Bull. Malays. Math. Sci. Soc. (2023) 46:44
https://doi.org/10.1007/s40840-022-01443-5

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Regularization Methods for the Split Equality Problems in Hilbert Spaces

Truong Minh Tuyen[1]

Received: 5 August 2022 / Revised: 16 November 2022 / Accepted: 21 November 2022 /
Published online: 5 December 2022

Abstract

We study the system of the split equality problems in Hilbert spaces. In order to solve this problem, we introduce several new iterative processes by using the Tikhonov regularization method. This approach is completely different from previous ones.

Keywords Hilbert space · Metric projection · Regularization · Split feasibility problem

Mathematics Subject Classification 47H05 · 47H09 · 49J53 · 90C25

1 Introduction

The *split equality problem* was first introduced and studied by Moudafi in [19]. This problem is formulated as follows: Let H, H_1 and H_2 be three real Hilbert spaces. Let C and Q be two nonempty closed and convex subsets of H_1 and H_2, respectively.

$$\text{Find } p \in C \text{ and } q \in Q \text{ such that } Ap - Bq = 0, \tag{1.1}$$

where $A: H_1 \to H$ and $B: H_2 \to H$, are two bounded linear operators.

The origins of Problem (1.1) can be found in [2]. The authors studied alternating minimization algorithms with costs-to-move. Next, a particular case of Problem (1.1) has been also studied by Attouch et al. in [3, 4] when C and Q are two minimizer point sets of two convex functions f and g, which are defined on H_1 and H_2, respectively.

Communicated by Rosihan M. Ali.

✉ Truong Minh Tuyen
tuyentm@tnus.edu.vn

1 Department of Mathematics and Informatics, Thai Nguyen University of Sciences, Thai Nguyen, Vietnam

We know that Problem (1.1) has some important applications in game theory, variational problems, partial differential equations, and intensity-modulated radiation therapy (see, e.g., [3, 9]). Thus, Problem (1.1) is an interesting topic of nonlinear analysis, which has attracted the attention of many mathematicians around the world. So, many algorithms have been presented for solving Problem (1.1) (see, e.g., [8, 11–14, 18, 20, 21, 25, 27–31]).

If $H = H_2$ and $B = I^{H_2}$, then Problem (1.1) becomes the *split convex feasibility problem*. Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. The split convex feasibility problem (SCFP for short) is formulated as follows:

$$\text{Find an element} \quad p \in C \text{ such that } Ap \in Q. \tag{1.2}$$

The SCFP was first introduced and studied by Censor and Elfving [10] for modeling certain inverse problems. It plays an important role in medical image reconstruction and in signal processing (see [6, 7]). Thus, all of the iterative methods or algorithms to approximate the solution to Problem (1.1) can be applied to find the solution to Problem (1.2).

We know that most applied problems are ill-posed problems in the sense, they do not satisfy at least one of the following three requirements: (1) The problem is solvable; (2) Its solution is unique; (3) The problem is stable in the sense, any small change in the input data leads to only small changes in the output data (the solution of the problem) (see, e.g., [17]). Thus SEP and its related problems are also ill-posed problems. One popular effective method for solving the ill-posed problems is the Tikhonov regularization method which is introduced by Tikhonov (see, e.g., [1], [26]).

In this paper, we study the system of the split equality problems. In order to find a solution to this problem, we first introduce a new implicit iterative method by using the Tikhonov regularization method. Next, we propose an explicit iterative regularization method and prove the strong convergence of it when the control parameters are chosen suitably.

2 Preliminaries

Let H be a real Hilbert space. We denote by $\langle x, y\rangle_H$ the inner product of two elements $x, y \in H$. The induced norm is denoted by $\|\cdot\|_H$, that is, $\|x\|_H = \sqrt{\langle x, x\rangle_H}$ for all $x \in H$.

Let C be a nonempty, closed and convex subset of H. It is well known that for each $x \in H$, there is unique point $P_C^H x \in C$ such that

$$\|x - P_C^H x\|_H = \inf_{u\in C} \|x - u\|_H. \tag{2.1}$$

The mapping $P_C^H : H \to C$ defined by (2.1) is called the *metric projection* of H onto C. We also recall (see, for example, Section 3 in [16]) that

$$\langle x - P_C^H x, y - P_C^H x\rangle_H \leq 0 \ \forall x \in H, \ \forall y \in C.$$

Lemma 2.1 *Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. Then for all $x, y \in H$, we have*

i) $\langle x - y, P_C^H x - P_C^H y\rangle_H \geq \|P_C^H x - P_C^H y\|_H^2$;
ii) $\langle x - y, (I - P_C^H)x - (I - P_C^H)y\rangle_H \geq \|(I - P_C^H)x - (I - P_C^H)y\|_H^2$.

Definition 2.2 Let $f : H \longrightarrow (-\infty, \infty]$ be a proper function. The *subdifferential* of f is the set-valued operator $\partial f : H \to 2^H$ which is defined by

$$\partial f(x) := \{v \in H : f(y) - f(x) \geq \langle v, y - x\rangle_H \ \ \forall y \in H\}.$$

Definition 2.3 An operator $S : H \to H$ is called

a) L-Lipschitz with Lipschitz constant $L \geq 0$ if for all $x, y \in H$, we have

$$\|Sx - Sy\|_H \leq L\|x - y\|_H.$$

In the above inequality, if $L \in [0, 1)$, then S is called a strict contraction and if $L = 1$, then it is said to be nonexpansive.

b) γ-strongly monotone with constant $\gamma > 0$ if the inequality

$$\langle x - y, Sx - Sy\rangle_H \geq \gamma\|x - y\|_H^2$$

holds for all $x, y \in H$.

c) hemicontinuous at a point $x_0 \in H$ if $S(x_0 + t_n x) \rightharpoonup Sx_0$ as $t_n \to 0^+$ where $\{t_n\}$ is any sequence of positive real numbers. The operator S is called hemicontinuous if it is hemicontinuous at each element $x \in H$.

d) coercive if there exists a function $c(t)$ defined for $t \geq 0$ such that $c(t) \to \infty$ as $t \to \infty$, and the inequality

$$\langle x, Sx\rangle_H \geq c(\|x\|_H)\|x\|_H$$

holds for all $x \in H$.

e) bounded on bounded sets if $S(M)$ is a bounded set for each bounded set $M \subset H$.

Remark 2.4 It follows from Lemma 2.1 that $I^H - P_C^H$ is a nonexpansive mapping.

In the sequel, we are going to use the following lemmas in the proofs of the main results of this paper.

Lemma 2.5 *Let H be a real Hilbert space. Suppose that $F : H \to H$ is a L-Lipschitz and γ-strongly monotone mapping. Then for any $\varepsilon \in (0, 2\gamma/L^2)$, we have $I^H - \varepsilon F$ is a strict contraction mapping with the contraction coefficient $\tau = \sqrt{1 - \varepsilon(2\gamma - \varepsilon L^2)}$.*

Proof For any $x, y \in H$, we have

$$\begin{aligned}\|(I^H-\varepsilon F)x-(I^H-\varepsilon F)y\|_H^2 &= \|(x-y)-\varepsilon(Fx-Fy)\|_H^2\\ &= \|x-y\|_H^2-2\varepsilon\langle x-y, Fx-Fy\rangle_H+\varepsilon^2\|Fx-Fy\|_H^2\\ &\le \|x-y\|_H^2-2\varepsilon\gamma\|x-y\|_H^2+\varepsilon^2L^2\|x-y\|_H^2\\ &= [1-\varepsilon(2\gamma-\varepsilon L^2)]\|x-y\|_H^2.\end{aligned}$$

Thus, it follows from the condition $\varepsilon \in (0, 2\gamma/L^2)$ that $I^H - \varepsilon F$ is strict contraction mapping with the contraction coefficient $\tau = \sqrt{1-\varepsilon(2\gamma-\varepsilon L^2)}$. □

Lemma 2.6 [15] *Let T be a nonexpansive self-mapping of a closed and convex subset C of a Hilbert space H. Then the mapping $I^H - T$ is demiclosed, that is, whenever $\{x_n\}$ is a sequence in C which weakly converges to some $x \in C$ and the sequence $\{(I-T)(x_n)\}$ strongly converges to some y, it follows that $(I^H - T)(x) = y$.*

Lemma 2.7 [24]

Let $\{\Gamma_n\}$ be a sequence of nonnegative numbers, $\{b_n\}$ be a sequence in $(0, 1)$ and let $\{c_n\}$ be a sequence of real numbers satisfying the following two conditions:

i) $\Gamma_{n+1} \le (1-b_n)\Gamma_n + b_n c_n$;
ii) $\sum_{n=0}^{\infty} b_n = \infty,\ \limsup_{n\to\infty} c_n \le 0$.

Then $\lim_{n\to\infty} \Gamma_n = 0$.

3 Main Results

Let H_1, H_2 and H, be three real Hilbert spaces. Let C_i and Q_i, be nonempty closed convex subsets of H_1 and H_2, respectively, $i = 1, 2, \ldots, N$. Let $A_i : H_1 \to H$ and $B_i : H_2 \to H$, $i = 1, 2, \ldots, N$, be bounded linear mappings and let b_i, $i = 1, 2, \ldots, N$, be N given elements in H. Suppose that

$$\Omega := \{(x, y) \in \cap_{i=1}^{N}(C_i \times Q_i) :\ A_i x - B_i y = b_i,\ i = 1, 2, \ldots, N\} \neq \emptyset.$$

Consider the problem of finding an element $(x^*, y^*) \in \Omega$, we denote this problem by (SSEP).

3.1 Implicit Iterative Method

First, we introduce a new Tikhonov regularization method type to approximate a solution of Problem (SSEP).

We define the sequences $\{x_n\}$ and $\{y_n\}$ by the following implicit iterative method:

$$\sum_{i=1}^{N}\Big((I^{H_1}-P_{C_i}^{H_1})x_n+A_i^*(A_ix_n-B_iy_n-b_i)\Big)+\alpha_nSx_n=0, \tag{3.1}$$

$$\sum_{i=1}^{N}\Big((I^{H_2}-P_{Q_i}^{H_2})y_n-B_i^*(A_ix_n-B_iy_n-b_i)\Big)+\alpha_nTy_n=0, \tag{3.2}$$

where $\{\alpha_n\}$ is a sequence of positive real numbers, $S : H_1 \to H_1$ and $T : H_2 \to H_2$ are two bounded on bounded sets, hemicontinuous and strongly monotone mappings with the constants γ_S and γ_T, respectively.

Theorem 3.1 *i) For each n, the system of equations (3.1)–(3.2) has a unique solution (x_n, y_n).*

ii) If $\lim_{n\to\infty}\alpha_n = 0$, then $x_n \to x^*$, $y_n \to y^*$ with $(x^*, y^*) \in \Omega$ and (x^*, y^*) is a unique solution to the following variational inequality

$$\langle x-x^*, Sx^*\rangle_{H_1}+\langle y-y^*, Ty^*\rangle_{H_2}\geq 0,\ \ \forall (x,y)\in\Omega. \tag{3.3}$$

Proof i) We rewrite the equations (3.1) and (3.2) in the following form

$$F_{\alpha_n}(x_n, y_n)=0, \tag{3.4}$$

where $F_{\alpha_n} : H_1 \times H_2 \to H_1 \times H_2$ defined by

$$F_{\alpha_n}(a)=\Bigg(\sum_{i=1}^{N}\Big((I^{H_1}-P_{C_i}^{H_1})x+A_i^*(A_ix-B_iy-b_i)\Big)+\alpha_nSx,$$
$$\sum_{i=1}^{N}\Big((I^{H_2}-P_{Q_i}^{H_2})y-B_i^*(A_ix-B_iy-b_i)\Big)+\alpha_nTy\Bigg),$$

for all $a = (x, y) \in H_1 \times H_2$.

We know that $H_1 \times H_2$ is a Hilbert spaces with the inner product

$$\langle (x_1,y_1),(x_2,y_2)\rangle_{H_1\times H_2}=\langle x_1,y_1\rangle_{H_1}+\langle x_2,y_2\rangle_{H_2}$$

for all (x_1, y_1) and (x_2, y_2) in $H_1 \times H_2$ and the norm on $H_1 \times H_2$ is defined by

$$\|(x,y)\|_{H_1\times H_2}=\sqrt{\|x\|_{H_1}^2+\|y\|_{H_2}^2}$$

for all $(x, y) \in H_1 \times H_2$ (see, e.g., [22, Proposition 2.2], [23, Proposition 2.4]).

We first show that F_{α_n} is a monotone mapping. Indeed, for any $a = (x_1, y_1)$ and $b = (x_2, y_2)$ in $H_1 \times H_2$, we have

$$
\begin{aligned}
&\langle a-b, F_{\alpha_n}(a)-F_{\alpha_n}(b)\rangle_{H_1\times H_2}\\
&\quad=\sum_{i=1}^{N}\langle x_1-x_2,(I^{H_1}-P_{C_i}^{H_1})x_1-(I^{H_1}-P_{C_i}^{H_1})x_2\rangle_{H_1}\\
&\qquad+\sum_{i=1}^{N}\langle x_1-x_2,A_i^*(A_i(x_1-x_2)-B_i(y_1-y_2))\rangle_{H_1}\\
&\qquad+\alpha_n\langle x_1-x_2,Sx_1-Sx_2\rangle_{H_1}\\
&\qquad+\sum_{i=1}^{N}\langle y_1-y_2,(I^{H_2}-P_{Q_i}^{H_2})y_1-(I^{H_2}-P_{Q_i}^{H_2})y_2\rangle_{H_2}\\
&\qquad-\sum_{i=1}^{N}\langle y_1-y_2,B_i^*(A_i(x_1-x_2)-B_i(y_1-y_2))\rangle_{H_2}\\
&\qquad+\alpha_n\langle y_1-y_2,Ty_1-Ty_2\rangle_{H_2}\\
&\quad=\sum_{i=1}^{N}\langle x_1-x_2,(I^{H_1}-P_{C_i}^{H_1})x_1-(I^{H_1}-P_{C_i}^{H_1})x_2\rangle_{H_1}\\
&\qquad+\sum_{i=1}^{N}\langle A_i(x_1-x_2),A_i(x_1-x_2)-B_i(y_1-y_2)\rangle_{H}\\
&\qquad+\alpha_n\langle x_1-x_2,Sx_1-Sx_2\rangle_{H_1}\\
&\qquad+\sum_{i=1}^{N}\langle y_1-y_2,(I^{H_2}-P_{Q_i}^{H_2})y_1-(I^{H_2}-P_{Q_i}^{H_2})y_2\rangle_{H_2}\\
&\qquad-\sum_{i=1}^{N}\langle B_i(y_1-y_2),A_i(x_1-x_2)-B_i(y_1-y_2)\rangle_{H}\\
&\qquad+\alpha_n\langle y_1-y_2,Ty_1-Ty_2\rangle_{H_2}\\
&\quad=\sum_{i=1}^{N}\langle x_1-x_2,(I^{H_1}-P_{C_i}^{H_1})x_1-(I^{H_1}-P_{C_i}^{H_1})x_2\rangle_{H_1}\\
&\qquad+\alpha_n\langle x_1-x_2,Sx_1-Sx_2\rangle_{H_1}\\
&\qquad+\sum_{i=1}^{N}\langle y_1-y_2,(I^{H_2}-P_{Q_i}^{H_2})y_1-(I^{H_2}-P_{Q_i}^{H_2})y_2\rangle_{H_2}\\
&\qquad+\sum_{i=1}^{N}\|A_i(x_1-x_2)-B_i(y_1-y_2)\|_H^2\\
&\qquad+\alpha_n\langle y_1-y_2,Ty_1-Ty_2\rangle_{H_2}.
\end{aligned}
$$

It follows from Lemma 2.1, the strongly monotone of S and T, and the above equality that

$$\begin{aligned}\langle a-b, F_{\alpha_n}(a)-F_{\alpha_n}(b)\rangle_{H_1\times H_2} &\geq \sum_{i=1}^{N}\|(I^{H_1}-P_{C_i}^{H_1})x_1-(I^{H_1}-P_{C_i}^{H_1})x_2\|_{H_1}^2\\ &\quad+\sum_{i=1}^{N}\|(I^{H_2}-P_{Q_i}^{H_2})y_1-(I^{H_2}-P_{Q_i}^{H_2})y_2\|_{H_2}^2\\ &\quad+\sum_{i=1}^{N}\|A_i(x_1-x_2)-B_i(y_1-y_2)\|_H^2\\ &\quad+\alpha_n(\gamma_S\|x_1-x_2\|_{H_1}^2+\gamma_T\|y_1-y_2\|_{H_2}^2)\\ &\geq 0. \end{aligned} \tag{3.5}$$

This implies that F_{α_n} is a monotone mapping. Moreover, we also have

$$\begin{aligned}\langle a-b, F_{\alpha_n}(a)-F_{\alpha_n}(b)\rangle_{H_1\times H_2} &\geq \alpha_n(\gamma_S\|x_1-x_2\|_{H_1}^2+\gamma_T\|y_1-y_2\|_{H_2}^2)\\ &\geq \alpha_n\min\{\gamma_S,\gamma_T\}(\|x_1-x_2\|_{H_1}^2+\|y_1-y_2\|_{H_2}^2)\\ &= c(\|a-b\|_{H_1\times H_2})\|a-b\|_{H_1\times H_2},\end{aligned}$$

with $c(t)=\alpha_n\min\{\gamma_S,\gamma_T\}t$ for all $t\geq 0$. This implies that F_{α_n} is a coercive mapping.

It follows from the hemicontinuity of S and T that F_{α_n} is a hemicontinuous mapping. We conclude that F_{α_n} is a single valued hemicontinuous monotone and coercive mapping on $H_1\times H_2$, and hence $R(F_{\alpha_n})=H_1\times H_2$ (see, e.g. [5, Proposition 1]). Thus Equation (3.4) is solvable.

Next, we prove the uniqueness of the solution (x_n, y_n) of Equation (3.4). Indeed, suppose that (u_n, v_n) is also another solution to Equation (3.4). Let $a=(x_n, y_n)$ and $b=(u_n, v_n)$, it follows from $F_{\alpha_n}(a)=F_{\alpha_n}(b)=0$ and (3.5) that

$$\gamma_S\|x_n-u_n\|_{H_1}^2+\gamma_T\|y_n-v_n\|_{H_2}^2\leq 0.$$

This implies that $x_n=u_n$ and $y_n=v_n$. So, Equation (3.4) has a unique solution, that is, the system of equations (3.1)-(3.2) has unique solution (x_n, y_n).

ii) Take any $c=(\bar{x},\bar{y})\in\Omega$, that is, $F_{\alpha_n}(c)=\alpha_n(S\bar{x}, T\bar{y})$. From (3.5), we have

$$\langle a-c, F_{\alpha_n}(a)-F_{\alpha_n}(c)\rangle_{H_1\times H_2}\geq \alpha_n(\gamma_S\|x_n-\bar{x}\|_{H_1}^2+\gamma_T\|y_n-\bar{y}\|_{H_2}^2).$$

On the other hand, we also have

$$\langle a-c, F_{\alpha_n}(a)-F_{\alpha_n}(c)\rangle_{H_1\times H_2}=-\alpha_n(\langle x_n-\bar{x}, S\bar{x}\rangle_{H_1}+\langle y_n-\bar{y}, T\bar{y}\rangle_{H_2}).$$

Thus, we obtain

$$\begin{aligned}&\|x_n-\bar{x}\|_{H_1}^2+\|y_n-\bar{y}\|_{H_2}^2\\ &\qquad\leq -\frac{1}{\min\{\gamma_S,\gamma_T\}}(\langle x_n-\bar{x}, S\bar{x}\rangle_{H_1}+\langle y_n-\bar{y}, T\bar{y}\rangle_{H_2})\end{aligned}$$

$$\le \frac{1}{\min\{\gamma_S, \gamma_T\}}(\|x_n - \bar{x}\|_{H_1}\|S\bar{x}\|_{H_1} + \|y_n - \bar{y}\|_{H_2}\|T\bar{y}\|_{H_2})$$
$$\le \frac{1}{\min\{\gamma_S, \gamma_T\}}\sqrt{\|x_n - \bar{x}\|^2_{H_1} + \|y_n - \bar{y}\|^2_{H_2}}\sqrt{\|S\bar{x}\|^2_{H_1} + \|T\bar{y}\|^2_{H_2}}. \tag{3.6}$$

This deduces that

$$\|x_n - \bar{x}\|^2_{H_1} + \|y_n - \bar{y}\|^2_{H_2} \le \frac{1}{(\min\{\gamma_S, \gamma_T\})^2}(\|S\bar{x}\|^2_{H_1} + \|T\bar{y}\|^2_{H_2}).$$

Thus we see that two sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Hence, there are two subsequences $\{x_{m_n}\}$ and $\{y_{m_n}\}$ of $\{x_n\}$ and $\{y_n\}$, respectively, such that $x_{m_n} \rightharpoonup x^*$ and $y_{m_n} \rightharpoonup y^*$, as $n \to \infty$.

It follows from Lemma 2.1, (3.1) and (3.2) that

$$\begin{aligned}
&-\alpha_n(\langle x_n - \bar{x}, Sx_n\rangle_{H_1} + \langle y_n - \bar{y}, Ty_n\rangle_{H_2}) \\
&\quad = \sum_{i=1}^{N}\langle x_n - \bar{x}, (I^{H_1} - P^{H_1}_{C_i})x_n - (I^{H_1} - P^{H_1}_{C_i})\bar{x}\rangle_{H_1} \\
&\qquad + \sum_{i=1}^{N}\langle y_n - \bar{y}, (I^{H_2} - P^{H_2}_{Q_i})y_n - (I^{H_2} - P^{H_2}_{Q_i})\bar{y}\rangle_{H_2} \\
&\qquad + \sum_{i=1}^{N}\langle x_n - \bar{x}, A_i^*(A_i x_n - B_i y_n - b_i)\rangle_{H_1} \\
&\qquad - \sum_{i=1}^{N}\langle y_n - \bar{y}, B_i^*(A_i x_n - B_i y_n - b_i)\rangle_{H_2} \\
&\quad = \sum_{i=1}^{N}\langle x_n - \bar{x}, (I^{H_1} - P^{H_1}_{C_i})x_n - (I^{H_1} - P^{H_1}_{C_i})\bar{x}\rangle_{H_1} \\
&\qquad + \sum_{i=1}^{N}\langle y_n - \bar{y}, (I^{H_2} - P^{H_2}_{Q_i})y_n - (I^{H_2} - P^{H_2}_{Q_i})\bar{y}\rangle_{H_2} \\
&\qquad + \sum_{i=1}^{N}\langle A_i x_n - B_i y_n - A_i\bar{x} + B_i\bar{y}, A_i x_n - B_i y_n - b_i\rangle_{H} \\
&\quad \ge \sum_{i=1}^{N}\|(I^{H_1} - P^{H_1}_{C_i})x_n\|^2_{H_1} + \sum_{i=1}^{N}\|(I^{H_2} - P^{H_2}_{Q_i})y_n\|^2_{H_2} \\
&\qquad + \sum_{i=1}^{N}\|A_i x_n - B_i y_n - b_i\|^2_{H}.
\end{aligned} \tag{3.7}$$

Since $\{x_n\}, \{y_n\}$ are bounded and S, T are two bounded on bounded sets mappings, there is a positive real number K such that

$$\max\{\sup_n \|x_n\|_{H_1}, \sup_n \|y_n\|_{H_2}, \sup_n \|Sx_n\|_{H_1}, \sup_n \|Ty_n\|_{H_2}\} \leq K.$$

From (3.7), we get

$$\begin{aligned}\sum_{i=1}^{N} \|(I^{H_1} - P_{C_i}^{H_1})x_n\|_{H_1}^2 + \sum_{i=1}^{N} \|(I^{H_2} - P_{Q_i}^{H_2})y_n\|_{H_2}^2 \\ + \sum_{i=1}^{N} \|A_i x_n - B_i y_n - b_i\|_H^2 \\ \leq K(2K + \|\bar{x}\|_{H_1} + \|\bar{y}\|_{H_2})\alpha_n.\end{aligned}$$

It follows from $\lim_{n\to\infty} \alpha_n = 0$ that

$$\lim_{n\to\infty} (I^{H_1} - P_{C_i}^{H_1})x_n\|_{H_1} = 0, \ \lim_{n\to\infty} \|(I^{H_2} - P_{Q_i}^{H_2})y_n\|_{H_2} = 0, \tag{3.8}$$

$$\lim_{n\to\infty} \|A_i x_n - B_i y_n - b_i\|_H = 0, \tag{3.9}$$

for all $i = 1, 2, \ldots, N$. In particular, we have that

$$\lim_{n\to\infty} (I^{H_1} - P_{C_i}^{H_1})x_{m_n}\|_{H_1} = 0, \ \lim_{n\to\infty} \|(I^{H_2} - P_{Q_i}^{H_2})y_{m_n}\|_{H_2} = 0, \tag{3.10}$$

$$\lim_{n\to\infty} \|A_i x_{m_n} - B_i y_{m_n} - b_i\|_H = 0. \tag{3.11}$$

From Lemma 2.6 and (3.10), we infer that $(x^*, y^*) \in C_i \times Q_i$. Since A_i and B_i are bounded linear mappings, and $x_{m_n} \rightharpoonup x^*$, $y_{m_n} \rightharpoonup y^*$, we obtain

$$A_i x_{m_n} - B_i y_{m_n} - b_i \rightharpoonup A_i x^* - B_i y^* - b_i.$$

This combines with (3.11), one has $A_i x^* - B_i y^* = b_i$. Thus, we have $(x^*, y^*) \in \Omega$.

Next, we show that (x^*, y^*) is a unique solution to the variational inequality (3.3). It follows from (3.6) that

$$\langle x_n - \bar{x}, S\bar{x}\rangle_{H_1} + \langle y_n - \bar{y}, T\bar{y}\rangle_{H_2} \leq 0.$$

Replacing n by m_n and letting $n \to \infty$, we get

$$\langle x^* - \bar{x}, S\bar{x}\rangle_{H_1} + \langle y^* - \bar{y}, T\bar{y}\rangle_{H_2} \leq 0, \tag{3.12}$$

for all $(\bar{x}, \bar{y}) \in \Omega$. Set $(x_t, y_t) = (x^*, y^*) + t(\bar{x} - x^*, \bar{y} - y^*)$ with $t \in (0, 1)$. Since Ω is a closed convex set, and $(x^*, y^*), (\bar{x}, \bar{y}) \in \Omega$, we have $(x_t, y_t) \in \Omega$. So, in (3.12), replacing $(\bar{x}, \bar{y})$ by (x_t, y_t), one has

$$\langle \bar{x} - x^*, S(x^* + t(\bar{x} - x^*))\rangle_{H_1} + \langle \bar{y} - y^*, T(y^* + t(\bar{y} - y^*))\rangle_{H_2} \geq 0, \ \forall t \in (0, 1).$$

Using the hemicontinuity of S, T and letting $t \to 0^+$, we obtain

$$\langle \bar{x} - x^*, Sx^*\rangle_{H_1} + \langle \bar{y} - y^*, Ty^*\rangle_{H_2} \geq 0, \ \forall (\bar{x}, \bar{y}) \in \Omega, \tag{3.13}$$

that is, (x^*, y^*) is a solution to the variational inequality (3.3).

We now establish the uniqueness of (x^*, y^*). Suppose that (u^*, v^*) is also another solution to the variational inequality (3.3), that is,

$$\langle \bar{x} - u^*, Su^*\rangle_{H_1} + \langle \bar{y} - u^*, Tu^*\rangle_{H_2} \geq 0, \ \forall (\bar{x}, \bar{y}) \in \Omega. \tag{3.14}$$

In (3.13) and (3.14), replacing $(\bar{x}, \bar{y})$ by (u^*, v^*) and (x^*, y^*), and adding the resulting inequalities, we obtain

$$\langle x^* - u^*, Sx^* - Sy^*\rangle_{H_1} + \langle y^* - v^*, Ty^* - Tv^*\rangle_{H_2} \leq 0.$$

Since S and T are strongly monotone with the constants γ_S and γ_T, it follows that

$$\gamma_S \|x^* - u^*\|^2_{H_1} + \gamma_T \|y^* - v^*\|^2_{H_2} \leq 0,$$

which implies that $u^* = x^*$ and $v^* = y^*$. Thus (x^*, y^*) is unique solution to the variational inequality (3.3).

It now follows from the uniqueness of (x^*, y^*) that $x_n \rightharpoonup x^*$ and $y_n \rightharpoonup y^*$. Finally, we show that $x_n \to x^*$ and $y_n \to y^*$. Indeed, in (3.6), replacing $(\bar{x}, \bar{y})$ by (x^*, y^*), one has

$$\|x_n - x^*\|^2_{H_1} + \|y_n - y^*\|^2_{H_2} \leq -\frac{1}{\min\{\gamma_S, \gamma_T\}}(\langle x_n - x^*, Sx^*\rangle_{H_1} + \langle y_n - y^*, Ty^*\rangle_{H_2}).$$

Letting $n \to \infty$, we infer that $x_n \to x^*$ and $y_n \to y^*$.

This completes the proof. □

We have the following theorem regarding to the distance between two solutions (x_n, y_n) and (x_m, y_m) of the system (3.1)–(3.2).

Theorem 3.2 *Let (x_n, y_n) and (x_m, y_m) be two solutions of the system (3.1)-(3.2) corresponding to α_n and α_m. Then we have the following estimate*

$$\sqrt{\|x_n - x_m\|^2_{H_1} + \|y_n - y_m\|^2_{H_2}} \leq K_1 \frac{|\alpha_n - \alpha_m|}{\alpha_n}, \tag{3.15}$$

where $K_1 = K\sqrt{2}/\min\{\gamma_S, \gamma_T\}$.

Proof Let $a = (x_n, y_n)$ and $b = (x_m, y_m)$. It follows from $F_{\alpha_n}(a) = F_{\alpha_m}(b) = 0$ and (3.5) that

$$
\begin{aligned}
0 &= \langle a-b, F_{\alpha_n}(a) - F_{\alpha_m}(b)\rangle_{H_1\times H_2} \\
&= \langle a-b, F_{\alpha_n}(a) - F_{\alpha_n}(b)\rangle_{H_1\times H_2} + \langle a-b, F_{\alpha_n}(b) - F_{\alpha_m}(b)\rangle_{H_1\times H_2} \\
&= \langle a-b, F_{\alpha_n}(a) - F_{\alpha_n}(b)\rangle_{H_1\times H_2} \\
&\quad + (\alpha_n - \alpha_m)(\langle x_n - x_m, Sx_m\rangle_{H_1} + \langle y_n - y_m, Ty_m\rangle_{H_2}) \\
&\geq \alpha_n(\gamma_S\|x_n - x_m\|^2_{H_1} + \gamma_T\|y_n - y_m\|^2_{H_2}) \\
&\quad + (\alpha_n - \alpha_m)(\langle x_n - x_m, Sx_m\rangle_{H_1} + \langle y_n - y_m, Ty_m\rangle_{H_2}).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\alpha_n(\gamma_S\|x_n - x_m\|^2_{H_1} + \gamma_T\|y_n - y_m\|^2_{H_2}) \\
&\qquad\qquad \leq K|\alpha_n - \alpha_m|(\|x_n - x_m\|_{H_1} + \|y_n - y_m\|_{H_2}).
\end{aligned}
$$

And hence, we have

$$
\begin{aligned}
&\|x_n - x_m\|^2_{H_1} + \|y_n - y_m\|^2_{H_2} \\
&\qquad \leq (K/\min\{\gamma_S, \gamma_T\})\frac{|\alpha_n - \alpha_m|}{\alpha_n}(\|x_n - x_m\|_{H_1} + \|y_n - y_m\|_{H_2}) \\
&\qquad \leq (K/\min\{\gamma_S, \gamma_T\})\frac{|\alpha_n - \alpha_m|}{\alpha_n}\sqrt{2(\|x_n - x_m\|^2_{H_1} + \|y_n - y_m\|^2_{H_2})},
\end{aligned}
$$

which shows that

$$
\sqrt{\|x_n - x_m\|^2_{H_1} + \|y_n - y_m\|^2_{H_2}} \leq K_1\frac{|\alpha_n - \alpha_m|}{\alpha_n},
$$

This completes the proof. □

3.2 Explicit Iterative Method

First, we have the following proposition.

Proposition 3.3 *Let H_1, H_2 and H, be three real Hilbert spaces. Let C_i and Q_i, be nonempty closed convex subsets of H_1 and H_2, respectively, $i = 1, 2, \dots, N$. Let $A_i : H_1 \to H$ and $B_i : H_2 \to H$, $i = 1, 2, \dots, N$, be bounded linear mappings and let b_i, $i = 1, 2, \dots, N$, be N given elements in H. Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be two strongly monotone mappings with constants γ_S, γ_T and Lipschitz mappings with the constants L_S, L_T, respectively. Then for each $\alpha > 0$, we have that*

$$
\begin{aligned}
F_\alpha(x, y) = \Bigg(&\sum_{i=1}^{N}\Big((I^{H_1} - P^{H_1}_{C_i})x + A_i^*(A_i x - B_i y - b_i)\Big) + \alpha Sx, \\
&\sum_{i=1}^{N}\Big((I^{H_2} - P^{H_2}_{Q_i})y - B_i^*(A_i x - B_i y - b_i)\Big) + \alpha Ty\Bigg),
\end{aligned}
$$

$$\forall (x,y) \in H_1 \times H_2$$

is a γ-strongly monotone and L-Lipschitz mapping on $H_1 \times H_2$ with $\gamma = \min\{\gamma_S, \gamma_T\}\alpha$ and $L = \sqrt{[(N+\alpha L_{S,T})^2 + \gamma_{A,B}^4](1+4N^2)}$, where $L_{S,T} = \max\{L_S, L_T\}$ and $\gamma_{A,B} = \max_{i=12,\dots,N}\{\|A_i\|, \|B_i\|\}$.

Proof For any $a = (x_1, y_1)$ and $b = (x_2, y_2)$ in $H_1 \times H_2$, it follows from (3.5) that

$$\begin{aligned}\langle a-b, F_\alpha(a) - F_\alpha(b)\rangle_{H_1\times H_2} &\geq \alpha(\gamma_S\|x_1 - x_2\|_{H_1}^2 + \gamma_T\|y_1 - y_2\|_{H_2}^2) \\ &\geq \alpha\min\{\gamma_S, \gamma_T\}\|a-b\|_{H_1\times H_2}^2.\end{aligned}$$

This implies that F_α is γ-strongly monotone mapping on $H_1 \times H_2$ with $\gamma = \alpha\min\{\gamma_S, \gamma_T\}$.

Next, we have

$$\begin{aligned}&\|F_\alpha(a) - F_\alpha(b)\|_{H_1\times H_2}^2 \\ &= \Big\| \sum_{i=1}^N \Big((I^{H_1} - P_{C_i}^{H_1})x_1 - (I^{H_1} - P_{C_i}^{H_1})x_2\Big) \\ &\quad + \sum_{i=1}^N A_i^*(A_i(x_1 - x_2) - B_i(y_1 - y_2)) + \alpha(Sx_1 - Sx_2)\Big\|_{H_1}^2 \\ &+ \Big\| \sum_{i=1}^N \Big((I^{H_2} - P_{Q_i}^{H_2})y_1 - (I^{H_2} - P_{Q_i}^{H_2})y_2\Big) \\ &\quad - \sum_{i=1}^N B_i^*(A_i(x_1 - x_2) - B_i(y_1 - y_2)) + \alpha(Ty_1 - Ty_2)\Big\|_{H_2}^2 \\ &\leq \Big[(N+\alpha L_S)\|x_1 - x_2\|_{H_1} + \sum_{i=1}^N \|A_i\|(\|A_i\|\|x_1 - x_2\|_{H_1} + \|B_i\|\|y_1 - y_2\|_{H_2})\Big]^2 \\ &\quad + \Big[(N+\alpha L_T)\|y_1 - y_2\|_{H_1} + \sum_{i=1}^N \|B_i\|(\|A_i\|\|x_1 - x_2\|_{H_1} + \|B_i\|\|y_1 - y_2\|_{H_2})\Big]^2 \\ &\leq \Big[(N+\alpha L_S)\|x_1 - x_2\|_{H_1} + \gamma_{A,B}^2 \sum_{i=1}^N (\|x_1 - x_2\|_{H_1} + \|y_1 - y_2\|_{H_2})\Big]^2 \\ &\quad + \Big[(N+\alpha L_T)\|y_1 - y_2\|_{H_1} + \gamma_{A,B}^2 \sum_{i=1}^N (\|x_1 - x_2\|_{H_1} + \|y_1 - y_2\|_{H_2})\Big]^2 \\ &\leq [(N+\alpha L_S)^2 + \gamma_{A,B}^4]\Bigg[\|x_1 - x_2\|_{H_1}^2 + \Big(\sum_{i=1}^N (\|x_1 - x_2\|_{H_1} + \|y_1 - y_2\|_{H_2})\Big)^2\Bigg] \\ &\quad + [(N+\alpha L_T)^2 + \gamma_{A,B}^4]\Bigg[\|y_1 - y_2\|_{H_2}^2 + \Big(\sum_{i=1}^N (\|x_1 - x_2\|_{H_1} + \|y_1 - y_2\|_{H_2})\Big)^2\Bigg]\end{aligned}$$

$$\leq [(N+\alpha L_{S,T})^2 + \gamma_{A,B}^4][\|a-b\|^2_{H_1\times H_2} + 2N^2(\|x_1-x_2\|_{H_1} + \|y_1-y_2\|_{H_2})^2]$$
$$\leq [(N+\alpha L_{S,T})^2 + \gamma_{A,B}^4][\|a-b\|^2_{H_1\times H_2} + 4N^2(\|x_1-x_2\|^2_{H_1} + \|y_1-y_2\|^2_{H_2})]$$
$$\leq [(N+\alpha L_{S,T})^2 + \gamma_{A,B}^4](1+4N^2)\|a-b\|^2_{H_1\times H_2},$$

where $L_{S,T} = \max\{L_S, L_T\}$ and $\gamma_{A,B} = \max\{\|A\|, \|B\|\}$. This implies that F_α is a Lipschitz mapping with the constant

$$L = \sqrt{[(N+\alpha L_{S,T})^2 + \gamma_{A,B}^4](1+4N^2)}.$$

This completes the proof. □

We now introduce the following explicit iterative regularization method for finding an element in Ω. For any $(d_0, e_0) \in H_1 \times H_2$, define the two sequences $\{d_n\}$ and $\{e_n\}$ as follows:

$$d_{n+1} = d_n - \varepsilon_n \Big[\sum_{i=1}^{N} \Big((I^{H_1} - P^{H_1}_{C_i})d_n + A_i^*(A_i d_n - B_i e_n - b_i)\Big) + \alpha_n S d_n\Big], \tag{3.16}$$

$$e_{n+1} = e_n - \varepsilon_n \Big[\sum_{i=1}^{N} \Big((I^{H_2} - P^{H_2}_{Q_i})e_n - B_i^*(A_i d_n - B_i e_n - b_i)\Big) + \alpha_n T e_n\Big], \tag{3.17}$$

where $\{\alpha_n\}$ and $\{\varepsilon_n\}$ are two sequences of positive real numbers.

Remark 3.4 The iterative method (3.16)–(3.17) can be rewritten in the following form.

$$(d_{n+1}, e_{n+1}) = (I^{H_1\times H_2} - \varepsilon_n F_{\alpha_n})(d_n, e_n), \ \forall n \geq 0. \tag{3.18}$$

The strong convergence of the sequences $\{d_n\}$ and $\{e_n\}$ generated by (3.16)-(3.17) are given in the following theorem.

Theorem 3.5 *Suppose that the mappings $S: H_1 \to H_1$ and $T: H_2 \to H_2$ are L_S and L_T Lipschitz, and γ_S and γ_T strongly monotone, respectively. If*

$$\varepsilon_n \in \left(0, \frac{2\min\{\gamma_S, \gamma_T\}\alpha_n}{[(N+\alpha_n L_{S,T})^2 + \gamma_{A,B}^4](1+4N^2)}\right), \ \forall n \geq 0 \quad (C),$$

and the following conditions hold

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \varepsilon_n \alpha_n = \infty$;
(C3) $\lim_{n\to\infty} \varepsilon_n/\alpha_n = 0$;
(C4) $\lim_{n\to\infty} \dfrac{|\alpha_{n+1} - \alpha_n|}{\varepsilon_n \alpha_n^2} = 0$,

then $d_n \to x^*$, $e_n \to y^*$ with $(x^*, y^*) \in \Omega$ and (x^*, y^*) is a unique solution to the variational inequality (3.3).

Proof We first show that the sequences $\{d_n\}$ and $\{e_n\}$ are bounded. Indeed, fixing $\bar{w} = (\bar{x}, \bar{y}) \in \Omega$, i.e., $F_{\alpha_n}(\bar{w}) = \alpha_n(S\bar{x}, T\bar{y})$. Let $w_n = (d_n, e_n)$. It follows from Lemma 2.5, Proposition 3.3 and the condition (C) that $I^{H_1\times H_2} - \varepsilon_n F_{\alpha_n}$ is a strict contraction mapping with the contraction coefficient

$$\tau_n = \sqrt{1 - \varepsilon_n(2\min\{\gamma_S, \gamma_T\}\alpha_n - \varepsilon_n L)},$$

where $L = \sqrt{[(N + \alpha L_{S,T})^2 + \gamma_{A,B}^4](1 + 4N^2)}$. Thus, from (3.18) we have

$$\begin{aligned}
\|w_{n+1} - \bar{w}\|_{H_1\times H_2} &= \|(I^{H_1\times H_2} - \varepsilon_n F_{\alpha_n})(w_n) - \bar{w}\|_{H_1\times H_2} \\
&\le \|(I^{H_1\times H_2} - \varepsilon_n F_{\alpha_n})(w_n) - (I^{H_1\times H_2} - \varepsilon_n F_{\alpha_n})(\bar{w})\|_{H_1\times H_2} \\
&\quad + \varepsilon_n \|F_{\alpha_n}(\bar{w})\|_{H_1\times H_2} \\
&= \|(I^{H_1\times H_2} - \varepsilon_n F_{\alpha_n})(w_n) - (I^{H_1\times H_2} - \varepsilon_n F_{\alpha_n})(\bar{w})\|_{H_1\times H_2} \\
&\quad + \varepsilon_n\alpha_n \|(S\bar{x}, T\bar{y})\|_{H_1\times H_2} \\
&\le \tau_n\|w_n - \bar{w}\|_{H_1\times H_2} + (1-\tau_n)\frac{\varepsilon_n\alpha_n\|(S\bar{x}, T\bar{y})\|_{H_1\times H_2}}{1-\tau_n}.
\end{aligned} \tag{3.19}$$

Next, since $\lim_{n\to\infty} \varepsilon_n/\alpha_n = 0$, it follows that

$$\begin{aligned}
\frac{\varepsilon_n\alpha_n}{1-\tau_n} &= \frac{\varepsilon_n\alpha_n(1+\tau_n)}{\varepsilon_n(2\min\{\gamma_S, \gamma_T\}\alpha_n - \varepsilon_n L)} \\
&= \frac{1+\tau_n}{(2\min\{\gamma_S, \gamma_T\} - L\varepsilon_n/\alpha_n)} \to 1/2\min\{\gamma_S, \gamma_T\}.
\end{aligned}$$

Thus, there is a positive real number K_2 such that $\sup_n \dfrac{\varepsilon_n\alpha_n}{1-\tau_n} \le K_2$. Hence, using (3.19), we get

$$\begin{aligned}
\|w_{n+1} - \bar{w}\|_{H_1\times H_2} &\le \tau_n\|w_n - \bar{w}\|_{H_1\times H_2} + (1-\tau_n)K_2\|(S\bar{x}, T\bar{y})\|_{H_1\times H_2} \\
&\le \max\{\|w_n - \bar{w}\|_{H_1\times H_2}, K_2\|(S\bar{x}, T\bar{y})\|_{H_1\times H_2}\} \\
&\ \ \vdots \\
&\le \max\{\|w_0 - \bar{w}\|_{H_1\times H_2}, K_2\|(S\bar{x}, T\bar{y})\|_{H_1\times H_2}\}.
\end{aligned}$$

This implies that the sequence $\{w_n\}$ is bounded, i.e., two sequences $\{d_n\}$ and $\{e_n\}$ are bounded.

Let $h_n = (x_n, y_n)$ which defined by the system of equations (3.1)-(3.2). It is easy to see that two mappings S and T satisfy all conditions in Theorem 3.1 and hence $x_n \to x^*$, $y_n \to y^*$ with $(x^*, y^*) \in \Omega$ and (x^*, y^*) is a unique solution to the variational inequality (3.3).

Now, in order to prove $d_n \to x^*$, $e_n \to y^*$, we will show that $d_n - x_n \to 0$ and $e_n - y_n \to 0$. Indeed, it follows from the inequality

$$\|x\|^2_{H_1\times H_2} \le \|y\|^2_{H_1\times H_2} + 2\langle x, x-y\rangle_{H_1\times H_2}$$

which holds for all $x, y \in H_1 \times H_2$, that

$$\|w_{n+1} - h_{n+1}\|^2_{H_1\times H_2} \le \|w_{n+1} - h_n\|^2_{H_1\times H_2} + 2\langle w_{n+1} - h_{n+1}, h_n - h_{n+1}\rangle_{H_1\times H_2},$$
$$\|w_{n+1} - h_n\|^2_{H_1\times H_2} \le \|w_n - h_n\|^2_{H_1\times H_2} + 2\langle w_{n+1} - h_n, w_{n+1} - w_n\rangle_{H_1\times H_2}.$$

This leads to

$$\begin{aligned}\|w_{n+1} - h_{n+1}\|^2_{H_1\times H_2} &\le \|w_n - h_n\|^2_{H_1\times H_2} + 2\langle w_{n+1} - h_n, w_{n+1} - w_n\rangle_{H_1\times H_2}\\ &\quad + 2\langle w_{n+1} - h_{n+1}, h_n - h_{n+1}\rangle_{H_1\times H_2}\\ &\le \|w_n - h_n\|^2_{H_1\times H_2}\\ &\quad + 2(\|w_{n+1}\|_{H_1\times H_2} + \|h_{n+1}\|_{H_1\times H_2})\|h_{n+1} - h_n\|_{H_1\times H_2}\\ &\quad + 2\langle w_{n+1} - h_n, w_{n+1} - w_n\rangle_{H_1\times H_2}. \end{aligned} \tag{3.20}$$

We now estimate the quantity $\langle w_{n+1} - h_n, w_{n+1} - w_n\rangle_{H_1\times H_2}$. It follows from $F_{\alpha_n}(h_n) = 0$ and (3.5) that

$$\begin{aligned}&\langle w_{n+1} - h_n, w_{n+1} - w_n\rangle_{H_1\times H_2}\\ &\quad= \langle w_n - \varepsilon_n F_{\alpha_n}(w_n) - h_n, w_n - \varepsilon_n F_{\alpha_n}(w_n) - w_n\rangle_{H_1\times H_2}\\ &\quad= \langle w_n - \varepsilon_n F_{\alpha_n}(w_n) - h_n, -\varepsilon_n F_{\alpha_n}(w_n)\rangle_{H_1\times H_2}\\ &\quad= -\varepsilon_n\langle w_n - h_n, F_{\alpha_n}(w_n)\rangle_{H_1\times H_2} + \varepsilon_n^2\|F_{\alpha_n}(w_n)\|^2_{H_1\times H_2}\\ &\quad= -\varepsilon_n\langle w_n - h_n, F_{\alpha_n}(w_n) - F_{\alpha_n}h_n\rangle_{H_1\times H_2} + \varepsilon_n^2\|F_{\alpha_n}(w_n)\|^2_{H_1\times H_2}\\ &\quad\le -\varepsilon_n\alpha_n(\gamma_S\|d_n - x_n\|^2_{H_1} + \gamma_T\|e_n - y_n\|^2_{H_2}) + \varepsilon_n^2\|F_{\alpha_n}(w_n)\|^2_{H_1\times H_2}\\ &\quad\le -\varepsilon_n\alpha_n\min\{\gamma_S, \gamma_T\}\|w_n - h_n\|^2_{H_1\times H_2} + \varepsilon_n^2\|F_{\alpha_n}(w_n)\|^2_{H_1\times H_2}. \end{aligned} \tag{3.21}$$

From the boundedness of $\{w_n\}$, it is easy to see that $\{F_{\alpha_n}(w_n)\}$ is bounded too. Thus, there exists a positive real number K_3 such that

$$\sup_n\{\|w_n\|_{H_1\times H_2}, \|F_{\alpha_n}(w_n)\|_{H_1\times H_2}\} \le K_3.$$

This combines with (3.15), (3.20) and (3.21), we obtain

$$\begin{aligned}\|w_{n+1} - h_{n+1}\|^2_{H_1\times H_2} &\le (1 - 2\min\{\gamma_S, \gamma_T\}\varepsilon_n\alpha_n)\|w_n - h_n\|^2_{H_1\times H_2}\\ &\quad + 2(K\sqrt{2} + K_3)K_1\frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n}\\ &\quad + 2K_3^2\varepsilon_n^2. \end{aligned} \tag{3.22}$$

Letting

$$\Gamma_n = \|w_n - h_n\|^2_{H_1\times H_2},$$
$$b_n = 2\min\{\gamma_S, \gamma_T\}\varepsilon_n\alpha_n,$$
$$c_n = \frac{1}{2\min\{\gamma_S, \gamma_T\}}\left(2(K\sqrt{2}+K_3)\frac{|\alpha_{n+1}-\alpha_n|}{\varepsilon_n\alpha_n^2} + 2K_3^2\frac{\varepsilon_n}{\alpha_n}\right).$$

we can rewrite the above inequality as follows:

$$\Gamma_{n+1} \le (1-b_n)\Gamma_n + b_n c_n.$$

It is not difficult to see that conditions (C1)–(C4) ensure that all the assumptions of Lemma 2.7 are satisfied. Therefore we immediately infer that $\Gamma_n \to 0$, that is, $\|w_n - h_n\|^2_{H_1\times H_2} \to 0$ or $\|d_n - x_n\|^2_{H_1} + \|e_n - y_n\|^2_{H_2} \to 0$. This shows that $\|d_n - x_n\|_{H_1} \to 0$, $\|e_n - y_n\|_{H_2} \to 0$ and hence $d_n \to x^*$, $e_n \to y^*$.
This completes the proof. □

Remark 3.6 It is easy to check that $\alpha_n = 1/\sqrt[4]{n}$ and

$$\varepsilon_n = \frac{2\min\{\gamma_S, \gamma_T\}\alpha_n^2}{[(N+\alpha_n L_{S,T})^2 + \gamma_{A,B}^4](1+4N^2)M},$$

with $M > 1$, satisfy all conditions of Theorem 3.5.

4 Relaxed Iterative Methods

In this section we consider Problem (SSFP) when C_i and Q_i, $i = 1, 2, \ldots, N$, are sublevel sets of the lower semicontinuous convex functions $c_i : H_1 \to \mathbb{R}$ and $q_i : H_2 \to \mathbb{R}$, $i = 1, 2, \ldots, N$, respectively. In other words,

$$C_i = \{x \in H_1 : c_i(x) \le 0\},$$
$$Q_i = \{y \in H_2 : q_i(y) \le 0\},\ i = 1, 2, \ldots, N.$$

Assume that c_i and q_i, $i = 1, 2, \ldots, N$, are subdifferentiable on H_1 and H_2, respectively, and that the subdifferentials ∂c_i and ∂q_i, $i = 1, 2, \ldots, N$, are bounded (on bounded sets). At a point $x_n \in H_1$ and $y_n \in H_2$, we define the subsets $C_{i,n}$ and $Q_{i,n}$ as follows:

$$C_{i,n} := \{x \in H_1 : c_i(x_n) \le \langle x_n - x, \xi_{i,n}\rangle_{H_1}\},$$
$$Q_{i,n} := \{y \in H_2 : q_i(y_n) \le \langle y_n - y, \eta_{i,n}\rangle_{H_2}\},$$

where $\xi_{i,n} \in \partial c_i(x_n)$ and $\eta_{i,n} \in \partial q_i(y_n)$ for all $i = 1, 2, \ldots, N$. The sets $C_{i,n}$ and $Q_{i,n}$ are called the relaxed sets of C_i and Q_i, respectively. It is not difficult to see that

$C_{i,n}$ and $Q_{i,n}$ are half-spaces of H_1 and H_2, and that $C_i \subset C_{i,n}$, $Q_i \subset Q_{i,n}$, for all $i = 1, 2, \ldots, N$.

It is well known that generally speaking, it is not easy to calculate the projections $P_{C_i}^{H_1} x$ and $P_{Q_i}^{H_2} y$. Therefore we introduce two relaxed iterative methods corresponding to the the iterative methods (3.1)–(3.2) and (3.16)–(3.17), where $P_{C_i}^{H_1}$ and $P_{Q_i}^{H_2}$ are replaced by the operators $P_{C_{i,n}}^{H_1}$ and $P_{Q_{i,n}}^{H_2}$, respectively, which are defined as follows:

$$P_{C_{i,n}}^{H_1} x = x - \max\left\{\frac{\langle x, \xi_{i,n}\rangle_{H_1} - \langle x_n, \xi_{i,n}\rangle_{H_1} + c_i(x_n)}{\|\xi_{i,n}\|_{H_1}^2}, 0\right\}\xi_{i,n},$$

$$P_{Q_{i,n}}^{H_2} y = y - \max\left\{\frac{\langle y, \eta_{i,n}\rangle_{H_2} - \langle y_n, \eta_{i,n}\rangle_{H_2} + q_i(y_n)}{\|\eta_{i,n}\|_{H_2}^2}, 0\right\}\eta_{i,n}.$$

We first state and prove the following theorem.

Theorem 4.1 *Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be two bounded on bounded sets, hemicontinuous and strongly monotone mappings with the constants γ_S and γ_T, respectively. Let $\{\alpha_n\}$ be a sequence of positive real numbers. Then the system of regularization equations*

$$\sum_{i=1}^{N}\Big((I^{H_1} - P_{C_{i,n}}^{H_1})x_n + A_i^*(A_i x_n - B_i y_n - b_i)\Big) + \alpha_n S x_n = 0, \tag{4.1}$$

$$\sum_{i=1}^{N}\Big((I^{H_2} - P_{Q_{i,n}}^{H_2})y_n - B_i^*(A_i x_n - B_i y_n - b_i)\Big) + \alpha_n T y_n = 0, \tag{4.2}$$

has a unique solution (x_n, y_n) for each $n \geq 1$. Moreover, if $\alpha_n \to 0$, then

$x_n \to x^*$, $y_n \to y^*$ with $(x^*, y^*) \in \Omega$ and (x^*, y^*) is a unique solution to the variational inequality (3.3).

Proof Following the proof of Theorem 3.1, the system (4.1)–(4.2) has unique solution (x_n, y_n). Moreover, the sequences $\{x_n\}$ and $\{y_n\}$ are bounded.

We now prove that all weak subsequential limits of the sequence $\{(x_n, y_n)\}$ belongs to Ω. Indeed, suppose that (x^*, y^*) is a weak subsequential limit of $\{(x_n, y_n)\}$. There are the subsequences $\{x_{p_n}\}$ and $\{y_{p_n}\}$ of $\{x_n\}$ and $\{y_n\}$, respectively, such that $x_{p_n} \rightharpoonup x^*$ and $y_{p_n} \rightharpoonup y^*$. Moreover, we also have

(1) $\lim_{n\to\infty} (I^{H_1} - P_{C_{i,n}}^{H_1})x_{p_n}\|_{H_1} = 0$, $\lim_{n\to\infty} \|(I^{H_2} - P_{Q_{i,n}}^{H_2})y_{p_n}\|_{H_2} = 0$,
(2) $\lim_{n\to\infty} \|A_i x_{p_n} - B_i y_{p_n} - b_i\|_H = 0$,

for all $i = 1, 2, \ldots, N$.

Since the subdifferential ∂c_i is assumed to be bounded on bounded sets and the sequence $\{x_n\}$ is bounded, there is a positive real number K_4 such that $\|\xi_{i,n}\| \leq K_4$ for all $n \geq 1$. It follows from $P_{C_{i,n}}^{H_1} x_{p_n} \in C_{i,n}$ and the definition of $C_{i,n}$ that

$$c_i(x_{p_n}) \leq \langle (I^{H_1} - P_{C_{i,n}}^{H_1})x_{p_n}, \xi_{i,p_n}\rangle_{H_1} \leq K_4\|(I^{H_1} - P_{C_{i,n}}^{H_1})x_{p_n}\|_{H_1} \to 0.$$

This implies that $\liminf_{n\to\infty} c_i(x_{p_n}) \leq 0$. By the lower semicontinuity of the function c, we have

$$c_i(x^*) \leq \liminf_{n\to\infty} c_i(x_{p_n}) \leq 0.$$

This implies that $x^* \in C_i$. By an argument similar to the one above, we also obtain that $y^* \in Q_i$. Furthermore, from $\lim_{n\to\infty} \|A_i x_{p_n} - B_i y_{p_n} - b_i\|_H = 0$, it is easy to deduce that $A_i x^* - B_i y^* = b_i$. Thus, we have $(x^*, y^*) \in \Omega$.

Using similar argument to the one employed in the proof of Theorem 3.1, we conclude that (x^*, y^*) is the unique weak subsequential limit of $\{(x_n, y_n)\}$, that it is the unique solution to the variational inequality (3.3) and that $x_n \to x^*$ and $y_n \to y^*$, as $n \to \infty$.
This completes the proof. □

Finally, by using a line of proof similar to the one in the proof of Theorem 3.5 and combining it with Theorem 4.1, we obtain the following theorem.

Theorem 4.2 *Suppose that the mappings $S : H_1 \to H_1$ and $T : H_2 \to H_2$ are L_S and L_T lipschitz, and γ_S and γ_T strongly monotone, respectively. Let $\{\varepsilon_n\}$ and $\{\alpha_n\}$ be two sequences of positive real numbers. For any $(d_0, e_0) \in H_1 \times H_2$, define the two sequences $\{d_n\}$ and $\{e_n\}$ as follows:*

$$d_{n+1} = d_n - \varepsilon_n \Big[\sum_{i=1}^{N} \Big((I^{H_1} - P^{H_1}_{C_{i,n}}) d_n + A_i^*(A_i d_n - B_i e_n - b_i) \Big) + \alpha_n S d_n \Big],$$

$$e_{n+1} = e_n - \varepsilon_n \Big[\sum_{i=1}^{N} \Big((I^{H_2} - P^{H_2}_{Q_{i,n}}) e_n - B_i^*(A_i d_n - B_i e_n - b_i) \Big) + \alpha_n T e_n \Big],$$

If conditions (C) and (C1)–(C4) *of Theorem* 3.5 *hold, then $d_n \to x^*$, $e_n \to y^*$ with $(x^*, y^*) \in \Omega$ and (x^*, y^*) is a unique solution to the variational inequality* (3.3).

Acknowledgements The author is very grateful to the editor and two anonymous referees for their helpful comments and useful suggestions.

Funding This research was supported by TNU-University of Sciences.

Availability of Data and Materials Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Competing interests The authors have no relevant financial or non-financial interests to disclose.

References

1. Alber, Y., Ryazantseva, I.: Nonlinear Ill-Posed Problems of Monotone Type. Springer (2006)
2. Attouch, H., Redont, P., Soubeyran, A.: A new class of alternating proximal minimization algorithms with costs-to-move. SIAM J. Optim. **18**, 1061–1081 (2007)

3. Attouch, H., Bolte, J., Redont, P., Soubeyran, A.: Alternating proximal algorithms for weakly coupled convex minimization problems, applications to dynamical games and PDE's. J. Convex Anal. **15**, 485–506 (2008)
4. Attouch, H., Cabot, A., Frankel, F., Peypouquet, J.: Alternating proximal algorithms for linearly constrained variational inequalities: application to domain decomposition for PDE's. Nonlinear Anal. **74**, 7455–7473 (2011)
5. Browder, F.E.: Nonlinear elliptic boundary value problems. Bull. Am. Math. Soc. **69**, 862–874 (1965)
6. Byrne, C.: Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Prob. **18**, 441–453 (2002)
7. Byrne, C.: A unified treatment of some iterative algorithms in signal processing and image reconstruction. Inverse Prob. **18**, 103–120 (2004)
8. Byrne, C., Moudafi, A.: Extensions of the CQ algorithm for the split feasibility and split equality problems. hal-00776640-version 1 (2013)
9. Censor, Y., Bortfeld, T., Martin, B., Trofimov, A.: A unified approach for inversion problems in intensity modulated radiation therapy. Phys. Med. Biol. **51**, 2353–2365 (2006)
10. Censor, Y., Elfving, T.: A multi projection algorithm using Bregman projections in a product space. Numer. Algorithms **8**, 221–239 (1994)
11. Chang, S.-S., Yang, L., Qin, L., Ma, Z.: Strongly convergent iterative methods for split equality variational inclusion problems in banach spaces. Acta Math. Sci. **36**, 1641–1650 (2016)
12. Chidume, C.E., Romanus, O.M., Nnyaba, U.V.: An iterative algorithm for solving split equality fixed point problems for a class of nonexpansive-type mappings in Banach spaces. Numer. Algorithms **82**, 987–1007 (2019)
13. Dong, Q.-L., He, S., Zhao, J.: Solving the split equality problem without prior knowledge of operator norms. Optimization **64**(9), 1887–1906 (2015)
14. Eslamian, M., Shehu, Y., Iyiola, O.S.: A strong convergence theorem for a general split equality problem with applications to optimization and equilibrium problem. Calcolo **55**(48) (2018)
15. Goebel, K., Kirk, W.A.: Topics in Metric Fixed Point Theory, Cambridge Studies in Advanced Mathematics, **28**, Cambridge University Press, Cambridge, UK (1990)
16. Goebel, K., Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Marcel Dekker, New York (1984)
17. Hadamard, J.: Le probléme de Cauchy et les équations aux dérivées partielles hyperboliques. Hermann, Paris (1932)
18. Kazmi, K.R., Ali, R., Furkan, M.: Common solution to a split equality monotone variational inclusion problem, a split equality generalized general variational-like inequality problem and a split equality fixed point problem. Fixed Point Theory **20**(1), 211–232 (2019)
19. Moudafi, A.: Alternating CQ-algorithms for convex feasibility and split fixed-point problems. J. Nonlinear Convex Anal. **15**, 809–818 (2014)
20. Moudafi, A., Al-Shemas, E.: Simultaneous iterative methods for split equality problems and application. Trans. Math. Prog. Appl. **1**, 1–11 (2013)
21. Nnakwe, M.O.: Solving split generalized mixed equality equilibrium problems and split equality fixed point problems for nonexpansive-type maps. Carpath. J. Math. **36**(1), 119–126 (2020)
22. Reich, S., Tuyen, T.M., Ha, M.T.T.: A product space approach to solving the split common fixed point problem in Hilbert spaces. J. Nonlinear Convex Anal. **21**(11), 2571–2588 (2021)
23. Reich, S., Tuyen, T.M.: A new approach to solving split equality problems in Hilbert spaces. Optimization (2021). https://doi.org/10.1080/02331934.2021.1945053
24. Saejung, S., Yotkaew, P.: Approximation of zeros of inverse strongly monotone operators in Banach spaces. Nonlinear Anal. **75**, 742–750 (2012)
25. Shehu, Y., Ogbuisi, F.U., Iyiola, O.S.: Strong convergence theorem for solving split equality fixed point problem which does not involve the prior knowledge of operator norms. Bull. Iran. Math. Soc. **43**(2), 349–371 (2017)
26. Tikhonov, A.N., Arsenin, V.Y.: Solutions of Ill-Posed Problems. Wiley, New York (1977)
27. Ugwunnadi, G.C.: Iterative algorithm for the split equality problem in Hilbert spaces. J. Appl. Anal. **22**(1) (2016)
28. Vuong, P.T., Strodiot, J.J., Nguyen, V.H.: A gradient projection method for solving split equality and split feasibility problems in Hilbert spaces. Optimization **64**(11), 2321–2341 (2015)
29. Zegeye, H.: The general split equality problem for Bregman quasi-nonexpansive mappings in Banach spaces, J. Fixed Point Theory Appl. **20**(6) (2018)

30. Zhao, J.: Solving split equality fixed point problem of quasi-nonexpansive mappings without prior knowledge of operator norms. Optimizations **64**, 2619–2630 (2015)
31. Zhao, J., Wang, S.: Viscosity approximation methods for the split equality common fixed point problem of quasi-nonexpansive operators. Acta Math. Sci. **36B**(5), 1474–1486 (2016)