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Asymptotic Behavior of Parabolic Nonlocal Equations in Cylinders Becoming Unbounded

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Abstract

The goal of this paper is to discuss the asymptotic behavior of weak solutions to a class of parabolic equations involving fractional Laplacian in cylindrical domains becoming unbounded in one direction. The results presented in this paper are new and extend some main results in the literature for local and nonlocal elliptic problems with Dirichlet boundary condition.

Keywords Fractional Laplacian · Parabolic fractional problem · Asymptotic behavior of solutions · Expanding cylindrical domains

Mathematics Subject Classification 35B40 · 35K55 · 35K59

1 Introduction and the Main Results

In this paper, we are interested in analyzing the asymptotic behavior of weak solutions to the following fractional parabolic problem when $\ell \to +\infty$

$$\begin{cases} \partial_t u_\ell(x,t) + (-\Delta)^s u_\ell(x,t) = f(x,t) & \text{in} \quad \Omega_\ell \times (0,T), \\ u_\ell(x,t) = 0 & \text{in} \ \left(\mathbb{R}^N \setminus \Omega_\ell\right) \times (0,T), \\ u_\ell(x,0) = u_0(x) & \text{in} \quad \Omega_\ell. \end{cases} \tag{1.1}$$

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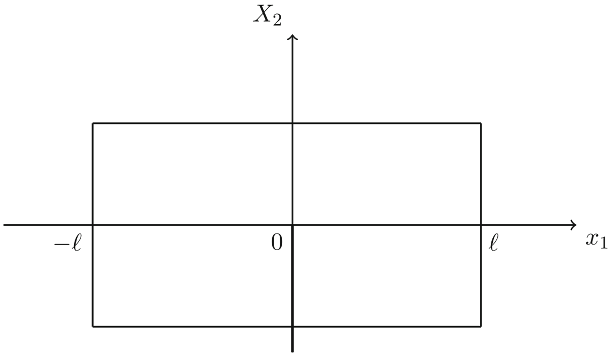

Fig. 1 the domain Ω_ℓ

Here $\partial_t = \partial/\partial t$, $T > 0$, and the leading operator $(-\Delta)^s$ is the fractional Laplace operator defined on smooth functions by

$$(-\Delta)^s \varphi(x) = C_{N,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}}\, dy \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $s \in (0, 1)$, $B_\epsilon(x)$ denotes the open ball in $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ with radius $\epsilon > 0$. The constant $C_{N,s}$ in (1.2) is given by

$$C_{N,s} = \frac{s 2^s \Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}} \Gamma(1-s)}, \tag{1.3}$$

where Γ denotes the usual Gamma function. The fractional Laplacian and the constant $C_{N,s}$ have been studied in detail in [2]. We point out that in the PDEs literature, the operator (1.2) is also known as the restricted fractional Laplacian (see, e.g., [9, 10]). In the setting of bounded domains, it was proven in [28] that the operator (1.2) is different from the spectral fractional Laplacian operator, whereas in the whole space $\mathbb{R}^N$, it was proven in [27] that the operator (1.2) has ten equivalent definitions.

In the sequel, we introduce some notations that we will use in the rest of the paper. For $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$, we set

$$x = (x_1, X_2), \quad X_2 = (x_2, x_3, \ldots, x_N). \tag{1.4}$$

Let $\ell > 0$, we shall denote by $\Omega_\ell = (-\ell, \ell) \times \omega \subset \mathbb{R}^N$ $(N \geq 2)$ the cylinder of length ℓ with the open bounded set $\omega \subset \mathbb{R}^{N-1}$ as the cross section. A schematic picture of the domain Ω_ℓ is shown in Fig. 1.

Recently, the study of fractional Laplacian and related problems has been received an increasing amount of attention because of their connection with many real-world phenomena. Indeed, the fractional Laplacian appears in many different areas, such as anomalous diffusion, quantum mechanics, finance, optimization, and game theory; see [1, 4, 15] and the references therein.

In the past, many problems of partial differential equations of the type "$\ell \to +\infty$" were studied by several researchers. In a seminal paper published in 2001, Chipot and Rougirel in [19] considered the following parabolic problem

$$\begin{cases} \partial_t u_\ell - \operatorname{div}(A(x,t)\nabla u_\ell) = f(X_2,t) & \text{in } \Omega_\ell \times (0,T), \\ u_\ell = 0 & \text{in } \partial\Omega_\ell \times (0,T), \\ u_\ell(x,0) = u_0(X_2) & \text{in } \quad \Omega_\ell, \end{cases} \tag{1.5}$$

where $\Omega_\ell := (-\ell,\ell)^p \times \omega$, ω is a bounded open subset of $\mathbb{R}^{N-p}$, $1 \le p < N$, $A(x,t) = (a_{ij})_{i,j=1,\ldots,N}$ is an $N \times N$ matrix satisfies some conditions. It was proved in that paper, for any fixed $\ell_0 > 0$ the unique weak solution u_ℓ of (1.5) converges to u_∞ in $L^2(0,T;L^2(\Omega_{\ell_0}))$ and $L^2(0,T;H^1(\Omega_{\ell_0}))$ with a speed larger than any power of $\frac{1}{\ell}$ where u_∞ is the unique weak solution of this cross section problem

$$\begin{cases} \partial_t u_\infty - \operatorname{div}(A_{2,2}(x,t)\nabla_{X_2} u_\infty) = f(X_2,t) & \text{in } \omega \times (0,T), \\ u_\infty = 0 & \text{in } \partial\omega \times (0,T), \\ u_\infty(x,0) = u_0(X_2) & \text{in } \quad \omega. \end{cases} \tag{1.6}$$

Moreover, in [19] it was studied the asymptotic behavior of solutions as $\ell \to +\infty$ to a class of quasilinear parabolic equations. Later on, Chipot and Rougirel [18] investigated the same question for a class of elliptic equations. Since then, there several interesting results have been established by many authors from different points of view. Among them we refer the reader to Guesmia [29, 30], Chipot and Xie [20], Chipot and Yeressian [21], Chipot and Mardare [22], Chipot et al. [25, 26], Esposito et al. [16] and the references there.

Next, in order to study the asymptotic behavior of (1.1) as $\ell \to +\infty$, we need to recall from [5] some properties of the fractional Sobolev spaces that will be used in the sequel. For any $s \in (0,1)$ the space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : [u]_{s,2,\mathbb{R}^N} < \infty \right\},$$

where $[u]_{s,2,\mathbb{R}^N}$ is the so-called Gagliardo semi-norm defined by

$$[u]^2_{s,2,\mathbb{R}^N} := \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy.$$

For every $u, v \in H^s(\mathbb{R}^N)$, we define

$$(u,v)_{H^s(\mathbb{R}^N)} = (u,v)_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy. \tag{1.7}$$

One can show easly that this is an inner product and the corresponding norm is denoted by $\|.\|_{H^s(\mathbb{R}^N)}$. It is well known that $H^s(\mathbb{R}^N)$ is a Hilbert space with respect to the inner product defined in (1.7) and $C_c^\infty(\mathbb{R}^N)$ is a dense subset of $H^s(\mathbb{R}^N)$. In what follows, we let Ω be an open bounded subset of $\mathbb{R}^N$ and define the space $H_0^s(\Omega)$ by

$$H_0^s(\Omega) = \left\{ u \in H^s(\mathbb{R}^N) : \ u = 0 \ \text{ a.e. in } \mathbb{R}^N \backslash \Omega \right\},$$

equipped with the norm

$$\|u\| := \|u\|_{H^s(\mathbb{R}^N)}.$$

We shall look for a solution to (1.1) in the following proper class of regularity

$$W(0, T; H_0^s(\Omega)) = \left\{ u \in L^2(0, T; H_0^s(\Omega)), \ u_t \text{ exists and } u_t \in L^2(0, T; L^2(\Omega) \right\}.$$

One can show easily that $W(0, T; H_0^s(\Omega))$ is a Banach space when endowed with the norm

$$\|u\|_W^2 = \int_0^T \|u(t)\|^2 \, dt + \int_0^T \|u_t(t)\|_{L^2(\Omega)}^2 \, dt.$$

Finally, we define the bilinear form $a : H_0^s(\Omega) \times H_0^s(\Omega) \to \mathbb{R}$ as

$$\begin{aligned} a(u, v) &= C_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))v(x)}{|x - y|^{N+2s}} \, dxdy \\ &= \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dxdy. \end{aligned}$$

It is important to point out that the asymptotic behavior of solutions to elliptic and parabolic equations involving fractional Laplacian in domain becoming unbounded in one or several directions is still not well studied in the literature and need further attention. As far as we know at this moment, there is still no research that focuses on the asymptotic of solutions as $\ell \to +\infty$ to linear or nonlinear parabolic equations driven by the fractional Laplacian. However, to the best of our knowledge, the asymptotic behavior of the solution of nonlocal elliptic problems in cylindrical domains becoming unbounded has been studied first by Yeressian [12] and more recently by Chowdhury and Roy [7] and Ambrosio et al. [31]. The author in [12] considered the following elliptic problem

$$\begin{cases} (-\Delta)^s u_\ell(x) = f(x) & \text{in } \quad \Omega_\ell, \\ u_\ell = 0 & \text{in } \mathbb{R}^N \backslash \Omega_\ell, \end{cases} \tag{1.8}$$

and established this result

Theorem *Let u_ℓ be the unique weak solution of (1.8) for $s = \frac{1}{2}$ with the condition that*

$$\text{support}(f) \subset \Omega_\ell \backslash \Omega_{\ell-1} \quad \text{and} \quad \|f\|_{L^2(\Omega_\ell)} \leq 1. \tag{1.9}$$

Then there holds

$$\int_{\Omega_1} u_\ell^2(x)\,dx \le \frac{C}{\ell^2} \quad \forall \ell > 0,$$

where $C > 0$ is a constant independent of ℓ.

Subsequently, Chowdhury and Roy [7] have extended Yeressian's result to the case where $s \in (0,1)$. Moreover, when the force term f is defined only on $\omega \subset \mathbb{R}^{N-1}$, namely, $f = f(x_2, x_3, \dots, x_N)$ the authors in [7] proved the following result:

Theorem *Let us assume $s \in \left(\frac{1}{2}, 1\right)$, $f(x_2, x_3, \dots, x_N) \in L^2(\omega)$. Let u_ℓ be the unique weak solution of (1.8) for each ℓ and u_∞ be the unique weak solution of the following equation on the cross section ω of the cylinder Ω_ℓ,*

$$\begin{cases} (-\Delta')^s u_\infty = f(x_2, x_3, \dots, x_N) \ \text{in} \quad \omega, \\ u_\infty = 0 \qquad\qquad\qquad\qquad\quad \text{in } \mathbb{R}^N \backslash \omega, \end{cases} \tag{1.10}$$

where $(-\Delta')^s$ denotes $N-1$-dimensional fractional Laplace operator. Then for each $\alpha \in (0,1)$, we have

$$\int_{\Omega_{\alpha\ell}} |u_\ell - u_\infty|^2\,dx \le \frac{1}{\ell^{2s-1}} \quad \forall \ell > 0.$$

In [31] Ambrosio et al. considered the following elliptic problem

$$\begin{cases} (-\Delta_{\mathbb{R}^{N+k}})^s u_\ell = f_\infty \ \text{in} \quad \Omega_\ell^{N+k}, \\ u_\ell = 0 \qquad\qquad\qquad\ \ \text{in } \mathbb{R}^{N+k} \backslash \Omega_\ell^{N+k}, \end{cases} \tag{1.11}$$

where $\Omega_\ell^{N+k} = \omega^N \times B_\ell^k \subset \mathbb{R}^N \times \mathbb{R}^k$, $k, N \ge 1$, ω^N is a given bounded and Lipschitz domain in $\mathbb{R}^N$, B_ℓ^k can be the rectangle $(-\ell, \ell)^k$ or the Euclidean ball of radius ℓ about the origin, and $(-\Delta_{\mathbb{R}^{N+k}})^s$ is $N+k$-dimensional fractional Laplace operator. The authors have proved in [31] the following results:

- $\displaystyle \lim_{\ell \to +\infty} \inf_{u \in H_0^s(\Omega_\ell^{N+k}) \backslash \{0\}} \frac{\langle (-\Delta_{\mathbb{R}^{N+k}})^s u, u \rangle}{\|u\|^2_{L^2(\Omega_\ell^{N+k})}} = \inf_{u \in H_0^s(\omega^N) \backslash \{0\}} \frac{\langle (-\Delta_{\mathbb{R}^N})^s u, u \rangle}{\|u\|^2_{L^2(\omega^N)}} = \lambda^s(\omega^N).$
- For a given $f_\infty \in L^2(\omega^N)$ and u_ℓ be the unique weak solution of (1.11). Then

$$\frac{1}{\ell^k B_1^k} \int_{B_\ell^k} u_\ell(x,t)\,dt \to u_\infty \ \text{ strongly in } H_0^s(\omega), \text{ as } \ell \to +\infty,$$

where u_∞ is the unique weak solution to the following problem

$$\begin{cases} (-\Delta_{\mathbb{R}^N})^s u_\infty = f_\infty \ \text{in} \quad \omega^N, \\ u_\infty = 0 \qquad\qquad\quad\ \ \text{in } \mathbb{R}^N \backslash \omega^N. \end{cases} \tag{1.12}$$

Djilali and Rougirel [17] investigated the existence and the asymptotic behavior of weak solutions to the following time fractional diffusion equations

$$\begin{cases} \mathbf{D}^{\alpha}_{0,t}u_\ell - \Delta u_\ell = f \text{ in } & \Omega_\ell \times (0,T), \\ u_\ell = 0 & \text{on } \partial\Omega_\ell \times (0,T), \\ (g_{1-\alpha} * u_\ell)(0) = v \text{ in } & \Omega_\ell. \end{cases} \tag{1.13}$$

where $\Omega_\ell = (-\ell,\ell)^p \times \omega \subset \mathbb{R}^p \times \mathbb{R}^{N-p}$, Δ is the classical Laplace operator, and $\mathbf{D}^{\alpha}_{0,t}$ is the so-called Riemann–Liouville fractional derivative of order $0 < \alpha < 1$ defined as

$$\mathbf{D}^{\alpha}_{0,t}v(t) := \frac{d}{dt}\,(v * g_\alpha)\,(t) = \frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}v(s)\,ds\right).$$

By using the Poincaré inequality and selecting a suitable test function, the authors proved the existence of two positive constants ϵ and C independent of ℓ such that

$$\int_0^T \int_{\Omega_{\ell/2}} |\nabla(u_\ell - u_\infty)|^2\, dxdt \leq Ce^{-\epsilon\ell},$$

where u_∞ is the unique weak solution of this problem

$$\begin{cases} \mathbf{D}^{\alpha}_{0,t}u_\infty - \Delta u_\infty = f \text{ in } & \omega \times (0,T), \\ u_\infty = 0 & \text{on } \partial\omega \times (0,T), \\ (g_{1-\alpha} * u_\infty)(0) = v \text{ in } & \omega. \end{cases} \tag{1.14}$$

Motivated by the above results especially by [7, 12], in this paper we shall discuss the asymptotic behavior of weak solutions as $\ell \to +\infty$ to the nonlocal parabolic problem (1.1). It is important to highlight that this is the first time the asymptotic behavior of weak solutions as $\ell \to +\infty$ to (1.1) is analyzed.

Before we go into details, we introduce various notations that would be used. We denote by $\langle\, ,\, \rangle_\Omega$ the duality between $H^{-s}(\Omega)$ and $H^s_0(\Omega)$. The $L^2(\Omega)$-norm will be denoted by $\|.\|_{2,\Omega}$. The short hand notation $u(t) = u(.,t)$ for any $t \in [0,T]$ will be used throughout of the paper.

Now we define the notion of weak solution to problem (1.1).

Definition 1.1 Let $u_0 \in L^2(\Omega_\ell)$ and $f \in L^2(0,T;L^2(\Omega_\ell))$. We say that $u_\ell \in W(0,T;H^s_0(\Omega_\ell))$ is a weak solution of (1.1), if for any $v \in H^s_0(\Omega_\ell)$ there holds

$$\int_{\Omega_\ell} \partial_t u_\ell(t) v\, dx + a(u_\ell(t), v) = \int_{\Omega_\ell} f(t) v\, dx \quad \text{for a.e. } t \in (0,T), \tag{1.15}$$

$$u_\ell(0) = u_0(X_2). \tag{1.16}$$

In the first of our main results, we shall show that the unique solution of (1.1) converges to the unique solution of the following problem in the L^2-norm

$$\begin{cases} \partial_t u_\infty + (-\Delta')^s u_\infty = f(X_2, t) & \text{in} \quad \omega \times (0, T), \\ u_\infty = 0 & \text{in} \left(\mathbb{R}^{N-1} \backslash \omega\right) \times (0, T), \\ u_\infty(0, x) = u_0(X_2) & \text{in} \quad \omega. \end{cases} \tag{1.17}$$

The weak solution of problem (1.17) can be defined in a similar way to Definition 1.1. However, for the reader's convenience we shall state it in full.

Definition 1.2 Let $u_0 \in L^2(\omega)$ and $f \in L^2(0, T; L^2(\omega))$. We shall say that $u_\infty \in W(0, T; H_0^s(\omega))$ is a weak solution of (1.1) if for all $v \in H_0^s(\omega)$ there holds

$$\int_\omega \partial_t u_\infty(t) v \, dX_2 + \frac{C_{N-1,s}}{2} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{(u_\infty(X_2, t) - u_\infty(Y_2, t))(v(X_2) - v(Y_2))}{|X_2 - Y_2|^{N-1+2s}} \, dX_2 dY_2 = \int_\omega f(X_2, t) v \, dX_2, \tag{1.18}$$

for a.e. $t \in (0, T)$, and

$$u_\infty(0) = u_0(X_2). \tag{1.19}$$

In order to solve problem (1.1), one can convert it into a first-order abstract Cauchy problem in $H = L^2(\Omega_\ell)$. To this aim we define the functional φ from H to $(-\infty, +\infty]$ by

$$\varphi(u_\ell) = \begin{cases} \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{(u_\ell(x,t) - u_\ell(y,t))^2}{|x-y|^{N+2s}} \, dx dy & \text{if} \quad u_\ell \in H_0^s(\Omega_\ell), \\ +\infty & \text{if} \quad u_\ell \in H \setminus H_0^s(\Omega_\ell). \end{cases}$$

It is easy to see that φ is convex, lower semi-continuous and proper. It was proven in [11, Theorem 2.3] that solving problem (1.1) in the sense of Definition 1.1 is equivalent to solve the following abstract Cauchy problem:

$$\begin{cases} \frac{du_\ell(t)}{dt} + \partial\varphi(u_\ell(t)) \ni f(t) \text{ in } H, \; 0 < t < T, \\ w(x, 0) = u_0 \end{cases} \tag{1.20}$$

where $\partial\varphi(u_\ell)$ denotes the subdifferential of φ at u_ℓ in the sense of convex analysis. Since $u_0 \in L^2(\Omega_\ell)$, then by [6, Theorem 3.6 and Lemma 3.3], for any $T > 0$, there exists a unique solution u_ℓ to the Cauchy problem (1.20).

The existence and uniqueness of weak solution to problem (1.17) can be obtained similarly by considering the functional ψ from $\mathcal{H} = L^2(\omega)$ to $(-\infty, +\infty]$ as

$$\psi(u_\infty) = \begin{cases} \frac{1}{2} \int_{\mathbb{R}^{N-1} \times \mathbb{R}^{N-1}} \frac{(u_\infty(X_2,t) - u_\infty(Y_2,t))^2}{|X_2 - Y_2|^{N-1+2s}} \, dX_2 dY_2 & \text{if} \quad u_\infty \in H_0^s(\omega), \\ +\infty & \text{if} \quad u_\infty \in \mathcal{H} \setminus H_0^s(\omega). \end{cases}$$

Our first result in this paper is the following:

Theorem 1.3 *Let u_ℓ be the unique weak solution of (1.1) and u_∞ be the unique weak solution of (1.17). Assume that the functions $f \in L^2(0, T; L^2(\omega))$ and $u_0 \in L^2(\omega)$ are independent of x_1, that is*

$$f(x,t) = f(X_2,t), \quad u_0(x) = u_0(X_2).$$

Moreover, if $s \in \left(\frac{1}{2}, 1\right)$ then for any fixed $\ell_0 > 0$ there exists $K > 0$ independent of ℓ such that

$$\begin{aligned}&\|u_\ell - u_\infty\|^2_{L^\infty(0,T;L^2(\Omega_{\ell_0}))} + \|u_\ell - u_\infty\|^2_{L^2(0,T;L^2(\Omega_{\ell_0}))}\\ &\quad\le \frac{K\left\{\|u_0\|^2_{2,\omega} + \|f\|_{L^2(0,T;L^2(\omega))}\right\}}{\ell^{2s-1}},\end{aligned}$$

for any large enough $\ell > 0$.

This result can be seen as an extension of the main result of Chowdhury and Roy stated above to the case of nonlocal parabolic equations. Here we will give a simpler proof than the one outlined in [7]. The proof of Theorem 1.3 will be based on some energy estimates and on a nonlocal Poincaré inequality established by Chowdhury-Csató-Roy-Sk in [8] as well as with a suitable choice of a test function in the weak formulation. We point out that it is still not clear if the result presented in Theorem 1.3 can be extended to the case of more general domain of the type $\Omega_\ell = (-\ell, \ell)^p \times \omega \subset \mathbb{R}^p \times \mathbb{R}^{N-p}$ where $p > 1$. However, from the point of view of applications, the most interesting case is when $p = 1$.

In the next theorem, we are interested in studying the asymptotic behavior of solutions to (1.1) as $\ell \to +\infty$ for every value $s \in (0, 1)$ and where the functions f and u_0 not only depend on X_2 but also on x_1.

Theorem 1.4 *Let u_ℓ be the weak solution of the problem (1.1) for $s \in (0, 1)$. Assume that f and u_0 satisfy the following conditions:*

$$support(f) \subset \Omega_\ell \backslash \Omega_{\ell-1}, \quad \|f\|_{L^2(0,T;L^2(\Omega_\ell \backslash \Omega_{\ell-1}))} \le 1, \tag{1.21}$$

and

$$support(u_0) \subset \Omega_\ell \backslash \Omega_{\ell-1}, \quad \|u_0\|_{L^2(\Omega_\ell \backslash \Omega_{\ell-1})} \le 1. \tag{1.22}$$

Then there exists $C > 0$ independent of ℓ such that for all $\ell > 0$ we have

$$\|u_\ell\|^2_{L^\infty(0,T;L^2(\Omega_{\ell^\alpha}))} + \|u_\ell\|^2_{L^2(0,T;L^2(\Omega_{\ell^\alpha}))} \le \frac{C}{\ell^{(1-\alpha)\epsilon}},$$

for each $\alpha \in [0, 1)$ and $\epsilon \in (0, 2)$.

As it was pointed out by Chipot in [23] and by Yeressian in [12] the estimates obtained in Theorems 1.3 and 1.4 are important in numerical computations of solutions in large domains, for example when one is only interested to compute the solution in a small subdomain. Furthermore, these estimates are also important in the proof the well-posedness of equations in unbounded domains with right-hand side non-decaying at infinity.

Before concluding, we mention some problems related to our results that we hope will inspire further research:

- Does Theorem 1.3 extend to case where $s \in (0, 1)$?.
- It would be interesting to see in Theorems 1.3 and 1.4 when it is possible to obtain an exponential rate of convergence.
- Under the assumptions of Theorem 1.3, is it possible to find estimates of $u_\ell - u_\infty$ in $H_0^s(\Omega_{\ell_0})$?.

The rest of this paper is organized as follows. In Sect. 2, we introduce some preliminaries which will play important roles in the proofs of the main results. In the other sections, we shall prove our main results.

1.1 Notations

Throughout this paper, the letters c, c_i, K, C, C_i, $i = 1, 2, \dots,$ denote positive constants which may vary from line to line but independent of ℓ.

2 Some Preliminary Results

In order to prove Theorems 1.3 and 1.4, we shall need several preliminary results.

Proposition 2.1 ([3, Proposition 1.2]) *Let V be a Banach space which is dense and continuously embedded in the Hilbert space H. We identify $H = H'$ so that $V \hookrightarrow H = H' \hookrightarrow V'$. Then the Banach space $W_p = \{u \in L^p(0, T, V),\ u' \in L^{p'}(0, T, V')\}$ is contained in $C([0, T], H)$. Moreover, if $u \in W_p$, then $\|u(t)\|_{L^2(\Omega)}$ is absolutely continuous on $[0, T]$, we have*

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} = 2\langle u_t(t), u(t)\rangle, \text{ a.e. on } [0, T],$$

and there is a constant $C > 0$ such that

$$\|u\|_{C(0,T,H)} \le C\|u\|_{W_p}, \text{ for all } u \in W_p.$$

In the next lemma, we state the fractional Pioncaré inequality due to Chowdhury-Csató-Roy-Sk [8, Theorem 1.2] where the authors established the best constant for this inequality in certain unbounded domains.

Lemma 2.2 (Pioncaré Inequality) *Consider the strip $D_\infty = \mathbb{R}^m \times \omega \subset \mathbb{R}^N$ with $1 \le m < N$, where ω is a bounded open subset of $\mathbb{R}^{N-m}$. Then for $0 < s < 1$, we* *have*

$$P^2_{N,s}(D_\infty) := \inf_{u\in H^s_0(D_\infty)\setminus\{0\}} \frac{[u]^2_{s,2,\mathbb{R}^N}}{\|u\|^2_{2,D_\infty}} = P^2_{N-m,s}(\omega) := \inf_{u\in H^s_0(\omega)\setminus\{0\}} \frac{[u]^2_{s,2,\mathbb{R}^{N-m}}}{\|u\|^2_{2,\omega}} > 0.$$

To the best of our knowledge, the fractional Poincaré inequality has been studied only in some papers [8, 12–14]. In [12], it was proved a fractional Pioncaré inequality for the spaces $H^s_0(\Omega)$ without driving the best constant. Later on, in [8] Chowdhury-Csató-Roy-Sk obtained the best constants for fractional Poincaré inequalities for the spaces $H^s_0(\Omega)$ in certain unbounded domains. Specifically, they proved that the best constant in the regional fractional Poincaré inequality $P^1_{N,s}(\mathbb{R}\times(-1,1))$ and the best constant in the fractional Poincaré inequality $P^2_{N,s}(\mathbb{R}^m\times\omega)$ are equal to those of the cross sections, that is to $P^1_{1,s}((-1,1))$ and $P^2_{N-m,s}(\omega)$, respectively, where ω is a bounded open subset of $\mathbb{R}^{N-m}$. Very recently, Mohanta and Sk in [14] proved a fractional Poincaré inequality in the setting of $W^{s,p}_0(\Omega)$- spaces for $1<p<\infty$ with establishing the best constant. Furthermore, they studied the asymptotic behavior of the first eigenvalue of the nonlocal Dirichlet p-Laplacian problem when the domain is becoming unbounded in several directions. Finally, in [13], the authors extended the results of [8] to the setting of fractional Orlicz-Sobolev spaces.

In the following lemma, we give an important relation between the constants $C_{N,s}$ and $C_{N-1,s}$ which appear in the definition of the fractional Laplacian.

Lemma 2.3 *For each $N\in\mathbb{N}$ and $s\in(0,1)$, let $C_{N,s}$ be the constant defined in (1.3). Then one has $C_{N,s}\Theta_N = C_{N-1,s}$, where*

$$\Theta_N = \int_{\mathbb{R}} \frac{dz}{(1+z^2)^{\frac{N+2s}{2}}}. \tag{2.1}$$

Proof The proof is somehow similar to that in [7] but we include it for completeness since it differs in some significant details. Indeed, from [32, Theorem 8.20] we have

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\, d\theta = \frac{\Gamma(p)\Gamma(q)}{2\Gamma(p+q)} \quad \text{for all } p,q>0, \tag{2.2}$$

where Γ denotes the gamma function. On the other hand, by using the change of variable $z=\tan(\theta)$ in (2.1), we obtain

$$\Theta_N = 2\int_0^{+\infty} \frac{dz}{(1+z^2)^{\frac{N+2s}{2}}} = 2\int_0^{\frac{\pi}{2}} (\cos\theta)^{N+2s-2}\, d\theta = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{N+2s-1}{2}\right)}{\Gamma\left(\frac{N+2s}{2}\right)}.$$

From this and by the definition of $C_{N,s}$ in (1.3), we deduce the desired result. □

Lemma 2.4 *Let u_ℓ be the weak solution to (1.1). Then, there exists $K>0$ a constant independent of ℓ such that*

$$\|u_\ell\|_{L^\infty(0,T;L^2(\Omega_\ell))} + \frac{C_{N,s}}{2}\int_0^T \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\ell(x,t)-u_\ell(y,t)|^2}{|x-y|^{N+2s}}\, dxdydt$$

$$\leq K\ell\left\{\|u_0\|_{2,\omega}^2 + \|f\|_{L^2(0,T;L^2(\omega))}^2\right\}. \tag{2.3}$$

Proof Taking $v = u_\ell$ in (1.15) and using the Young's inequality, we obtain for a.e. $t \in (0, T)$

$$\frac{1}{2}\frac{d}{dt}\|u_\ell(t)\|_{2,\Omega_\ell}^2 + \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\ell(x,t)-u_\ell(y,t)|^2}{|x-y|^{N+2s}}\,dxdy \\ \leq \|f(t)\|_{2,\Omega_\ell}\|u_\ell(t)\|_{2,\Omega_\ell}.$$

Using the fractional Pioncaré's inequality and the Young's inequality we derive

$$\frac{1}{2}\frac{d}{dt}\|u_\ell(t)\|_{2,\Omega_\ell}^2 + \frac{C_{N,s}}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\ell(x,t)-u_\ell(y,t)|^2}{|x-y|^{N+2s}}\,dxdy \leq C_1\|f(t)\|_{2,\Omega_\ell}^2,$$

where $C_1 > 0$ independent of ℓ. Further, integrating on $(0, t)$ we obtain for a.e. t

$$\|u_\ell(t)\|_{2,\Omega_\ell}^2 + \frac{C_{N,s}}{2}\int_0^t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\ell(x,\sigma)-u_\ell(y,\sigma)|^2}{|x-y|^{N+2s}}\,dxdyd\sigma \\ \leq \|u_0\|_{2,\Omega_\ell}^2 + 2C_1\int_0^T\|f(t)\|_{2,\Omega_\ell}^2\,dt, \\ \leq K\ell\left\{\|u_0\|_{2,\omega}^2 + \|f\|_{L^2(0,T;L^2(\omega))}^2\right\},$$

where $K = \max\{2C_1, 1\}$. Thus the proof is now complete. □

The following lemma will be a key in proving Theorem 1.3.

Lemma 2.5 *Let $x = (x_1, X_2) \in \Omega_\ell$ and define $u^*(x,t) = u_\infty(X_2,t)$ where u_∞ is the unique weak solution to (1.17). Then, for any $v \in H_0^s(\Omega_\ell)$ there holds*

$$\frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u^*(x,t)-u^*(y,t))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy \\ = -\int_{\Omega_\ell}\partial_t u^*(x,t)v\,dx + \int_{\Omega_\ell} f(X_2,t)v\,dx, \quad a.e.\ t \in (0,T). \tag{2.4}$$

Proof Through direct calculations, we have

$$\int_{\Omega_\ell} f(X_2,t)v(x)\,dx - \int_{\Omega_\ell}\partial_t u^*(t)v(x)\,dx \\ = \int_{\mathbb{R}}\int_\omega f(X_2,t)v(x_1,X_2)\,dX_2dx_1 - \int_{\mathbb{R}}\int_\omega \partial_t u_\infty(X_2,t)v(x_1,X_2)\,dX_2dx_1 \\ \overset{\text{by (1.18)}}{=} \frac{C_{N,s}\theta_N}{C_{N-1,s}}C_{N-1,s}\int_{\mathbb{R}}\int_{\mathbb{R}^{N-1}} \\ \int_{\mathbb{R}^{N-1}}\frac{(u_\infty(X_2,t)-u_\infty(Y_2,t))v(x_1,X_2)}{|X_2-Y_2|^{N-1+2s}}\,dY_2dX_2dx_1$$

$$\overset{\text{by Lemma 2.3}}{=} C_{N,s}\theta_N \int_{\mathbb{R}}\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}^{N-1}} \frac{(u_\infty(X_2,t)-u_\infty(Y_2,t))v(x_1,X_2)}{|X_2-Y_2|^{N-1+2s}}\,dY_2dX_2dx_1$$
$$= C_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u^*(x,t)-u^*(y,t))v(x)}{|X_2-Y_2|^{N+2s}\left(1+\frac{|x_1-y_1|^2}{|X_2-Y_2|^2}\right)^{\frac{N+2s}{2}}}\,dydx$$
$$= C_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u^*(x,t)-u^*(y,t))v(x)}{|x-y|^{N+2s}}\,dydx$$
$$= a(u^*(t),v) \qquad \text{a.e. } t\in(0,T).$$

Thus this completes the proof. □

3 Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. In $\mathbb{R}$, we consider a function $\rho=\rho(x_1)$ whose graph is given in Fig. 2.

Clearly this function satisfies

$$0\le\rho\le 1,\quad \rho=1 \text{ on } \left(\frac{-1}{2},\frac{1}{2}\right),\quad \rho=0 \text{ on } \mathbb{R}\backslash(-1,1),\quad |\rho'|\le 2. \tag{3.1}$$

Before starting the proof, we prepare some technical lemmas.

Lemma 3.1 *Let u_∞ be the unique weak solution of (1.17) and u_ℓ be the unique weak solution to (1.1). Then, for a.e. $t\in(0,T)$,*

$$(u_\ell(t)-u_\infty(t))\rho_\ell^2(x_1)\in H_0^s(\Omega_\ell). \tag{3.2}$$

where $\rho_\ell(x_1)=\rho\left(\frac{x_1}{\ell}\right)$.

Proof In order to show this lemma, it is enough to show that $\psi_\ell(x,t) := u_\infty(X_2,t)\rho^2\left(\frac{x_1}{\ell}\right)\in H_0^s(\Omega_\ell)$ a.e. $t\in(0,T)$. The proof of $u_\ell(x,t)\rho^2\left(\frac{x_1}{\ell}\right)\in H_0^s(\Omega_\ell)$ for a.e. $t\in(0,T)$ can be done similarly. Indeed, it is obvious that $\psi_\ell(t)\in L^2(\Omega_\ell)$

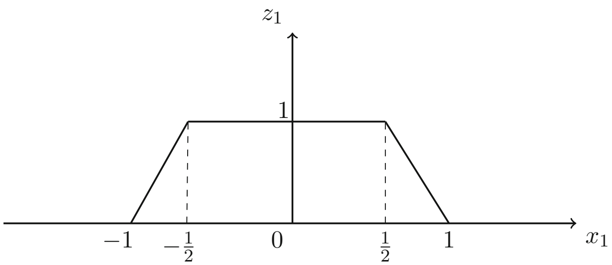

Fig. 2 the function ρ

a.e. $t \in (0, T)$. Hereafter, we consider the Gagliardo semi-norm of $\psi_\ell(t)$

$$\begin{aligned}
[\psi_\ell(t)]^2_{s,2,\mathbb{R}^N} &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\psi_\ell(x,t)-\psi_\ell(y,t))^2}{|x-y|^{N+2s}}\,dxdy \\
&= \int_{\Omega_\ell}\int_{\mathbb{R}^N} \frac{(\psi_\ell(x,t)-\psi_\ell(y,t))^2}{|x-y|^{N+2s}}\,dxdy \\
&\quad + \int_{\mathbb{R}^N\setminus\Omega_\ell}\int_{\mathbb{R}^N} \frac{(\psi_\ell(x,t)-\psi_\ell(y,t))^2}{|x-y|^{N+2s}}\,dxdy \\
&\le 2\int_{\Omega_\ell}\int_{\mathbb{R}^N} \frac{(\psi_\ell(x,t)-\psi_\ell(y,t))^2}{|x-y|^{N+2s}}\,dxdy \\
&\quad + \underbrace{\int_{\mathbb{R}^N\setminus\Omega_\ell}\int_{\mathbb{R}^N\setminus\Omega_\ell} \frac{(\psi_\ell(x,t)-\psi_\ell(y,t))^2}{|x-y|^{N+2s}}\,dxdy}_{=0} \\
&= I[t].
\end{aligned}$$

Next it is sufficient to prove that $I[t]$ is finite a.e. $t \in (0, T)$. Indeed, by the symmetry of the integral in the Gagliardo semi-norm with respect to x and y, one has

$$\begin{aligned}
I[t] &= 2\int_{\Omega_\ell}\int_{\mathbb{R}^N} \frac{(\psi_\ell(x,t)-\psi_\ell(y,t))^2}{|x-y|^{N+2s}}\,dxdy \\
&= 2\int_{\Omega_\ell}\int_{\mathbb{R}^N} \frac{(\psi_\ell(x,t)-\psi_\ell(y,t))^2}{|x-y|^{N+2s}}\,dydx \\
&\le 4\int_{\Omega_\ell}\int_{\mathbb{R}^N} \frac{u_\infty^2(X_2,t)(\rho_\ell^2(x_1)-\rho_\ell^2(y_1))^2}{|x-y|^{N+2s}}\,dydx \\
&\quad +4\int_{\Omega_\ell}\int_{\mathbb{R}^N} \frac{\rho_\ell^4(y_1)(u_\infty(X_2,t)-u_\infty(Y_2,t))^2}{|x-y|^{N+2s}}\,dydx \\
&= I_1[t] + I_2[t].
\end{aligned}$$

In what follows, we show that

$$\int_{\Omega_\ell}\int_{\mathbb{R}^N} \frac{|u_\infty(X_2,t)-u_\infty(Y_2,t)|^2}{|x-y|^{N+2s}}\,dydx < \infty.$$

Through direct calculations, we obtain

$$\begin{aligned}
&\int_{\Omega_\ell}\int_{\mathbb{R}^N} \frac{|u_\infty(X_2,t)-u_\infty(Y_2,t)|^2}{|x-y|^{N+2s}}\,dydx \\
&\quad = \int_{-\ell}^{\ell}\int_{\omega}\int_{\mathbb{R}^N} \frac{|u_\infty(X_2,t)-u_\infty(Y_2,t)|^2}{|x-y|^{N+2s}}\,dydX_2dx_1
\end{aligned}$$

$$= \int_{-\ell}^{\ell}\int_{\omega}\int_{\mathbb{R}^N} \frac{|u_\infty(X_2,t)-u_\infty(Y_2,t)|^2}{|X_2-Y_2|^{N+2s}\left(1+\frac{|x_1-y_1|^2}{|X_2-Y_2|^2}\right)^{\frac{N+2s}{2}}}\, dydX_2dx_1$$

$$= \int_{-\ell}^{\ell}\int_{\omega}\int_{\mathbb{R}^{N-1}} \frac{|u_\infty(X_2,t)-u_\infty(Y_2,t)|^2}{|X_2-Y_2|^{N+2s}} \left(\int_{\mathbb{R}} \frac{dy_1}{\left(1+\frac{|x_1-y_1|^2}{|X_2-Y_2|^2}\right)^{\frac{N+2s}{2}}}\right) dY_2dX_2dx_1.$$

With a simple change of variable, we obtain

$$\begin{aligned} &\int_{\Omega_\ell}\int_{\mathbb{R}^N} \frac{|u_\infty(X_2,t)-u_\infty(Y_2,t)|^2}{|x-y|^{N+2s}}\, dydx \\ &\leq \theta_N \int_{-\ell}^{\ell}\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}^{N-1}} \frac{|u_\infty(X_2,t)-u_\infty(Y_2,t)|^2}{|X_2-Y_2|^{N-1+2s}}\, dY_2dX_2dx_1, \\ &\leq 2\ell\theta_N \|u_\infty(t)\|^2_{H^s(\mathbb{R}^{N-1})}, \quad \text{a.e. } t\in(0,T). \end{aligned} \tag{3.3}$$

where θ_N is given as in Lemma 2.3. Hence, this combined with the fact that $0\leq\rho_\ell\leq 1$, yields

$$I_2[t] \leq 8\ell\theta_N\|u_\infty(t)\|^2_{H^s(\mathbb{R}^{N-1})}, \quad \text{a.e. } t\in(0,T). \tag{3.4}$$

On the other hand, we have

$$\begin{aligned} I_1[t] &= 4\int_{\Omega_\ell}\int_{|x-y|<1} \frac{u_\infty^2(X_2,t)(\rho_\ell^2(x_1)-\rho_\ell^2(y_1))^2}{|x-y|^{N+2s}}\, dydx \\ &\quad +4\int_{\Omega_\ell}\int_{|x-y|\geq 1} \frac{u_\infty^2(X_2,t)(\rho_\ell^2(x_1)-\rho_\ell^2(y_1))^2}{|x-y|^{N+2s}}\, dydx \\ &\leq \frac{C}{\ell^2}\int_{\Omega_\ell}\int_{|x-y|<1} \frac{u_\infty^2(X_2,t)|x-y|^2}{|x-y|^{N+2s}}\, dydx \\ &\quad +C\int_{\Omega_\ell}\int_{|x-y|\geq 1} \frac{u_\infty^2(X_2,t)}{|x-y|^{N+2s}}\, dydx \\ &= \frac{C}{\ell}\|u_\infty(t)\|^2_{2,\omega}\int_{B(0,1)} \frac{1}{|z|^{N+2s-2}}\, dz + C\ell\|u_\infty(t)\|^2_{2,\omega} \\ &\quad \int_{\mathbb{R}^N\setminus B(0,1)} \frac{1}{|z|^{N+2s}}\, dz < \infty \ \text{ a.e. } t\in(0,T). \end{aligned}$$

Thus the proof is now complete. □

We note that by Proposition 2.1, we have the following identity

$$\int_0^t \int_{\Omega_\ell} \frac{du_\ell}{d\sigma}(u_\ell - u_\infty)\rho_\ell^2 - \frac{du_\infty}{d\sigma}(u_\ell - u_\infty)\rho_\ell^2 \, dx d\sigma$$
$$= \int_0^t \int_{\Omega_\ell} \frac{d(u_\ell - u_\infty)\rho_\ell}{d\sigma}(u_\ell - u_\infty)\rho_\ell \, dx d\sigma$$
$$= \frac{1}{2}\|(u_\ell - u_\infty)\rho_\ell\|_{2,\Omega_\ell}^2. \tag{3.5}$$

Here we have used the fact that $u_\ell(0) - u_\infty(0) = 0$.

Now we are in position to finish the proof of Theorem 1.3. Denoting $v_\ell := u_\ell - u_\infty$, we obtain by subtracting the identities (1.15) and (2.4) that

$$\int_{\Omega_\ell} \partial_t v_\ell(t) w \, dx + \frac{C_{N,s}}{2} \int_{\mathbb{R}^N}$$
$$\int_{\mathbb{R}^N} \frac{(v_\ell(x,t) - v_\ell(y,t))(w(x) - w(y))}{|x-y|^{N+2s}} \, dx dy = 0 \quad \text{a.e. } t \in (0,T), \tag{3.6}$$

for each $w \in H_0^s(\Omega_\ell)$. According to Lemma 3.1, we know that $v_\ell \rho_\ell^2 \in H_0^s(\Omega_\ell)$, and hence by taking $w = v_\ell \rho_\ell^2$ in (3.6) yields

$$\int_{\Omega_\ell} \partial_t v_\ell(t) v_\ell(t) \rho_\ell^2 \, dx + \frac{C_{N,s}}{2} \int_{\mathbb{R}^N}$$
$$\int_{\mathbb{R}^N} \frac{(v_\ell(x,t) - v_\ell(y,t))(v_\ell(x,t)\rho_\ell^2(x_1) - v_\ell(y,t)\rho_\ell^2(y_1))}{|x-y|^{N+2s}} \, dx dy = 0 \text{a.e. } t \in (0,T). \tag{3.7}$$

Integrating the latter identity over $[0,t]$ and using (3.5), we get

$$\frac{1}{2}\|v_\ell(t)\rho_\ell\|_{2,\Omega_\ell}^2 + \frac{C_{N,s}}{2} \int_0^t \int_{\mathbb{R}^N} \int_{\mathbb{R}^N}$$
$$\frac{(v_\ell(x,\sigma) - v_\ell(y,\sigma))(v_\ell(x,\sigma)\rho_\ell^2(x_1) - v_\ell(y,\sigma)\rho_\ell^2(y_1))}{|x-y|^{N+2s}} \, dx dy d\sigma = 0, \tag{3.8}$$

which implies through simple manipulations

$$\frac{1}{2}\|v_\ell(t)\rho_\ell\|_{2,\Omega_\ell}^2 + \frac{C_{N,s}}{2} \int_0^t \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v_\ell(x,\sigma) - v_\ell(y,\sigma))^2 \rho_\ell^2(x_1)}{|x-y|^{N+2s}} \, dx dy d\sigma$$
$$= -\frac{C_{N,s}}{2} \int_0^t \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{v_\ell(y,\sigma)(v_\ell(x,\sigma) - v_\ell(y,\sigma))(\rho_\ell^2(x_1) - \rho_\ell^2(y_1))}{|x-y|^{N+2s}} \, dx dy d\sigma$$
$$\le \frac{C_{N,s}\epsilon}{2} \int_0^t \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v_\ell(x,\sigma) - v_\ell(y,\sigma))^2 (\rho_\ell(x_1) + \rho_\ell(y_1))^2}{|x-y|^{N+2s}} \, dx dy d\sigma$$

$$+\frac{C_{N,s}}{2\epsilon}\int_0^t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_\ell^2(y,\sigma)(\rho_\ell(x_1)-\rho_\ell(y_1))^2}{|x-y|^{N+2s}}\,dxdyd\sigma$$
$$\leq 2C_{N,s}\epsilon\int_0^t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(v_\ell(x,\sigma)-v_\ell(y,\sigma))^2\rho_\ell^2(x_1)}{|x-y|^{N+2s}}\,dxdyd\sigma$$
$$+\frac{C_{N,s}}{2\epsilon}\int_0^t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_\ell^2(y,\sigma)(\rho_\ell(x_1)-\rho_\ell(y_1))^2}{|x-y|^{N+2s}}\,dxdyd\sigma \qquad \forall\epsilon>0.$$

Further, we may choose ϵ small enough to guarantee the existence of a constant $C>0$ independent of ℓ such that

$$\|v_\ell(t)\rho_\ell\|_{2,\Omega_\ell}^2+\int_0^t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(v_\ell(x,\sigma)-v_\ell(y,\sigma))^2\rho_\ell^2(x_1)}{|x-y|^{N+2s}}\,dxdyd\sigma$$
$$\leq C\int_0^t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_\ell^2(y,\sigma)(\rho_\ell(x_1)-\rho_\ell(y_1))^2}{|x-y|^{N+2s}}\,dxdyd\sigma. \tag{3.9}$$

Using the nonlocal Poincaré inequality (see, Lemma 2.2) for the function $v_\ell\rho_\ell$, we obtain

$$\int_{\Omega_\ell}v_\ell^2(x,t)\rho^2(x_1)\,dx$$
$$\leq C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(v_\ell(x,t)\rho_\ell(x_1)-v_\ell(y,t)\rho_\ell(y_1))^2}{|x-y|^{N+2s}}\,dxdy$$
$$\leq 2C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(v_\ell(x,t)-v_\ell(y,t))^2\rho_\ell^2(x_1)}{|x-y|^{N+2s}}\,dxdy$$
$$+2C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\rho_\ell(x_1)-\rho_\ell(y_1))^2v_\ell^2(y,t)}{|x-y|^{N+2s}}\,dxdy. \tag{3.10}$$

Combining (3.9) and (3.10) yields

$$\|v_\ell(t)\rho_\ell\|_{2,\Omega_\ell}^2+\int_0^t\|v_\ell(\sigma)\rho_\ell\|_{2,\Omega_\ell}^2\,d\sigma$$
$$\leq C_1\int_0^t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_\ell^2(y,\sigma)(\rho_\ell(x_1)-\rho_\ell(y_1))^2}{|x-y|^{N+2s}}\,dxdyd\sigma.$$
$$\overset{\text{by Fubini's theorem}}{=} C_1\prod_{k=2}^N\theta_k\int_0^t\int_{\mathbb{R}^N}v_\ell^2(y,\sigma)\int_{\mathbb{R}}\frac{(\rho_\ell(x_1)-\rho_\ell(y_1))^2}{|x_1-y_1|^{1+2s}}\,dx_1dyd\sigma$$
$$=C_1\prod_{k=2}^N\theta_k\int_0^t\int_{\mathbb{R}^N}v_\ell^2(y,t)g_\ell(y_1)dyd\sigma, \tag{3.11}$$

where

$$g_\ell(y_1)=\int_{\mathbb{R}}\frac{(\rho_\ell(x_1)-\rho_\ell(y_1))^2}{|x_1-y_1|^{1+2s}}\,dx_1. \tag{3.12}$$

Next, we examine the term $g_\ell(y_1)$. Indeed, we observe that if $|y_1| \geq 2\ell$ then

$$\begin{aligned} g_\ell(y_1) &= \int_{\mathbb{R}} \frac{(\rho_\ell(x_1))^2}{|x_1 - y_1|^{1+2s}}\, dx_1 \overset{\text{from Fig.2.}}{\leq} \int_{-\ell}^{\ell} \frac{1}{|x_1 - y_1|^{1+2s}}\, dx_1 \\ &\leq \frac{1}{s}\left(\frac{1}{|-\ell + y_1|^{2s}} + \frac{1}{|\ell + y_1|^{2s}}\right). \end{aligned}$$

On the other hand, if we assume that $|y_1| < 2\ell$, then from the fact that $|\rho_\ell'| \leq 2$ we have

$$\begin{aligned} g_1(y_1) &= \int_{-\ell}^{\ell} \frac{(\rho_\ell(x_1 + y_1) - \rho_\ell(y_1))^2}{|x_1|^{1+2s}}\, dx_1 + \int_{\mathbb{R}\backslash(-\ell,\ell)} \frac{(\rho_\ell(x_1 + y_1) - \rho_\ell(y_1))^2}{|x_1|^{1+2s}}\, dx_1 \\ &\leq \frac{4}{\ell^2}\int_{-\ell}^{\ell} \frac{|x_1|^2}{|x_1|^{1+2s}}\, dx_1 + 2\int_{\mathbb{R}\backslash(-\ell,\ell)} \frac{1}{|x_1|^{1+2s}}\, dx_1, \\ &\leq \frac{C_2}{\ell^{2s}}, \end{aligned}$$

where $C_2 > 0$ independent of ℓ. Therefore, we conclude that

$$g_\ell(y_1) \leq \begin{cases} \frac{C_2}{\ell^{2s}} & \text{if } \quad y_1 \in (-2\ell, 2\ell), \\ \frac{1}{s}\left(\frac{1}{|-\ell+y_1|^{2s}} + \frac{1}{|\ell+y_1|^{2s}}\right) & \text{if } y_1 \in \mathbb{R}\backslash(-2\ell, 2\ell). \end{cases} \tag{3.13}$$

Since $v_\ell(x, t) = u_\ell(x, t) - u_\infty(x, t)$, then from (3.11) we have

$$\begin{aligned} &\|v_\ell(t)\rho_\ell\|^2_{2,\Omega_\ell} + \int_0^t \|v_\ell(\sigma)\rho_\ell\|^2_{2,\Omega_\ell}\, d\sigma \leq C_1 \prod_{k=2}^{N} \theta_k \int_0^t \int_{\mathbb{R}^N} v_\ell^2(y, \sigma) g_\ell(y_1) dy d\sigma \\ &\leq 2C_1 \prod_{k=2}^{N} \theta_k \int_0^T \int_{\mathbb{R}^N} u_\ell^2(y, t) g_\ell(y_1) dy dt \\ &\quad +2C_1 \prod_{k=2}^{N} \theta_k \int_0^T \int_{\mathbb{R}^N} u_\infty^2(Y_2, t) g_\ell(y_1) dy dt = I_1 + I_2. \end{aligned} \tag{3.14}$$

In view of (3.13) and by using the fact that $u_\ell(y, t) = 0$ in $\mathbb{R}^N\backslash\Omega_\ell$, we infer that

$$\begin{aligned} I_1 &= 2C_1 \prod_{k=2}^{N} \theta_k \int_0^T \int_{\mathbb{R}^N} u_\ell^2(y, t) g_\ell(y_1) dy dt \\ &= 2C_1 \prod_{k=2}^{N} \theta_k \int_0^T \int_{\Omega_\ell} u_\ell^2(y, t) g_\ell(y_1) dy dt \leq \frac{C_2}{\ell^{2s}} \int_0^T \int_{\Omega_\ell} u_\ell^2(y, t) dy dt. \end{aligned} \tag{3.15}$$

From this and according to Lemma 2.4, there exists $C_3 > 0$ independent of ℓ such that

$$I_1 \leq \frac{C_3}{\ell^{2s-1}} \left\{ \|u_0\|_{2,\omega}^2 + \|f\|_{L^2(0,T;L^2(\omega))}^2 \right\}. \tag{3.16}$$

Next we note that from (3.13) and the fact that $s > \frac{1}{2}$ we obtain

$$\begin{aligned}
I_2 &= 2C_1 \prod_{k=2}^{N} \theta_k \int_0^T \int_{\mathbb{R}^N} u_\infty^2(Y_2, t) g_\ell(y_1) dy dt \\
&= 2C_1 \prod_{k=2}^{N} \theta_k \int_0^T \|u_\infty(t)\|_{2,\omega}^2 \, dt \int_{-\infty}^{+\infty} g_\ell(y_1)\, dy_1 \\
&\leq 2C_1 \prod_{k=2}^{N} \theta_k \int_0^T \|u_\infty(t)\|_{2,\omega}^2 \, dt \\
&\quad \left\{ \int_{-2\ell}^{2\ell} g_\ell(y_1)\, dy_1 + \int_{2\ell}^{+\infty} g_\ell(y_1)\, dy_1 + \int_{-\infty}^{-2\ell} g_\ell(y_1)\, dy_1 \right\} \\
&= \frac{2C_4 \prod_{k=2}^{N} \theta_k \int_0^T \|u_\infty(t)\|_{2,\omega}^2 \, dt}{\ell^{2s-1}}.
\end{aligned}$$

Now by taking $v = u_\infty$ in (1.15) and using the the Young's inequality and through direct computations we derive

$$\|u_\infty(t)\|_{2,\omega}^2 \leq e^T \left(\|u_0\|_{2,\omega}^2 + 2\|f\|_{L^2(0,T;L^2(\omega))} \right). \tag{3.17}$$

This combined with the last inequality proves the existence of $C_5 > 0$ independent of ℓ such that

$$I_2 \leq \frac{C_5 \left(\|u_0\|_{2,\omega}^2 + 2\|f\|_{L^2(0,T;L^2(\omega))} \right)}{\ell^{2s-1}}. \tag{3.18}$$

Hence, by gathering (3.14), (3.16), and (3.18) we may obtain a constant $K > 0$ independent of ℓ such that

$$\begin{aligned}
&\int_{\Omega_\ell} (u_\ell(t) - u_\infty(t))^2 \rho_\ell(x_1)\, dx + \int_0^T \int_{\Omega_\ell} (u_\ell(t) - u_\infty(t))^2 \rho_\ell(x_1)\, dx dt \\
&\leq \frac{K \left\{ \|u_0\|_{2,\omega}^2 + \|f\|_{L^2(0,T;L^2(\omega))} \right\}}{\ell^{2s-1}}.
\end{aligned} \tag{3.19}$$

Since $\rho_\ell = 1$ on $\Omega_{\ell/2}$ this leads to

$$\|u_\ell(t) - u_\infty(t)\|_{2,\Omega_{\ell/2}}^2 + \int_0^T \|u_\ell(t) - u_\infty(t)\|_{2,\Omega_{\ell/2}}^2 \, dt$$

$$\leq \frac{K\left\{\|u_0\|_{2,\omega}^2 + \|f\|_{L^2(0,T;L^2(\omega))}\right\}}{\ell^{2s-1}}. \tag{3.20}$$

We choose ℓ large enough so that $\ell/2 > \ell_0$, we obtain

$$\|u_\ell - u_\infty\|_{L^\infty(0,T;L^2(\Omega_{\ell_0}))}^2 + \|u_\ell - u_\infty\|_{L^2(0,T;L^2(\Omega_{\ell_0}))}^2$$
$$\leq \frac{K\left\{\|u_0\|_{2,\omega}^2 + \|f\|_{L^2(0,T;L^2(\omega))}\right\}}{\ell^{2s-1}}.$$

Hence the proof is now complete.

4 Proof of Theorem 1.4

This section is devoted to prove Theorem 1.4. The next proposition is the key argument to prove Theorem 1.4.

Proposition 4.1 *let u_ℓ be the weak solution of (1.1). If for some $0 < \gamma < \frac{1}{10}$, there exists a non-negative bounded Lipschitz continuous function ρ such that*

$$S(\rho)(x) := \int_{\mathbb{R}^N} \frac{(\sqrt{\rho(x+y)} - \sqrt{\rho(x)})^2}{|y|^{N+2s}}\, dy \leq \frac{\gamma}{C_p}\rho(x) \quad \forall x \in \mathbb{R}^N, \tag{4.1}$$

where $C_p > 0$ is the Poincaré constant for the domain Ω_ℓ. Then, there exists $C_{N,s,\gamma} > 0$ independent of ℓ such that

$$\|u_\ell\sqrt{\rho}\|_{L^\infty(0,T;L^2(\Omega_\ell))}^2 + \|u_\ell\sqrt{\rho}\|_{L^2(0,T;L^2(\Omega_\ell))}^2$$
$$\leq C_{N,s,\gamma}\left\{\int_0^T \int_{\Omega_\ell} f^2(x,t)\rho(x)\, dxdt + \int_{\Omega_\ell} u_0^2(x)\rho(x)\, dx\right\}. \tag{4.2}$$

Proof From (4.1) and [12, Lemma 3], we have the weighted Poincaré inequality

$$\int_{\Omega_\ell} u_\ell^2(x,t)\rho(x)\, dx \leq \frac{2C_p}{1-2\gamma}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_\ell(x,t)-u_\ell(y,t))^2\rho(x)}{|x-y|^{N+2s}}\, dxdy \forall t \in (0,T). \tag{4.3}$$

In a similar fashion as above, one can show that $u_\ell(t)\rho \in H_0^s(\Omega_\ell)$ a.e. $t \in (0,T)$. Thus, by taking $v = u_\ell\rho$ in (1.15) yields

$$\int_{\Omega_\ell} \partial_t u_\ell(t) u_\ell(t)\rho\, dx + a(u_\ell(t), u_\ell(t)\rho) = \int_{\Omega_\ell} f(x,t) u_\ell(t)\rho\, dx \quad \text{a.e. } t \in (0,T). \tag{4.4}$$

Using Proposition 2.1, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_\ell(t)\sqrt{\rho}\|^2_{2,\Omega_\ell} + a(u_\ell(t), u_\ell(t)\rho) = \int_{\Omega_\ell} f(x,t)u_\ell(t)\rho\,dx \quad \text{a.e. } t \in (0,T). \tag{4.5}$$

Next we note that

$$\begin{aligned}
&a(u_\ell(t), u_\ell(t)\rho)\\
&= \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_\ell(x,t)-u_\ell(x,t))(u_\ell(x,t)\rho(x)-u_\ell(y,t)\rho(y))}{|x-y|^{N+2s}}\,dxdy\\
&= \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_\ell(x,t)-u_\ell(y,t))^2\rho(x)}{|x-y|^{N+2s}}\,dxdy\\
&\quad + \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_\ell(x,t)-u_\ell(y,t))(\rho(x)-\rho(y))u_\ell(y,t)}{|x-y|^{N+2s}}\,dxdy.
\end{aligned} \tag{4.6}$$

This combined with (4.5) gives

$$\begin{aligned}
&\frac{d}{dt}\|u_\ell(t)\sqrt{\rho}\|^2_{2,\Omega_\ell} + C_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_\ell(x,t)-u_\ell(y,t))^2\rho(x)}{|x-y|^{N+2s}}\,dxdy0\\
&= 2\int_{\Omega_\ell} f(x,t)u_\ell(t)\rho\,dx\\
&\quad - C_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_\ell(x,t)-u_\ell(y,t))(\rho(x)-\rho(y))u_\ell(y,t)}{|x-y|^{N+2s}}\,dxdy \text{a.e.} t \in (0,T).
\end{aligned} \tag{4.7}$$

The first term on the right-hand side can be estimated as follows:

$$\begin{aligned}
2\int_{\Omega_\ell} f(x,t)u_\ell(t)\rho\,dx &\le \frac{1}{\varepsilon}\int_{\Omega_\ell} f^2(x,t)\rho(x)\,dx\\
&+\varepsilon\int_{\Omega_\ell} u_\ell^2(x,t)\rho(x)\,dx \quad \forall\varepsilon > 0.
\end{aligned} \tag{4.8}$$

While the second term on the right-hand side can be estimated in this way

$$\begin{aligned}
&-C_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_\ell(x,t)-u_\ell(y,t))(\rho(x)-\rho(y))u_\ell(y,t)}{|x-y|^{N+2s}}\,dxdy\\
&= \frac{-C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_\ell(x,t)-u_\ell(y,t))(\sqrt{\rho(x)}+\sqrt{\rho(y)})(\sqrt{\rho(x)}-\sqrt{\rho(y)})u_\ell(y,t)}{|x-y|^{N+2s}}\,dxdy\\
&\le \frac{C_{N,s}}{2\nu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_\ell(x,t)-u_\ell(y,t))^2(\sqrt{\rho(x)}+\sqrt{\rho(y)})^2}{|x-y|^{N+2s}}\,dxdy\\
&\quad + \frac{\nu C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\sqrt{\rho(x)}-\sqrt{\rho(y)})^2u_\ell^2(y,t)}{|x-y|^{N+2s}}\,dxdy
\end{aligned}$$

$$
\begin{aligned}
&\overset{\text{by (4.1)}}{\leq} \frac{2C_{N,s}}{\nu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_\ell(x,t)-u_\ell(y,t))^2(\sqrt{\rho(x)}+\sqrt{\rho(y)})^2}{|x-y|^{N+2s}}\,dxdy\\
&+\frac{\nu C_{N,s}}{2}\int_{\mathbb{R}^N} S(\rho)u_\ell^2(x,t)\,dx\\
&\overset{\text{by (4.1)}}{\leq} \frac{2C_{N,s}}{\nu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_\ell(x,t)-u_\ell(y,t))^2\rho(x)}{|x-y|^{N+2s}}\,dxdy\\
&+\frac{\nu C_{N,s}\gamma}{2C_p}\int_{\mathbb{R}^N} u_\ell^2(x,t)\rho(x)\,dx\\
&\overset{\text{by(4.3)}}{\leq} \frac{2C_{N,s}}{\nu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_\ell(x,t)-u_\ell(y,t))^2\rho(x)}{|x-y|^{N+2s}}\,dxdy\\
&+\frac{\nu C_{N,s}\gamma}{1-2\gamma}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_\ell(x,t)-u_\ell(y,t))^2\rho(x)}{|x-y|^{N+2s}}\,dxdy \quad \forall\nu>0.
\end{aligned}
$$

Now, by taking $\nu=\left(\frac{2(1-2\gamma)}{\gamma}\right)^{1/2}$ we infer that

$$
\begin{aligned}
&-C_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_\ell(x,t)-u_\ell(y,t))(\rho(x)-\rho(y))u_\ell(y,t)}{|x-y|^{N+2s}}\,dxdy\\
&\quad\leq C_{N,s}\left(\frac{8\gamma}{1-2\gamma}\right)^{1/2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_\ell(x,t)-u_\ell(y,t))^2\rho(x)}{|x-y|^{N+2s}}\,dxdy.
\end{aligned}
$$

This combined with (4.7) and (4.8) yields

$$
\begin{aligned}
&\frac{d}{dt}\|u_\ell(t)\sqrt{\rho}\|^2_{2,\Omega_\ell}+C_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_\ell(x,t)-u_\ell(y,t))^2\rho(x)}{|x-y|^{N+2s}}\,dxdy\\
&\leq\frac{1}{\varepsilon}\int_{\Omega_\ell}f^2(x,t)\rho(x)\,dx+\varepsilon\int_{\Omega_\ell}u_\ell^2(x,t)\rho(x)\,dx\\
&\quad+C_{N,s}\left(\frac{8\gamma}{1-2\gamma}\right)^{1/2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_\ell(x,t)-u_\ell(y,t))^2\rho(x)}{|x-y|^{N+2s}}\,dxdy\\
&\leq\frac{1}{\varepsilon}\int_{\Omega_\ell}f^2(x,t)\rho(x)\,dx+\left(\frac{2C_p\varepsilon}{1-2\gamma}+C_{N,s}\left(\frac{8\gamma}{1-2\gamma}\right)^{1/2}\right)\int_{\mathbb{R}^N}\\
&\quad\int_{\mathbb{R}^N}\frac{(u_\ell(x,t)-u_\ell(y,t))^2\rho(x)}{|x-y|^{N+2s}}\,dxdy.
\end{aligned}\tag{4.9}
$$

Since $0<\gamma<\frac{1}{10}$, we have

$$
C_{N,s}\left(\frac{8\gamma}{1-2\gamma}\right)^{1/2}<C_{N,s},
$$

thus by choosing ε small enough, we get

$$
\frac{2C_p\varepsilon}{1-2\gamma}+C_{N,s}\left(\frac{8\gamma}{1-2\gamma}\right)^{1/2}<C_{N,s}.
$$

From this and (4.9), it turns out that

$$\frac{d}{dt}\|u_\ell(t)\sqrt{\rho}\|^2_{2,\Omega_\ell} + K_{N,s,\gamma}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_\ell(x,t)-u_\ell(y,t))^2\rho(x)}{|x-y|^{N+2s}}\,dxdy \leq \frac{1}{\varepsilon}\int_{\Omega_\ell} f^2(x,t)\rho(x)\,dx. \tag{4.10}$$

where $K_{s,N,\gamma} = C_{N,s} - \left(\frac{2C_p\varepsilon}{1-2\gamma} + C_{N,s}\left(\frac{8\gamma}{1-2\gamma}\right)^{1/2}\right) > 0$. Using (4.3) and integrating (4.10) over $[0,t]$ yields

$$\|u_\ell(t)\sqrt{\rho}\|^2_{2,\Omega_\ell} + \frac{K_{N,s,\gamma}(1-2\gamma)}{2C_p}\int_0^t\int_{\Omega_\ell} u_\ell^2(x,\sigma)\rho(x)\,dxd\sigma \leq \frac{1}{\varepsilon}\int_0^T\int_{\Omega_\ell} f^2(x,t)\rho(x)\,dxdt + \int_{\Omega_\ell} u_0^2(x)\rho(x)\,dx,$$

which can be rewritten as

$$\|u_\ell(t)\sqrt{\rho}\|^2_{2,\Omega_\ell} + \int_0^t\int_{\Omega_\ell} u_\ell^2(x,\sigma)\rho(x)\,dxd\sigma \leq C_{N,s,\gamma}\left\{\int_0^T\int_{\Omega_\ell} f^2(x,t)\rho(x)\,dxdt + \int_{\Omega_\ell} u_0^2(x)\rho(x)\,dx\right\},$$

where $C_{N,s,\gamma} := \frac{1}{\min\left\{1, \frac{K_{N,s,\gamma}(1-2\gamma)}{2C_p}\right\}\varepsilon}$. Hence the proof of Proposition 4.1 is now complete. □

For each $\sigma \in \mathbb{R}$ and $\epsilon \in (0,2)$ let us define

$$\rho_\epsilon(\sigma) = \frac{1}{1+|\sigma|^\epsilon}.$$

Obviously, the function ρ is non-negative, bounded, and Lipschitz continuous. Then, for each $\lambda > 0$ the function ρ_λ on $\mathbb{R}^N$ defined by

$$\rho_{\epsilon,\lambda}(x) := \rho_\epsilon\left(\frac{x_1}{\lambda}\right) \tag{4.11}$$

is also non-negative bounded and Lipschitz continuous. The proof of the following lemma is quite similar to that of Theorem 5.1 and Remark 5.1 in [7], so we omit it.

Lemma 4.2 *For each $\lambda > 0$ there exists a constant $C_\epsilon > 0$ such that*

$$S(\rho_{\epsilon,\lambda})(x) := \int_{\mathbb{R}^N}\frac{(\sqrt{\rho_{\epsilon,\lambda}(x+y)}-\sqrt{\rho_{\epsilon,\lambda}(x)})^2}{|y|^{N+2s}}\,dy \leq \frac{C_\epsilon}{\lambda^{2s}}\rho_{\epsilon,\lambda}(x) \quad \forall x \in \mathbb{R}^N.$$

Now we are in position to finish the proof of Theorem 1.4. By choosing λ large enough so that

$$\frac{C_\epsilon}{\lambda^{2s}} < \frac{\gamma}{C_p} \text{ with } 0 < \gamma < \frac{1}{10},$$

we can apply Proposition 4.1 to obtain

$$\begin{aligned} &\int_{\Omega_{\ell\alpha}} u_\ell^2(x,t)\rho_{\epsilon,\lambda}(x)\,dx + \int_0^T \int_{\Omega_{\ell\alpha}} u_\ell^2(x,t)\rho_{\epsilon,\lambda}(x)\,dxdt \\ &\leq C_{N,s,\gamma}\left\{\int_0^T \int_{\Omega_\ell} f^2(x,t)\rho_{\epsilon,\lambda}(x)\,dxdt + \int_{\Omega_\ell} u_0^2(x)\rho_{\epsilon,\lambda}(x)\,dx\right\}. \end{aligned} \tag{4.12}$$

for any $\alpha \in [0, 1)$ and ℓ large enough. Next, using assumptions (1.21)–(1.22), and the definition of $\rho_{\epsilon,\lambda}$ we obtain

$$\begin{aligned} &\frac{1}{\lambda^\epsilon + \ell^{\epsilon\alpha}}\left\{\int_{\Omega_{\ell\alpha}} u_\ell^2(x,t)\,dx + \int_0^T \int_{\Omega_{\ell\alpha}} u_\ell^2(x,t)\,dxdt\right\} \\ &\leq \frac{\lambda^\epsilon}{(\ell-1)^\epsilon}\left\{\|f\|^2_{L^2(0,T;L^2(\Omega_\ell\backslash\Omega_{\ell-1}))} + \|u_0\|^2_{L^2(\Omega_\ell\backslash\Omega_{\ell-1})}\right\}, \end{aligned}$$

which implies for large ℓ that

$$\begin{aligned} &\int_{\Omega_{\ell\alpha}} u_\ell^2(x,t)\,dx + \int_0^T \int_{\Omega_{\ell\alpha}} u_\ell^2(x,t)\,dxdt \\ &\leq \frac{\lambda^\epsilon\,(\lambda^\epsilon + \ell^{\epsilon\alpha})}{(\ell-1)^\epsilon}\left\{\|f\|^2_{L^2(0,T;L^2(\Omega_\ell\backslash\Omega_{\ell-1}))} + \|u_0\|^2_{L^2(\Omega_\ell\backslash\Omega_{\ell-1})}\right\}, \\ &\leq \frac{2\lambda^\epsilon\,(\lambda^\epsilon + \ell^{\epsilon\alpha})}{(\ell-1)^\epsilon} \leq \frac{C}{\ell^{(1-\alpha)\epsilon}}. \end{aligned}$$

Hence, the proof is now complete.

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References

1. Applebaum, D.: Lévy processes-from probability to finance quantum groups. Not. Am. Math. Soc. **51**, 1336–1347 (2004)
2. Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker's Guide to the fractional Sobolev spaces. Bull. Sci. Math. **136**, 519–527 (2012)
3. Showalter, E.R.: *Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations* (Mathematical surveys and monographs, vol. 49) American Mathematical Society, Providence (2013)
4. Valdinoci, E.: From the long jump random walk to the fractional Laplacian. Bol. Soc. Esp. Mat. Apl. **49**, 33–44 (2009)
5. G. Molica Bisci, G., V. Rdulescu, R. Servadei, : Variational Methods for Nonlocal Fractional Equations. Encyclopedia of Mathematics and its Applications, vol. 162. Cambridge University Press, Cambridge (2016)
6. Brézis, H.: Opérateurs Maximaux Monotones et semi-groupes des contractions dans les espaces de Hilbert. North-Holland/American Elsevier, Amsterdam/London/New York (1971)
7. Chowdhury, I., Roy, P.: On the asymptotic analysis of problems involving fractional Laplacian in cylindrical domains tending to infinity. Commun. Contemp. Math. **19**(5), 21 (2017)
8. Chowdhury, I., Csató, G., Roy, P., Sk, F.: Study of fractional Poincaré inequalities on unbounded domains. Discrete Contin. Dyn. Syst. **41**(6), 2993–3020 (2021)
9. Vázquez, J.L.: The mathematical theories of diffusion. Nonlinear and Fractional Diffusion. Springer Lecture Notes in Mathematics, CIME Subseries (2017)
10. Vázquez, J.L.: Recent progress in the theory of nonlinear diffusion with fractional Laplacian operators. Discrete Contin. Dyn. Syst. Ser. S **7**(4), 857–885 (2014). https://doi.org/10.3934/dcdss.2014.7.857
11. Mazón, J.M., Rossi, J.D., Toledo, J.: Fractional p-Laplacian evolution equations. J. Math. Pures Appl. **105**(6), 810–844 (2016)
12. Yeressian, K.: Asymptotic behavior of elliptic nonlocal equations set in cylinders. Asymptotic. Anal. **89**(1–2), 21–35 (2014)
13. Bal, K., Mohanta, K., Roy, P., Sk, F.: Hardy and Poincaré inequalities in fractional Orlicz-Sobolev space. (2020) arXiv preprint arXiv:2009.07035
14. Mohanta, K., Sk, F.: On the best constant in fractional p-Poincaré inequalities on cylindrical domains. (2021) arXiv preprint arXiv:2013.16845v2
15. Caffarelli, L.: Non-local diffusions, drifts and games. Nonlinear Partial Differ. Equ. **7**, 37–52 (2012)
16. Esposito, L., Roy, P., Sk, F.: On the asymptotic behavior of the eigenvalues of nonlinear elliptic problems in domains becoming unbounded. Asymptot. Anal. **123**(1–2), 79–94 (2021)
17. Djilali, L., Rougirel, A.: Galerkin method for time fractional diffusion equations. J. Elliptic Parabol. Equ. **4**(2), 349368 (2018)
18. Chipot, M., Rougirel, A.: On the asymptotic behavior of the solution of elliptic problems in cylindrical domains becoming unbounded. Commun. Contemp. Math. **4**(1), 15–44 (2002)
19. Chipot, M., Rougirel, A.: On the asymptotic behavior of the solution of parabolic problems in cylindrical domains of large size in some directions. Discrete Contin. Dyn. Syst. Ser. B **1**(3), 319–338 (2001)
20. Chipot, M., Xie, Y.: On the asymptotic behavior of the p-Laplace equation in cylinders becoming unbounded, Nonlinear partial differential equations and their applications, pp. 16–27, GAKUTO Internat. Ser. Math. Sci. Appl. 20, Gakkotosho, Tokyo, (2004)
21. Chipot, M., Yeressian, K.: Asymptotic behavior of the solution to variational inequalities with joint constraints on its value and its gradient. Contemp. Math. **594**, 137–154 (2013)
22. Chipot, M., Mardare, S.: Asymptotic behavior of the Stokes problem in cylinders becoming unbounded in one direction. J. Math. Pures Appl. **90**(2), 133–159 (2013)
23. Chipot, M.: ℓ Goes to Plus Infinity. Birkhäuser, Basel (2002)
24. Chipot, M.: Elements of Nonlinear Analysis. Brihäuser Vergel, Basel (2000)
25. Chipot, M., Roy, P., Shafrir, I.: Asymptotics of eigenstates of elliptic problems with mixed boundary data on domains tending to infinity. Asymptot. Anal. **85**(3–4), 199–227 (2013)
26. Chipot, M., Mojsic, A., Roy, P.: On some variational problems set on domains tending to infinity. Discrete Contin. Dyn. Syst. **36**(7), 3603–3621 (2016)
27. Kwaśnicki, M.: Ten equivalent definitions of the fractional Laplace operator. Fract. Calc. Appl. Anal. **20**(1), 7–51 (2015)
28. Servadei, R., Valdinoci, E.: On the spectrum of two different fractional operators. Proc. R. Soc. Edinb. Sect. A **144**(4), 831–855 (2014)

29. Guesmia, S.: Some convergence results for quasilinear parabolic boundary value problems in cylindrical domains of large size. Nonlinear Anal. **70**(9), 3320–3331 (2009)
30. Guesmia, S.: Some results on the asymptotic behavior for hyperbolic problems in cylindrical domains becoming unbounded. J. Math. Anal. Appl. **341**(2), 1190–1212 (2008)
31. Ambrosio, V., Freddi, L., Musina, R.: Asymptotic analysis of the Dirichlet fractional Laplacian in domains becoming unbounded. J. Math. Anal. Appl. **485**(2), 123845 (2020)
32. Rudin, W.: Principles of Mathematical Analysis. International Series in Pure and Applied Mathematics, 3rd edn. McGraw-Hill, New York (1976)