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Sharp Bounds on Hermitian Toeplitz Determinants for Sakaguchi Classes

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Abstract

The main purpose of this paper is to derive the sharp lower and upper bounds on Hermitian Toeplitz determinants for starlike and convex functions with respect to symmetric points. Some of the results provide improvements (or corrections) to several recent results.

Keywords Analytic function · Sakaguchi class · Hermitian Toeplitz determinant

Mathematics Subject Classification 30C55 · 30C45

1 Introduction

Let $\mathcal{A}$ denote the class of functions *analytic* in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

We denote $\mathcal{S}$ by the subclass of $\mathcal{A}$ whose elements are univalent functions.

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Sakaguchi [40] (see also [38, 43, 46]) once introduced a class $\mathcal{S}_s^*$ of starlike functions with respect to symmetric points, it consists of functions $f \in \mathcal{S}$ satisfying

$$\Re\left(\frac{2zf'(z)}{f(z)-f(-z)}\right) > 0 \quad (z \in \mathbb{D}).$$

In a later paper, Das and Singh [13] discussed a class $\mathcal{K}_s$ of convex functions with respect to symmetric points, it consists of functions $f \in \mathcal{S}$ satisfying

$$\Re\left(\frac{2\left(zf'(z)\right)'}{\left(f(z)-f(-z)\right)'}\right) > 0 \quad (z \in \mathbb{D}).$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_s^*(\alpha)$, consisting of starlike functions of order α with respect to symmetric points, if it satisfies the following condition

$$\Re\left(\frac{2zf'(z)}{f(z)-f(-z)}\right) > \alpha \quad (0 \le \alpha < 1;\ z \in \mathbb{D}).$$

Denote $\mathcal{K}_s(\alpha)$ by the class of convex functions of order α with respect to symmetric points, which satisfy the condition

$$\Re\left(\frac{2\left(zf'(z)\right)'}{\left(f(z)-f(-z)\right)'}\right) > \alpha \quad (0 \le \alpha < 1;\ z \in \mathbb{D}).$$

We note that $\mathcal{S}_s^*(0) =: \mathcal{S}_s^*$ and $\mathcal{K}_s(0) =: \mathcal{K}_s$.

Recently, Cunda et al. [12] (see also [26]) introduced the notation of Hermitian Toeplitz determinants for the class $\mathcal{A}$, and some of its subclasses. Hermitian Toeplitz matrices play important roles in functional analysis, applied mathematics as well as in physics and technical sciences, e.g., the Szegö theory, the stochastic filtering, the signal processing, the biological information processing and other engineering problems.

Given $q, n \in \mathbb{N}$, the Hermitian Toeplitz matrix $T_{q,n}(f)$ of a function $f \in \mathcal{A}$ of the form (1.1) is defined by

$$T_{q,n}(f) = \begin{pmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ \overline{a}_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{a}_{n+q-1} & \overline{a}_{n+q-2} & \cdots & a_n \end{pmatrix},$$

where $\overline{a}_k := \overline{a_k}$. For convenience, we let $\det(T_{q,n})(f)$ denote the determinant of $T_{q,n}(f)$.

By the definition, $\det(T_{2,1})(f)$, $\det(T_{3,1})(f)$ and $\det(T_{4,1})(f)$ are given by

$$\det(T_{2,1})(f) = \begin{vmatrix} a_1 & a_2 \\ \overline{a}_2 & a_1 \end{vmatrix}, \quad \det(T_{3,1})(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ \overline{a}_2 & a_1 & a_2 \\ \overline{a}_3 & \overline{a}_2 & a_1 \end{vmatrix},$$

and

$$\det(T_{4,1})(f) = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ \overline{a}_2 & a_1 & a_2 & a_3 \\ \overline{a}_3 & \overline{a}_2 & a_1 & a_2 \\ \overline{a}_4 & \overline{a}_3 & \overline{a}_2 & a_1 \end{vmatrix},$$

respectively. Note that for $f \in \mathcal{A}, a_1 = 1, \det(T_{2,1})(f), \det(T_{3,1})(f)$ and $\det(T_{4,1})(f)$ reduce to

$$\det(T_{2,1})(f) = 1 - |a_2|^2, \tag{1.2}$$
$$\det(T_{3,1})(f) = 1 - 2|a_2|^2 - |a_3|^2 + 2\Re\big(a_2^2\overline{a}_3\big), \tag{1.3}$$

and

$$\begin{aligned}\det(T_{4,1})(f) = {} & 1 - 3|a_2|^2 + |a_2|^4 - 2|a_2|^2|a_3|^2 - 2|a_3|^2 + |a_3|^4 + |a_2|^2|a_4|^2 - |a_4|^2 \\ & + 4\Re\big(a_2^2\overline{a}_3\big) + 4\Re\big(a_2a_3\overline{a}_4\big) - 2\Re\big(a_2^3\overline{a}_4\big) - 2\Re\big(a_2\overline{a}_3^2a_4\big),\end{aligned} \tag{1.4}$$

respectively.

In recent years, many investigations have been devoted to finding bounds of determinants, whose elements are coefficients of functions in $\mathcal{A}$, or its subclasses. Hankel matrices, i.e., square matrices which have constant entries along the reverse diagonal, and the symmetric Toeplitz determinants are of particular interest (see [1]).

The sharp upper bounds on the second Hankel determinants were obtained by [2, 6, 10, 15, 16, 31, 34], for various classes of analytic functions. We refer to [4, 5, 7, 9, 18, 20, 22, 25, 28, 37, 39, 42, 44, 45, 47] for discussions on the upper bounds of the third or fourth Hankel determinants for various classes of univalent functions. However, some of these results are far from sharpness. In a recent paper, Kwon *et al.* [21] found such a formula of expressing c_4 by Carathéodory functions, the sharp results of the third Hankel determinants are found for some classes of univalent functions.

The Hermitian Toeplitz determinants in relation to normalized analytic functions is a natural concept to study. By the work of [12], the study of the Hermitian Toeplitz determinants on classes of normalized analytic functions has been initiated. We refer to [3, 11, 12, 17, 23, 26, 27, 30, 35] for discussions on the sharp bounds of Hermitian Toeplitz determinants for various classes of univalent functions.

Recently, Krishna et al. [19] (see also [8, 14, 36]) obtained upper bounds of the third Hankel determinants for the classes of starlike and convex functions with respect to symmetric points. Moreover, Kumar and Kumar [24] obtained the following sharp bounds of the second- and third-order Hermitian Toeplitz determinants for starlike and convex functions of order α with respect to symmetric points.

Theorem A. *Let* $\alpha \in [0, 1)$. *If* $f \in \mathcal{S}_s^*(\alpha)$, *then*

$$(2-\alpha)\alpha \le \det(T_{2,1})(f) \le 1,$$

and

$$(3-2\alpha)\alpha^2 \le \det(T_{3,1})\,(f) \le 1.$$

All inequalities are sharp.

Theorem B. *Let* $\alpha \in [0, 1)$. *If* $f \in \mathcal{K}_s(\alpha)$, *then*

$$(2-\alpha)\alpha \le \det(T_{2,1})(f) \le 1,$$

and

$$\frac{1}{9}(-4+20\alpha-\alpha^2-6\alpha^3) \le \det(T_{3,1})(f) \le 1.$$

All inequalities are sharp.

We note that the proof of Theorem B exists several errors, and the lower bounds are not true. For the sake of completeness, we give the corrected proofs in the next section.

In this paper, we aim at deriving the sharp bounds on the second and third-order Hermitian Toeplitz determinants for the class of convex functions of order α with respect to symmetric points. We observe that the problem of finding sharp estimates of the Hermitian Toeplitz determinants $\det(T_{q,1})(f)$ for $q \ge 4$ is technically much more difficult, and few sharp bounds have been obtained. Recently, Lecko et al. [29] obtained the sharp bounds on $\det(T_{4,1})(f)$ of the class of convex functions. We shall find the sharp bounds on $\det(T_{4,1})(f)$ of the class $\mathcal{S}_s^*$.

Denote $\mathcal{P}$ by the class of *Carathéodory functions* p normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{D}) \tag{1.5}$$

and satisfy the condition $\Re\big(p(z)\big) > 0$.

The following results will be required in the proof of our main results.

Lemma 1.1 (See [32, 33]) *If* $p \in \mathcal{P}$, *then*

$$2c_2 = c_1^2 + (4-c_1^2)\zeta \tag{1.6}$$

and

$$4c_3 = c_1^3 + \left(4-c_1^2\right) c_1\zeta(2-\zeta) + 2\left(4-c_1^2\right)\left(1-|\zeta|^2\right)\eta \tag{1.7}$$

for some $\zeta, \eta \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$.

Lemma 1.2 *Let* $\alpha \in [0, 1)$. *If* $f \in \mathcal{A}$ *and*

$$\frac{2zf'(z)}{f(z)-f(-z)} = \frac{1+(1-2\alpha)z}{1-z} \quad (0 \le \alpha < 1;\ z \in \mathbb{D}), \tag{1.8}$$

then $f \in \mathcal{S}_s^*(\alpha)$ *and*

$$a_{2n+1} = a_{2n} = \prod_{k=1}^{n}\left(1-\frac{\alpha}{k}\right) \quad (n = 1, 2, 3, \cdots). \tag{1.9}$$

Proof For the function $f \in \mathcal{A}$ given by (1.1), in view of (1.8), we get $f \in \mathcal{S}_s^*(\alpha)$, that is

$$\Re\left(\frac{2zf'(z)}{f(z)-f(-z)}\right) = \Re\left(\frac{1+(1-2\alpha)z}{1-z}\right) > \alpha \quad (0 \leq \alpha < 1;\ z \in \mathbb{D}).$$

By (1.8) and elementary calculations, we have

$$(1-z)\left(z+\sum_{n=2}^{\infty} na_n z^n\right) = \left[1+(1-2\alpha)z\right]\left(z+\sum_{n=2}^{\infty} a_{2n-1}z^{2n-1}\right). \tag{1.10}$$

It follows from (1.10) that

$$2na_{2n} - (2n-1)a_{2n-1} = (1-2\alpha)a_{2n-1} \quad (n = 1, 2, 3, \cdots),$$

and

$$(2n+1)a_{2n+1} - 2na_{2n} = a_{2n+1} \quad (n = 1, 2, 3, \cdots).$$

Therefore, by virtue of the above relationships, we get

$$\begin{aligned} a_{2n+1} = a_{2n} &= \left(1-\frac{\alpha}{n}\right)a_{2n-1} = \left(1-\frac{\alpha}{n}\right)a_{2n-2} \\ &= \left(1-\frac{\alpha}{n}\right)\left(1-\frac{\alpha}{n-1}\right)a_{2n-3} = \left(1-\frac{\alpha}{n}\right)\left(1-\frac{\alpha}{n-1}\right)a_{2n-4} \\ &= \cdots\cdots \\ &= \left(1-\frac{\alpha}{n}\right)\left(1-\frac{\alpha}{n-1}\right)\cdots\left(1-\frac{\alpha}{2}\right)\left(1-\frac{\alpha}{1}\right)a_1 \\ &= \prod_{k=1}^{n}\left(1-\frac{\alpha}{k}\right) \quad (n = 1, 2, 3, \cdots). \end{aligned}$$

□

In view of Lemma 1.2 and the relationship

$$f(z) \in \mathcal{K}_s(\alpha) \Longleftrightarrow zf'(z) \in \mathcal{S}_s^*(\alpha) \quad (0 \leq \alpha < 1;\ z \in \mathbb{D}), \tag{1.11}$$

we know that

$$f_1(z) = z + \sum_{n=1}^{\infty} \left[\prod_{k=1}^{n} \left(1 - \frac{\alpha}{k}\right)\left(z^{2n} + z^{2n+1}\right) \right] \quad (0 \le \alpha < 1;\ z \in \mathbb{D}) \tag{1.12}$$

belongs to the class $\mathcal{S}_s^*(\alpha)$, and

$$f_2(z) = z + \sum_{n=1}^{\infty} \left[\prod_{k=1}^{n} \left(1 - \frac{\alpha}{k}\right)\left(\frac{z^{2n}}{2n} + \frac{z^{2n+1}}{2n+1}\right) \right] \quad (0 \le \alpha < 1;\ z \in \mathbb{D}) \tag{1.13}$$

belongs to the class $\mathcal{K}_s(\alpha)$.

2 Main results

We begin by determining the sharp bounds for $\det(T_{2,1})(f)$ and $\det(T_{3,1})(f)$ in the class of convex functions of order α with respect to symmetric points. By observing that the coefficient a_2 of $f \in \mathcal{K}_s(\alpha)$ in [24, Formula (2.7)] was written as

$$a_2 = \frac{1}{2}(1 - \alpha)c_1,$$

thus, the lower bounds of $\det(T_{2,1})(f)$ [24, p.1046, line 17] and $\det(T_{3,1})(f)$ [24, Theorem 2.4] are not true. Theorems 2.1 and 2.2 are the corrected versions of $\det(T_{2,1})(f)$ and $\det(T_{3,1})(f)$ for the class $\mathcal{K}_s(\alpha)$, respectively.

Theorem 2.1 *Let $\alpha \in [0, 1)$. If $f \in \mathcal{K}_s(\alpha)$, then*

$$1 - \frac{1}{4}(1 - \alpha)^2 \le \det(T_{2,1})(f) \le 1.$$

Both inequalities are sharp with equalities attained by $f_2(z)$ defined by (1.13), and by the identity function $f(z) = z$, respectively.

Proof For the function $f \in \mathcal{K}_s(\alpha)$ given by (1.1), we know that there exists an analytic function $p \in \mathcal{P}$ in the unit disk $\mathbb{D}$ with $p(0) = 1$ and $\Re\big(p(z)\big) > 0$ such that

$$\frac{2\big(zf'(z)\big)'}{\big(f(z) - f(-z)\big)'} = (1 - \alpha)p(z) + \alpha \quad (z \in \mathbb{D}).$$

By elementary calculations, we have

$$1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1} = \left(1 + (1 - \alpha)\sum_{n=1}^{\infty} c_n z^n\right)\left(1 + \sum_{n=2}^{\infty}(2n - 1)a_{2n-1} z^{2n-2}\right). \tag{2.1}$$

It follows from (2.1) that

$$a_2 = \frac{1}{4}(1-\alpha)c_1, \quad a_3 = \frac{1}{6}(1-\alpha)c_2. \tag{2.2}$$

Since the class $\mathcal{K}_s(\alpha)$ and $\det(T_{2,1})(f)$ are rotationally invariants, we may assume that $c := c_1 \in [0, 2]$. By using (1.2) and $|c_1| \le 2$, we see that

$$\det(T_{2,1})(f) = 1 - \frac{1}{16}(1-\alpha)^2|c_1|^2 \in \left[1 - \frac{1}{4}(1-\alpha)^2, 1\right].$$

Obviously, the sharp estimates are attained by the extremal function $f_2(z)$ defined by (1.13), and identity function $f(z) = z$, respectively. □

Theorem 2.2 *Let* $\alpha \in [0, 1)$. *If* $f \in \mathcal{K}_s(\alpha)$, *then*

$$1 - \frac{1}{18}(1-\alpha)^2(8+3\alpha) \le \det(T_{3,1})(f) \le 1. \tag{2.3}$$

Both inequalities are sharp with equalities attained by $f_2(z)$ *defined by (1.13), and by the identity function* $f(z) = z$, *respectively.*

Proof Let $f \in \mathcal{K}_s(\alpha)$ be given by (1.1). Since the class $\mathcal{K}_s(\alpha)$ and $\det(T_{3,1})(f)$ are rotationally invariants, we may assume that $c := c_1 \in [0, 2]$. Thus, (1.3) and (2.2) show that

$$\det(T_{3,1})(f) = 1 - \frac{1}{8}(1-\alpha)^2c^2 - \frac{1}{36}(1-\alpha)^2\left|c_2\right|^2 + \frac{1}{48}(1-\alpha)^3c^2\Re(c_2).$$

By virtue of (1.6), we conclude that

$$\begin{aligned}\det(T_{3,1})(f) = 1 &- \frac{1}{8}(1-\alpha)^2c^2 + \frac{1}{288}(1-\alpha)^2(1-3\alpha)c^4 \\ &- \frac{1}{288}(1-\alpha)^2(1+3\alpha)c^2\left(4-c^2\right)\Re(\zeta) \\ &- \frac{1}{144}(1-\alpha)^2\left(4-c^2\right)^2|\zeta|^2 \\ =: &\ \Psi(c, |\zeta|, \Re(\zeta))\end{aligned} \tag{2.4}$$

for some $c \in [0, 2]$ and $\zeta \in \overline{\mathbb{D}}$.

A. We first prove the right-side inequality in (2.3).

By means of (2.4), we get

$$\begin{aligned}\det(T_{3,1})(f) &= \Psi(c, |\zeta|, \Re(\zeta)) \le \Psi(c, |\zeta|, -|\zeta|) \\ &=: P\left(c^2, |\zeta|\right) \quad ((c, |\zeta|) \in [0, 2] \times [0, 1]),\end{aligned} \tag{2.5}$$

where $P:\ [0,4]\times[0,1]\to\mathbb{R}$ is defined by

$$P(x,y)=1-\frac{1}{8}(1-\alpha)^2x+\frac{1}{288}(1-\alpha)^2(1-3\alpha)x^2 \\ +\frac{1}{288}(1-\alpha)^2(1+3\alpha)(4-x)xy-\frac{1}{144}(1-\alpha)^2(4-x)^2y^2.$$

A1. For the case $x=0$, we have

$$P(0,y)=1-\frac{1}{9}(1-\alpha)^2y^2\leq 1\quad (y\in[0,1]).$$

A2. For the case $x=4$, we obtain

$$P(4,y)=1-\frac{1}{18}(1-\alpha)^2(8+3\alpha)\quad (y\in[0,1]).$$

A3. For the case $y=0$, we get

$$P(x,0)=1-\frac{1}{8}(1-\alpha)^2x+\frac{1}{288}(1-\alpha)^2(1-3\alpha)x^2\quad (x\in[0,4]).$$

For $\alpha=\frac{1}{3}$, we find that

$$P(x,0)=1-\frac{1}{18}x\leq P(0,0)\quad (x\in[0,4]).$$

For $0<\alpha<\frac{1}{3}$, we know that

$$\begin{aligned}P(x,0)&=\frac{1}{288}(1-\alpha)^2(1-3\alpha)\left(x-\frac{18}{1-3\alpha}\right)^2-\frac{(1+3\alpha)^2}{1-3\alpha}\\&\leq P(0,0)\quad (x\in[0,4]).\end{aligned}$$

For $\frac{1}{3}<\alpha<1$, we see that

$$\begin{aligned}P(x,0)&=\frac{1}{288}(1-\alpha)^2(1-3\alpha)\left(x-\frac{18}{1-3\alpha}\right)^2-\frac{(1+3\alpha)^2}{1-3\alpha}\\&\leq P(0,0)\quad (x\in[0,4]).\end{aligned}$$

Thus, for all $0\leq\alpha<1$, we have

$$P(x,0)\leq P(0,0)=1\quad (x\in[0,4]).$$

A4. For the case $y = 1$, we see that

$$
\begin{aligned}
P(x,1) &= 1 - \frac{1}{9}(1-\alpha)^2 - \frac{1}{72}(1-\alpha)^2(4-3\alpha)x - \frac{1}{144}(1-\alpha)^2(1+3\alpha)x^2 \\
&= -\frac{1}{144}(1-\alpha)^2(1+3\alpha)\left(x - \frac{3\alpha-4}{1+3\alpha}\right)^2 \\
&\quad + 1 - \frac{1}{9}(1-\alpha)^2 + \frac{1}{144}\frac{[(1-\alpha)(4-3\alpha)]^2}{1+3\alpha} \\
&\le P(0,1) = 1 \quad (x \in [0,4]).
\end{aligned}
$$

A5. Let $(x, y) \in (0, 4) \times (0, 1)$. Then

$$
\frac{\partial P}{\partial y} = \frac{1}{288}(1-\alpha)^2(4-x)\big[(1+3\alpha)x - 4(4-x)y\big] = 0
$$

if and only if

$$
y_0 = \frac{(1+3\alpha)x}{4(4-x)}.
$$

Therefore, we see that

$$
\frac{\partial P}{\partial x}\big(x, y_0\big) = 0
$$

if and only if

$$
(1-\alpha)^2x^2 - 4[(1-\alpha)^2 + 4]x + 64 = 0. \tag{2.6}
$$

For $0 \le \alpha < 1$, the equation (2.6) has solutions:

$$
x_1 = 4, \quad x_2 = \frac{16}{(1-\alpha)^2} \ge 16.
$$

Thus, $P(x, y)$ has no critical point in $(0, 4) \times (0, 1)$.

It now follows from **A**1-**A**5 that

$$
\det(T_{3,1})(f) \le \max\left\{1,\ 1 - \frac{1}{18}(1-\alpha)^2(8+3\alpha)\right\} = 1
$$

for $0 \le \alpha < 1$ and $(x, y) \in [0, 4] \times [0, 1]$. By virtue of (2.5), we deduce that the upper bound in (2.3) holds.

B. We now prove the left-side inequality in (2.3).

Note that

$$\det(T_{3,1})(f) = \Psi(c, |\zeta|, \Re(\zeta)) \geq \Psi(c, |\zeta|, |\zeta|) \\ =: Q(c^2, |\zeta|) \quad ((c, |\zeta|) \in [0, 2] \times [0, 1]), \tag{2.7}$$

where $Q: [0, 4] \times [0, 1] \longrightarrow \mathbb{R}$ is defined by

$$Q(x, y) = 1 - \frac{1}{8}(1-\alpha)^2 x + \frac{1}{288}(1-\alpha)^2(1-3\alpha)x^2 \\ - \frac{1}{288}(1-\alpha)^2(1+3\alpha)(4-x)xy - \frac{1}{144}(1-\alpha)^2(4-x)^2 y^2.$$

B1. For the case $x = 0$, we obtain

$$Q(0, y) = 1 - \frac{1}{9}(1-\alpha)^2 y^2 \geq 1 - \frac{1}{9}(1-\alpha)^2 \quad (y \in [0, 1]).$$

B2. For the case $x = 4$, we get

$$Q(4, y) = 1 - \frac{1}{18}(1-\alpha)^2(8+3\alpha) \quad (y \in [0, 1]).$$

B3. For the case $y = 0$, we have

$$Q(x, 0) = 1 - \frac{1}{8}(1-\alpha)^2 x + \frac{1}{288}(1-\alpha)^2(1-3\alpha)x^2 \quad (x \in [0, 4]).$$

For $\alpha = \frac{1}{3}$, we know that

$$Q(x, 0) = 1 - \frac{1}{18}x \geq Q(4, 0) \quad (x \in [0, 4]).$$

For $0 < \alpha < \frac{1}{3}$, we find that

$$Q(x, 0) = \frac{1}{288}(1-\alpha)^2(1-3\alpha)\left(x - \frac{18}{1-3\alpha}\right)^2 - \frac{(1+3\alpha)^2}{1-3\alpha} \\ \geq Q(4, 0) \quad (x \in [0, 4]).$$

For $\frac{1}{3} < \alpha < 1$, we see that

$$Q(x, 0) = \frac{1}{288}(1-\alpha)^2(1-3\alpha)\left(x - \frac{18}{1-3\alpha}\right)^2 - \frac{(1+3\alpha)^2}{1-3\alpha} \\ \geq Q(4, 0) \quad (x \in [0, 4]).$$

Thus, for all $0 \le \alpha < 1$, we have

$$Q(x,0) \ge Q(4,0) = 1 - \frac{1}{18}(1-\alpha)^2(8+3\alpha) \quad (x \in [0,4]).$$

B4. For the case $y = 1$, we find that

$$\begin{aligned} Q(x,1) &= 1 - \frac{1}{9}(1-\alpha)^2 - \frac{1}{24}(1-\alpha)^2(2+\alpha)x \\ &\ge Q(4,1) = 1 - \frac{1}{18}(1-\alpha)^2(8+3\alpha) \quad (x \in [0,4]). \end{aligned}$$

B5. Let $(x, y) \in (0,4) \times (0,1)$. Then

$$\frac{\partial Q}{\partial y} = -\frac{1}{288}(1-\alpha)^2(4-x)\big[(1+3\alpha)x + 4(4-x)y\big] = 0$$

if and only if

$$y_0 = -\frac{(1+3\alpha)x}{4(4-x)}.$$

Obviously, for $0 \le \alpha < 1$ and $x \in (0,4)$, we have $y_0 < 0$. Thus, $Q(x, y)$ has no critical point in $(0,4) \times (0,1)$.

In summary, when $0 \le \alpha < 1$ and $(x, y) \in [0,4] \times [0,1]$, parts **B**1-**B**5 imply that

$$\begin{aligned} Q(x,y) &\ge \min\left\{1 - \frac{1}{9}(1-\alpha)^2,\ 1 - \frac{1}{18}(1-\alpha)^2(8+3\alpha)\right\} \\ &= 1 - \frac{1}{18}(1-\alpha)^2(8+3\alpha). \end{aligned}$$

Furthermore, by virtue of (2.7), we get the lower bound in (2.3).

At last, we show that inequalities (2.3) are sharp.

In view of part **A** and part **B**, the right-side equality in (2.3) clearly holds for the identity function $f(z) = z$. By means of the relationships (1.11) and (1.8), the sharp function f_2 is given by (1.13) with

$$a_2 = \frac{1}{2}(1-\alpha), \quad a_3 = \frac{1}{3}(1-\alpha),$$

and yields the left-side equality in (2.3). This completes the proof of Theorem 2.2. □

By choosing $\alpha = 0$ in Theorem 2.1 and Theorem 2.2, we get the following corollary.

Corollary 2.1 *If $f \in \mathcal{K}_s$, then*

$$\frac{3}{4} \le \det(T_{2,1})(f) \le 1,$$

and

$$\frac{5}{9} \leq \det(T_{3,1})(f) \leq 1.$$

All the inequalities are sharp.

Now, we consider the sharp bounds of $\det(T_{4,1})(f)$ for the class $\mathcal{S}_s^*$ of starlike functions with respect to symmetric points.

Theorem 2.3 *If $f \in \mathcal{S}_s^*$ be of the form (1.1), then*

$$-\frac{27}{256} \leq \det(T_{4,1})(f) \leq 1. \tag{2.8}$$

Both inequalities are sharp with equalities attained by $f(z)$ defined by (2.22), and by the identity $f(z) = z$, respectively.

Proof For the function $f \in \mathcal{S}_s^*$ given by (1.1), we know that there exists an analytic function $p \in \mathcal{P}$ in the unit disk $\mathbb{D}$ with $p(0) = 1$ and $\Re\big(p(z)\big) > 0$ such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = p(z) \quad (z \in \mathbb{D}). \tag{2.9}$$

By elementary calculations, we have

$$z + \sum_{n=2}^{\infty} na_n z^n = \left(1 + \sum_{n=1}^{\infty} c_n z^n\right)\left(z + \sum_{n=2}^{\infty} a_{2n-1} z^{2n-1}\right). \tag{2.10}$$

It follows from (2.10) that

$$a_2 = \frac{1}{2}c_1, \quad a_3 = \frac{1}{2}c_2, \quad a_4 = \frac{1}{8}(2c_3 + c_1c_2). \tag{2.11}$$

Since the class $\mathcal{S}_s^*$ and $\det(T_{4,1})(f)$ are rotationally invariant, we may assume that $c := c_1 \in [0, 2]$. Thus, (1.4) and (2.11) give

$$\begin{aligned}
\det(T_{4,1})(f) =& 1 - \frac{3}{4}c^2 + \frac{1}{16}c^4 - \frac{1}{2} \cdot |c_2|^2 + \frac{1}{16} \cdot |c_2|^4 \\
& - \frac{1}{256}c^2\left(4 - c^2\right) \cdot |c_2|^2 - \frac{1}{64}\left(4 - c^2\right) \cdot |c_3|^2 \\
& + \frac{1}{2}c^2 \cdot \Re\big(c_2\big) - \frac{1}{32}c^4 \cdot \Re\big(c_2\big) - \frac{1}{32}c^2 \cdot |c_2|^2 \cdot \Re\big(c_2\big) \\
& - \frac{1}{16}c^3 \cdot \Re\big(c_3\big) + \frac{1}{64}c\left(c^2 + 12\right) \cdot \Re\big(c_2\overline{c_3}\big) - \frac{1}{16}c \cdot \Re\big(c_2^2\overline{c_3}\big).
\end{aligned}$$

Hence, by using (1.6) and (1.7), we get

$$\begin{aligned}\det(T_{4,1})(f) = \frac{1}{1024}\left(4-c^2\right)^3 \cdot \Big[16 - (32+c^2)\cdot|\zeta|^2 + 2c^2\cdot|\zeta|^2\cdot\Re\big(\zeta\big) \\ + \left(16-c^2\right)\cdot|\zeta|^4 + 4c\cdot\left(1-|\zeta|^2\right)\cdot\Re\big(\overline{\zeta}\eta\big) \\ - 4c\cdot\left(1-|\zeta|^2\right)\cdot\Re\big(\overline{\zeta^2}\eta\big) - 4\cdot\left(1-|\zeta|^2\right)^2\cdot|\eta|^2\Big]\end{aligned} \tag{2.12}$$

for some $c \in [0,2]$ and $\zeta,\ \eta \in \overline{\mathbb{D}}$.

We now consider the lower and upper bounds for the class $\mathcal{S}_s^*$ for various cases.

A. Suppose that $\zeta = 0$. Then

$$0 \le \det(T_{4,1})(f) = \frac{1}{1024}\left(4-c^2\right)^3 \cdot \left(16 - 4|\eta|^2\right) \le 1 \tag{2.13}$$

for all $c \in [0,2]$ and $\eta \in \overline{\mathbb{D}}$.

B. Suppose that $\eta = 0$. Then

$$\begin{aligned}\det(T_{4,1})(f) &= \frac{1}{1024}\left(4-c^2\right)^3 \cdot \Big[16 - \left(32+c^2\right)\cdot|\zeta|^2 + 2c^2\cdot|\zeta|^2\cdot\Re\big(\zeta\big) \\ &\quad + \left(16-c^2\right)\cdot|\zeta|^4\Big].\end{aligned} \tag{2.14}$$

It follows that

$$\begin{aligned}\det(T_{4,1})(f) &\ge \frac{1}{1024}\left(4-c^2\right)^3 \cdot \Big[16 - \left(32+c^2\right)\cdot|\zeta|^2 - 2c^2\cdot|\zeta|^3 \\ &\quad + \left(16-c^2\right)\cdot|\zeta|^4\Big] \\ &= \frac{1}{1024}\left(4-c^2\right)^3 \cdot \Big[16\big(1-|\zeta|^2\big)^2 - c^2\cdot|\zeta|^2\cdot(1+|\zeta|)^2\Big] \\ &=: P(c^2, |\zeta|),\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}\det(T_{4,1})(f) &\le \frac{1}{1024}(4-c^2)^3 \cdot \Big[16 - (32+c^2)\cdot|\zeta|^2 + 2c^2\cdot|\zeta|^3 \\ &\quad + (16-c^2)\cdot|\zeta|^4\Big] \\ &= \frac{1}{1024}(4-c^2)^3 \cdot \Big[16\big(1-|\zeta|^2\big)^2 - c^2\cdot|\zeta|^2\cdot(1-|\zeta|)^2\Big] \\ &=: Q(c^2, |\zeta|),\end{aligned} \tag{2.16}$$

where $P,\ Q:\ [0,4]\times[0,1]\longrightarrow\mathbb{R}$ is defined by

$$P(u,x)=\frac{1}{1024}(4-u)^3\cdot\left[16(1-x^2)^2-ux^2(1+x)^2\right],$$

and

$$Q(u,x)=\frac{1}{1024}(4-u)^3\cdot\left[16(1-x^2)^2-ux^2(1-x)^2\right],$$

respectively.

B1. We discuss the lower bound of $P(u,x)$.

(i) On the vertices of $[0,4]\times[0,1]$, we have

$$P(0,0)=1,\ \ P(0,1)=P(4,0)=P(4,1)=0.$$

(ii) On the side $u=0$, we get

$$P(0,x)=(1-x^2)^2\geq 0\quad (x\in[0,1]).$$

(iii) On the side $u=4$, we obtain

$$P(4,x)=0\quad (x\in[0,1]).$$

(iv) On the side $x=0$, we see that

$$P(u,0)=\frac{1}{64}(4-u)^3\geq 0\quad (u\in[0,4]).$$

(v) On the side $x=1$, we know that

$$P(u,1)=-\frac{1}{256}u(4-u)^3=:\phi(u)\quad (u\in[0,4]).$$

Note that

$$\phi'(u)=-\frac{1}{64}(4-u)^2(1-u)\quad (u\in[0,4]),$$

for $0\leq u\leq 1$, we know that $\phi'(u)\leq 0$, which implies that

$$\phi(u)\geq\phi(1)=-\frac{27}{256},$$

and for $1\leq u\leq 4$, we find that $\phi'(u)\geq 0$, which shows that

$$\phi(u)\geq\phi(1)=-\frac{27}{256}.$$

Thus, we deduce that

$$P(u,1)=\phi(u)\geq-\frac{27}{256}\quad(u\in[0,4]).$$

(vi) In the interior of $(0,4)\times(0,1)$, since the equation

$$\frac{\partial P}{\partial x}=-\frac{1}{512}x(1+x)(4-u)^3[32(1-x)+u(1+2x)]=0$$

has no solution in $(0,4)\times(0,1)$, we see that P has no critical point in the interior of $(0,4)\times(0,1)$.

Therefore, from (2.15), it follows that

$$\det(T_{4,1})(f)\geq P\left(c^2,|\zeta|\right)\geq-\frac{27}{256}\quad((c^2,|\zeta|)\in[0,4]\times[0,1]).$$

B2. We next discuss the upper bound of $Q(u,x)$.

(i) On the vertices of $[0,4]\times[0,1]$, we get

$$Q(0,0)=1,\ \ Q(0,1)=Q(4,0)=Q(4,1)=0.$$

(ii) On the side $u=0$, we have

$$Q(0,x)=\left(1-x^2\right)^2\leq1\quad(x\in[0,1]).$$

(iii) On the side $u=4$, we obtain

$$Q(4,x)=0\quad(x\in[0,1]).$$

(iv) On the side $x=0$, we know that

$$Q(u,0)=\frac{1}{64}(4-u)^3\leq1\quad(u\in[0,4]).$$

(v) On the side $x=1$, we find that

$$Q(u,1)=0\quad(u\in[0,4]).$$

(vi) In the interior of $(0,4)\times(0,1)$, since the equation

$$\frac{\partial Q}{\partial x}=-\frac{1}{512}x(1-x)(4-u)^3[32+u+2(16-u)x]=0$$

has no solution in $(0,4)\times(0,1)$, we know that P has no critical point in the interior of $(0,4)\times(0,1)$. Thus, it follows from (2.16) that

$$\det(T_{4,1})(f) \le Q(c^2,|\zeta|) \le 1 \quad \left(\left(c^2,|\zeta|\right) \in [0,4]\times[0,1]\right).$$

C. Suppose that $\zeta,\ \eta \in \overline{\mathbb{D}}\setminus\{0\}$. Then, there exist unique θ and φ in $[0,2\pi)$ such $\zeta = xe^{i\theta}$ and $\eta = ye^{i\varphi}$, where $x := |\zeta| \in (0,1]$ and $y := |\eta| \in (0,1]$. Thus, from (2.12), we get

$$\det(T_{4,1})(f) = \frac{1}{1024}\left(4-c^2\right)^3 \cdot F(c,x,y,\theta,\varphi), \tag{2.17}$$

where

$$\begin{aligned} F(c,x,y,\theta,\varphi) = 16 &- \left(32+c^2\right)x^2 + 2c^2x^3\cos\theta + \left(16-c^2\right)x^4 \\ &+ 4c\left(1-x^2\right)xy\cos(\theta-\varphi) - 4c\left(1-x^2\right)x^2y\cos(2\theta-\varphi) \\ &- 4\left(1-x^2\right)^2 y^2. \end{aligned}$$

For $c\in[0,2]$ and $x,\ y\in(0,1]$, we have

$$G(c,x,y) \le F(c,x,y,\theta,\varphi) \le H(c,x,y), \tag{2.18}$$

where

$$\begin{aligned} &G(c,x,y) := F(c,x,y,\pi,0) \\ &= 16-(32+c^2)x^2-2c^2x^3+(16-c^2)x^4-4c(1-x^2)(1+x)xy \\ &\quad -4(1-x^2)^2y^2 \\ &= 16(1-x^2)^2-c^2x^2(1+x)^2-4c(1-x^2)(1+x)xy-4(1-x^2)^2y^2, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &H(c,x,y) \\ &= 16-(32+c^2)x^2+2c^2x^3+(16-c^2)x^4+4c(1-x^2)(1+x)xy \\ &\quad -4(1-x^2)^2y^2 \\ &= 16(1-x^2)^2-c^2x^2(1-x)^2+4c(1-x^2)(1+x)xy-4(1-x^2)^2y^2. \end{aligned} \tag{2.20}$$

C1. We discuss the lower bound of $G(c,x,y)$.
Let $x=1$. Then

$$G(c,1,y) = -4c^2 \quad (c\in[0,2];\ y\in(0,1]).$$

From (2.17), (2.18) and part **B**1 (v), it follows that

$$\det(T_{4,1})(f) \geq -\frac{1}{256}c^2(4-c^2)^3 \geq -\frac{27}{256}.$$

Let $x \in (0, 1)$. Then

$$y_\omega = -\frac{cx}{2(1-x)} \leq 0, \quad -4(1-x^2)^2 < 0.$$

We now find that

$$G(c, x, y) \geq G(c, x, 1) = 12(1-x^2)^2 - c^2x^2(1+x)^2 - 4c(1-x^2)(1+x)x$$
$$=: \psi(c, x) \quad ((c, x) \in [0, 2] \times (0, 1)).$$

(i) On the side $x = 0$, we obtain

$$\psi(c, 0) = 12 \quad (c \in [0, 2]).$$

(ii) On the side $x = 1$, we get

$$\psi(c, 1) = -4c^2 \quad (c \in [0, 2]).$$

It follows from (2.17), (2.18) and part **B**1 (v) that

$$\det(T_{4,1})(f) \geq -\frac{1}{256}c^2(4-c^2)^3 \geq -\frac{27}{256}.$$

(iii) On the side $c = 0$, we have

$$\psi(0, x) = 12(1-x^2)^2 \geq 0 \quad (x \in [0, 1]).$$

(iv) On the side $c = 2$, we get

$$\psi(2, x) = 4(1+x)^2(3-2x)(1-2x) \quad (x \in [0, 1]),$$

thus, from (2.17) and (2.18), we deduce that

$$\det(T_{4,1})(f) \geq \frac{1}{1024} \cdot (4-c^2)^3\big|_{c=2} \cdot \psi(2, x) = 0.$$

(v) It remains to consider the interior of $(0, 2) \times (0, 1)$. Since the system of equations

$$\frac{\partial \psi}{\partial c} = -2x(1+x)^2[cx + 2(1-x)] = 0$$

has no solution in $(0, 2) \times (0, 1)$, we see that ψ has no critical point in the interior of $(0, 2) \times (0, 1)$.

From part **C**1, we find that

$$\det(T_{4,1})(f) \geq \frac{1}{1024}(4-c^2)^3 \cdot G(c,x,y) \geq -\frac{27}{256}.$$

C2. We next discuss the upper bound of $H(c,x,y)$.
Let $x=1$. Then

$$H(c,1,y)=0 \quad (c \in [0,2],\ y \in (0,1]).$$

Let $x \in (0,1)$. Then

$$y_0 = \frac{cx}{2(1-x)} \geq 0, \quad -4(1-x^2)^2 < 0.$$

Therefore, we need to consider the following two cases.
C2.1. Assume that $y_0 < 1$, i.e.,

$$x \in \left(0, \frac{2}{c+2}\right) \subset [0,1]$$

for all $c \in [0,2]$. Let

$$\triangle_1 := \left\{(c,x):\ 0 \leq c \leq 2,\ 0 \leq x \leq x_0(c) = \frac{2}{c+2}\right\}.$$

Then

$$H(c,x,y) \leq H(c,x,y_0) =: h(c,x) \quad ((c,x) \in \triangle_1,\ y \in (0,1]),$$

where

$$h(c,x) = 16(1-x^2)^2 + 4c^2x^3 \quad ((c,x) \in \triangle_1).$$

(i) On the vertices of $\triangle_1$, we know that

$$h(0,0)=16, \quad h(0,x_0(0)) = h(0,1) = 0,$$
$$h(2,0)=16, \quad h(2,x_0(2)) = h(2,1/2) = 11.$$

(ii) On the side $x=0$, we get

$$h(c,0) = 16 \quad (c \in (0,2)).$$

(iii) On the side $x = x_0(c)$ with $c \in (0,2)$, we have

$$h(c,x_0(c)) = \frac{16c^2}{(2+c)^4}(c^2+10c+20) =: \gamma(c).$$

By noting that

$$\gamma'(c) = \frac{-32c}{(2+c)^5}(c^2 - 10c - 40) = \frac{32c}{(2+c)^5}\left[65 - (c-5)^2\right] > 0 \quad (c \in (0,2)),$$

it shows that γ is an increasing function for $c \in (0, 2)$. Thus,

$$h(c, x_0(c)) \le \gamma(2) = 11 \quad (c \in (0,2)).$$

(iv) On the side $c = 0$, we have

$$h(0, x) = 16(1 - x^2)^2 \le 16 \quad (x \in [0, 1)).$$

(v) on the side $c = 2$, we get

$$h(2, x) = 16 - 16x^2(x+2)(1-x) \le 16 \quad (x \in [0, 1/2]).$$

(vi) It remains to consider the interior of $\triangle_1$. Since the system of equations

$$\begin{cases} \partial h/\partial c = 8cx^3 = 0 \\ \partial h/\partial x = -64x + 12c^2x^2 + 64x^3 = 0 \end{cases}$$

has solutions $(0, 0)$, $(0, 1)$ and $(0, -1)$, we know that h has no critical point in the interior of $\triangle_1$.

C2.2. Assume that $y_0 \ge 1$, i.e., $x \in [x_0(c),\ 1]$ for all $c \in [0, 2]$. Let

$$\triangle_2 := \left\{(c, x) :\ 0 \le c \le 2,\ x_0(c) = \frac{2}{c+2} \le x \le 1\right\}.$$

Then

$$H(c, x, y) \le H(c, x, 1) =: g(c, x) \quad ((c, x) \in \triangle_2,\ y \in (0, 1]),$$

where for $(c, x) \in \triangle_2$,

$$\begin{aligned} g(c, x) &= 12(1 - x^2)^2 - c^2x^2(1-x)^2 + 4c(1-x^2)(1+x)x \\ &= 12 + 4cx - (c^2 - 4c + 24)x^2 + 2(c^2 - 2c)x^3 - (c^2 + 4c - 12)x^4. \end{aligned}$$

(i) On the vertices of $\triangle_2$, we get

$$g(0, x_0(0)) = g(0, 1) = 0, \quad g(2, x_0(2)) = g(2, 1/2) = 11, \quad g(2, 1) = 0.$$

(ii) On the side $x = x_0(c)$, see the case **C**2.1 (iii).

(iii) On the side $x = 1$, we have

$$g(c, 1) = 0 \quad (c \in (0, 2)).$$

(iv) On the side $c = 2$, we obtain

$$g(2, x) = 12 + 8x - 20x^2 \leq (12 + 8x - 20x^2)\big|_{x=1/2} = 11 \quad (x \in [1/2,\ 1)).$$

(v) It remains to consider the interior of $\triangle_2$. Since the system of equations

$$\begin{cases} \partial g/\partial c = 4x - 2(c-2)x^2 + 4(c-1)x^3 - 2(c+2)x^4 = 0 \\ \partial g/\partial x = 4c - 2(c^2 - 4c + 24)x + 6(c^2 - 2c)x^2 - 4(c^2 + 4c - 12)x^3 = 0 \end{cases} \tag{2.21}$$

has solution $c = x = 0$ evidently. Let $x \neq 0$. From the first equation of (2.21), we get

$$c_0 = \frac{2(1+x)^2}{x(1-x)},$$

but $c_0 \notin [0, 2]$ for $x \in [1/2,\ 1)$. Thus, g has no critical point in the interior of $\triangle_2$.

It follows from part **C**2 that

$$\det(T_{4,1})(f) = \frac{1}{1024}(4 - c^2)^3 \cdot F(c, x, y, \theta, \varphi) \leq \frac{1}{1024} \cdot 4^3 \cdot 16 = 1.$$

For the sharpness of (2.8), in view of parts **B**1 (v) and **C**1 (ii), let

$$p(z) = \frac{1 - z - z^2 + z^3}{1 - 2z + 2z^2 - z^3} = 1 + z - z^2 - 2z^3 + \cdots \quad (z \in \mathbb{D}),$$

which belongs to the class $\mathcal{P}$, for the extremal function given by

$$\frac{2zf'(z)}{f(z) - f(-z)} = \frac{1 - z - z^2 + z^3}{1 - 2z + 2z^2 - z^3} \quad (z \in \mathbb{D}) \tag{2.22}$$

with

$$a_2 = \frac{1}{2}, \ \ a_3 = -\frac{1}{2}, \ \ a_4 = -\frac{5}{8}.$$

Thus, we find that

$$\det(T_{4,1})(f) = -\frac{27}{256}.$$

It is clear that equality for the upper bound in (2.8) holds for the identity function. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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