequality yields

$$\begin{aligned}&\int_{[0,1]^d}\Big(\sum_{k_2\in\mathbb{Z}^d}\Big|\sum_{l\in\mathbb{Z}^d}f_x(y+l)\phi_{k_1-x}(k_2-y-l)\Big|^q\Big)^{\frac{1}{q}}\mathrm{d}y\\&\leq\Big(\sum_{l\in\mathbb{Z}^d}\int_{[0,1]^d}|f_x(y+l)|^q\mathrm{d}y\Big)^{\frac{1}{q}}\Big(\sum_{l\in\mathbb{Z}^d}\operatorname*{ess\,sup}_{y\in[0,1]^d}|\phi_{k_1-x}(y+l)|\Big)\\&=\|f(x,\cdot)\|_{L^q}\|\phi(k_1-x,\cdot)\|_{W(L^1)}.\end{aligned}\tag{5.3}$$

Substituting (5.3) into (5.1), one obtains that

$$\begin{aligned}\|f*\phi\|_{l^{p,q}}&\leq\Bigg[\sum_{k_1\in\mathbb{Z}}\Big(\int_{\mathbb{R}}\|f(x,\cdot)\|_{L^q}\|\phi(k_1-x,\cdot)\|_{W(L^1)}\mathrm{d}x\Big)^p\Bigg]^{\frac{1}{p}}\\&=\Bigg[\sum_{k_1\in\mathbb{Z}}\Big(\sum_{n\in\mathbb{Z}}\int_{[0,1]}\|f(x+n,\cdot)\|_{L^q}\|\phi(k_1-x-n,\cdot)\|_{W(L^1)}\mathrm{d}x\Big)^p\Bigg]^{\frac{1}{p}}.\end{aligned}$$

Denote $a_x(n):=\|f(x+n,\cdot)\|_{L^q}$ and $b_x(n):=\|\phi(n-x,\cdot)\|_{W(L^1)}$. Then by the generalized Minkowski inequality again,

$$\|f*\phi\|_{l^{p,q}}\leq\Big\|\int_{[0,1]}\sum_{n\in\mathbb{Z}}a_x(n)b_x(\cdot-n)\mathrm{d}x\Big\|_{l^p}\leq\int_{[0,1]}\|a_x*b_x\|_{l^p}\mathrm{d}x.$$

Using Young's inequality and the Hölder inequality with $\frac{1}{p}+\frac{1}{p'}=1$, one gets

$$\begin{aligned}\|f*\phi\|_{l^{p,q}}&\leq\int_{[0,1]}\|a_x\|_{l^p}\|b_x\|_{l^1}\mathrm{d}x\leq\Big(\int_{[0,1]}\|a_x\|_{l^p}^p\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\int_{[0,1]}\|b_x\|_{l^1}^{p'}\mathrm{d}x\Big)^{\frac{1}{p'}}\\&\leq\Big(\int_{[0,1]}\sum_{n\in\mathbb{Z}}\|f(x+n,\cdot)\|_{L^q}^p\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\sum_{n\in\mathbb{Z}}\operatorname*{ess\,sup}_{x\in[0,1]}\|\phi(n-x,\cdot)\|_{W(L^1)}\Big)\\&=\|f\|_{L^{p,q}}\|\phi\|_{W(L^{1,1})},\end{aligned}$$

which is the desired conclusion. The proof is completed. □

Theorem 5.1 *Let $\Phi\in W_0(L^1)^{(r)}(\mathbb{R}^{d+1})$, $\Gamma=\{\gamma_{j,k},\ (j,k)\in\mathbb{J}\}$ be a relatively separated sampling set and $\{\varepsilon_{j,k}^l,\ (j,k)\in\mathbb{J},\ l=1,\cdots,s\}$ be random variables satisfying*

$$E\left(\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}|\varepsilon_{j,k}^l|\right)<N\ \text{for}\ 0<N<\infty\ \text{and}\ l=1,\cdots,s.$$

Then for any initial data $\{\langle f, \psi_a^l(\cdot-\gamma_{j,k})\rangle+\varepsilon_{j,k}^l\}_{(j,k)\in\mathbb{J}}$ *or* $\{\langle f, \psi_{j,k}^l\rangle+\varepsilon_{j,k}^l\}_{(j,k)\in\mathbb{J}}$,

$$E\|f_\infty - f\|_{L^{p,q}} \le \frac{sCN}{1-\alpha}\|\widetilde{\Phi}\|\cdot\|\Phi\|_{W(L^1)},$$

where $C = C(d, p, q, \delta_5, \delta_6)$, $\alpha = \alpha_1$ *for algorithm* (4.7)*; or* $C = C(d, p, q, \delta_7, \delta_8)$, $\alpha = \alpha_2$ *for algorithm* (4.10)*, and the positive constant* $\|\widetilde{\Phi}\|$ *is given in Lemma* 3.4.

Proof Denote $A := \begin{cases} A_{\Gamma,a}, & \textit{for algorithm } (4.7); \\ A_\Gamma, & \textit{for algorithm } (4.10). \end{cases}$ Then Lemmas 4.2 and 4.3 tell $\alpha = \|I - PA\|_{\text{op}} < 1$, which implies

$$\left(I + \sum_{k=1}^{\infty}(I - PA)^k\right)PA = I. \tag{5.4}$$

Let $h_0 := \sum_{l=1}^s \sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}\varepsilon_{j,k}^l P(u_{j,k})$, and the initial data be $\{\langle f, \psi_a^l(\cdot - \gamma_{j,k})\rangle+\varepsilon_{j,k}^l\}_{(j,k)\in\mathbb{J}}$ or $\{\langle f, \psi_{j,k}^l\rangle+\varepsilon_{j,k}^l\}_{(j,k)\in\mathbb{J}}$, respectively. Then the original signal f can be recovered by the following iterative algorithm

$$\begin{cases} \widetilde{f}_1 = PAf + h_0, \\ f_n = \widetilde{f}_1 - PAf_{n-1} + f_{n-1}, \quad n \ge 2, \end{cases}$$

based on Theorem 4.1 and Theorem 4.2, respectively. Moreover, due to (5.4), $f_n = \widetilde{f}_1 + (I-PA)f_{n-1} = \widetilde{f}_1 + (I-PA)\left(\widetilde{f}_1 + (I-PA)f_{n-2}\right) = \cdots = \left(I+\sum_{k=1}^{n-1}(I-PA)^k\right)\widetilde{f}_1$ and $f = \left(I+\sum_{k=1}^{\infty}(I-PA)^k\right)PAf = \left(I+\sum_{k=1}^{\infty}(I-PA)^k\right)(\widetilde{f}_1 - h_0)$, i.e.,

$$\begin{cases} f_n = \left(I + \sum_{k=1}^{n-1}(I-PA)^k\right)\widetilde{f}_1, \\ f = \left(I + \sum_{k=1}^{\infty}(I-PA)^k\right)(\widetilde{f}_1 - h_0). \end{cases} \tag{5.5}$$

According to $\alpha = \|I - PA\|_{\text{op}} < 1$, one obtains

$$\begin{aligned} \|f_\infty - f\|_{L^{p,q}} &= \left\|\left(I + \sum_{k=1}^{\infty}(I-PA)^k\right)h_0\right\|_{L^{p,q}} \le \left\|\sum_{k=0}^{\infty}(I-PA)^k\right\|_{\text{op}} \|h_0\|_{L^{p,q}} \\ &\le \frac{1}{1-\alpha}\|h_0\|_{L^{p,q}}. \end{aligned}$$

By the definition of h_0 and the Minkowski inequality, the above inequality reduces to

$$\|f_\infty - f\|_{L^{p,q}} \le \frac{1}{1-\alpha}\left\|\sum_{l=1}^{s}\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}\varepsilon_{j,k}^l P(u_{j,k})\right\|_{L^{p,q}}$$

$$\leq \frac{1}{1-\alpha}\left(\sum_{l=1}^{s}\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}|\varepsilon_{j,k}^{l}|\cdot\|P(u_{j,k})\|_{L^{p,q}}\right). \tag{5.6}$$

It follows from supp $u_{j,k} \subset B(\gamma_{j,k}, \sqrt{\delta_5^2+\delta_6^2})$, $x \in B(x_j, \sqrt{\delta_5^2+\delta_6^2})$ and $y \in B(y_k, \sqrt{\delta_5^2+\delta_6^2})$ that

$$\begin{aligned}\|u_{j,k}\|_{L^{p,q}}^{p} &\leq \int_{B(x_j,\sqrt{\delta_5^2+\delta_6^2})}\left(\int_{B(y_k,\sqrt{\delta_5^2+\delta_6^2})}|u_{j,k}(x,y)|^q \mathrm{d}y\right)^{\frac{p}{q}}\mathrm{d}x\\ &\leq \int_{B(x_j,\sqrt{\delta_5^2+\delta_6^2})}\left(V_d(\delta_5^2+\delta_6^2)^{\frac{d}{2}}\right)^{\frac{p}{q}}\mathrm{d}x = 2\sqrt{\delta_5^2+\delta_6^2}\cdot\left(V_d(\delta_5^2+\delta_6^2)^{\frac{d}{2}}\right)^{\frac{p}{q}}\end{aligned}$$

thanks to $|u_{j,k}(x,y)| \leq 1$ for all x, y. This implies that $u_{j,k}$ belongs to $L^{p,q}(\mathbb{R}^{d+1})$ based on the first kind of sampling functional. Similarly, one can derive $u_{j,k} \leq 2\sqrt{\delta_7^2+\delta_8^2}\cdot\left(V_d(\delta_7^2+\delta_8^2)^{\frac{d}{2}}\right)^{\frac{p}{q}}$, i.e., $u_{j,k}$ also belongs to $L^{p,q}(\mathbb{R}^{d+1})$ based on the second kind of sampling functional.

Furthermore, according to the definition of the operator P and Lemma 2.1, one finds that

$$\begin{aligned}\|P(u_{j,k})\|_{L^{p,q}} &= \left\|\sum_{i=1}^{r}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^d}\langle u_{j,k},\ \tilde{\phi}_i(\cdot-k_1,\cdot-k_2)\rangle\phi_i(x-k_1,y-k_2)\right\|_{L^{p,q}}\\ &\leq \sum_{i=1}^{r}\left\|\left\{\langle u_{j,k},\ \tilde{\phi}_i(\cdot-k_1,\cdot-k_2)\rangle\right\}_{k_1,k_2}\right\|_{l^{p,q}}\|\phi_i\|_{W(L^{1,1})}.\end{aligned}$$

This with Lemma 5.1 leads to

$$\begin{aligned}\|P(u_{j,k})\|_{L^{p,q}} &\leq \sum_{i=1}^{r}\|u_{j,k}\|_{L^{p,q}}\|\tilde{\phi}_i(-\cdot)\|_{W(L^{1,1})}\|\phi_i\|_{W(L^1)}\\ &\leq C\|\widetilde{\Phi}\|\cdot\|\Phi\|_{W(L^1)},\end{aligned} \tag{5.7}$$

where C is given by

$$C := \begin{cases} 2\sqrt{\delta_5^2+\delta_6^2}\cdot\left(V_d(\delta_5^2+\delta_6^2)^{\frac{d}{2}}\right)^{\frac{p}{q}} & \textit{for algorithm}(4.7);\\ 2\sqrt{\delta_7^2+\delta_8^2}\cdot\left(V_d(\delta_7^2+\delta_8^2)^{\frac{d}{2}}\right)^{\frac{p}{q}} & \textit{for algorithm}(4.10).\end{cases}$$

Combining (5.6) with (5.7) and $E\Big(\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}|\varepsilon^l_{j,k}|\Big) < N$ $(l = 1, \cdots, s)$, one concludes that

$$E\|f_\infty - f\|_{L^{p,q}} \leq \frac{sCN}{1-\alpha}\|\widetilde{\Phi}\| \cdot \|\Phi\|_{W(L^1)},$$

which completes the proof. □

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Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

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