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Stability of a General Functional Equation in *m*-Banach Spaces

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Abstract

In this note, the Ulam stability of a general functional equation in several variables is investigated. It is shown that this equation is Ulam stable in m-Banach spaces. Since a particular case of the considered equation is, among others, a functional equation introduced by Ji et al. and Zhao et al. for a characterization of the so-called multi-quadratic mapping, a result on its stability is also presented. Moreover, some other applications are provided.

Keywords Ulam stability · Functional equation · Multi-quadratic mapping · m-Banach space

Mathematics Subject Classification 39B82 · 39B52 · 41A65

1 Background and Motivation

1.1 Functional Equations

It is well known that one of the most important functional equations is the *Jordan–von Neumann equation*

$$q(x+y)+q(x-y)=2q(x)+2q(y). \tag{1}$$

This functional equation is useful, among others, in some characterizations of inner product spaces, and its solutions are called *quadratic mappings*. For more information

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about these mappings, other applications and the Ulam stability of equation (1) we refer the reader, for example, to the books [19, 20].

Denote by $\mathbb{N}$, as usual, the set of all positive integers and fix an $n \in \mathbb{N}$ with $n \geq 2$.

Let us recall (see [6]) that a function $Q : G^n \to X$, where G and X are abelian groups, is called *n-quadratic* (roughly, *multi-quadratic*) if it is quadratic in each variable, i.e.

$$\begin{aligned} &Q(x_1, \ldots, x_{i-1}, x_i + y_i, x_{i+1}, \ldots, x_n) \\ &\quad + Q(x_1, \ldots, x_{i-1}, x_i - y_i, x_{i+1}, \ldots, x_n) \\ &\quad = 2Q(x_1, \ldots, x_n) + 2Q(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \end{aligned}$$

for any $i \in \{1, \ldots, n\}$, $x_1, \ldots, x_{i-1}, x_i, y_i, x_{i+1}, \ldots, x_n \in G$.

In papers [18, 26], the above system of n equations was reduced to a single functional equation. Namely, we have the following characterization.

Proposition 1 *Assume that $n \in \mathbb{N}$ is such that $n \geq 2$, G is an abelian group and X is a linear space over a field of the characteristic different from 2. Then a mapping $Q : G^n \to X$ is n-quadratic if and only if it satisfies the functional equation*

$$\begin{aligned} &\sum_{i_1,\ldots,i_n \in \{-1,1\}} Q(x_{11} + i_1 x_{12}, \ldots, x_{n1} + i_n x_{n2}) \\ &\quad = \sum_{j_1,\ldots,j_n \in \{1,2\}} 2^n Q(x_{1j_1}, \ldots, x_{nj_n}) \end{aligned} \tag{2}$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in G$.

Assume that $n \in \mathbb{N}$, X is a linear space over the field $\mathbb{F}$, and Y is a linear space over the field $\mathbb{K}$. Let, moreover, $a_{1,i_1,\ldots,i_n}, \ldots, a_{n,i_1,\ldots,i_n} \in \mathbb{F}$ for $i_1, \ldots, i_n \in \{-1, 1\}$ and $A_{j_1,\ldots,j_n} \in \mathbb{K}$ for $j_1, \ldots, j_n \in \{1, 2\}$ be given scalars.

In this paper, we deal with the Ulam stability of the following functional equation in several variables

$$\begin{aligned} &\sum_{i_1,\ldots,i_n \in \{-1,1\}} f(a_{1,i_1,\ldots,i_n}(x_{11} + i_1 x_{12}), \ldots, a_{n,i_1,\ldots,i_n}(x_{n1} + i_n x_{n2})) \\ &\quad = \sum_{j_1,\ldots,j_n \in \{1,2\}} A_{j_1,\ldots,j_n} f(x_{1j_1}, \ldots, x_{nj_n}). \end{aligned} \tag{3}$$

This equation was recently introduced and investigated in [9, 11] as a natural generalization of (2) (it is obvious that (3) with $a_{1,i_1,\ldots,i_n} = \ldots = a_{n,i_1,\ldots,i_n} = 1$ for $i_1, \ldots, i_n \in \{-1, 1\}$ and $A_{j_1,\ldots,j_n} = 2^n$ for $j_1, \ldots, j_n \in \{1, 2\}$ leads to Eq. (2)). Let us also mention that some special cases of (3) (one of them is clearly Jordan–von Neumann functional equation (1)) have been considered for years by several authors (see, for example, [8, 12, 13, 18–20, 26] and the references therein).

We will show that Eq. (3) is Ulam stable in m-Banach spaces. This complements some results from [11] as well as generalizes those from [9]. Moreover, as corollaries from our main result, we get several outcomes on approximate solutions of a few known functional equations being special cases of (3). Let us finally mention that in papers [9, 11] a variant of the fixed point method is applied, while we use another technique: the direct/Hyers method.

1.2 Ulam Stability

One of the approaches to the question about an error we commit replacing an object possessing some properties by an object fulfilling them only approximately is the notion of the Ulam stability.

Let us recall that an equation is said to be *Ulam stable* in a class of mappings provided each mapping from this class fulfilling our equation "approximately" is "near" to its actual solution.

It is well known that the problem of the stability of homomorphisms of metric groups (in other words, the Cauchy functional equation) was posed by S.M. Ulam in 1940 (a year later, its solution in the case of Banach spaces was presented by D.H. Hyers). Since then the Ulam type stability of various objects (including functional, difference and differential equations, isometries, operators, groups, $C*$-algebras, etc.) has been studied by many researchers (see for instance [1–4, 6–9, 11, 12, 14, 15, 17–19, 22, 24–26]).

In this note, the Ulam stability of Eq. (3) is shown. Furthermore, we apply our main result (Theorem 2) to get some stability outcomes on a few known functional equations.

2 Main Result

In this section, we prove our main outcome, i.e. we show that functional equation (3) is Ulam stable in m-Banach spaces. Let us mention that such spaces were defined in 1989 by A. Misiak (see [23]) as a generalization of the notion of 2-normed spaces, which was introduced by S. Gähler a quarter of a century earlier.

Assume that $m \in \mathbb{N}$ is such that $m \geq 2$ and Y is an at least m-dimensional real linear space. If a mapping $\|\cdot, \dots, \cdot\| : Y^m \to \mathbb{R}$ fulfils the following four conditions:

(i) $\|x_1, \dots, x_m\| = 0$ if and only if $x_1, \dots, x_m$ are linearly dependent,

(ii) $\|x_1, \dots, x_m\|$ is invariant under permutation,

(iii) $\|\alpha x_1, \dots, x_m\| = |\alpha| \|x_1, \dots, x_m\|$,

(iv) $\|x + y, x_2, \dots, x_m\| \leq \|x, x_2, \dots, x_m\| + \|y, x_2, \dots, x_m\|$,

for any $\alpha \in \mathbb{R}$ and $x, y, x_1, \dots, x_m \in Y$, then it is said to be an *m-norm* on Y, whereas the pair $(Y, \|\cdot, \dots, \cdot\|)$ is called an *m-normed space*.

The two standard examples of m-norms are the following.

The Euclidean m-norm $\|x_1, \ldots, x_m\|$ on $\mathbb{R}^m$:

$$\|x_1, \ldots, x_m\| = |\det(x_{ij})| = \mathrm{abs}\left(\begin{vmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mm} \end{vmatrix}\right),$$

where $x_i = (x_{i1}, \ldots, x_{im}) \in \mathbb{R}^m$ for $i \in \{1, \ldots, m\}$.

Let $(X, < \cdot, \cdot >)$ be an at least m-dimensional real inner product space. The standard m-norm on X is given by

$$\|x_1, \ldots, x_m\| = \begin{vmatrix} < x_1, x_1 > & \cdots & < x_1, x_m > \\ \vdots & \ddots & \vdots \\ < x_m, x_1 > & \cdots & < x_m, x_m > \end{vmatrix}^{1/2},$$

where $x_i \in X$ for $i \in \{1, \ldots, m\}$.

Let $(y_k)_{k\in\mathbb{N}}$ be a sequence of elements of an m-normed space $(Y, \|\cdot, \ldots, \cdot\|)$. We say that it is a *Cauchy sequence* provided

$$\lim_{p,l\to\infty} \|y_p - y_l, x_2, \ldots, x_m\| = 0, \qquad x_2, \ldots, x_m \in Y.$$

On the other hand, the sequence $(y_k)_{k\in\mathbb{N}}$ is called *convergent* if there is a $y \in Y$ such that

$$\lim_{k\to\infty} \|y_k - y, x_2, \ldots, x_m\| = 0, \qquad x_2, \ldots, x_m \in Y.$$

The element y is said then to be the *limit* of $(y_k)_{k\in\mathbb{N}}$ and it is denoted by $\lim_{k\to\infty} y_k$.

By an *m-Banach space* we mean an m-normed space such that each its Cauchy sequence is convergent.

The above definitions, properties of m-normed spaces as well as information on some problems investigated in such spaces can be found, for example, in [3, 5, 16, 21, 23].

Now, we can formulate and prove our result on the stability of functional equation (3).

Theorem 2 *Let $m \in \mathbb{N}$, Y be an $(m+1)$-Banach space,*

$$\Big|\sum_{j_1,\ldots,j_n\in\{1,2\}} A_{j_1,\ldots,j_n}\Big| > 1, \tag{4}$$

and $\varepsilon > 0$. If $f : X^n \to Y$ is a mapping such that $f(x_{11}, \ldots, x_{n1}) = 0$ for any $(x_{11}, \ldots, x_{n1}) \in X^n$ with at least one component which is equal to zero and

$$\Big\| \sum_{i_1,\dots,i_n\in\{-1,1\}} f(a_{1,i_1,\dots,i_n}(x_{11}+i_1x_{12}),\dots,a_{n,i_1,\dots,i_n}(x_{n1}+i_nx_{n2})) - \sum_{j_1,\dots,j_n\in\{1,2\}} A_{j_1,\dots,j_n} f(x_{1j_1},\dots,x_{nj_n}), z\Big\| \le \varepsilon \tag{5}$$

for $x_{11}, x_{12}, \dots, x_{n1}, x_{n2} \in X$ *and* $z \in Y^m$, *then there exists a function* $F : X^n \to Y$ *satisfying Eq. (3) and the condition*

$$\|f(x_1,\dots,x_n) - F(x_1,\dots,x_n), z\| \le \frac{\varepsilon}{|\sum_{j_1,\dots,j_n\in\{1,2\}} A_{j_1,\dots,j_n}| - 1} \tag{6}$$

for $x_1, \dots, x_n \in X$ *and* $z \in Y^m$.

Proof Put

$$A := \sum_{j_1,\dots,j_n\in\{1,2\}} A_{j_1,\dots,j_n}.$$

Let us first note that (5) with $x_{i2} = x_{i1}$ for $i \in \{1,\dots,n\}$ gives

$$\|f(2a_{1,1,\dots,1}x_{11},\dots,2a_{n,1,\dots,1}x_{n1}) - Af(x_{11},\dots,x_{n1}), z\| \le \varepsilon,$$

$$(x_{11},\dots,x_{n1}) \in X^n,\ z \in Y^m,$$

and consequently

$$\Big\| \frac{f((2a_{1,1,\dots,1})^{k+1}x_{11},\dots,(2a_{n,1,\dots,1})^{k+1}x_{n1})}{A^{k+1}} - \frac{f((2a_{1,1,\dots,1})^{k}x_{11},\dots,(2a_{n,1,\dots,1})^{k}x_{n1})}{A^{k}}, z\Big\| \le \frac{\varepsilon}{|A|^{k+1}}, \tag{7}$$

$$(x_{11},\dots,x_{n1}) \in X^n,\ z \in Y^m,\ k \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}.$$

Fix $l, p \in \mathbb{N}_0$ such that $l < p$ and observe that we have

$$\Big\| \frac{f((2a_{1,1,\dots,1})^{p}x_{11},\dots,(2a_{n,1,\dots,1})^{p}x_{n1})}{A^{p}} - \frac{f((2a_{1,1,\dots,1})^{l}x_{11},\dots,(2a_{n,1,\dots,1})^{l}x_{n1})}{A^{l}}, z\Big\| \le \sum_{j=l}^{p-1} \frac{\varepsilon}{|A|^{j+1}},$$

$$(x_{11},\dots,x_{n1}) \in X^n,\ z \in Y^m. \tag{8}$$

Therefore, for each $(x_{11},\dots,x_{n1}) \in X^n$, $\left(\frac{f((2a_{1,1,\dots,1})^{k}x_{11},\dots,(2a_{n,1,\dots,1})^{k}x_{n1})}{A^{k}}\right)_{k\in\mathbb{N}_0}$ is a Cauchy sequence. Using the fact that Y is an $(m+1)$-Banach space, we conclude that this sequence is convergent, which allows us to define

$$F(x_{11},\ldots,x_{n1}) := \lim_{k\to\infty} \frac{f((2a_{1,1,\ldots,1})^k x_{11},\ldots,(2a_{n,1,\ldots,1})^k x_{n1})}{A^k}, \quad (x_{11},\ldots,x_{n1}) \in X^n. \tag{9}$$

Putting $l = 0$ and letting $p \to \infty$ in (8) we get

$$\|f(x_{11},\ldots,x_{n1}) - F(x_{11},\ldots,x_{n1}), z\| \le \frac{\varepsilon}{|A|-1},$$

$$(x_{11},\ldots,x_{n1}) \in X^n,\ z \in Y^m,$$

i.e. (6) holds true.

Let us next observe that from (5) we get

$$\Bigg\| \sum_{i_1,\ldots,i_n\in\{-1,1\}} \frac{f((2a_{1,1,\ldots,1})^k a_{1,i_1,\ldots,i_n}(x_{11}+i_1x_{12}),\ldots,(2a_{n,1,\ldots,1})^k a_{n,i_1,\ldots,i_n}(x_{n1}+i_nx_{n2}))}{A^k} - \sum_{j_1,\ldots,j_n\in\{1,2\}} A_{j_1,\ldots,j_n} \frac{f((2a_{1,1,\ldots,1})^k x_{1j_1},\ldots,(2a_{n,1,\ldots,1})^k x_{nj_n})}{A^k}, z \Bigg\| \le \frac{\varepsilon}{|A|^k}$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$, $z \in Y^m$ and $k \in \mathbb{N}_0$. Letting now $k \to \infty$ and applying definition (9) we see that

$$\Bigg\| \sum_{i_1,\ldots,i_n\in\{-1,1\}} F(a_{1,i_1,\ldots,i_n}(x_{11}+i_1x_{12}),\ldots,a_{n,i_1,\ldots,i_n}(x_{n1}+i_nx_{n2})) - \sum_{j_1,\ldots,j_n\in\{1,2\}} A_{j_1,\ldots,j_n} F(x_{1j_1},\ldots,x_{nj_n}), z \Bigg\| \le 0$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$ and $z \in Y^m$, and thus the mapping $F : X^n \to Y$ is a solution of the functional equation (3). □

3 Some Applications

Theorem 2 with $a_{1,i_1,\ldots,i_n} = \ldots = a_{n,i_1,\ldots,i_n} = 1$ for $i_1,\ldots,i_n \in \{-1,1\}$ and $A_{j_1,\ldots,j_n} = 2^n$ for $j_1,\ldots,j_n \in \{1,2\}$ immediately yields the following outcome on the Ulam stability of Eq. (2).

Corollary 3 *Let $m \in \mathbb{N}$, Y be an $(m+1)$-Banach space and $\varepsilon > 0$. If $f : X^n \to Y$ is a function such that $f(x_{11}, \dots, x_{n1}) = 0$ for any $(x_{11}, \dots, x_{n1}) \in X^n$ with at least one component which is equal to zero and*

$$\left\| \sum_{i_1,\dots,i_n \in \{-1,1\}} f(x_{11} + i_1 x_{12}, \dots, x_{n1} + i_n x_{n2}) - \sum_{j_1,\dots,j_n \in \{1,2\}} 2^n f(x_{1j_1}, \dots, x_{nj_n}), z \right\| \leq \varepsilon$$

for $x_{11}, x_{12}, \dots, x_{n1}, x_{n2} \in X$ and $z \in Y^m$, then there exists a solution $F : X^n \to Y$ of Eq. (2) with

$$\|f(x_1, \dots, x_n) - F(x_1, \dots, x_n), z\| \leq \frac{\varepsilon}{4^n - 1}, \qquad (x_1, \dots, x_n) \in X^n,\ z \in Y^m.$$

Let us also mention that the stability of this equation was previously investigated in [12, 18, 26].

Another particular case of Eq. (3), i.e. the functional equation

$$\begin{aligned} &f(x_{11} + x_{12}, x_{21} + x_{22}) + f(x_{11} + x_{12}, x_{21} - x_{22}) \\ &\quad + f(x_{11} - x_{12}, x_{21} + x_{22}) + f(x_{11} - x_{12}, x_{21} - x_{22}) \\ &= A_{1,1} f(x_{11}, x_{21}) + A_{1,2} f(x_{11}, x_{22}) \\ &\quad + A_{2,1} f(x_{12}, x_{21}) + A_{2,2} f(x_{12}, x_{22}), \end{aligned} \tag{10}$$

with $A_{1,1}, A_{1,2}, A_{2,1}, A_{2,2} \geq 0$, was very recently investigated in [13], where its characterizations and representations of set-valued solutions are obtained.

As for the Ulam stability of Eq. (10), Theorem 2 with $n = 2$, $a_{1,1,1} = a_{2,1,1} = a_{1,1,-1} = a_{2,1,-1} = a_{1,-1,1} = a_{2,-1,1} = a_{1,-1,-1} = a_{2,-1,-1} = 1$ yields the following.

Corollary 4 *Assume that $m \in \mathbb{N}$, Y is an $(m+1)$-Banach space,*

$$|A_{1,1} + A_{1,2} + A_{2,1} + A_{2,2}| > 1,$$

and $\varepsilon > 0$. If $f : X^2 \to Y$ is a function such that

$$f(x_1, 0) = 0 = f(0, x_2), \qquad x_1, x_2 \in X \tag{11}$$

holds and

$$\begin{aligned} &\| f(x_{11} + x_{12}, x_{21} + x_{22}) + f(x_{11} + x_{12}, x_{21} - x_{22}) \\ &\ + f(x_{11} - x_{12}, x_{21} + x_{22}) + f(x_{11} - x_{12}, x_{21} - x_{22}) \\ &\ - A_{1,1} f(x_{11}, x_{21}) - A_{1,2} f(x_{11}, x_{22}) \end{aligned}$$

$$-A_{2,1}f(x_{12},x_{21}) - A_{2,2}f(x_{12},x_{22}), z\Big\| \le \varepsilon$$

for $x_{11}, x_{12}, x_{21}, x_{22} \in X$ *and* $z \in Y^m$*, then there exists a solution* $F : X^2 \to Y$ *of Eq.* (10) *fulfilling*

$$\|f(x_1,x_2) - F(x_1,x_2), z\| \le \frac{\varepsilon}{|A_{1,1}+A_{1,2}+A_{2,1}+A_{2,2}|-1}$$

for $x_1, x_2 \in X$ *and* $z \in Y^m$.

Next, we derive from Theorem 2 the Ulam stability of the functional equation

$$\begin{aligned} &f(x_{11}+x_{12}, x_{21}+x_{22}) + f(x_{11}-x_{12}, x_{21}-x_{22}) \\ &\quad = A_{1,1}f(x_{11},x_{21}) + A_{1,2}f(x_{11},x_{22}). \end{aligned} \tag{12}$$

Set-valued solutions of this equation were very recently investigated in [10]. Let us also mention that a particular case (with $A_{1,1} = A_{1,2} = 2$) of (12) is the functional equation

$$\begin{aligned} &f(x_{11}+x_{12}, x_{21}+x_{22}) + f(x_{11}-x_{12}, x_{21}-x_{22}) \\ &\quad = 2f(x_{11},x_{21}) + 2f(x_{11},x_{22}). \end{aligned}$$

Its stability was studied in [9, 11, 17], and our result complements and generalizes the corresponding outcomes obtained in these two papers.

To get the mentioned result, we apply Theorem 2 with $n = 2$, $a_{1,1,1} = a_{2,1,1} = a_{1,-1,-1} = a_{2,-1,-1} = 1$, $a_{1,1,-1} = a_{2,1,-1} = a_{1,-1,1} = a_{2,-1,1} = 0$ and $A_{2,1} = A_{2,2} = 0$. In consequence, we have the following.

Corollary 5 *Let* $m \in \mathbb{N}$*,* Y *be an* $(m+1)$*-Banach space,* $|A_{1,1}+A_{1,2}| > 1$*, and* $\varepsilon > 0$*. If* $f : X^2 \to Y$ *is a function such that condition* (11) *holds and*

$$\begin{aligned} &\Big\|f(x_{11}+x_{12}, x_{21}+x_{22}) + f(x_{11}-x_{12}, x_{21}-x_{22}) \\ &\quad -A_{1,1}f(x_{11},x_{21}) - A_{1,2}f(x_{11},x_{22}), z\Big\| \le \varepsilon \end{aligned}$$

for $x_{11}, x_{12}, x_{21}, x_{22} \in X$ *and* $z \in Y^m$*, then there exists a solution* $F : X^2 \to Y$ *of Eq.* (12) *fulfilling*

$$\|f(x_1,x_2) - F(x_1,x_2), z\| \le \frac{\varepsilon}{|A_{1,1}+A_{1,2}|-1}, \qquad x_1, x_2 \in X,\ z \in Y^m.$$

Let us finally consider the functional equation

$$\begin{aligned} &f(x_{11}+x_{12}, x_{21}+x_{22}) + f(x_{11}+x_{12}, x_{21}-x_{22}) \\ &\quad = 2f(x_{11},x_{21}) + 2f(x_{11},x_{22}) + 2f(x_{12},x_{21}) + 2f(x_{12},x_{22}), \end{aligned} \tag{13}$$

which was introduced and studied in [25] (see also [9] for a stability result in 2-Banach spaces).

Using Theorem 2 with $n = 2$, $a_{1,1,1} = a_{2,1,1} = a_{1,1,-1} = a_{2,1,-1} = 1$, $a_{1,-1,1} = a_{2,-1,1} = a_{1,-1,-1} = a_{2,-1,-1} = 0$, $A_{1,1} = A_{1,2} = A_{2,1} = A_{2,2} = 2$ we get the following outcome.

Corollary 6 *Let $m \in \mathbb{N}$, Y be an $(m+1)$-Banach space and $\varepsilon > 0$. If $f : X^2 \to Y$ is a function such that (11) holds and*

$$\begin{aligned}&\|f(x_{11}+x_{12}, x_{21}+x_{22}) + f(x_{11}+x_{12}, x_{21}-x_{22})\\&\quad -2f(x_{11}, x_{21}) - 2f(x_{11}, x_{22}) - 2f(x_{12}, x_{21}) - 2f(x_{12}, x_{22}), y\| \le \varepsilon\end{aligned}$$

for $x_{11}, x_{12}, x_{21}, x_{22} \in X$ and $y \in Y^m$, then there exists a solution $F : X^2 \to Y$ of Eq. (13) fulfilling

$$\|f(x_1, x_2) - F(x_1, x_2), y\| \le \frac{\varepsilon}{7}, \qquad x_1, x_2 \in X,\ y \in Y^m.$$

4 Conclusions

In the paper, we have shown that Eq. (3) is Ulam stable in m-Banach spaces. This complements some outcomes from [11] (where the case of Banach spaces was studied) as well as generalizes those from [9] (where 2-Banach spaces were considered). In the proof of our main result (Theorem 2), we have used the direct method, while in [9, 11] a variant of the fixed point method was applied.

From Theorem 2, we have derived some stability results on functional equations (2), (10), (12) and (13), which were previously investigated in [10, 12, 13, 18, 25, 26].

We finish the paper with three problems.

Problem 1 Does the assertion of Theorem 2 hold without the assumption that $f(x_{11}, \ldots, x_{n1}) = 0$ for any $(x_{11}, \ldots, x_{n1}) \in X^n$ with at least one component which is equal to zero?

Problem 2 Is Eq. (3) stable for

$$\Big|\sum_{j_1,\ldots,j_n \in \{1,2\}} A_{j_1,\ldots,j_n}\Big| \le 1?$$

Problem 3 Find a general solution of Eq. (3).

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Declarations

Conflict of interest The author declares that he has no conflict of interest.

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