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Infinitely Many Solutions for a Fractional Schrödinger Equation in $\mathbb{R}^N$ with Combined Nonlinearities

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Abstract

This paper is devoted to the following class of nonlinear fractional Schrödinger equations:

$$(-\Delta)^s u + V(x)u = f(x,u) + \lambda g(x,u), \quad \text{in } \mathbb{R}^N,$$

where $s \in (0,1)$, $N > 2s$, $(-\Delta)^s$ stands for the fractional Laplacian, $\lambda \in \mathbb{R}$ is a parameter, $V \in C(\mathbb{R}^N, \mathbb{R})$, $f(x,u)$ is superlinear and $g(x,u)$ is sublinear with respect to u, respectively. We prove the existence of infinitely many high energy solutions of the aforementioned equation by means of the Fountain theorem. Some recent results are extended and sharply improved.

Keywords Fractional Schrödinger equation · Fountain theorem · Infinitely many solutions

Mathematics Subject Classification 35R11 · 35A15 · 35B38

1 Introduction

Consider the following fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

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where $s \in (0, 1)$, $N > 2s$ and $(-\Delta)^s$ stands for the fractional Laplacian which can be defined for a sufficiently smooth function u as

$$(-\Delta)^s u(x) = C(N, s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B(x,\varepsilon)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \mathrm{d}y, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $B(x, \varepsilon) = \{x \in \mathbb{R}^N : |x| < \varepsilon\}$ and $C(N, s) > 0$ is a dimensional constant that depends on N and s (see [6]).

Equation (1.1) arises in the study of the following fractional Schrödinger equation

$$i\partial_t \Psi(x, t) + (-\Delta)^s \Psi(x, t) + (V(x) - \omega)\Psi(x, t) = h(|\Psi(x, t)|)\Psi(x, t),$$

when looking for standing waves, that is, solutions of the form $\Psi(x, t) = \exp(-i\omega t)u(x)$. The fractional Schrödinger equation was introduced by Laskin [14, 15] in the context of fractional quantum mechanics, as a result of extending the Feynman path integral from the Brownian-like to Lévy-like quantum mechanical paths. It is also appeared in several subjects such as plasma physics, image processing, finance and stochastic models, see for instance [1, 4, 10, 16].

In recent years, Eq. (1.1) has been extensively studied under various assumptions on V and f and there are many interesting results in the literature on the existence and multiplicity of solutions to problem (1.1) has been obtained via variational approaches, we refer the readers to [3, 5, 7–9, 11–13, 17, 19–23]. In particular, the existence of infinitely many high or small energy solutions to problem (1.1) was established in [5, 7, 9, 11–13, 17, 20] by the aid of variant fountain theorems (see [25]) or the symmetric mountain pass theorem (see [24]). However, there are few papers concern with the existence of infinitely many (high or small) energy solutions to problem (1.1) in the case where $f(x, u)$ is a combination of sublinear and superlinear terms at infinity with respect to u, see for instance [7, 17, 21].

In [7], Du and Tian considered the following class of fractional Schrödinger equations with concave and critical nonlinearities

$$(-\Delta)^s u + V(x)u = \mu a(x)|u|^{q-2}u + |u|^{2_s^*-2}u, \quad x \in \mathbb{R}^N, \tag{1.3}$$

where (and in the sequel) $2_s^* = \frac{2N}{N-2s}$ is the critical Sobolev exponent, $\mu > 0$ is a parameter, $1 < q < 2$, $a(x)$ is positive continuous functions satisfying $a(x) \in L^{\frac{2}{2-q}}\left(\mathbb{R}^N\right) \cap L^{\frac{2_s^2}{2s}-q}\left(\mathbb{R}^N\right)$ and $V(x)$ satisfies the following assumptions

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$, where V_0 is a constant. Moreover, there exists $r_0 > 0$ such that

$$\lim_{|y|\to\infty} \text{meas}\{x \in \mathbb{R}^N : |x - y| \leq r_0,\ V(x) \leq M\} = 0, \quad \forall M > 0,$$

where meas(.) is the Lebesgue measure on $\mathbb{R}^N$. The authors proved that there exists $\mu^* > 0$ such that, for any $0 < \mu < \mu^*$, problem (1.3) possesses infinitely many small energy solutions by using the Dual fountain theorem.

In [21], Timoumi established infinitely many small energy solutions to the problem

$$(-\Delta)^s u + V(x)u = g(x,u) + h(x,u), \quad \text{in } \mathbb{R}^N,$$

by means of the Dual Fountain Theorem (see [24]), where $V(x)$ satisfies assumptions (V), $g(x,u)$ is sublinear in u and $h(x,u)$ is superlinear in u.

Li and Shang [17] studied the following problem

$$(-\Delta)^s u + V(x)u = f(x,u) + \lambda h(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N \tag{1.4}$$

where $\lambda > 0$ is a parameter, $p \in [1,2)$, $h \in L^{\frac{2}{2-p}}\left(\mathbb{R}^N\right)$ and V and f satisfies the following assumptions:

(V') $V(x) \in C\left(\mathbb{R}^N, \mathbb{R}\right)$, $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$ and $\lim_{|x|\to\infty} V(x) = \infty$;

(f_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exist constants $a_1, a_2 \geq 0$, $q \in \left[2, \frac{2N+4s}{N}\right)$ with $\frac{a_1}{2S_2^2} + \frac{a_2}{qS_q^q} < \frac{1}{2}$ such that

$$|f(x,u)| \leq a_1|u| + a_2|u|^{q-1}, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R},$$

where S_q is the best constant for the embedding of $X \subset L^q\left(\mathbb{R}^N\right)$ and

$$X = \left\{u \in L^2\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(z)|^2}{|x-z|^{N+2s}} \mathrm{d}x\mathrm{d}z + \int_{\mathbb{R}^N} V(x)u(x)^2 \mathrm{d}x < +\infty\right\};$$

(f_2) $\lim\limits_{t\to\infty} \frac{F(x,t)}{|t|^2} = \infty$ uniformly in $x \in \mathbb{R}^N$ and there exists $r_1 > 0$ such that $F(x,u) \geq 0$, for any $x \in \mathbb{R}^N$, $u \in \mathbb{R}$ and $|u| \geq r_1$, where $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$;

(f_3) $2F(x,u) < f(x,u)u$, $\forall (x,u) \in \mathbb{R}^N \times \mathbb{R}$.

(F_4) $f(x,-u) = -f(x,u)$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$

By using the symmetric mountain pass theorem, the authors showed that there exists a constant $\lambda_0 > 0$ such that, for any $\lambda \in (0, \lambda_0)$, problem (1.4) possesses infinitely many high energy solutions.

Motivated by these works, in the present paper we are concerned with the existence of infinitely many high energy solutions to the following class of fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x,u) + \lambda g(x,u), \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $\lambda \in \mathbb{R}$ is a parameter, $V(x)$ satisfies assumptions (V) and f and g satisfy the following assumptions

(F_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exist constants $c_1, c_2 > 0$ and $p \in (2, 2_s^*)$ such that

$$|f(x,u)| \leq c_1|u| + c_2|u|^{p-1}, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R},$$

where $2_s^* = \frac{2N}{N-2s}$ is the critical Sobolev exponent.

(F_2) $\lim\limits_{|u|\to\infty} \frac{F(x,u)}{u^2} = +\infty$ a.e. $x \in \mathbb{R}^{\mathbb{N}}$, where $F(x,u) = \int_0^u f(x,t)\mathrm{d}t$ and there exists $r_1 > 0$ such that

$$\inf_{x\in\mathbb{R}^N, |u|\geq r_1} F(x,u) \geq 0;$$

(F_3) There exist constants $\mu > 2$, $c_3 > 0$ and $a_0 > 0$ such that such that

$$\mu F(x,u) \leq f(x,u)u + c_3|u|^2, \quad \forall (x,|u|) \in \mathbb{R}^N \times [a_0,\infty).$$

(g_1) There exist constants $1 < \delta_1 < \delta_2 < 2$ and positive functions $\xi_i \in L^{\frac{2}{2-\delta_i}}(\mathbb{R}^N)$ $(i = 1,2)$ such that

$$|g(x,u)| \leq \xi_1(x)|u|^{\delta_1-1} + \xi_2(x)|u|^{\delta_2-1}, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

(g_2) $g(x,-u) = -g(x,u)$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$;

By using the Fountain theorem (i.e., [24, Theorem 3.6]), we prove the existence of an unbounded sequence of nontrivial solutions $\{u_k\}$ to problem (1.5) under assumptions (V), (F_1)–(F_4) and (g_1)–(g_2). Our result extends and sharply improves that in [17].

The remainder of this paper is organized as follows. In Sect. 2, we prepare the variational framework of the studied problem. In Sect. 3, employing the fountain theorem (3.1), we establish the existence of infinitely many high energy solutions to problem (1.5).

2 Variational Setting and Main Result

In this section, for the reader's convenience, we shall introduce some notations and we revise some known results about the fractional Sobolev spaces which can be found in [6].

As usual, for $1 \leq p < +\infty$, we define

$$\|u\|_{L^p} := \|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p \mathrm{d}x\right)^{\frac{1}{p}}, \quad u \in L^p(\mathbb{R}^N),$$

The fractional Sobolev space $H^s(\mathbb{R}^N) = W^{s,2}(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) \ : \ \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+s}} \in L^2\left(\mathbb{R}^N \times \mathbb{R}^N\right) \right\}$$

with the inner product and the norm

$$\langle u,v\rangle_{H^s} = \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} u(x)v(x)\mathrm{d}x,$$

$$\|u\|_{H^s}^2 = \langle u, u\rangle_{H^s} = \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} |u(x)|^2\mathrm{d}x,$$

where the norm

$$[u]_{H^s}^2 = \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y$$

is the so-called Gagliardo semi-norm of u.

Let $\mathscr{S}(\mathbb{R}^N)$ be the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^N$. We recall that the Fourier transform of a function $u \in \mathscr{S}(\mathbb{R}^N)$ is defined as

$$\mathscr{F}u(\xi) := \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{-ix\xi}u(x)\mathrm{d}x.$$

By Plancherel's theorem, we have

$$\|u\|_2 = \|\mathscr{F}u\|_2, \quad \forall u \in \mathscr{S}(\mathbb{R}^N).$$

Let $s \in (0, 1)$, the fractional Laplacian $(-\Delta)^s$ of a function $u \in \mathscr{S}(\mathbb{R}^N)$ is defined by means of the Fourier transform as

$$\mathscr{F}\left((-\Delta)^s u\right)(\xi) = |\xi|^{2s}\mathscr{F}u(\xi), \quad \forall s \in (0,1).$$

The space $H^s(\mathbb{R}^N)$ can also be described via the Fourier transform as follows

$$H^s(\mathbb{R}^N) := \left\{u \in L^2(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N}\left(1+|\xi|^2\right)^s |\mathscr{F}u(\xi)|^2\mathrm{d}\xi < \infty\right\},$$

and the norm is defined by

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^N}\left(1+|\xi|^2\right)^s |\mathscr{F}u(\xi)|^2\mathrm{d}\xi\right)^{\frac{1}{2}}.$$

For the problem (1.5), we define the following Hilbert space

$$H := \left\{u \in H^s(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} V(x)|u|^2\mathrm{d}x < \infty\right\},$$

endowed with the inner product

$$\langle u, v\rangle := \langle u, v\rangle_H = \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)u(x)v(x)\mathrm{d}x.$$

Then, the norm on H is given by

$$\|u\| := \left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u|^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

Obviously, by assumptions (V), the embedding $H \hookrightarrow H^s(\mathbb{R}^N)$ is continuous.

From [6], the embeddings $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is continuous for $p \in [2, 2_s^*]$. Therefore, $H \hookrightarrow L^p(\mathbb{R}^N),\ 2 \leq p \leq 2_s^*$ is continuous, namely, there exist constants $\eta_p > 0$ such that

$$\|u\|_p \leq \eta_p \|u\|, \quad \forall u \in H,\ p \in [2, 2_s^*], \tag{2.1}$$

Moreover, from [20], we know that the embedding $H \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $2 \leq p < 2_s^*$ under condition (V).

For the fractional Schrödinger equation (1.5), the associated energy functional is defined on H as follows

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x - \lambda \int_{\mathbb{R}^N} G(x,u)\mathrm{d}x. \tag{2.2}$$

By hypotheses (V), (F_1) and (g_1), the functional I is well define and of class $C^1(H,\mathbb{R})$ with

$$\begin{aligned} \langle I_\lambda'(u), v\rangle = &\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &+ \int_{\mathbb{R}^N} V(x)uv\mathrm{d}x - \int_{\mathbb{R}^N} f(x,u)v\mathrm{d}x - \lambda \int_{\mathbb{R}^N} g(x,u)v\mathrm{d}x, \end{aligned} \tag{2.3}$$

for all $v \in H$. Besides, the critical points of I in H are solutions of problem (1.5). Now, we are ready to state the main result of this paper as follows.

Theorem 2.1 *Assume that conditions (V), (F_1)–(F_4) and (g_1)–(g_2) hold. Then there exists $\overline{\lambda} > 0$ such that problem (1.5) possesses infinitely many nontrivial solutions $\{u_k\}$ provided $|\lambda| \leq \overline{\lambda}$. Moreover, there holds*

$$I_\lambda(u_k) \to \infty \text{ as } k \to \infty.$$

Remark 2.1 Since the problem (1.5) is defined on the entire space $\mathbb{R}^N$, the main difficulty of this problem is the lack of compactness for Sobolev embedding theorem. In the context of studying of the existence of solutions for the classical Schrödinger equation

$$-\Delta u + V(x)u = f(x,u), \quad x \in R^N,$$

Bartsch et al. [2] presented the general conditions (V) which guarantee the compactness of the embeddings $\left\{u \in H^1\left(R^N\right) : \int_{R^N} V(x)|u|^2 dx < +\infty\right\} \hookrightarrow$

$L^p\left(\mathbb{R}^N\right)$, $p \in [2, \frac{2N}{N-2}]$. Furthermore, conditions (V) are weaker than the coercivity condition (V') used in [17].

Remark 2.2 Firstly, comparing with Theorem 1.1 in [17], our assumptions (F_1)–(F_3) are more general than (f_1)–(f_3). Indeed, let $f(x,u) = au + b|u|^{p-2}u$, where $a > 2S_2^2, b > qS_q^q$ and $p \in (2, 2_2^*)$. Then, clearly f satisfies (F_1) but not (f_1) since $\frac{a}{2S_2^2} + \frac{b}{qS_q^q} > 2$. Secondly, let

$$f(x,t) = 3|t|t - \frac{15}{2}|t|^{1/2}t + t, \quad t \in \mathbb{R}.$$

Then,

$$F(x,t) = |t|^3 - 3|t|^{5/2} + \frac{1}{2}t^2.$$

It is easy to verify that the above function f satisfies (F_1), (F_2), (F_4) and (F_3) (which was initially gave in [18]) with $\mu = \frac{5}{2}$. However, f does not satisfy (f_3), in fact we have

$$f(x,t)t - 2F(x,t) = |t|^3 - \frac{3}{2}|t|^{5/2} \le 0, \quad \forall t \in \left[-\frac{9}{4}, \frac{9}{4}\right].$$

This shows that (f_3) is not satisfied. Finally, it is easy to see that $\widetilde{g}(x,u) = \lambda h(x)|u|^{p-2}u$ considered in (1.4) is a special case of $g(x,u)$ considered in this paper. Furthermore, unlike (1.4), the parameter λ in (1.5) is allowed to be sign-changing. Consequently, Theorem 2.1 generalizes and sharply improves Theorem 1.1 in [17].

Remark 2.3 When $s = 1$, Eq. (1.5) becomes the classical Schrödinger equation

$$-\Delta u + V(x)u = f(x,u) + \lambda g(x,u), \quad x \in R^N,$$

As far as we know, our result is new even for the case $s = 1$.

3 Proof of the Main Result

Hereafter, we shall use c_i, C_i, $i = 1, 2, \ldots$ to denote various positive constants which may change from line to line. We start this section by introducing some variational preliminaries and abstract results that we need to prove our main results.

Definition 3.1 ((PS)-condition)

- A sequence $\{u_n\} \subset H$ is said to be a Palais–Smale sequence at level $c \in \mathbb{R}$ ($(\mathrm{PS})_c$ sequence for short) if $I(u_n) \to c$ and $I'(u_n) \to 0$ in H^* the dual space of H.
- The functional I satisfies the Palais–Smale condition at the level c ($(\mathrm{PS})_c$ condition for short) if any $(\mathrm{PS})_c$ sequence has a convergent subsequence.

Lemma 3.1 *Under the assumptions of Theorem 2.1, the functional I_λ satisfies the $(PS)_c$ condition for any $c > 0$.*

Proof Let $\{u_n\} \subset H$ be any (PS) sequence of I_λ, that is,

$$I_\lambda(u_n) \to c > 0, \quad I'_\lambda(u_n) \to 0 \text{ in } H^*. \tag{3.1}$$

First, we prove that $\{u_n\}$ is bounded in H. Arguing indirectly, suppose that $\|u_n\| \to \infty$ as $n \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$, thus $\{v_n\}$ is bounded in H. Using assumption (F_1) we have

$$\begin{aligned} |F(x,u)| = |F(x,u) - F(x,0)| &= \left| \int_0^1 f(x,tu)u\mathrm{d}t \right| \\ &\leq \int_0^1 \left(c_1|u|^2 t + c_2|u|^p t^{p-1} \right) \mathrm{d}t \\ &= \frac{c_1}{2}|u|^2 + \frac{c_2}{p}|u|^p, \qquad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}. \end{aligned} \tag{3.2}$$

Set $\mathcal{F}(x,u_n) = f(x,u_n)u_n - \mu F(x,u_n)$. Therefore, for $x \in \mathbb{R}^N$ and $|u(x)| < a_0$, by (3.2), we have

$$\begin{aligned} |f(x,u)u - \mu F(x,u)| &\leq |f(x,u)u| + \mu|F(x,u)| \\ &\leq \left(c_1|u|^2 + c_2|u|^p \right) + \left(c_1\frac{\mu}{2}|u|^2 + c_2\frac{\mu}{p}|u|^p \right) \\ &\leq \left(\frac{2+\mu}{2}c_1 + \frac{p+\mu}{p}c_2 a_0^{p-2} \right)|u|^2 \\ &= c_3|u|^2, \end{aligned}$$

where μ and $a_0 > 0$ are given in (F_3). Combining the above inequality with (F_3), we conclude that there exists $c_4 > 0$ such that

$$\mathcal{F}(x,u) = f(x,u)u - \mu F(x,u) \geq -c_4|u|^2, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.3}$$

By (2.2), (2.3) and (3.3) , we have

$$\begin{aligned} \mu I_\lambda(u_n) - \langle I'_\lambda(u_n), u_n \rangle &= \frac{\mu-2}{2}\|u_n\|^2 + \int_{\mathbb{R}^N} \mathcal{F}(x,u_n)\mathrm{d}x - \lambda \int_{\mathbb{R}^N} \mathcal{G}(x,u_n)\mathrm{d}x \\ &\geq \frac{\mu-2}{2}\|u_n\|^2 - c_4 \int_{\mathbb{R}^N} |u_n|^2\mathrm{d}x - \lambda \int_{\mathbb{R}^N} \mathcal{G}(x,u_n)\mathrm{d}x, \end{aligned} \tag{3.4}$$

where $\mathcal{G}(x,u) := g(x,u)u - \mu G(x,u)$. By (g_1) one has

$$\begin{aligned} |\mathcal{G}(x,u)| &= |g(x,u)u - \mu G(x,u)| \\ &\le |g(x,u)u| + \mu|G(x,u)| \\ &\le \xi_1(x)|u|^{\delta_1} + \xi_2(x)|u|^{\delta_2} + \frac{\mu}{\delta_1}\xi_1(x)|u|^{\delta_1} + \frac{\mu}{\delta_2}\xi_2(x)|u|^{\delta_2} \\ &:= \gamma_1\xi_1(x)|u|^{\delta_1} + \gamma_2\xi_2(x)|u|^{\delta_2}, \end{aligned} \tag{3.5}$$

where $\gamma_i = \frac{\mu+\delta_i}{\delta_i}$ $(i=1,2)$. Since $\xi_i \in L^{\frac{2}{2-\delta_i}}(\mathbb{R}^N)$, it follows from (3.5), the Hölder's inequality and (2.1)

$$\begin{aligned} &\left|\int_{\mathbb{R}^N} \mathcal{G}(x,u_n)\mathrm{d}x\right| \\ &\le \int_{\mathbb{R}^N} |\mathcal{G}(x,u_n)|\mathrm{d}x \\ &\le \gamma_1 \int_{\mathbb{R}^N} \xi_1(x)|u_n|^{\delta_1}\mathrm{d}x + \gamma_2 \int_{\mathbb{R}^N} \xi_2(x)|u_n|^{\delta_2}\mathrm{d}x \\ &\le \sum_{i=1}^{2} \gamma_i \left(\int_{\mathbb{R}^N} |\xi_i(x)|^{\frac{2}{2-\delta_i}}\mathrm{d}x\right)^{\frac{2-\delta_i}{2}} \left(\int_{\mathbb{R}^N} |u_n|^2\mathrm{d}x\right)^{\frac{\delta_i}{2}} \\ &\le \gamma_1\|\xi_1\|_{\frac{2}{2-\delta_1}}\|u_n\|_2^{\delta_1} + \gamma_2\|\xi_2\|_{\frac{2}{2-\delta_2}}\|u_n\|_2^{\delta_2} \\ &\le \gamma_1\eta_2^{\delta_1}\|\xi_1\|_{\theta_1}\|u_n\|^{\delta_1} + \gamma_2\eta_2^{\delta_2}\|\xi_2\|_{\theta_2}\|u_n\|^{\delta_2} \\ &:= C_1\|\xi_1\|_{\theta_1}\|u_n\|^{\delta_1} + C_2\|\xi_2\|_{\theta_2}\|u_n\|^{\delta_2}, \end{aligned} \tag{3.6}$$

where $C_i = \gamma_i\eta_2^{\delta_i}$ and $\theta_i = \frac{2}{2-\delta_i}$, $i=1,2$. Combining (3.1) with (3.4) and (3.6), for sufficiently large $n \in \mathbb{N}$, there exists a constant $C_3 > 0$ such that

$$\begin{aligned} C_3 &\ge \mu I_\lambda(u_n) - \langle I'_\lambda(u_n), u_n\rangle \\ &\ge \frac{\mu-2}{2}\|u_n\|^2 - c_4\int_{\mathbb{R}^N}|u_n|^2\mathrm{d}x - \lambda\int_{\mathbb{R}^N}\mathcal{G}(x,u_n)\mathrm{d}x \\ &\ge \frac{\mu-2}{2}\|u_n\|^2 - c_4\|u_n\|_2^2 - |\lambda|\left(C_1\|\xi_1\|_{\theta_1}\|u_n\|^{\delta_1} + C_2\|\xi_2\|_{\theta_2}\|u_n\|^{\delta_2}\right), \end{aligned}$$

which yields

$$\begin{aligned} \frac{\|u_n\|_2^2}{\|u_n\|^2} \ge \frac{\mu-2}{2c_4} - \frac{1}{c_4}\Bigg[&\frac{C_3}{\|u_n\|^2} + |\lambda|\frac{C_1\|\xi_1\|_{\theta_1}}{\|u_n\|^{2-\delta_1}} \\ &+|\lambda|\frac{C_2\|\xi_2\|_{\theta_2}}{\|u_n\|^{2-\delta_2}}\Bigg]. \end{aligned}$$

Since $1 < \delta_1 < \delta_2 < 2$ and $\|u_n\| \to \infty$, we can choose a large $n \in \mathbb{N}$ so that

$$\frac{C_3}{\|u_n\|^2} + |\lambda| \frac{C_1 \|\xi_1\|_{\theta_1}}{\|u_n\|^{2-\delta_1}} + |\lambda| \frac{C_2 \|\xi_2\|_{\theta_2}}{\|u_n\|^{2-\delta_2}} \le \frac{\mu - 2}{4},$$

we then conclude

$$\|v_n\|_2^2 = \frac{\|u_n\|_2^2}{\|u_n\|^2} \ge \frac{\mu - 2}{4c_4} > 0. \tag{3.7}$$

Set $\Omega_n = \{x \in \mathbb{R}^N : |u_n(x)| \le r_1\}$ and $A_n = \{x \in \mathbb{R}^N : v_n(x) \neq 0\}$, then $\text{meas}(A_n) > 0$ due to (3.7). Besides, since $\|u_n\| \to \infty$ as $n \to \infty$, we obtain

$$|u_n(x)| \to \infty \quad \text{as} \quad n \to \infty, \quad \forall x \in A_n. \tag{3.8}$$

Hence, $A_n \subseteq \mathbb{R}^N \setminus \Omega_n$ for $n \in \mathbb{N}$ large enough.

Similarly to (3.6), by (g_1), (2.1) and Hölder's inequality, we derive that

$$\begin{aligned} \int_{\mathbb{R}^N} G(x, u_n)\mathrm{d}x &\le \int_{\mathbb{R}^N} \xi_1(x)|u_n|^{\delta_1}\mathrm{d}x + \int_{\mathbb{R}^N} \xi_2(x)|u_n|^{\delta_2}\mathrm{d}x \\ &\le \|\xi_1\|_{\theta_1}\|u_n\|_2^{\delta_1} + \|\xi_2\|_{\theta_2}\|u_n\|_2^{\delta_2} \\ &\le \|\xi_1\|_{\theta_1}\eta_2^{\delta_1}\|u_n\|^{\delta_1} + \|\xi_2\|_{\theta_2}\eta_2^{\delta_2}\|u_n\|^{\delta_2} \end{aligned} \tag{3.9}$$

Therefore

$$\int_{\mathbb{R}^N} \frac{|G(x, u_n)|}{\|u_n\|^2}\mathrm{d}x \le \frac{\|\xi_1\|_{\theta_1}\|u_n\|^{\delta_1} + \|\xi_2\|_{\theta_2}\|u_n\|^{\delta_2}}{\|u_n\|^2} \longrightarrow 0, \quad \text{as } n \to \infty, \tag{3.10}$$

in view of $\|u_n\| \to \infty$ and $1 < \delta_1 < \delta_2 < 2$. Hence, by (3.2), (2.1), (2.2), (3.1), (3.8), (3.10) and Fatou's lemma, we obtain

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{I_\lambda(u_n)}{\|u_n\|^2} \\ &= \lim_{n\to\infty} \left[\frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(x, u_n)}{\|u_n\|^2}\mathrm{d}x - \lambda \int_{\mathbb{R}^N} \frac{G(x, u_n)}{\|u_n\|^2}\mathrm{d}x\right] \\ &= \frac{1}{2} + \lim_{n\to\infty}\left[-\int_{\Omega_n} \frac{F(x, u_n)}{u_n^2}v_n^2\mathrm{d}x - \int_{\mathbb{R}^N\setminus\Omega_n} \frac{F(x, u_n)}{u_n^2}v_n^2\mathrm{d}x\right] \\ &\le \frac{1}{2} + \limsup_{n\to\infty}\left[\left(c_1 + c_2 r_1^{p-2}\right)\int_{\mathbb{R}^N} |v_n|^2\mathrm{d}x - \int_{\mathbb{R}^N\setminus\Omega_n} \frac{F(x, u_n)}{u_n^2}v_n^2\mathrm{d}x\right] \\ &\le \frac{1}{2} + \left(c_1 + c_2 r_1^{p-2}\right)\eta_2^2 - \liminf_{n\to\infty}\int_{\mathbb{R}^N\setminus\Omega_n} \frac{F(x, u_n)}{u_n^2}v_n^2\mathrm{d}x \\ &\le C_4 - \int_{A_n} \liminf_{n\to\infty} \frac{F(x, u_n)}{u_n^2}v_n^2\mathrm{d}x \end{aligned}$$

$$= C_4 - \int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{F(x,u_n)}{u_n^2}[\chi_{A_n}(x)]v_n^2 \mathrm{d}x \longrightarrow -\infty, \quad \text{as} \quad n \to \infty. \tag{3.11}$$

This is an obvious contradiction. Consequently, $\{u_n\}$ is bounded in H.

Since $\{u_n\}$ is bounded in H, then there exists a constant $M > 0$ such that

$$\|u_n\| \le M, \quad \forall n \in \mathbb{N}. \tag{3.12}$$

Furthermore, passing to a subsequence, there is $u \in H$ such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } H;\\ &u_n \to u \text{ in } L^p(\mathbb{R}^N), \quad 2 \le p < 2_s^*;\\ &u_n \to u \text{ a.e. in } \mathbb{R}^N. \end{aligned} \tag{3.13}$$

By (F_1), (2.1), (3.12), the Hölder's inequality and (3.13), it has

$$\begin{aligned} &\int_{\mathbb{R}^N} |f(x,u_n) - f(x,u)|\,(u_n-u)\mathrm{d}x\\ &\le \int_{\mathbb{R}^N} |f(x,u_n)|(u_n-u)\mathrm{d}x + \int_{\mathbb{R}^N} |f(x,u)|(u_n-u)\mathrm{d}x\\ &\le c_1 \int_{\mathbb{R}^N} (|u_n| + |u|)\,(u_n-u)\mathrm{d}x + c_2 \int_{\mathbb{R}^N} (|u_n|^{p-1} + |u|^{p-1})(u_n-u)\mathrm{d}x\\ &\le C_5\|u_n-u\|_2 + C_6\|u_n-u\|_p = o_n(1), \end{aligned} \tag{3.14}$$

where $C_5 = c_1(\eta_2 M + \|u\|_2)$, $C_6 = c_2\left(\eta_p^{p-1}M^{p-1} + \|u\|_p^{p-1}\right)$ and $o_n(1) \to 0$ as $n \to \infty$.

On the other hand, it follows from (g_1), (2.1), (3.12), Hölder's inequality and (3.13) that

$$\begin{aligned} &\int_{\mathbb{R}^N} |g(x,u_n) - g(x,u)|(u_n-u)\mathrm{d}x\\ &\le \int_{\mathbb{R}^N} |g(x,u_n)|(u_n-u)\mathrm{d}x + \int_{\mathbb{R}^N} |g(x,u)|(u_n-u)\mathrm{d}x\\ &\le \int_{\mathbb{R}^N} \xi_1(x)|u_n|^{\delta_1-1}(u_n-u)\mathrm{d}x + \int_{\mathbb{R}^N} \xi_2(x)|u_n|^{\delta_2-1}(u_n-u)\mathrm{d}x\\ &\quad + \int_{\mathbb{R}^N} \xi_1(x)|u|^{\delta_1-1}(u_n-u)\mathrm{d}x + \int_{\mathbb{R}^N} \xi_2(x)|u|^{\delta_2-1}(u_n-u)\mathrm{d}x\\ &\le \|\xi_1\|_{\frac{2}{2-\delta_1}}\left(\|u_n\|_2^{\delta_1-1} + \|u\|_2^{\delta_1-1}\right)\|u_n-u\|_2\\ &\quad + \|\xi_2\|_{\frac{2}{2-\delta_2}}\left(\|u_n\|_2^{\delta_2-1} + \|u\|_2^{\delta_2-1}\right)\|u_n-u\|_2\\ &\le (M_1+M_2)\|u_n-u\|_2 = o_n(1), \end{aligned} \tag{3.15}$$

where $M_i = \|\xi_i\|_{\frac{2}{2-\delta_i}} \left(\eta_2^{\delta_i - 1} M^{\delta_i - 1} + \|u\|_2^{\delta_i - 1}\right)$, $i = 1, 2$. Then, combining (2.3), (3.1), (3.14) and (3.15), for $n \in \mathbb{N}$ large enough, we have

$$\begin{aligned} o_n(1) &= \langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u \rangle \\ &= \|u_n - u\|^2 - \int_{\mathbb{R}^N} [f(x, u_n) - f(x, u)] (u_n - u) \mathrm{d}x \\ &\quad - \lambda \int_{\mathbb{R}^N} [g(x, u_n) - g(x, u)] (u_n - u) \mathrm{d}x \\ &= \|u_n - u\|^2 + o_n(1). \end{aligned}$$

Consequently, $u_n \to u$ strongly in H as $n \to \infty$. Thus, the functional I satisfies the $(\mathrm{PS})_c$ condition for any $c > 0$. The proof is completed. □

Let $(X, \|\cdot\|)$ be a Banach space such that $X = \overline{\oplus_{i=1}^{\infty} X_i}$ with $\dim X_i < +\infty$ for each $i \in \mathbb{N}$. Set

$$Y_k = \bigoplus_{i=1}^{k} X_i, \quad Z_k = \bigoplus_{i=k}^{\infty} X_i.$$

In order to prove Theorem 2.1, we shall use the following Fountain Theorem.

Theorem 3.1 [24, Theorem 3.6] *Let X be an infinite dimensional Banach space. Assume that $\varphi \in C^1(X, \mathbb{R})$, $\varphi(-u) = \varphi(u)$ for all $u \in X$. If, for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that*

(A_1) *φ satisfies the $(PS)_c$ condition for every $c > 0$;*
(A_2) $a_k := \max\limits_{u \in Y_k, \|u\| = \rho_k} \varphi(u) \leq 0.$
(A_3) $b_k := \inf\limits_{u \in Z_k, \|u\| = r_k} \varphi(u) \to +\infty$ *as* $k \to \infty$.

Then φ has a sequence of critical points $\{u_k\}$ such that $\varphi(u_k) \to +\infty$.

Since $H \hookrightarrow L^2(\mathbb{R}^N)$ is compact under assumptions (V) and $L^2(\mathbb{R}^N)$ is a separable Hilbert space, then H possesses is a countable orthonormal basis $\{e_j\}_{j=1}^{\infty}$. Define

$$X_j = \mathbb{R} e_j, \quad Y_k = \bigoplus_{j=1}^{k} X_j, \quad Z_k = \overline{\bigoplus_{j=k+1}^{\infty} X_j}, \quad k \in \mathbb{N}^*.$$

Then, $H = \overline{\bigoplus_{j=1}^{\infty} X_j}$ and Y_k is finite dimensional.

Lemma 3.2 *Assume that (V), (F_1) and (g_1) hold, then there exist $\overline{\lambda} > 0$ and $r_k > 0$ such that*

$$\inf_{u \in Z_k, \|u\| = r_k} I_\lambda(u) \to +\infty \quad \text{as } k \to \infty$$

whenever $|\lambda| \leq \overline{\lambda}$.

Proof Similar to Lemma 3.8 in [24], for any $2 \leq p < 2_s^*$, we have

$$\beta_k(p) := \sup_{u \in Z_k, \|u\|=1} \|u\|_p \to 0, \tag{3.16}$$

as $k \to \infty$.

By (2.2), (3.2), (3.9) and (3.16) we obtain

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x - \lambda \int_{\mathbb{R}^N} G(x,u) \\ &\geq \frac{1}{2}\|u\|^2 - \frac{c_1}{2}\|u\|_2^2 - \frac{c_2}{p}\|u\|_p^p - \lambda \left(\|\xi_1\|_{\theta_1} \eta_2^{\delta_1} \|u\|^{\delta_1} + \|\xi_2\|_{\theta_2} \eta_2^{\delta_2} \|u\|^{\delta_2} \right) \\ &\geq \frac{1}{2}\|u\|^2 - \frac{c_1}{2}\beta_k^2(2)\|u\|^2 - \frac{c_2}{p}\beta_k^p(p)\|u\|^p - |\lambda| \\ &\quad \left(\|\xi_1\|_{\theta_1} \eta_2^{\delta_1} \|u\|^{\delta_1} + \|\xi_2\|_{\theta_2} \eta_2^{\delta_2} \|u\|^{\delta_2} \right). \end{aligned}$$

According to (3.16), we can choose a large $k_0 > 1$ so that

$$\beta_k^2(2) \leq \frac{1}{2c_1}, \quad \forall k \geq k_0.$$

This provides

$$I_\lambda(u) \geq \frac{1}{4}\|u\|^2 - \frac{c_2}{p}\beta_k^p(p)\|u\|^p - |\lambda| \left(\|\xi_1\|_{\theta_1} \eta_2^{\delta_1} \|u\|^{\delta_1} + \|\xi_2\|_{\theta_2} \eta_2^{\delta_2} \|u\|^{\delta_2} \right).$$

For any $u \in Z_k$ satisfying $\|u\| \geq 1$, we have

$$\|\xi_1\|_{\theta_1} \eta_2^{\delta_1} \|u\|^{\delta_1} + \|\xi_2\|_{\theta_2} \eta_2^{\delta_2} \|u\|^{\delta_2} \leq \left(\|\xi_1\|_{\theta_1} \eta_2^{\delta_1} + \|\xi_2\|_{\theta_2} \eta_2^{\delta_2} \right) \|u\|^{\delta_2},$$

since $1 < \delta_1 < \delta_2 < 2$. Hence, we obtain

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{4}\|u\|^2 - \frac{c_2}{p}\beta_k^p(p)\|u\|^p - |\lambda| \left(\|\xi_1\|_{\theta_1} \eta_2^{\delta_1} \|u\|^{\delta_1} + \|\xi_2\|_{\theta_2} \eta_2^{\delta_2} \|u\|^{\delta_2} \right) \\ &\geq \frac{1}{4}\|u\|^2 - \frac{c_2}{p}\beta_k^p(p)\|u\|^p - |\lambda| \left(\|\xi_1\|_{\theta_1} \eta_2^{\delta_1} + \|\xi_2\|_{\theta_2} \eta_2^{\delta_2} \right) \|u\|^{\delta_2} \\ &= \|u\|^{\delta_2} \left[\frac{1}{4}\|u\|^{2-\delta_2} - \frac{c_2}{p}\beta_k^p(p)\|u\|^{p-\delta_2} - |\lambda| K \right], \end{aligned} \tag{3.17}$$

where $K = \|\xi_1\|_{\theta_1} \eta_2^{\delta_1} + \|\xi_2\|_{\theta_2} \eta_2^{\delta_2}$. For each $k \in \mathbb{N}$ sufficiently large, taking

$$r_k := \left(\frac{p}{8c_2\beta_k^p(p)} \right)^{1/(p-2)}.$$

Then, by virtue of (3.16) we obtain

$$r_k \to +\infty \quad \text{as} \quad k \to \infty.$$

Then, there exists $k_1 > 1$ such that $r_k \geq 1$ when $k \geq k_1$. By (3.17), for $u \in Z_k$, $\|u\| = r_k$, we have

$$\begin{aligned} I_\lambda(u) &\geq r_k^{\delta_2}\left(\frac{1}{4}r_k^{2-\delta_2} - \frac{c_2}{p}\beta_k^p(p)r_k^{p-\delta_2} - |\lambda|K\right) \\ &= r_k^{\delta_2}\left(\frac{1}{4}r_k^{2-\delta_2} - \frac{c_2}{p}\beta_k^p(p)r_k^{p-\delta_2} - |\lambda|K\right) \\ &= r_k^{\delta_2}\left(\frac{1}{8}r_k^{2-\delta_2} - |\lambda|K\right). \end{aligned} \tag{3.18}$$

Putting $\lambda_k = \dfrac{r_k^{2-\delta_2}}{16K}$, then, $\lambda_k > 0$ and $\lambda_k \to \infty$ as $k \to \infty$. Let

$$\overline{\lambda} = \inf_{k \geq \overline{k}} \lambda_k$$

where $\overline{k} = \max\{k_0, k_1\}$, therefore, for any $\lambda \in \mathbb{R}$ satisfying $|\lambda| \leq \overline{\lambda}$ we get from (3.18)

$$I_\lambda(u) \geq \frac{r_k^2}{16}, \quad u \in Z_k, \ \|u\| = r_k.$$

Hence, for $k \geq \overline{k}$ we deduce

$$\inf_{u \in Z_k, \|u\| = r_k} I_\lambda(u) \geq \frac{r_k^2}{16} \to +\infty \quad \text{as} \quad k \to \infty$$

whenever $|\lambda| \leq \overline{\lambda}$. This completes the proof. □

Lemma 3.3 *For any finite dimensional subspace $Y_k \subset H$, there holds*

$$\max_{u \in Y_k, \|u\| = \rho_k} I_\lambda(u) \leq 0.$$

Proof Let Y_k be any finite dimensional subspace of H, we claim that there exists a constant $R_k = R(Y_k) > 0$ such that $I_\lambda(u) \leq 0$ $\|u\| \geq R_k$. Otherwise, there is a sequence $\{u_n\} \subset Y_k$ such that

$$\|u_n\| \to \infty \quad \text{and} \quad I_\lambda(u_n) \geq 0. \tag{3.19}$$

Set $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Therefore, by the Sobolev embedding theorem, up to a subsequence, we can assume $v_n \rightharpoonup v$ in H, $v_n \to v$ in $L^p(\mathbb{R}^N)$ $(2 \leq p < 2_s^*)$ and

$v_n \to v$ a.e. in $\mathbb{R}^N$. Set $E = \{x \in \mathbb{R}^N \ : \ v(x) \neq 0\}$. Since on the finite dimensional subspace Y_k all norms are equivalent, there exists a constant $\alpha_k > 0$ such that

$$\|u\|_p \geq \alpha_k \|u\|, \quad \forall u \in Y_k,$$

and then

$$\int_{\mathbb{R}^N} |u_n|^p \, \mathrm{d}x \geq \alpha_k^p \, \|u_n\|^p \, , \quad \forall n \in \mathbb{N},$$

which yields

$$\alpha_k^p \leq \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^p \, \mathrm{d}x = \|v_n\|_p^p.$$

Hence meas$(E) > 0$and then $|u_n(x)| \to \infty$ for all $x \in E$. Using (2.2) and (3.19) we obtain

$$\frac{1}{2}\|u_n\|^2 \geq \int_{\mathbb{R}^N} F(x, u_n)\mathrm{d}x + \lambda \int_{\mathbb{R}^N} G(x, u_n)\mathrm{d}x, \quad \forall x \in \mathbb{R}^N.$$

Therefore,

$$\frac{1}{2} \geq \int_{\mathbb{R}^N} \frac{F(x, u_n)}{\|u_n\|^2}\mathrm{d}x + \lambda \int_{\mathbb{R}^N} \frac{G(x, u_n)}{\|u_n\|^2}\mathrm{d}x.$$

Then, by (3.10) and Fatou's Lemma we deduce

$$\begin{aligned} \frac{1}{2} &\geq \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x, u_n)}{|u_n|^2}|v_n|^2\mathrm{d}x \\ &\geq \liminf_{n\to\infty} \int_E \frac{F(x, u_n)}{|u_n|^2}|v_n|^2\mathrm{d}x \geq \int_E \liminf_{n\to\infty} \frac{F(x, u_n)}{|u_n|^2}|v_n|^2\mathrm{d}x = +\infty. \end{aligned}$$

We have a contradiction. This shows that there exists a constant $R_k = R(Y_k) > 0$ such that $I(u) \leq 0$ for all $u \in Y_k \setminus B_{R_k}(0)$. Hence, choosing $\rho_k > \max\{R_k, r_k\}$, we conclude that

$$\max_{u \in Y_k, \|u\| = \rho_k} I_\lambda(u) \leq 0.$$

□

Proof of Theorem 2.1 We have $I_\lambda \in C^1(H, \mathbb{R})$ is even in view of (F_4) and (g_2). On the other hand, by Lemmas 3.1 and 3.3, the functional I_λ satisfies the conditions (A_1)–(A_2) of the Fountain Theorem 3.1, respectively. Moreover, condition (A_3) is satisfied whenever $|\lambda| \leq \overline{\lambda}$ due to Lemma 3.2. Thus, the functional I_λ has a sequence of critical points $\{u_k\} \subset H$ such that $I_\lambda(u_k) \to \infty$ as $k \to \infty$, whenever $|\lambda| \leq \overline{\lambda}$, that is, Eq. (1.5) possesses infinitely many solutions. □

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Declarations

Conflict of interest I have no conflicts of interest to disclose.

References

1. Applebaum, D.: Lévy processes-from probability to finance and quantum groups. Not. Am. Math. Soc. **51**, 1336–1347 (2004)
2. Bartsch, T., Wang, Z.Q., Willem, M.: Chapter 1—the Dirichlet problem for superlinear elliptic equations. Handb Differ Equ Stationary Part Differ Equ **2**, 1–55 (2005)
3. Bieganowski, B., Secchi, S.: Non-local to local transition for ground states of fractional Schrödinger equations on bounded domains. J. Fixed Point Theory Appl. **22**, 76 (2020)
4. Canneori, G.M., Mugnai, D.: On fractional plasma problems. Nonlinearity **31**, 3251–3283 (2018)
5. Chen, C.: Infinitely many solutions for fractional Schrödinger equations in $\mathbb{R}^N$. Electron. J. Differ. Equ. **88**, 1–15 (2016)
6. Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhikers guide to the fractional Sobolev spaces. Bull. Sci. Math. **136**(5), 521–573 (2012)
7. Du, M., Tian, L.: Infinitely many solutions of the nonlinear fractional Schrödinger equation. Discrete Contin. Dyn. Syst. **21**(10), 3407–3428 (2016)
8. Felmer, P., Torres, C.: Radial symmetry of ground states for a regional fractional nonlinear Schrödinger equation. Commun. Pure Appl. Anal. **13**(6), 2395–2406 (2014)
9. Ge, B.: Multiple solutions of nonlinear Schrödinger equation with fractional Laplacian. Nonlinear Anal. Real World Appl. **30**, 236–247 (2016)
10. Gilboa, G., Osher, S.: Nonlocal operators with applications to image processing. Multiscale Model. Simul. **7**(3), 1005–1028 (2008)
11. Hou, G.L., Ge, B., Lu, J.F.: Infinitely many solutions for sublinear fractional Schrödinger-type equation with general potential. Electron. J. Differ. Equ. **97**, 1–13 (2018)
12. Khoutir, S., Chen, H.: Existence of infinitely many high energy solutions for a fractional Schrödinger equation in $\mathbb{R}^N$. Appl. Math. Lett. **61**, 156–162 (2016)
13. Khoutir, S.: Multiplicity results for a fractional Schrödinger equation with potentials. Rocky Mt. J. Math. **49**(7), 2205–2226 (2019)
14. Laskin, N.: Fractional quantum mechanics and Lévy path integrals. Phys. Lett. A **268**(4), 298–305 (2000)
15. Laskin, N.: Fractional Schrödinger equation. Phys. Rev. E **66**(5), 056108 (2002)
16. Laskin, N.: Principles of Fractional Quantum Mechanics, Fractional Dynamics, pp. 393–427. World Science, Hackensack (2011)
17. Li, P., Shang, Y.: Infinitely many solutions for fractional Schrödinger equations with perturbation via variational methods. Open Math. **15**, 578–586 (2017)
18. Lin, X., Tang, X.H.: Existence of infinitely many solutions for p-Laplacian equations in $\mathbb{R}^N$. Nonlinear Anal. Theory Methods Appl. **92**, 72–81 (2013)
19. Shen, Z., Han, Z., Zhang, Q.: Ground states of nonlinear Schrödinger equations with fractional Laplacians. Discrete Contin. Dyn. Syst. S **12**(7), 2115–2125 (2019)
20. Teng, K.: Multiple solutions for a class of fractional Schrödinger equations in $\mathbb{R}^N$. Nonlinear Anal. Real World Appl. **21**, 76–86 (2015)
21. Timoumi, M.: Infinitely many solutions for fractional Schrödinger equations with superquadratic conditions or combined nonlinearities. J. Korean Math. Soc. **57**(4), 825–844 (2020)
22. Wang, Z., Zhou, H.S.: Radial sign-changing solution for fractional Schrödinger equation. Discrete Contin. Dyn. Syst. **36**(1), 499–508 (2016)
23. Wang, Q.: Multiple positive solutions of fractional elliptic equations involving concave and convex nonlinearities in $\mathbb{R}^N$. Commun. Pure Appl. Anal. **15**(5), 1671–1688 (2016)
24. Willem, M.: Minimax Theorems. Birkhäuser, Boston (1996)

25. Zou, W.: Variant fountain theorems and their applications. Manuscr. Math. **104**, 343–358 (2001)

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