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Remarks on Blowup of Solutions for Compressible Navier–Stokes Equations with Revised Maxwell's Law

Jianwei Dong[1]

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Abstract

In this note, we study the blowup of classical solutions to the three-dimensional compressible Navier–Stokes equations with revised Maxwell's law. First, we improve the previous blowup result with initial density away from vacuum by removing three restrictions. Next, we present a blowup result for the classical solutions with decay at far fields when the shear relaxation time is zero by introducing a new averaged quantity.

Keywords Compressible Navier–Stokes equations · Revised Maxwell's law · Classical solutions · Blowup

Mathematics Subject Classification 35Q30 · 35Q35 · 35B44

1 Introduction

In this note, we consider the following three-dimensional compressible Navier–Stokes equations with revised Maxwell's law (see [10] for instance):

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \operatorname{div}\mathbb{S}, \end{cases} \tag{1.1}$$

where the fluid density $\rho = \rho(x, t)$, the fluid velocity $\mathbf{u} = \mathbf{u}(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ and the stress tensor $\mathbb{S} = \mathbb{S}(x, t)$ are the unknown functions with

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✉ Jianwei Dong
dongjianweiccm@163.com

1 School of Mathematics, Zhengzhou University of Aeronautics, Zhengzhou 450015, People's Republic of China

$(x, t) \in \mathbb{R}^3 \times \mathbb{R}^+$. The function $p(\rho) = a\rho^\gamma$ represents the pressure with $a > 0$ and $\gamma > 1$ being two constants, γ is called the adiabatic exponent. The stress tensor $\mathbb{S}$ is assumed to satisfy the revised Maxwell's law: $\mathbb{S} = \mathbb{S}_1 + S_2\mathbb{I}$, where $\mathbb{S}_1$ and S_2 are given by the following equations

$$\tau_1(\mathbb{S}_1)_t + \mathbb{S}_1 = \mu[(\nabla \mathbf{u} + \nabla \mathbf{u}^T) - \frac{2}{3}\text{div}\mathbf{u}\mathbb{I}], \quad (1.2)$$

$$\tau_2(S_2)_t + S_2 = \lambda \text{div}\mathbf{u}, \quad (1.3)$$

where μ and λ are positive constants, τ_1 and τ_2 denote the shear relaxation time and the compressible relaxation time, respectively, $\mathbb{I}$ represents 3×3 identity matrix. Equations (1.2) and (1.3) were first proposed by Yong [14] from some mathematical point of view, later Chakraborty and Sader [2] showed the fact that the division of $\mathbb{S}$ into two parts has its physical meanings, where the importance of this model for describing high frequency limits is underlined together with the presentation of numerical experiments.

There were some works about the mathematical analysis of the Navier–Stokes equations with revised Maxwell's law in the literature. Yong proved that system (1.1)–(1.3) is symmetric hyperbolic system, which implies local well-posedness immediately. Again due to the symmetric hyperbolic property of system (1.1)–(1.3), the important property of finite propagation speed is available, which allows one to define some averaged quantities as in [8] and shows finite time blowup of solutions by establishing a Riccati-type inequality, see [10]. Precisely, Wang and Hu [10] considered system (1.1)–(1.3) with initial data

$$(\rho, \mathbf{u}, \mathbb{S}_1, S_2)(x, 0) = (\rho_0, \mathbf{u}_0, \mathbb{S}_{10}, S_{20})(x), \quad (1.4)$$

they showed that if the initial data $(\rho_0 - \overline{\rho}, \mathbf{u}_0, \mathbb{S}_{10}, S_{20})$ are compactly supported in $B_R := \{x \in \mathbb{R}^3 | |x| \leq R\}$ for some $R > 0$ with $\overline{\rho}$ being any positive constant, then there exists a constant σ such that

$$(\rho(\cdot, t) - \overline{\rho}, \mathbf{u}(\cdot, t), \mathbb{S}_1(\cdot, t), S_2(\cdot, t)) = (0, 0, 0, 0) \quad (1.5)$$

on $D = \{x \in \mathbb{R}^3 | |x| > R + \sigma t\}$ for C^1 solutions $(\rho(\cdot, t), \mathbf{u}(\cdot, t), \mathbb{S}_1(\cdot, t), S_2(\cdot, t))$ to the Cauchy problem (1.1)–(1.4) on $[0, T]$, see Proposition 2.1 of [10]. This guarantees that the following averaged quantities are finite:

$$m(t) := \int_{\mathbb{R}^3} (\rho(x, t) - \overline{\rho}) dx, \quad (1.6)$$

$$A(t) := \int_{\mathbb{R}^3} S_2(x, t) dx, \quad (1.7)$$

$$F(t) := \int_{\mathbb{R}^3} x \cdot (\rho \mathbf{u})(x, t) dx. \quad (1.8)$$

By establishing a Riccati-type inequality of $F(t)$, the authors of Wang and Hu [10] proved that the life span of any C^1 solution to (1.1)–(1.4) must be finite for some special large initial data. Unfortunately, the following three restrictions are required in [10]:

$$m(0) \geq 0, \quad \mathrm{Trace}\{\mathbb{S}_{10}(x)\} = 0, \quad A(0) \leq 0. \tag{1.9}$$

In this note, our first aim is to give a blowup result for the C^1 solutions to (1.1)–(1.4) without the conditions (1.9).

When $\tau_1 = \tau_2 = 0$, the system (1.1)–(1.3) is reduced to the classical compressible Navier–Stokes system. Many authors have studied the blowup of smooth solutions to the classical compressible Navier–Stokes system, see [1, 3, 5–7, 9, 12, 13] and the references therein. Particularly, the authors of Jiu [5] and Wang et al. [9] investigated the blowup of classical solutions to the classical compressible Navier–Stokes system when the density and the velocity decay at far fields. Motivated by Jiu [5] and Wang et al. [9], we can present a blowup result of classical solutions to (1.1)–(1.4) when $\tau_1 = 0$ and the density and the velocity decay at far fields. This is our second aim of this note. Different from Jiu [5] and Wang et al. [9], for the problem (1.1)–(1.4) we cannot obtain the conversation of the total energy or the decrease with time of the total energy. This makes us difficult to get the upper bound of the momentum of inertia, which is crucial to establish the Riccati-type inequality of a weighted momentum. To overcome this difficulty, we introduce a new averaged quantity $J(t)$ [see (3.3) below], which decreases with time and is related to the total energy.

Remark 1.1 For the results about the non-isentropic Navier–Stokes equations with revised Maxwell's law, we can refer to [4, 11].

2 Improvement on the Blowup Result of Wang and Hu [10]

Our result in this section is stated as follows.

Theorem 2.1 *Let* $(\rho, u, \mathbb{S}_1, S_2)$ *be a* C^1 *solution to the Cauchy problem* (1.1)–(1.4) *for* $0 \leq t \leq T_1$ *with the initial data* $(\rho_0 - \overline{\rho}, \mathbf{u}_0, \mathbb{S}_{10}, S_{20})$ *being compactly supported in* $B_R := \{x \in \mathbb{R}^3 | |x| \leq R\}$. *For any fixed* $t_1^* > 0$, *if*

$$F(0) > \sqrt{2[m(0) + \overline{\rho}|B(t_1^*)|][3\overline{p}|B(t_1^*)| + 3|A(0)| + |G(0)|](R + \sigma t_1^*)} \tag{2.1}$$

and

$$F(0) > \left\{ \int_0^{t_1^*} \frac{dt}{2(R + \sigma t)^2 [m(0) + \overline{\rho}|B(t)|]} \right\}^{-1}, \tag{2.2}$$

then $T_1 < t_1^*$, *where*

$$|B(t_1^*)| = \frac{4}{3}\pi(R + \sigma t_1^*)^3, \quad \overline{p} = p(\overline{\rho}), \quad G(0) = \int_{\mathbb{R}^3} \mathrm{Trace}\{\mathbb{S}_{10}\} dx. \tag{2.3}$$

Proof By (1.6), $(1.1)_1$ and (1.5), we get

$$m'(t) = \int_{\mathbb{R}^3} \rho_t dx = -\int_{\mathbb{R}^3} \mathrm{div}(\rho \mathbf{u}) dx = 0, \tag{2.4}$$

which implies that

$$m(t) = m(0). \tag{2.5}$$

In view of (1.7), (1.3) and (1.5), one has

$$A'(t) = -\frac{1}{\tau_2}\int_{\mathbb{R}^3} S_2 dx + \frac{\lambda}{\tau_2}\int_{\mathbb{R}^3} \mathrm{div}\mathbf{u} dx = -\frac{1}{\tau_2}\int_{\mathbb{R}^3} S_2 dx = -\frac{1}{\tau_2}A(t), \tag{2.6}$$

so we obtain

$$|A(t)| = |A(0)|e^{-\frac{t}{\tau_2}} \leq |A(0)|. \tag{2.7}$$

By (1.2), we know that

$$\tau_1 \partial_t \mathrm{Trace}\{\mathbb{S}_1\} + \mathrm{Trace}\{\mathbb{S}_1\} = 0 \tag{2.8}$$

due to

$$\mathrm{Trace}\{(\nabla \mathbf{u} + \nabla \mathbf{u}^T) - \frac{2}{3}\mathrm{div}\mathbf{u}\mathbb{I}\} = 0. \tag{2.9}$$

Define

$$G(t) = \int_{\mathbb{R}^3} \mathrm{Trace}\{\mathbb{S}_1(\mathrm{x}, \mathrm{t})\} dx, \tag{2.10}$$

then by (2.8), we have

$$|G(t)| = |G(0)|e^{-\frac{t}{\tau_1}} \leq |G(0)|. \tag{2.11}$$

It follows from (1.8), $(1.1)_2$, (1.5), (2.7) and (2.11) that

$$\begin{aligned}
F'(t) &= \int_{\mathbb{R}^3} x \cdot (\rho \mathbf{u})_t dx \\
&= -\int_{\mathbb{R}^3} x \cdot \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) dx - \int_{\mathbb{R}^3} x \cdot \nabla (p(\rho) - \overline{p}) dx + \int_{\mathbb{R}^3} x \cdot \mathrm{div}\mathbb{S} dx \\
&= \int_{\mathbb{R}^3} \rho |\mathbf{u}|^2 dx + 3\int_{\mathbb{R}^3} (p(\rho) - \overline{p}) dx - 3\int_{\mathbb{R}^3} S_2 dx - \int_{\mathbb{R}^3} \mathrm{Trace}\{\mathbb{S}_1\} dx \\
&= \int_{B(t)} \rho |\mathbf{u}|^2 dx + 3\int_{B(t)} (p(\rho) - \overline{p})) dx - 3A(t) - G(t)
\end{aligned}$$

$$\geq \int_{B(t)} \rho|\mathbf{u}|^2 dx - 3\overline{p}|B(t)| - 3|A(0)| - |G(0)|, \tag{2.12}$$

where

$$B(t) = \{x \in \mathbb{R}^3 | |x| \leq R + \sigma t\}, \quad |B(t)| = \frac{4}{3}\pi(R+\sigma t)^3. \tag{2.13}$$

By the Schwarz inequality, it holds

$$F(t)^2 = \left(\int_{B(t)} x \cdot \rho \mathbf{u} dx\right)^2 \leq \int_{B(t)} |x|^2 \rho dx \cdot \int_{B(t)} \rho|\mathbf{u}|^2 dx. \tag{2.14}$$

By (1.6) and (2.5), we know that

$$\begin{aligned} \int_{B(t)} |x|^2 \rho dx &\leq (R+\sigma t)^2 \int_{B(t)} \rho dx = (R+\sigma t)^2 \left[\int_{B(t)} (\rho - \overline{\rho}) dx + \int_{B(t)} \overline{\rho} dx\right] \\ &= (R+\sigma t)^2 \left[m(t) + \overline{\rho}|B(t)|\right] = (R+\sigma t)^2 \left[m(0) + \overline{\rho}|B(t)|\right]. \end{aligned} \tag{2.15}$$

Combining (2.12), (2.14) and (2.15), we obtain

$$F'(t) \geq \frac{F(t)^2}{(R+\sigma t)^2 \left[m(0) + \overline{\rho}|B(t)|\right]} - 3\overline{p}|B(t)| - 3|A(0)| - |G(0)|. \tag{2.16}$$

For any fixed $t_1^* > 0$, when $0 \leq t \leq t_1^*$, it follows from (2.16) that

$$\begin{aligned} F'(t) \geq & \frac{F(t)^2}{2(R+\sigma t)^2 \left[m(0) + \overline{\rho}|B(t)|\right]} \\ & + \left\{ \frac{F(t)^2}{2(R+\sigma t_1^*)^2 \left[m(0) + \overline{\rho}|B(t_1^*)|\right]} - 3\overline{p}|B(t_1^*)| - 3|A(0)| - |G(0)| \right\}. \end{aligned} \tag{2.17}$$

By (2.1), we know that

$$\frac{F(0)^2}{2(R+\sigma t_1^*)^2 \left[m(0) + \overline{\rho}|B(t_1^*)|\right]} - 3\overline{p}|B(t_1^*)| - 3|A(0)| - |G(0)| > 0, \tag{2.18}$$

which together with (2.17) imply that $F'(0) > 0$, so $F(t) > F(0) > 0$ holds at least for a small time $t > 0$, then by (2.17) we derive that

$$F'(t) \geq \frac{F(t)^2}{2(R+\sigma t)^2 \left[m(0) + \overline{\rho}|B(t)|\right]} \tag{2.19}$$

and

$$\frac{F(t)^2}{2(R+\sigma t_1^*)^2\left[m(0)+\overline{\rho}|B(t_1^*)|\right]}-3\overline{p}|B(t_1^*)|-3|A(0)|-|G(0)|>0 \quad (2.20)$$

hold whenever $F(t)$ exists and $0<t\le t_1^*$. We divide (2.19) by $F(t)^2$ and integrate the resultant inequality over $[0, t_1^*]$ to have

$$\frac{1}{F(0)}>\frac{1}{F(0)}-\frac{1}{F(t_1^*)}\ge\int_0^{t_1^*}\frac{dt}{2(R+\sigma t)^2\left[m(0)+\overline{\rho}|B(t)|\right]}. \quad (2.21)$$

On the other hand, by (2.2) we know that

$$\frac{1}{F(0)}<\int_0^{t_1^*}\frac{dt}{2(R+\sigma t)^2\left[m(0)+\overline{\rho}|B(t)|\right]}, \quad (2.22)$$

which together with (2.21) imply that $T_1<t_1^*$. We complete the proof of Theorem 2.1. □

Remark 2.1 In Theorem 2.1 above, we show the blowup of classical solutions to the problem (1.1)–(1.4) without the three restrictions $m(0)\ge 0$, $\mathrm{Trace}\{\mathbb{S}_{10}(x)\}=0$ and $A(0)\le 0$, so we have improved the blowup result of [10].

3 Blowup for Solutions Decay at Far Fields

In this section, we need the following averaged quantities:

$$I(t):=\frac{1}{2}\int_{\mathbb{R}^3}|x|^2\rho(x,t)dx, \quad (3.1)$$

$$E(t):=\frac{1}{2}\int_{\mathbb{R}^3}\rho|\mathbf{u}|^2dx+\frac{a}{\gamma-1}\int_{\mathbb{R}^3}\rho^\gamma dx=E_k(t)+E_i(t), \quad (3.2)$$

$$J(t):=E(t)+\frac{\tau_2}{2\lambda}\int_{\mathbb{R}^3}S_2^2dx, \quad (3.3)$$

where $I(t)$, $E(t)$, $E_k(t)$ and $E_i(t)$ represent the momentum of inertia, the total energy, the kinetic energy and the internal energy, respectively, which have been used in [5, 9]. We remark that the averaged quantity $J(t)$ is new in the literature. In this section, we only consider the classical solutions with decay at far fields. Precisely, for any $T>0$, we require that the solutions $(\rho, \mathbf{u}, \mathbb{S}_1, S_2)$ satisfy that

$$\rho|\mathbf{u}|^3,\ \rho^\gamma|\mathbf{u}|,\ |S_2\mathbf{u}|,\ |\mathbf{u}\nabla\mathbf{u}|,\ |x|\rho|\mathbf{u}|^2,\ |x|\rho^\gamma,\ |x\nabla\mathbf{u}|,\ |xS_2|,\ |\mathbf{u}|\in L^\infty(0,T;L^1(\mathbb{R}^3)). \quad (3.4)$$

We should remark that the condition (3.4) guarantees that the integration by parts in our calculations makes sense (see also [5, 9]).

The main result of this section is stated as follows.

Theorem 3.1 *Let* $(\rho, \mathbf{u}, \mathbb{S}_1, S_2)$ *be a* C^1 *solution to the Cauchy problem* (1.1)–(1.4) *with* $\tau_1 = 0$ *satisfying* (3.4) *for* $0 \le t \le T_2$ *with the initial data satisfying* $0 < |A(0)|, I(0), J(0) < +\infty$. *For any fixed* $t_2^* > 0$, *if*

$$F(0) > \sqrt{6|A(0)|\{[\max\{2, 3(\gamma-1)\}J(0) + 3|A(0)|]t_2^{*2} + 2F(0)t_2^* + 2I(0)\}} \tag{3.5}$$

and

$$F(0) > \left\{ \int_0^{t_2^*} \frac{dt}{2\{[\max\{2, 3(\gamma-1)\}J(0) + 3|A(0)|]t^2 + 2F(0)t + 2I(0)\}} \right\}, \tag{3.6}$$

then $T_2 < t_2^*$.

Proof We multiply $(1.1)_2$ by $\mathbf{u}$ and integrate it over $\mathbb{R}^3$ to have

$$\begin{aligned}&\int_{\mathbb{R}^3} (\rho\mathbf{u})_t \cdot \mathbf{u}dx + \int_{\mathbb{R}^3} \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) \cdot \mathbf{u}dx + \int_{\mathbb{R}^3} \nabla p(\rho) \cdot \mathbf{u}dx \\ &\quad = \mu \int_{\mathbb{R}^3} \operatorname{div}[(\nabla\mathbf{u} + \nabla\mathbf{u}^T) - \frac{2}{3}\operatorname{div}\mathbf{u}\mathbb{I}] \cdot \mathbf{u}dx + \int_{\mathbb{R}^3} \nabla S_2 \cdot \mathbf{u}dx,\end{aligned} \tag{3.7}$$

where we have used $\mathbb{S} = \mathbb{S}_1 + S_2\mathbb{I}$ and (1.2) with $\tau_1 = 0$. We use integration by part and $(1.1)_1$ to obtain

$$\begin{aligned}\int_{\mathbb{R}^3} \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) \cdot \mathbf{u}dx &= \int_{\mathbb{R}^3} \operatorname{div}(\rho\mathbf{u})|\mathbf{u}|^2 dx + \int_{\mathbb{R}^3} \rho\mathbf{u} \cdot \nabla\mathbf{u} \cdot \mathbf{u}dx \\ &= \int_{\mathbb{R}^3} \operatorname{div}(\rho\mathbf{u})|\mathbf{u}|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \rho\mathbf{u} \cdot \nabla(|\mathbf{u}|^2)dx \\ &= \frac{1}{2}\int_{\mathbb{R}^3} \operatorname{div}(\rho\mathbf{u})|\mathbf{u}|^2 dx \\ &= -\frac{1}{2}\int_{\mathbb{R}^3} \rho_t|\mathbf{u}|^2 dx.\end{aligned} \tag{3.8}$$

Similarly,

$$\begin{aligned}\int_{\mathbb{R}^3} \nabla p(\rho) \cdot \mathbf{u}dx &= a\gamma \int_{\mathbb{R}^3} \rho^{\gamma-1}\nabla\rho \cdot \mathbf{u}dx \\ &= \frac{a\gamma}{\gamma-1}\int_{\mathbb{R}^3} \nabla(\rho^{\gamma-1}) \cdot (\rho\mathbf{u})dx \\ &= -\frac{a\gamma}{\gamma-1}\int_{\mathbb{R}^3} \rho^{\gamma-1}\operatorname{div}(\rho\mathbf{u})dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{a\gamma}{\gamma-1}\int_{\mathbb{R}^3}\rho^{\gamma-1}\rho_t dx \\
&= \frac{d}{dt}\int_{\mathbb{R}^3}\frac{\rho^\gamma}{\gamma-1}dx, \qquad (3.9)
\end{aligned}
$$

$$
\int_{\mathbb{R}^3}\nabla S_2\cdot\mathbf{u}dx = -\int_{\mathbb{R}^3} S_2\mathrm{div}\mathbf{u}dx. \qquad (3.10)
$$

We estimate the integral $\mu\int_{\mathbb{R}^3}\mathrm{div}[(\nabla\mathbf{u}+\nabla\mathbf{u}^T)-\frac{2}{3}\mathrm{div}\mathbf{u}\mathbb{I}]\cdot\mathbf{u}dx$ as follows. Noticing that

$$
\begin{aligned}
\mathrm{div}(\nabla\mathbf{u}+\nabla\mathbf{u}^T) = \Bigg(&2\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right), \\
&\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)+2\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right), \\
&\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)+2\frac{\partial}{\partial x_3}\left(\frac{\partial u_3}{\partial x_3}\right)\Bigg),
\end{aligned} \qquad (3.11)
$$

one has

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\mathrm{div}(\nabla\mathbf{u}+\nabla\mathbf{u}^T)\cdot\mathbf{u}dx \\
&= \int_{\mathbb{R}^3}u_1\left[2\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)\right]dx \\
&\quad + \int_{\mathbb{R}^3}u_2\left[\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)+2\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)\right]dx \\
&\quad + \int_{\mathbb{R}^3}u_3\left[\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)+2\frac{\partial}{\partial x_3}\left(\frac{\partial u_3}{\partial x_3}\right)\right]dx \\
&= -2\int_{\mathbb{R}^3}\sum_{i=1}^3\left(\frac{\partial u_i}{\partial x_i}\right)^2dx - \int_{\mathbb{R}^3}\left[\left(\frac{\partial u_1}{\partial x_2}\right)^2+\left(\frac{\partial u_1}{\partial x_3}\right)^2\right]dx \\
&\quad - \int_{\mathbb{R}^3}\left[\left(\frac{\partial u_2}{\partial x_1}\right)^2+\left(\frac{\partial u_2}{\partial x_3}\right)^2\right]dx - \int_{\mathbb{R}^3}\left[\left(\frac{\partial u_3}{\partial x_1}\right)^2+\left(\frac{\partial u_3}{\partial x_2}\right)^2\right]dx \\
&\quad - 2\int_{\mathbb{R}^3}\left(\frac{\partial u_1}{\partial x_2}\cdot\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\cdot\frac{\partial u_3}{\partial x_1}+\frac{\partial u_2}{\partial x_3}\cdot\frac{\partial u_3}{\partial x_2}\right)dx \\
&= -2\int_{\mathbb{R}^3}\sum_{i=1}^3\left(\frac{\partial u_i}{\partial x_i}\right)^2dx - \int_{\mathbb{R}^3}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)^2dx \\
&\quad - \int_{\mathbb{R}^3}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)^2dx - \int_{\mathbb{R}^3}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)^2dx.
\end{aligned} \qquad (3.12)
$$

$$
\frac{2}{3}\int_{\mathbb{R}^3}\mathrm{div}(\mathrm{div}\mathbf{u}\mathbb{I})\cdot\mathbf{u}dx = -\frac{2}{3}\int_{\mathbb{R}^3}(\sum_{i=1}^3\frac{\partial u_i}{\partial x_i})^2dx. \qquad (3.13)
$$

Consequently,

$$\int_{\mathbb{R}^3} \operatorname{div}\left[(\nabla \mathbf{u} + \nabla \mathbf{u}^T) - \frac{2}{3}\operatorname{div}\mathbf{u}\mathbb{I}\right] \cdot \mathbf{u} dx$$
$$= -2\int_{\mathbb{R}^3} \sum_{i=1}^{3} \left(\frac{\partial u_i}{\partial x_i}\right)^2 dx + \frac{2}{3}\int_{\mathbb{R}^3} (\sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i})^2 dx$$
$$- \int_{\mathbb{R}^3} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)^2 dx - \int_{\mathbb{R}^3} \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)^2 dx - \int_{\mathbb{R}^3} \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)^2 dx$$
$$\leq - \int_{\mathbb{R}^3} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)^2 dx - \int_{\mathbb{R}^3} \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)^2 dx$$
$$- \int_{\mathbb{R}^3} \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)^2 dx. \tag{3.14}$$

Combining (3.7)–(3.10), (3.14) and (3.2), we obtain

$$\frac{d}{dt}E(t) \leq - \int_{\mathbb{R}^3} S_2 \operatorname{div}\mathbf{u} dx - \int_{\mathbb{R}^3} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)^2 dx$$
$$- \int_{\mathbb{R}^3} \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)^2 dx - \int_{\mathbb{R}^3} \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)^2 dx. \tag{3.15}$$

We multiply (1.3) by $\frac{S_2}{\lambda}$ and integrate the resultant equation over $\mathbb{R}^3$ to have

$$\frac{\tau_2}{2\lambda}\frac{d}{dt}\int_{\mathbb{R}^3} S_2^2 dx = -\frac{1}{\lambda}\int_{\mathbb{R}^3} S_2^2 dx + \int_{\mathbb{R}^3} S_2 \operatorname{div}\mathbf{u} dx. \tag{3.16}$$

In view of (3.15), (3.16) and (3.3), it holds

$$\frac{d}{dt}J(t) \leq - \int_{\mathbb{R}^3} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)^2 dx - \int_{\mathbb{R}^3} \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)^2 dx$$
$$- \int_{\mathbb{R}^3} \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)^2 dx \leq 0, \tag{3.17}$$

which implies that

$$J(t) \leq J(0), \quad 0 \leq t \leq T_2. \tag{3.18}$$

By (3.1), $(1.1)_1$ and (1.8), we know that

$$I'(t) = \frac{1}{2}\int_{\mathbb{R}^3} |x|^2 \rho_t dx = -\frac{1}{2}\int_{\mathbb{R}^3} |x|^2 \operatorname{div}(\rho\mathbf{u}) dx = \int_{\mathbb{R}^3} x \cdot \rho\mathbf{u} dx = F(t). \tag{3.19}$$

In view of (1.8) and $(1.1)_2$, we get

$$\begin{aligned} F'(t) &= \int_{\mathbb{R}^3} x\cdot(\rho\mathbf{u})_t dx \\ &= -\int_{\mathbb{R}^3} x\cdot \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u})dx - \int_{\mathbb{R}^3} x\cdot\nabla p(\rho)dx \\ &\quad +\mu\int_{\mathbb{R}^3} x\cdot \mathrm{div}[(\nabla\mathbf{u}+\nabla\mathbf{u}^T)-\frac{2}{3}\mathrm{div}\mathbf{u}\mathbb{I}]dx + \int_{\mathbb{R}^3} x\cdot\nabla S_2 dx, \end{aligned} \tag{3.20}$$

where we have used $\mathbb{S}=\mathbb{S}_1+S_2\mathbb{I}$ and (1.2) with $\tau_1=0$. Using (3.4) and integration by part, we obtain

$$\begin{aligned} -\int_{\mathbb{R}^3} x\cdot\mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u})dx &= -\int_{\mathbb{R}^3} x_1\left[\frac{\partial}{\partial x_1}(\rho u_1^2)+\frac{\partial}{\partial x_2}(\rho u_2 u_1)+\frac{\partial}{\partial x_3}(\rho u_3 u_1)\right]dx \\ &\quad -\int_{\mathbb{R}^3} x_2\left[\frac{\partial}{\partial x_1}(\rho u_1 u_2)+\frac{\partial}{\partial x_2}(\rho u_2^2)+\frac{\partial}{\partial x_3}(\rho u_3 u_2)\right]dx \\ &\quad -\int_{\mathbb{R}^3} x_3\left[\frac{\partial}{\partial x_1}(\rho u_1 u_3)+\frac{\partial}{\partial x_2}(\rho u_2 u_3)+\frac{\partial}{\partial x_3}(\rho u_3^2)\right]dx \\ &= \int_{\mathbb{R}^3}\rho(u_1^2+u_2^2+u_3^2)dx = \int_{\mathbb{R}^3}\rho|\mathbf{u}|^2 dx, \end{aligned} \tag{3.21}$$

$$-\int_{\mathbb{R}^3} x\cdot\nabla p(\rho)dx = 3a\int_{\mathbb{R}^3}\rho^\gamma dx, \tag{3.22}$$

$$\begin{aligned} &\mu\int_{\mathbb{R}^3} x\cdot\mathrm{div}\left[(\nabla\mathbf{u}+\nabla\mathbf{u}^T)-\frac{2}{3}\mathrm{div}\mathbf{u}\mathbb{I}\right]dx \\ &= -\mu\int_{\mathbb{R}^3}\mathrm{Trace}\left[(\nabla\mathbf{u}+\nabla\mathbf{u}^T)-\frac{2}{3}\mathrm{div}\mathbf{u}\mathbb{I}\right]dx = 0, \end{aligned} \tag{3.23}$$

$$\int_{\mathbb{R}^3} x\cdot\nabla S_2 dx = -3\int_{\mathbb{R}^3} S_2 dx. \tag{3.24}$$

It follows from (3.20) to (3.24) that

$$F'(t) = \int_{\mathbb{R}^3}\rho|\mathbf{u}|^2 dx + 3a\int_{\mathbb{R}^3}\rho^\gamma dx - 3\int_{\mathbb{R}^3} S_2 dx, \tag{3.25}$$

which together with (3.2) lead to

$$F'(t) = 2E_k(t) + 3(\gamma-1)E_i(t) - 3\int_{\mathbb{R}^3} S_2 dx. \tag{3.26}$$

We integrate (1.3) over $\mathbb{R}^3$ and use the condition $|\mathbf{u}| \in L^\infty(0, T; L^1(\mathbb{R}^3))$ in (3.4) to have

$$\tau_2 \frac{d}{dt} \int_{\mathbb{R}^3} S_2 dx + \int_{\mathbb{R}^3} S_2 dx = \lambda \int_{\mathbb{R}^3} \mathrm{div}\mathbf{u} dx = 0, \tag{3.27}$$

which together with (1.7) and (3.26) imply that

$$\begin{aligned} F'(t) &= 2E_k(t) + 3(\gamma - 1)E_i(t) - 3A(t) \\ &= 2E_k(t) + 3(\gamma - 1)E_i(t) - 3A(0)e^{-\frac{t}{\tau_2}}. \end{aligned} \tag{3.28}$$

By (3.19), (3.28), (3.2), (3.3) and (3.18), we know that

$$\begin{aligned} I''(t) = F'(t) &\leq \max\{2, 3(\gamma - 1)\}J(t) + 3|A(0)| \\ &\leq \max\{2, 3(\gamma - 1)\}J(0) + 3|A(0)|. \end{aligned} \tag{3.29}$$

We integrate (3.29) over $\mathbb{R}^3$ twice and use (3.19) to obtain

$$I(t) \leq \frac{1}{2}[\max\{2, 3(\gamma - 1)\}J(0) + 3|A(0)|]t^2 + F(0)t + I(0). \tag{3.30}$$

By the Schwarz inequality, (3.1) and (3.2), it holds

$$F(t)^2 = \left(\int_{\mathbb{R}^3} x \cdot \rho\mathbf{u} dx\right)^2 \leq \int_{\mathbb{R}^3} |x|^2 \rho dx \cdot \int_{\mathbb{R}^3} \rho|\mathbf{u}|^2 dx = 4I(t)E_k(t), \tag{3.31}$$

which together with (3.28) and (3.30) imply that

$$\begin{aligned} F'(t) &\geq 2E_k(t) - 3|A(0)| \geq \frac{F(t)^2}{2I(t)} - 3|A(0)| \\ &\geq \frac{F(t)^2}{[\max\{2, 3(\gamma - 1)\}J(0) + 3|A(0)|]t^2 + 2F(0)t + 2I(0)} - 3|A(0)|. \end{aligned} \tag{3.32}$$

This corresponds to (2.16), so the rest proof is similar to the one of Theorem 2.1, we omit the details. □

In fact, if we assume $A(0) \leq 0$, we can obtain a refined blowup result for the Cauchy problem (1.1)–(1.4) with solutions decay at far fields.

Corollary 3.1 *Denote*

$$c_1 = \max\{2, 3(\gamma - 1)\}, \quad c_2 = \sqrt{2[c_1 J(0) - 3A(0)]I(0) - F(0)^2}. \tag{3.33}$$

Under the assumptions of Theorem 3.1, if we further assume that $A(0) \le 0$, $F(0) > 0$ and

$$\frac{c_2}{F(0)} + \arctan\frac{F(0)}{c_2} < \frac{\pi}{2}, \tag{3.34}$$

then the life span T_2 of the classical solutions to (1.1)–(1.4) satisfies that

$$T_2 < \frac{c_2}{c_1 J(0) - 3A(0)} \tan\left[\frac{c_2}{F(0)} + \arctan\frac{F(0)}{c_2}\right] - \frac{F(0)}{c_1 J(0) - 3A(0)} \tag{3.35}$$

Proof By the condition $A(0) \le 0$ and (3.28)–(3.30), we know that

$$F'(t) \ge 2E_k(t) + 3(\gamma - 1)E_i(t) \tag{3.36}$$

and

$$I(t) \le \frac{1}{2}[\max\{2, 3(\gamma - 1)\}J(0) - 3A(0)]t^2 + F(0)t + I(0). \tag{3.37}$$

Then (3.32) becomes

$$F'(t) \ge \frac{F(t)^2}{[\max\{2, 3(\gamma - 1)\}J(0) - 3A(0)]t^2 + 2F(0)t + 2I(0)}. \tag{3.38}$$

The inequality (3.36) implies that $F(t)$ increases with time, which together with $F(0) > 0$ lead to the fact that $F(t) > 0$ for $t \in [0, T_2]$. We use (3.33) to rewrite (3.38) as

$$F'(t) \ge \frac{F(t)^2}{[c_1 J(0) - 3A(0)]\left\{\left(t + \frac{F(0)}{c_1 J(0) - 3A(0)}\right)^2 + \frac{c_2^2}{[c_1 J(0) - 3A(0)]^2}\right\}}. \tag{3.39}$$

Noticing that

$$\begin{aligned} c_2^2 = 2[c_1 J(0) - 3A(0)]I(0) - F(0)^2 \ge 2c_1 J(0)I(0) \\ - F(0)^2 > 4E_k(0)I(0) - F(0)^2 \ge 0, \end{aligned} \tag{3.40}$$

we have $c_2 > 0$. So we divide (3.39) by $F(t)^2$ and integrate the resultant inequality over $[0, T_2]$ to obtain

$$\begin{aligned} \frac{1}{F(0)} &> \frac{1}{F(0)} - \frac{1}{F(T_2)} \\ &\ge \frac{1}{c_2}\left[\arctan\frac{[c_1 J(0) - 3A(0)]\left(T_2 + \frac{F(0)}{c_1 J(0) - 3A(0)}\right)}{c_2} - \arctan\frac{F(0)}{c_2}\right], \end{aligned} \tag{3.41}$$

which can be solved as (3.35). The proof of Corollary 3.1 is finished. □

Remark 3.1 The condition (3.34) in Corollary 3.1 is used to ensure that the upper bound in (3.35) is positive. In fact, by using (3.34) and the monotonicity of the function $\tan x$ in $(0, \frac{\pi}{2})$, we have

$$\frac{c_2}{c_1 J(0) - 3A(0)} \tan\left(\frac{c_2}{F(0)} + \arctan\frac{F(0)}{c_2}\right) > \frac{c_2}{c_1 J(0) - 3A(0)} \cdot \frac{F(0)}{c_2} = \frac{F(0)}{c_1 J(0) - 3A(0)}. \tag{3.42}$$

Remark 3.2 If $\tau_1 > 0$, we do not know how to treat the integral $\int_{\mathbb{R}^3} \mathrm{div}\mathbb{S}_1 \cdot \mathbf{u} dx$, which is crucial to obtain the inequality like (3.18). Perhaps we need to construct some new averaged quantities for the case of $\tau_1 > 0$.

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Declarations

Conflict of interest The author declares that this work does not have any conflicts of interest.

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