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Sobolev-Type Inequalities on Musielak–Orlicz–Morrey Spaces of an Integral Form

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Abstract

We give Sobolev-type inequalities for variable Riesz potentials $I_{\alpha(\cdot)}f$ of functions in Musielak–Orlicz–Morrey spaces of an integral form $\mathcal{L}^{\Phi,\omega}(G)$. As a corollary, we give Sobolev-type inequalities on $\mathcal{L}^{\Phi,\omega}(G)$ for double phase functions $\Phi(x,t) = t^{p(x)} + a(x)t^{q(x)}$.

Keywords Riesz potentials · Maximal functions · Sobolev's inequality · Musielak–Orlicz–Morrey spaces · Double phase functions

Mathematics Subject Classification 46E30 · 42B25

1 Introduction

Let G be an open bounded set in $\mathbf{R}^N$. Let $\alpha(\cdot)$ be a measurable function on G such that

$$0 < \inf_{x \in G} \alpha(x) \leq \sup_{x \in G} \alpha(x) < N.$$

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We define the Riesz potential of variable order $\alpha(\cdot)$ for a locally integrable function f on G by

$$I_{\alpha(\cdot)} f(x) = \int_G |x-y|^{\alpha(x)-N} f(y)\, dy;$$

when $\alpha(\cdot)$ is a constant α, this is simply written as $I_\alpha f$.

Sobolev-type inequalities for $I_\alpha f$ have been established on various function spaces by many researchers. Sobolev-type inequalities were studied on variable exponent Lebesgue spaces $L^{p(\cdot)}$ in [7, 9, 11], on two variable exponent Lebesgue spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$ in [12, 25], on variable exponent Morrey spaces $L^{p(\cdot),\nu}$ in [2, 13, 14, 22, 23, 28], on Musielak–Orlicz–Morrey spaces $L^{\Phi,\kappa}$ in [19, 20].

In the previous paper [32], we gave Sobolev-type inequalities for $I_{\alpha(\cdot)} f$ of functions in variable exponent Morrey spaces of an integral form $\mathcal{L}^{p(\cdot),\omega}(G)$, as an extension of [29, Theorem 5.4] from Morrey spaces of an integral form.

In this paper, we establish a Sobolev-type inequality for $I_{\alpha(\cdot)} f$ of functions in Musielak–Orlicz–Morrey spaces of an integral form $\mathcal{L}^{\Phi,\omega}(G)$ defined by general functions $\Phi(x,t)$ and $\omega(x,r)$ satisfying certain conditions (Theorem 4.5), as an extension of [32, Theorem 4.4]. To do this, we apply Hedberg's method ([17]) and the boundedness of the maximal operator M in $\mathcal{L}^{\Phi,\omega}(G)$ (Theorem 3.4) which is an extension of [32, Theorem 3.5].

As an application of our general theory, we give Sobolev-type inequalities (Theorem 5.3) in the framework of double phase functions $\Phi(x,t)$ with variable exponents given by

$$\Phi(x,t) = t^{p(x)} + a(x)t^{q(x)},$$

where $p(\cdot)$ and $q(\cdot)$ satisfy log-Hölder conditions, $p(x) < q(x)$ for $x \in G$ and $a(\cdot)$ is nonnegative, bounded and Hölder continuous of order $\theta \in (0,1]$. For the studies by Mingione and collaborators, see [3, 4, 8]. We refer to [20, 27] for Sobolev's inequality and to, e.g., [6, 10, 16, 24, 30, 33] for the recent results.

Throughout the paper, we let C denote various constants independent of the variables in question and $C(a, b, \cdots)$ be a constant that depends on $a, b, \cdots$ only. The symbol $g \sim h$ means that $C^{-1}h \le g \le Ch$ for some constant $C > 0$.

2 Musielak–Orlicz–Morrey Spaces of an Integral Form

To define the norm of Musielak–Orlicz–Morrey spaces of an integral form, let us consider a function

$$\Phi(x,t) : G \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions $(\Phi1)$ – $(\Phi3)$:

$(\Phi1)$ $\Phi(\,\cdot\,, t)$ is measurable on G for each $t \ge 0$ and $\Phi(x, \,\cdot\,)$ is continuous on $[0,\infty)$ for each $x \in G$;

(Φ2) there exists a constant $A_1 \ge 1$ such that

$$A_1^{-1} \le \Phi(x,1) \le A_1 \quad \text{for all } x \in G;$$

(Φ3) $t \mapsto \Phi(x,t)/t$ is uniformly almost increasing on $(0,\infty)$, namely there exists a constant $A_2 \ge 1$ such that

$$\Phi(x,t_1)/t_1 \le A_2\Phi(x,t_2)/t_2 \quad \text{for all } x \in G \text{ whenever } 0 < t_1 < t_2.$$

We write

$$\bar{\phi}(x,t) = \sup_{0<s\le t} (\Phi(x,s)/s)$$

and

$$\overline{\Phi}(x,t) = \int_0^t \bar{\phi}(x,r)\,dr$$

for $x \in G$ and $t \ge 0$. Then $\overline{\Phi}(x,\cdot)$ is convex and

$$\Phi(x,t/2) \le \overline{\Phi}(x,t) \le A_2\Phi(x,t) \tag{2.1}$$

for all $x \in G$ and $t \ge 0$ since $\bar{\phi}(x,\cdot)$ is increasing on $(0,\infty)$ for each $x \in G$.

For $x \in \mathbf{R}^N$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x with radius r and $d_G = \sup\{|x-y| : x, y \in G\}$. For a set $E \subset \mathbf{R}^N$, $|E|$ denotes the Lebesgue measure of E.

We also consider a weight function $\omega(x,r) : G \times (0,\infty) \to (0,\infty)$ satisfying the following conditions:

(ω0) $\omega(\,\cdot\,,r)$ is measurable on G for each $r > 0$ and $\omega(x,\,\cdot\,)$ is continuous on $(0,\infty)$ for each $x \in G$;

(ω1) $r \mapsto \omega(x,r)$ is uniformly almost increasing on $(0,\infty)$, namely there exists a constant $\tilde{c}_1 \ge 1$ such that

$$\omega(x,r_1) \le \tilde{c}_1\omega(x,r_2)$$

for all $x \in G$ whenever $0 < r_1 < r_2 < \infty$;

(ω2) there exists a constant $\tilde{c}_2 > 1$ such that

$$\tilde{c}_2^{-1}\omega(x,r) \le \omega(x,2r) \le \tilde{c}_2\omega(x,r)$$

for all $x \in G$ whenever $r > 0$;

($\omega3; \omega_0$) there exist constants $\omega_0 > 0$ and $\tilde{c}_3 \ge 1$ such that

$$\tilde{c}_3^{-1}r^{\omega_0} \le \omega(x,r) \le \tilde{c}_3$$

for all $x \in G$ and $0 < r \le 2d_G$.

Let $f^- := \inf_{x\in G} f(x)$ and $f^+ := \sup_{x\in G} f(x)$ for a measurable function f on G. Let us write that $L_c(t) = \log(c+t)$ for $c > 1$ and $t \geq 0$, $L_c^{(1)}(t) = L_c(t)$, $L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))$.

Example 2.1 Let $\sigma(\cdot)$ and $\beta(\cdot)$ be measurable functions on G such that $0 < \sigma^- \leq \sigma^+ \leq \omega_0$ and $-c(\omega_0 - \sigma(x)) \leq \beta(x) \leq c$ for all $x \in G$ and some constant $c > 0$. Then

$$\omega(x,r) = r^{\sigma(x)} L_e(1/r)^{\beta(x)}$$

satisfies (ω0), (ω1), (ω2) and (ω3; ω_0).

Given $\Phi(x,t)$ and $\omega(x,r)$ as above, we define the $\mathcal{L}^{\Phi,\omega}$ norm by

$$\|f\|_{\mathcal{L}^{\Phi,\omega}(G)} = \inf\Bigg\{\lambda > 0\,; \sup_{x\in G}\left(\int_0^{2d_G} \frac{\omega(x,r)}{|B(x,r)|}\left(\int_{G\cap B(x,r)} \overline{\Phi}\,(y,|f(y)|/\lambda)\,dy\right)\frac{dr}{r}\right) \leq 1\Bigg\},$$

which is the Luxemburg norm ([18]). The space of all measurable functions f on G with $\|f\|_{\mathcal{L}^{\Phi,\omega}(G)} < \infty$ is denoted by $\mathcal{L}^{\Phi,\omega}(G)$. The space $\mathcal{L}^{\Phi,\omega}(G)$ is called a Musielak–Orlicz–Morrey space of an integral form. Here note that $2d_G$ can be replaced by κd_G with $\kappa > 1$. In case $\Phi(x,t) = t^{p(x)}$, $\mathcal{L}^{\Phi,\omega}(G)$ is denoted by $\mathcal{L}^{p(\cdot),\omega}(G)$ for simplicity. If $p(\cdot) \equiv p$, then we write $\mathcal{L}^{p(\cdot),\omega}(G) = \mathcal{L}^{p,\omega}(G)$.

Remark 2.2 If there exists a constant $C_0 > 0$ such that

$$\int_0^{2d_G} \omega(x,r)\,\frac{dr}{r} \leq C_0$$

for all $x \in G$, then we see that $\mathcal{L}^{\Phi,\omega}(G) \neq \{0\}$ since

$$\int_0^{2d_G} \frac{\omega(x,r)}{|B(x,r)|}\left(\int_{G\cap B(x,r)} \overline{\Phi}\,(y,1)\,dy\right)\frac{dr}{r} \leq A_1 A_2 \int_0^{2d_G} \omega(x,r)\,\frac{dr}{r} \leq A_1 A_2 C_0$$

for all $x \in G$ by (2.1) and (Φ2). See also [5, Lemma 1].

We shall also consider the following conditions for $\Phi(x,t)$: Let $p \geq 1$, $q \geq 1$ and $\nu > 0$ be given.

(Φ3; 0; p) $t \mapsto t^{-p}\Phi(x,t)$ is uniformly almost increasing on $(0,1]$, namely there exists a constant $A_{2,0,p} \geq 1$ such that

$$t_1^{-p}\Phi(x,t_1) \leq A_{2,0,p}\, t_2^{-p}\,\Phi(x,t_2) \quad \text{for all } x \in G \text{ whenever } 0 < t_1 < t_2 \leq 1;$$

($\Phi3; \infty; q$) $t \mapsto t^{-q}\Phi(x,t)$ is uniformly almost increasing on $[1,\infty)$, namely there exists a constant $A_{2,\infty,q} \ge 1$ such that

$$t_1^{-q}\Phi(x,t_1) \le A_{2,\infty,q}\, t_2^{-q}\Phi(x,t_2) \quad \text{for all } x \in G \text{ whenever } 1 \le t_1 < t_2;$$

($\Phi5; \nu$) for every $\gamma > 0$, there exists a constant $B_{\gamma,\nu} \ge 1$ such that

$$\Phi(x,t) \le B_{\gamma,\nu}\Phi(y,t)$$

whenever $x, y \in G$, $|x-y| \le \gamma t^{-\nu}$ and $t \ge 1$.

Remark 2.3 We refer to [1, p. 2544] and [15, Section 7.3] for ($\Phi5; \nu$). If $\Phi(x,t)$ satisfies ($\Phi3; \infty; q$), then it satisfies ($\Phi3; \infty; q'$) for $1 \le q' \le q$. If $\Phi(x,t)$ satisfies ($\Phi5; \nu$), then it satisfies ($\Phi5; \nu'$) for all $\nu' \ge \nu$.

We give some examples of $\Phi(x,t)$.

Example 2.4 Let $p(\cdot)$ and $q_j(\cdot)$, $j = 1,\dots,k$ be given measurable functions on G such that $1 < p^- \le p^+ < \infty$ and $-\infty < q_j^- \le q_j^+ < \infty$, $j = 1, \dots k$. Then,

$$\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t) = t^{p(x)}\prod_{j=1}^{k}\left(L_e^{(j)}(t)\right)^{q_j(x)}$$

satisfies ($\Phi1$), ($\Phi2$) and ($\Phi3$). This function satisfies ($\Phi3; \infty; q$) for $1 \le q < p^-$ in general and for $1 \le q \le p^-$ in case $q_j^- \ge 0$ for all $j = 1,\dots,k$.

Moreover, we see that $\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t)$ satisfies ($\Phi5; \nu$) for every $\nu > 0$ if $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x)-p(y)| \le \frac{C_p}{L_e(1/|x-y|)} \quad (x,\ y \in G)$$

with a constant $C_p \ge 0$ and $q_j(\cdot)$ is $(j+1)$-log-Hölder continuous, namely

$$|q_j(x)-q_j(y)| \le \frac{C_j}{L_e^{(j+1)}(1/|x-y|)} \quad (x,\ y \in G)$$

with constants $C_j \ge 0$ for each $j = 1,\dots k$.

Example 2.5 Theorem 3.4 applies, e.g., to the following nondoubling functions

$$\Phi_1(t) = e^{p(x)t} - p(x)t - 1,\ \Phi_2(t) = e^t t^{p(x)},\ \Phi_3(t) = e^{t^{p(x)}} - 1$$

which satisfy ($\Phi1$), ($\Phi2$) and ($\Phi3$). We refer to [21, Examples 3-5] for the conditions on p and q which ($\Phi3; 0; p$) and ($\Phi3; \infty; q$) hold.

Example 2.6 The double phase function with variable exponents

$$\Phi(x,t) = t^{p(x)} + a(x)t^{q(x)},\ x \in G,\ t \geq 0,$$

where $p(x) < q(x)$ for $x \in G$, $a(\cdot)$ is a nonnegative, bounded and Hölder continuous function of order $\theta \in (0, 1]$, was studied in [20]. We refer to [20, Lemma 5.1] and Section 5 for the conditions on $p(\cdot)$ and $q(\cdot)$ which (Φ1), (Φ2), (Φ3), ($\Phi3; 0; p^-$), ($\Phi; \infty; p^-$) and ($\Phi5; \nu$) hold.

3 Boundedness of the Maximal Operator

For a locally integrable function f on G, the Hardy-Littlewood maximal function Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{G\cap B(x,r)} |f(y)|\, dy.$$

We know the boundedness of M on $\mathcal{L}^{p,\omega}(G)$.

Lemma 3.1 *([32, Lemma 3.2]) Suppose*

($\omega1'$) *$r \mapsto r^{-\varepsilon_1}\omega(x,r)$ is uniformly almost increasing in $(0, d_G]$ for some $\varepsilon_1 > 0$.*

If $p > 1$, then there is a constant $C > 0$ such that

$$\|Mf\|_{\mathcal{L}^{p,\omega}(G)} \leq C\|f\|_{\mathcal{L}^{p,\omega}(G)}$$

for all $f \in \mathcal{L}^{p,\omega}(G)$.

Remark 3.2 Note that ($\omega1'$) implies ($\omega1$).

Let $\omega(x,r) = r^{\sigma(x)}L_e(1/r)^{\beta(x)}$ be as in Example 2.1. Then note that ($\omega1'$) holds for $0 < \varepsilon_1 < \sigma^-$.

Lemma 3.3 *Suppose $\Phi(x,t)$ satisfies ($\Phi3; 0; p$), ($\Phi3; \infty; q$) and ($\Phi5; \nu$) for $p \geq 1$, $q \geq 1$ and $\nu > 0$ satisfying $\nu \leq q/\omega_0$. Set*

$$I(f;x,r) = \frac{1}{|B(x,r)|} \int_{G\cap B(x,r)} f(y)\, dy$$

and

$$J(f;x,r) = \frac{1}{|B(x,r)|} \int_{G\cap B(x,r)} \Phi\big(y, f(y)\big)^{1/p_0}\, dy$$

for $x \in G$ and $0 < r \leq d_G$, where $1 \leq p_0 \leq \min(p,q)$. Then, given $L \geq 1$, there exist constants $C_1 = C(L) \geq 2$ and $C_2 > 0$ such that

$$\Phi\big(x, I(f;x,r)/C_1\big)^{1/p_0} \leq C_2 J(f;x,r)$$

for all $x \in G$, $0 < r \le d_G$ and for all nonnegative measurable functions f on G such that $f(y) \ge 1$ or $f(y) = 0$ for each $y \in G$ and

$$\sup_{z\in G}\left(\int_0^{2d_G}\frac{\omega(z,t)}{|B(z,t)|}\left(\int_{G\cap B(z,t)}\Phi\left(y,f(y)\right)dy\right)\frac{dt}{t}\right)\le L. \tag{3.1}$$

Proof Given f as in the statement of the lemma, $x \in G$ and $0 < r < d_G$, set $I = I(f; x, r)$ and $J = J(f; x, r)$. Taking f, note that (3.1) implies

$$\begin{aligned}&\frac{\omega(x,r)}{|B(x,r)|}\int_{G\cap B(x,r)}\Phi\left(y,f(y)\right)dy\\&\le C_0\int_r^{2r}\frac{\omega(x,t)}{|B(x,t)|}\left(\int_{G\cap B(x,t)}\Phi\left(y,f(y)\right)dy\right)\frac{dt}{t}\le C_0L,\end{aligned}$$

so that

$$J \le C_0^{1/p_0}\omega(x,r)^{-1/p_0}L^{1/p_0}. \tag{3.2}$$

We treat only the case $J > 1$. Since $\Phi(x,t)^{1/p_0} \to \infty$ as $t \to \infty$ by $(\Phi3; \infty; q)$ and $p_0 \le q$, there exists $K > 1$ such that

$$\Phi(x,K)^{1/p_0} = \Phi(x,1)^{1/p_0}J. \tag{3.3}$$

With this K, we have by $(\Phi3; \infty; q)$ and $p_0 \le q$

$$\int_{G\cap B(x,r)} f(y)\,dy \le K|B(x,r)| + A_{2,\infty,p_0}^{1/p_0}K\int_{G\cap B(x,r)}\frac{\Phi\big(y,f(y)\big)^{1/p_0}}{\Phi(y,K)^{1/p_0}}\,dy.$$

Since $K > 1$, by $(\Phi3; \infty; q)$, we have

$$\Phi(x,1)^{1/p_0}J = \Phi(x,K)^{1/p_0} \ge A_{2,\infty,q}^{-1/p_0}K^{q/p_0}\Phi(x,1)^{1/p_0},$$

so that, in view of (3.2) and $(\omega3; \omega_0)$,

$$K^q \le A_{2,\infty,q}J^{p_0} \le C_0A_{2,\infty,q}\,\omega(x,r)^{-1}L \le C_0A_{2,\infty,q}\tilde{c}_3Lr^{-\omega_0}$$

or $r \le \gamma K^{-q/\omega_0}$ with $\gamma = (C_0A_{2,\infty,q}\tilde{c}_3L)^{1/\omega_0}$. Thus, if $|x-y| \le r$, then

$$|x-y| \le \gamma K^{-q/\omega_0} \le \gamma K^{-\nu}$$

since $\nu \le q/\omega_0$. Hence, by $(\Phi5; \nu)$ with $B_{\gamma,\nu}^{1/p_0} = \beta$

$$\int_{G\cap B(x,r)} f(y)\,dy \le K|B(x,r)|\left\{1 + \left(A_1A_{2,\infty,p_0}\right)^{1/p_0}\beta\right\}$$

as in the proof of [20, Lemma 3.3]. See [21, Lemma 9] and [20, Lemma 3.3] for details. □

In view of Lemmas 3.1 and 3.3, we show the boundedness of M on $\mathcal{L}^{\Phi,\omega}(G)$ as an extension of [32, Theorem 3.5].

Theorem 3.4 *Suppose* $\Phi(x,t)$ *satisfies* $(\Phi3;0;p)$, $(\Phi3;\infty;q)$ *and* $(\Phi5;\nu)$ *for* $p>1$, $q>1$ *and* $\nu>0$ *satisfying* $\nu \le q/\omega_0$. *Assume that* $(\omega1')$ *holds. Then there is a constant* $C>0$ *such that*

$$\|Mf\|_{\mathcal{L}^{\Phi,\omega}(G)} \le C\|f\|_{\mathcal{L}^{\Phi,\omega}(G)}$$

for all $f \in \mathcal{L}^{\Phi,\omega}(G)$.

Proof Set $p_0 = \min(p,q)$. Then $p_0 > 1$. Consider the function

$$\Phi_0(x,t) = \Phi(x,t)^{1/p_0}.$$

Let f be a nonnegative measurable function on G with $\|f\|_{\mathcal{L}^{\Phi,\omega}(G)} \le 1/2$. Let $f_1 = f\chi_{\{x\in G: f(x)\ge 1\}}$, $f_2 = f - f_1$. Applying Lemma 3.3 to f_1 and $L=1$, there exist constants $C_1 \ge 2$ and $C_2 > 0$ such that

$$\Phi_0\big(x, Mf_1(x)/C_1\big) \le C_2 M[\Phi_0\big(\cdot, f_1(\cdot)\big)](x),$$

so that

$$\Phi\big(x, Mf_1(x)/C_1\big) \le C_2^{p_0}\,[M[\Phi_0\,(\cdot, f(\cdot))](x)]^{p_0} \tag{3.4}$$

for all $x \in G$.

On the other hand, since $Mf_2 \le 1$, we have by $(\Phi2)$ and $(\Phi3)$

$$\Phi\big(x, Mf_2(x)/C_1\big) \le A_1A_2 \tag{3.5}$$

for all $x \in G$.

By (2.1), (3.4), (3.5) and Lemma 3.1, we obtain

$$\begin{aligned}
&\int_0^{2d_G} \frac{\omega(z,r)}{|B(z,r)|}\left(\int_{G\cap B(z,r)} \overline{\Phi}\,(x, Mf(x)/(2C_1))\,dx\right)\frac{dr}{r} \\
&\le \frac{A_2}{2}\left\{\int_0^{2d_G} \frac{\omega(z,r)}{|B(z,r)|}\left(\int_{G\cap B(z,r)} \Phi\,(x, Mf_1(x)/C_1)\,dx\right)\frac{dr}{r}\right. \\
&\quad \left. + \int_0^{2d_G} \frac{\omega(z,r)}{|B(z,r)|}\left(\int_{G\cap B(z,r)} \Phi\,(x, Mf_2(x)/C_1)\,dx\right)\frac{dr}{r}\right\} \\
&\le C\left\{\int_0^{2d_G} \frac{\omega(z,r)}{|B(z,r)|}\left(\int_{G\cap B(z,r)} [M[\Phi_0\,(\cdot, f(\cdot))](x)]^{p_0}\,dx\right)\frac{dr}{r} + \int_0^{2d_G} \omega(z,r)\,\frac{dr}{r}\right\} \\
&\le C
\end{aligned}$$

for all $z \in G$ since there exists a constant $C_3 > 0$ such that

$$\int_0^{2d_G} \omega(z,r)\,\frac{dr}{r} = \int_0^{2d_G} r^{-\varepsilon_1}\omega(z,r)\cdot r^{\varepsilon_1}\,\frac{dr}{r} \le C\int_0^{2d_G} r^{\varepsilon_1}\,\frac{dr}{r} \le C_3 \quad (3.6)$$

for all $z \in G$ by ($\omega1'$) and ($\omega3$; ω_0). Thus, this theorem is proved. □

4 Sobolev-Type Inequality

We recall the following lemma from [19].

Lemma 4.1 *([19, Lemma 5.1]) Let $F(x,t)$ be a positive function on $G \times (0,\infty)$ satisfying the following conditions:*

(F1) *$F(x,\,\cdot)$ is continuous on $(0,\infty)$ for each $x \in G$;*
(F2) *there exists a constant $K_1 \ge 1$ such that*

$$K_1^{-1} \le F(x,1) \le K_1 \quad \textit{for all } x \in G;$$

(F3) *$t \mapsto t^{-\varepsilon'}F(x,t)$ is uniformly almost increasing for some $\varepsilon' > 0$; namely there exists a constant $K_2 \ge 1$ such that*

$$t_1^{-\varepsilon'}F(x,t_1) \le K_2 t_2^{-\varepsilon'}F(x,t_2) \quad \textit{for all } x \in G \textit{ whenever } 0 < t_1 < t_2.$$

Set

$$F^{-1}(x,s) = \sup\{t > 0\,;\ F(x,t) < s\}$$

for $x \in G$ and $s > 0$. Then:

(1) *$F^{-1}(x,\cdot)$ is nondecreasing.*
(2)

$$F^{-1}(x,\lambda t) \le (K_2\lambda)^{1/\varepsilon'}F^{-1}(x,t) \quad (4.1)$$

for all $x \in G$, $t > 0$ and $\lambda \ge 1$.
(3)

$$F(x, F^{-1}(x,t)) = t$$

for all $x \in G$ and $t > 0$.
(4)

$$K_2^{-1/\varepsilon'}t \le F^{-1}(x, F(x,t)) \le K_2^{2/\varepsilon'}t$$

for all $x \in G$ and $t > 0$.

(5)

$$\min\left\{1, \left(\frac{s}{K_1 K_2}\right)^{1/\varepsilon'}\right\} \le F^{-1}(x,s) \le \max\{1, (K_1 K_2 s)^{1/\varepsilon'}\}$$

for all $x \in G$ *and* $s > 0$.

Remark 4.2 Note that $F(x,t) = \Phi(x,t)$ is a function satisfying (F1), (F2) and (F3) with $K_1 = A_1$, $K_2 = A_2$ and $\varepsilon' = 1$.

We consider the following condition:

($\Phi\omega\alpha$) there exist constants $\varepsilon_2 > 0$ and $A_4 \ge 1$ such that

$$r_2^{\varepsilon_2+\alpha(x)}\Phi^{-1}(x, \omega(x, r_2)^{-1}) \le A_4 r_1^{\varepsilon_2+\alpha(x)}\Phi^{-1}(x, \omega(x, r_1)^{-1})$$

for all $x \in G$ whenever $0 < r_1 < r_2 < d_G$.

Lemma 4.3 *Suppose* $\Phi(x,t)$ *satisfies* ($\Phi3; \infty; q$) *and* ($\Phi5; \nu$) *for* $q \ge 1$ *and* $\nu > 0$ *satisfying* $\nu \le q/\omega_0$. *Assume that* ($\Phi\omega\alpha$) *holds. Then there exists a constant* $C > 0$ *such that*

$$\int_{G\setminus B(x,\delta)} |x-y|^{\alpha(x)-N} f(y)\,dy \le C\delta^{\alpha(x)}\Phi^{-1}(x, \omega(x,\delta)^{-1})$$

for all $x \in G$, $0 < \delta < d_G/2$ *and nonnegative* $f \in \mathcal{L}^{\Phi,\omega}(G)$ *with* $\|f\|_{\mathcal{L}^{\Phi,\omega}(G)} \le 1$.

Proof Let f be a nonnegative measurable function with $\|f\|_{\mathcal{L}^{\Phi,\omega}(G)} \le 1/2$. Let $x \in G$ and $0 < \delta < d_G/2$. By ($\Phi3$) and ($\Phi3; \infty; q$),

$$\min\{1, (A_1A_2)^{-1}s\} \le F^{-1}(x,s) \le \max\{1, (A_1A_{2,\infty,q}s)^{1/q}\};$$

cf. Lemma 4.1 (5). Set

$$c_1 = \max\left\{A_1A_2\tilde{c}_3,\ (A_1A_{2,\infty,q}\tilde{c}_3)^{-1}d_G^{\omega_0}\right\}.$$

Then we have by ($\omega3; \omega_0$), Lemma 4.1 and the condition $\nu \le q/\omega_0$

$$\Phi^{-1}\left(x, c_1\omega(x,|x-y|)^{-1}\right) \ge \min\{1, (A_1A_2)^{-1}c_1\tilde{c}_3^{-1}\} \ge 1$$

and

$$\begin{aligned}\Phi^{-1}\left(x, c_1\omega(x,|x-y|)^{-1}\right) &\le \max\{1, (A_1A_{2,\infty,q}c_1\tilde{c}_3|x-y|^{-\omega_0})^{1/q}\}\\ &= (A_1A_{2,\infty,q}c_1\tilde{c}_3d_G^{-\omega_0})^{1/q}(|x-y|/d_G)^{-\omega_0/q}\\ &\le (A_1A_{2,\infty,q}c_1\tilde{c}_3d_G^{-\omega_0})^{1/q}(|x-y|/d_G)^{-1/\nu}\end{aligned}$$

for all $x, y \in G$. Hence,

$$|x-y| \le c_2 \left\{\Phi^{-1}\left(x, c_1\omega(x,|x-y|)^{-1}\right)\right\}^{-\nu}$$

for all $x, y \in G$, where $c_2 = d_G(A_1 A_{2,\infty,q} c_1 \tilde{c}_3 d_G^{-\omega_0})^{\nu/q}$. We find by ($\Phi$3), ($\Phi5;\nu$) and Lemma 4.1 (3)

$$\begin{aligned}
&\int_{G\setminus B(x,\delta)} |x-y|^{\alpha(x)-N} f(y)\,dy \\
&\le \int_{G\setminus B(x,\delta)} |x-y|^{\alpha(x)-N} \Phi^{-1}\left(x, c_1\omega(x,|x-y|)^{-1}\right) dy \\
&+ A_2 \int_{G\setminus B(x,\delta)} |x-y|^{\alpha(x)-N} f(y) \\
&\times \frac{f(y)^{-1}\Phi(y,f(y))}{\left\{\Phi^{-1}\left(x, c_1\omega(x,|x-y|)^{-1}\right)\right\}^{-1}\Phi\left(y, \Phi^{-1}\left(x, c_1\omega(x,|x-y|)^{-1}\right)\right)}\,dy \\
&\le \int_{G\setminus B(x,\delta)} |x-y|^{\alpha(x)-N} \Phi^{-1}\left(x, c_1\omega(x,|x-y|)^{-1}\right) dy \\
&+ C\int_{G\setminus B(x,\delta)} |x-y|^{\alpha(x)-N}\omega(x,|x-y|)\Phi^{-1}\left(x, c_1\omega(x,|x-y|)^{-1}\right)\Phi(y,f(y))\,dy \\
&= I_1 + CI_2.
\end{aligned}$$

Let j_0 be the smallest integer such that $2^{j_0}\delta \ge d_G$. By (ω1), (ω2), (4.1) and ($\Phi\omega\alpha$), we obtain

$$\begin{aligned}
I_1 &= \sum_{j=1}^{j_0} \int_{G\cap(B(x,2^j\delta)\setminus B(x,2^{j-1}\delta))} |x-y|^{\alpha(x)-N}\Phi^{-1}\left(x, c_1\omega(x,|x-y|)^{-1}\right) dy \\
&\le C\sum_{j=1}^{j_0} (2^j\delta)^{\alpha(x)}\Phi^{-1}\left(x, \omega(x,2^j\delta)^{-1}\right) \\
&\le C\delta^{\alpha(x)}\Phi^{-1}(x,\omega(x,\delta)^{-1})
\end{aligned}$$

as in the proof of [29, Lemma 4.2].

For I_2, it follows from ($\Phi\omega\alpha$), (4.1), (ω1) and (ω2) that

$$\begin{aligned}
I_2 &\le C\delta^{\alpha(x)}\Phi^{-1}(x,\omega(x,\delta)^{-1}) \int_{G\setminus B(x,\delta)} \frac{\omega(x,|x-y|)}{|B(x,|x-y|)|}\Phi(y,f(y))\,dy \\
&\le C\delta^{\alpha(x)}\Phi^{-1}(x,\omega(x,\delta)^{-1}) \sum_{j=1}^{j_0} \frac{\omega(x,2^j\delta)}{|B(x,2^j\delta)|}\int_{G\cap B(x,2^j\delta)} \Phi(y,f(y))\,dy \\
&\le C\delta^{\alpha(x)}\Phi^{-1}(x,\omega(x,\delta)^{-1})
\end{aligned}$$

as in the proof of [29, Lemma 4.2]. Thus, the present lemma is proved. □

To state our main theorem, we consider a function

$$\Psi(x,t) : G \times [0,\infty) \to [0,\infty)$$

that satisfies (Φ1) – (Φ3) and

($\Psi\Phi$) there exists a constant $A' \geq 1$ such that

$$\Psi\left(x, t\left(\omega^{-1}\left(x, \Phi(x,t)^{-1}\right)\right)^{\alpha(x)}\right) \leq A'\Phi(x,t)$$

for all $x \in G$ and $t \geq 1$.

Remark 4.4 In [26], we considered the condition like ($\Psi\Phi$) for Musielak–Orlicz spaces.

We give a Sobolev-type inequality for $I_{\alpha(\cdot)}f$ of functions in $\mathcal{L}^{\Phi,\omega}(G)$ by Theorem 3.4, as an extension of [32, Theorem 4.4].

Theorem 4.5 *Suppose* $\Phi(x,t)$ *satisfies* ($\Phi3; 0; p$), ($\Phi3; \infty; q$) *and* ($\Phi5; \nu$) *for* $p > 1$, $q > 1$ *and* $\nu > 0$ *satisfying* $\nu \leq q/\omega_0$. *Assume that* ($\omega1'$) *and* ($\Phi\omega\alpha$) *hold. Then there exists a constant* $C > 0$ *such that*

$$\|I_{\alpha(\cdot)}f\|_{\mathcal{L}^{\Psi,\omega}(G)} \leq C\|f\|_{\mathcal{L}^{\Phi,\omega}(G)}$$

for all $f \in \mathcal{L}^{\Phi,\omega}(G)$.

Proof Let f be a nonnegative measurable function on G such that $\|f\|_{\mathcal{L}^{\Phi,\omega}(G)} \leq 1$. We may assume that

$$\sup_{z\in G}\int_0^{2d_G} \frac{\omega(z,r)}{|B(z,r)|}\left(\int_{G\cap B(z,r)} \Phi\left(x, Mf(x)\right)\,dx\right)\frac{dr}{r} \leq 1 \tag{4.2}$$

by Theorem 3.4. Let $x \in G$ and $0 < \delta < d_G/2$. By Lemma 4.3, we find

$$\begin{aligned} I_{\alpha(\cdot)}f(x) &= \int_{G\cap B(x,\delta)} |x-y|^{\alpha(x)-N} f(y)\,dy + \int_{G\setminus B(x,\delta)} |x-y|^{\alpha(x)-N} f(y)\,dy \\ &\leq C\left\{\delta^{\alpha(x)}Mf(x) + \delta^{\alpha(x)}\Phi^{-1}(x, \omega(x,\delta)^{-1})\right\}. \end{aligned}$$

If $\omega^{-1}\left(x, \Phi(x, Mf(x))^{-1}\right) \geq d_G/2$, then, taking $\delta = d_G/2$, we have $I_{\alpha(\cdot)}f(x) \leq C$ by Lemma 4.1, (ω1) and ($\omega3; \omega_0$). If $\omega^{-1}\left(x, \Phi(x, Mf(x))^{-1}\right) < d_G/2$, then take $\delta = \omega^{-1}\left(x, \Phi(x, Mf(x))^{-1}\right)$. Then we have

$$I_{\alpha(\cdot)}f(x) \leq CMf(x)\left(\omega^{-1}\left(x, \Phi(x, Mf(x))^{-1}\right)\right)^{\alpha(x)}$$

by Lemma 4.1. Therefore, we obtain

$$I_{\alpha(\cdot)}f(x) \le C_1' \max\left\{ Mf(x)\left(\omega^{-1}\left(x, \Phi(x, Mf(x))^{-1}\right)\right)^{\alpha(x)}, 1\right\},$$

so that by $(\Psi\Phi)$, we have

$$\begin{aligned}\Psi\left(x, I_{\alpha(\cdot)}f(x)/C_1'\right) &\le C\left\{\Psi\left(x, Mf(x)\left(\omega^{-1}\left(x, \Phi(x, Mf(x))^{-1}\right)\right)^{\alpha(x)}\right) + 1\right\} \\ &\le C\left\{\Phi\big(x, Mf(x)\big) + 1\right\}.\end{aligned}$$

Hence, it follows from (4.2) and (3.6) that

$$\begin{aligned}&\int_0^{2d_G} \frac{\omega(z,r)}{|B(z,r)|}\left(\int_{G\cap B(z,r)} \Psi\left(x, I_{\alpha(\cdot)}f(x)/C_1'\right) dx\right)\frac{dr}{r} \\ &\le C\left\{\int_0^{2d_G} \frac{\omega(z,r)}{|B(z,r)|}\left(\int_{G\cap B(z,r)} \Phi\left(x, Mf(x)\right) dx\right)\frac{dr}{r} + \int_0^{2d_G} \omega(z,r)\frac{dr}{r}\right\} \\ &\le C\end{aligned}$$

for all $z \in G$. Thus, we complete the proof. □

Remark 4.6 When $\Phi(x,t) = t^{p(x)}$, Theorem 4.5 was proved in [32, Theorem 4.4].

Remark 4.7 Let $\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t) = t^{p(x)}\prod_{j=1}^{k}\big(L_e^{(j)}(t)\big)^{q_j(x)}$ and $\omega(x,r) = r^{\sigma(x)}L_e(1/r)^{\beta(x)}$.

Set

$$\Psi(x,t) = \left[\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t)\right]^{p^*(x)/p(x)} L_e(t)^{p^*(x)\alpha(x)\beta(x)/\sigma(x)},$$

where $1/p^*(x) = 1/p(x) - \alpha(x)/\sigma(x)$. Then $\Psi(x,t)$ satisfies condition $(\Psi\Phi)$ (see [31, Remark 3.14]).

5 Double Phase Functions with Variable Exponents

In this section, let

$$\omega(x,r) = r^{\sigma(x)}L_e(1/r)^{\beta(x)}$$

be as in Example 2.1 (Remark 3.2) and let $p(\cdot)$ and $q(\cdot)$ be real valued measurable functions on G such that

(P1) $1 \le p^- \le p^+ < \infty$,
(Q1) $1 \le q^- \le q^+ < \infty$.

We assume that

(P2) $p(\cdot)$ is log-Hölder continuous, that is,

$$|p(x) - p(y)| \le \frac{C_p}{L_e(1/|x-y|)} \quad (x,\ y \in G)$$

with a constant $C_p \ge 0$, and

(Q2) $q(\cdot)$ is log-Hölder continuous, that is,

$$|q(x) - q(y)| \le \frac{C_q}{L_e(1/|x-y|)} \quad (x,\ y \in G)$$

with a constant $C_q \ge 0$.

As an example and application, we consider the case where $\Phi(x,t)$ is a double phase function with variable exponents given by

$$\Phi(x,t) = t^{p(x)} + a(x)t^{q(x)},\ x \in G,\ t \ge 0,$$

where $p(x) < q(x)$ for $x \in G$, $a(\cdot)$ is nonnegative, bounded and Hölder continuous of order $\theta \in (0,1]$ (cf. [20, 33]).

This $\Phi(x,t)$ satisfies (Φ1), (Φ2), ($\Phi3; 0; p^-$) and ($\Phi3; \infty; p^-$). $\Phi(x,t)$ also satisfies ($\Phi5; \nu$) for $\nu \ge \sup_{x \in G_0}(q(x) - p(x))/\theta$; see [20, Lemma 5.1].

Let $G_0 = \{x \in G : a(x) > 0\}$.

In view of Theorem 3.4, we have the boundedness of the maximal operator on $\mathcal{L}^{\Phi,\omega}(G)$ in the framework of double phase functions Φ.

Theorem 5.1 *If $p^- > 1$ and $\sup_{x \in G_0}(q(x) - p(x))/\theta \le p^-/\omega_0$, then there exists a constant $C > 0$ such that*

$$\|Mf\|_{\mathcal{L}^{\Phi,\omega}(G)} \le C\|f\|_{\mathcal{L}^{\Phi,\omega}(G)}$$

for all $f \in \mathcal{L}^{\Phi,\omega}(G)$.

Let $p^*(x)$ and $q^*(x)$ be defined by

$$\frac{1}{p^*(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{\sigma(x)}$$

when $1/p(x) - \alpha(x)/\sigma(x) > 0$, and

$$\frac{1}{q^*(x)} = \frac{1}{q(x)} - \frac{\alpha(x)}{\sigma(x)}$$

when $1/q(x) - \alpha(x)/\sigma(x) > 0$. In this section, set

$$\Psi(x,t) = t^{p^*(x)} L_e(t)^{\alpha(x)p^*(x)\beta(x)/\sigma(x)} + \left(a(x)^{1/q(x)} t\right)^{q^*(x)} L_e\left(a(x)^{1/q(x)} t\right)^{\alpha(x)q^*(x)\beta(x)/\sigma(x)}$$

for $x \in G$ and $t \ge 0$.

Lemma 5.2 *([20, Lemma 5.6 (1), (3)])*

(1) *If* $\inf_{x\in G_0}(\sigma(x)/q(x) - \alpha(x)) > 0$ *and* $\inf_{x\in G\setminus G_0}(\sigma(x)/p(x) - \alpha(x)) > 0$, *then* $(\Phi\omega\alpha)$ *holds.*
(2) $\Psi(x,t)$ *satisfies* $(\Psi\Phi)$.

Finally, by Lemma 5.2 and Theorem 4.5, we obtain a Sobolev inequality in our setting.

Theorem 5.3 *If* $p^- > 1$, $\inf_{x\in G_0}(\sigma(x)/q(x) - \alpha(x)) > 0$, $\inf_{x\in G\setminus G_0}(\sigma(x)/p(x) - \alpha(x)) > 0$ *and* $\sup_{x\in G_0}(q(x)-p(x))/\theta \le p^-/\omega_0$, *then there exists a constant* $C > 0$ *such that*

$$\|I_{\alpha(\cdot)} f\|_{\mathcal{L}^{\Psi,\omega}(G)} \le C\|f\|_{\mathcal{L}^{\Phi,\omega}(G)}$$

for all $f \in \mathcal{L}^{\Phi,\omega}(G)$.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Ahmida, Y., Chlebicka, I., Gwiazda, P., Youssfi, A.: Gossez's approximation theorems in Musielak-Orlicz-Sobolev spaces. J. Funct. Anal. **275**(9), 2538–2571 (2018)
2. Almeida, A., Hasanov, J., Samko, S.: Maximal and potential operators in variable exponent Morrey spaces. Georgian Math. J. **15**, 195–208 (2008)
3. Baroni, P., Colombo, M., Mingione, G.: Regularity for general functionals with double phase. Calc. Var. Partial Differ. Equs. **57**(2), 62 (2018)
4. Baroni, P., Colombo, M., Mingione, G.: Non-autonomous functionals, borderline cases and related function classes. St Petersburg Math. J. **27**, 347–379 (2016)
5. Burenkov, V.I., Guliyev, H.V.: Necessary and sufficient conditions for boundedness of the maximal operator in local Morrey-type spaces. Studia Math. **163**(2), 157–176 (2004)
6. Byun, S.S., Lee, H.S.: Calderón-Zygmund estimates for elliptic double phase problems with variable exponents. J. Math. Anal. Appl. **501**, 124015 (2021)
7. Capone, C., Cruz-Uribe, D., Fiorenza, A.: The fractional maximal operator and fractional integrals on variable L^p spaces. Rev. Mat. Iberoamericana **23**(3), 743–770 (2007)
8. Colombo, M., Mingione, G.: Regularity for double phase variational problems. Arch. Ration. Mech. Anal. **215**(2), 443–496 (2015)
9. Diening, L.: Riesz potentials and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$. Math. Nachr. **263**(1), 31–43 (2004)
10. De Filippis, C., Mingione, G.: Lipschitz bounds and nonautonomous integrals. Arch. Ration. Mech. Anal. **242**, 973–1057 (2021)
11. Futamura, T., Mizuta, Y., Shimomura, T.: Sobolev embeddings for Riesz potential space of variable exponent. Math. Nachr. **279**(13–14), 1463–1473 (2006)
12. Futamura, T., Mizuta, Y., Shimomura, T.: Integrability of maximal functions and Riesz potentials in Orlicz spaces of variable exponent. J. Math. Anal. Appl. **366**, 391–417 (2010)
13. Guliyev, V.S., Hasanov, J., Samko, S.: Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces. Math. Scand. **107**, 285–304 (2010)

14. Guliyev, V.S., Hasanov, J., Samko, S.: Boundedness of the maximal potential and Singular integral operators in the generalized variable exponent Morrey type spaces. J. Math. Sci. **170**(4), 423–443 (2010)
15. Harjulehto, P., Hästö, P.: Orlicz spaces and generalized Orlicz spaces. Lecture Notes in Mathematics, vol. 2236. Springer, Cham (2019)
16. Hästö, P., Ok, J.: Maximal regularity for local minimizers of non-autonomous functionals. J. Eur. Math. Soc. **24**(4), 1285–1334 (2022)
17. Hedberg, L.I.: On certain convolution inequalities. Proc. Amer. Math. Soc. **36**, 505–510 (1972)
18. Krasnoesl'skii, M.A., Rutickii, Ya.B.: Convex Functions and Orlicz Spaces. P. Noordhoff Ltd., Groningen (1961)
19. Maeda, F.-Y., Mizuta, Y., Ohno, T., Shimomura, T.: Boundedness of maximal operators and Sobolev's inequality on Musielak-Orlicz-Morrey spaces. Bull. Sci. Math. **137**, 76–96 (2013)
20. Maeda, F.-Y., Mizuta Y., Ohno T., Shimomura T.: Sobolev's inequality for double phase functionals with variable exponents, Forum Math. **31** 517–527 (2019)
21. Maeda, F.-Y., Ohno, T., Shimomura, T.: Boundedness of the maximal operator on Musielak-Orlicz-Morrey spaces. Tohoku Math. J. **69**, 483–495 (2017)
22. Mizuta, Y., Nakai, E., Ohno, T., Shimomura, T.: Riesz potentials and Sobolev embeddings on Morrey spaces of variable exponent. Complex Vari. Elliptic Equ. **56**(7–9), 671–695 (2011)
23. Mizuta, Y., Nakai, E., Ohno, T., Shimomura, T.: Maximal functions, Riesz potentials and Sobolev embeddings on Musielak-Orlicz-Morrey spaces of variable exponent in$\mathbf{R}^n$. Rev. Mat. Complut. **25**(2), 413–434 (2012)
24. Mizuta Y., Nakai E., Ohno T., Shimomura T.: Campanato-Morrey spaces for the double phase functionals with variable exponents, Nonlinear Anal. **197**, 111827, 19 (2020)
25. Mizuta, Y., Ohno, T., Shimomura, T.: Sobolev's inequalities and vanishing integrability for Riesz potentials of functions in the generalized Lebesgue space $L^{p(\cdot)}(\log L)^{q(\cdot)}$. J. Math. Anal. Appl. **345**, 70–85 (2008)
26. Mizuta, Y., Ohno, T., Shimomura, T.: Sobolev inequalities for Musielak-Orlicz spaces. Manuscripta Math. **155**, 209–227 (2018)
27. Mizuta, Y., Ohno, T., Shimomura, T.: Sobolev's theorem for double phase functionals. Math. Ineq. Appl. **23**, 17–33 (2020)
28. Mizuta, Y., Shimomura, T.: Sobolev embeddings for Riesz potentials of functions in Morrey spaces of variable exponent. J. Math. Soc. Japan **60**, 583–602 (2008)
29. Mizuta, Y., Shimomura, T.: Sobolev's inequality for Riesz potentials of functions in Morrey spaces of integral form. Math. Nachr. **283**(9), 1336–1352 (2010)
30. Mizuta, Y., Shimomura, T.: Hardy-Sobolev inequalities in the unit ball for double phase functionals, J. Math. Anal. Appl. **501** 124133, 17 (2021)
31. Ohno, T., Shimomura, T.: Sobolev's inequality for Musielak-Orlicz-Morrey spaces over metric measure spaces. J. Aust. Math. Soc. **110**, 371–385 (2021)
32. Ohno, T., Shimomura, T.: Sobolev-type inequalities on variable exponent Morrey spaces of an integral form. Ricerche Mat. **71**, 189–204 (2022)
33. Ragusa, M.A., Tachikawa, A.: Regularity for minimizers for functionals of double phase with variable exponents. Adv. Nonlinear Anal. **9**(1), 710–728 (2020)