Bull. Malays. Math. Sci. Soc. (2023) 46:48
https://doi.org/10.1007/s40840-022-01423-9

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Subordination of Cesàro Means of Convex Functions

Manju Yadav[1] · **Sushma Gupta**[1] · **Sukhjit Singh**[1]

Received: 16 June 2022 / Revised: 28 September 2022 / Accepted: 14 October 2022 /
Published online: 10 December 2022

Abstract

For all $n \in \mathbb{N}$, we find sharp constants $\mu_n^{(2)}$ and $\mu_n^{(3)}$ such that $\mu_n^{(2)}\sigma_n^{(2)}(z, f) \prec f(z)$ and $\mu_n^{(3)}\sigma_n^{(3)}(z, f) \prec f(z)$ in the open unit disc $\mathbb{D}$ for all f- normalized convex univalent functions in $\mathbb{D}$. Here $\sigma_n^{(\alpha)}(z, f)$ stands for nth Cesàro mean of order α, $\alpha \geq 0$, of $f(z) = \sum_{k=1}^{\infty} a_k z^k$ defined by $\sigma_n^{(\alpha)}(z, f) := \binom{n+\alpha-1}{n-1}^{-1} \sum_{k=1}^{n} \binom{n+\alpha-k}{n-k} a_k z^k$ and the symbol $'\prec'$ stands for subordination between two analytic functions. Among other things, a generalization of an earlier known result related to subordination is also presented.

Keywords Cesàro means · De la Vallée Poussin means · Subordination · Hadamard product

Mathematics Subject Classification 30C45 · 30C80

1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane and $\mathcal{S}$ the class of functions $f(z) = z + a_2 z^2 + \ldots$ which are analytic and univalent in $\mathbb{D}$. Denote by $\mathcal{S}^*$ and $\mathcal{K}$ the usual subclasses of $\mathcal{S}$ consisting of functions which map $\mathbb{D}$ onto starlike (w.r.t. the origin) and convex domains, respectively. Let $\mathcal{S}^*(1/2) \subset \mathcal{S}$ be the

Communicated by V. Ravichandran.

✉ Manju Yadav
manjuyadav1197@gmail.com

Sushma Gupta
sushmagupta1@yahoo.com

Sukhjit Singh
sukhjit_d@yahoo.com

[1] Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal, Punjab 148106, India

class of functions which are starlike of order 1/2. It is known that $\mathcal{K} \subset \mathcal{S}^*(1/2)$. Further, by $\mathcal{C}$ we denote the class of functions $f \in \mathcal{S}$ for which there exist $g \in \mathcal{K}$ such that $\Re\left\{f'(z)/g'(z)\right\} > 0$, $z \in \mathbb{D}$. Functions in the class $\mathcal{C}$ are called close-to-convex.

The convolution or Hadamard product of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is denoted by $(f * g)(z)$ and is defined as

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n.$$

A function f is said to be subordinate to a function g (in symbols $f(z) \prec g(z)$) in $|z| < r$ $(0 < r < 1)$ if g is univalent in $|z| < r$, $f(0) = g(0)$ and $f(|z| < r) \subset g(|z| < r)$.

For a given function $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $n \in \mathbb{N}$, $\alpha \geq 0$, let

$$s_n(z, f) = \sum_{k=1}^{n} a_k z^k,$$

$$\sigma_n^{(\alpha)}(z, f) = \binom{n+\alpha-1}{n-1}^{-1} \sum_{k=1}^{n} \binom{n+\alpha-k}{n-k} a_k z^k$$

and

$$V_n(z, f) = \binom{2n}{n}^{-1} \sum_{k=1}^{n} \binom{2n}{n+k} a_k z^k$$

denote, the nth partial sum, the nth Cesàro mean of order α and the nth de la Vallée Poussin mean of f, respectively.

Pólya and Schoenberg ([7], Theorem 2, p.298) showed that the de la Vallée Poussin means $V_n(z, f)$ are convex (starlike) if and only if f is convex (starlike) and also, established the following fascinating result : □

Theorem 1.1 *If $f \in \mathcal{K}$, then $V_n(z, f) \prec f(z)$ in $\mathbb{D}$.*

Robertson [8] further extended this result by proving that if f is univalent in $\mathbb{D}$, then converse of Theorem 1.1 is also true.

It has been a long tradition to study mapping properties of Cesàro means (for example, see [4], [3], [9], [2], [6], [11], [12]). In 1995, Singh and Singh [14] proved the following analogues of above theorem of Pólya and Schoenberg for certain transformations of the nth partial sum, $s_n(z, f)$, and the nth Cesàro mean of first order, $\sigma_n^{(1)}(z, f)$, of $f \in \mathcal{K}$.

Theorem 1.2 *If $f \in \mathcal{K}$, then*
(i) $(1/z)\int_0^z s_n(t, f)dt \prec f(z)$ in $\mathbb{D}$ for every $n \in \mathbb{N}$.
(ii) $(n/(n+1))\sigma_n^{(1)}(z, f) \prec f(z)$ in $\mathbb{D}$. This result is sharp for every $n \in \mathbb{N}$.

□

In the same paper, i.e., [14], Singh and Singh also proved the following result :

Theorem 1.3 *For every* $f \in \mathscr{S}^*(1/2)$ *and for every positive integer n, we have*

$$\Re \frac{V_n(z,f)}{\sigma_n^{(2)}(z,f)} > 0, \quad z \in \mathbb{D}.$$

□

In the present note, we establish the analogues of Theorem 1.1 for certain transformations of the nth Cesàro means $\sigma_n^{(2)}(z,f)$ and $\sigma_n^{(3)}(z,f)$, of $f \in \mathscr{K}$. We also prove an analogue of Theorem 1.3 by replacing $\sigma_n^{(2)}(z,f)$ there with $\sigma_n^{(3)}(z,f)$. A generalization of a result of Singh and Singh [13] related to subordination between $V_2(z,f)$ and $\sigma_2^{(1)}(z,f)$ of $f \in \mathscr{K}$ is also presented.

2 Preliminaries

In this section, we collect following definition and results which shall be needed to prove our results in this paper.

Definition 2.1 A sequence $\{b_n\}_1^\infty$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z) = \sum_{n=1}^\infty a_n z^n$, $a_1 = 1$, is univalent and convex in $\mathbb{D}$, we have

$$\sum_{n=1}^\infty a_n b_n z^n \prec f(z).$$

□

Lemma 2.2 *[15] A sequence* $\{b_n\}_1^\infty$ *of complex numbers is a subordinating factor sequence if and only if*

$$\Re\left[1 + 2\sum_{n=1}^\infty b_n z^n\right] > 0, \quad z \in \mathbb{D}.$$

□

Lemma 2.3 *([5], p.3) If* $h(z) = nz + (n-1)z^2 + \ldots + z^n$, *then*

$$h(e^{i\theta}) = -\frac{n+1}{2} + \frac{\sin^2(n+1)\theta/2}{2\sin^2\theta/2} + \frac{(n+1)\sin\theta - \sin(n+1)\theta}{4\sin^2\theta/2}.$$

□

Lemma 2.4 *[10] Let f and g belong to $\mathscr{S}^*(1/2)$. Then for each function F analytic in $\mathbb{D}$ and satisfying $\Re F(z) > 0, z \in \mathbb{D}$, we have*

$$\Re \frac{f(z) * F(z)g(z)}{f(z) * g(z)} > 0, \quad z \in \mathbb{D}.$$

□

Lemma 2.5 *[1] Suppose that b_0, b_1, b_2 are complex numbers, $b_2 \neq 0$, and let $P(z) = b_0 + b_1 z + b_2 z^2$. Then the zeros of P(z) lie on $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ if, and only if*
(i) $|b_0| \leq |b_2|$ and
(ii) $|b_0\bar{b}_1 - b_1\bar{b}_2| \leq |b_2|^2 - |b_1|^2$.

Lemma 2.6 *[10] Let ϕ and ψ be convex functions in $\mathbb{D}$ and suppose that f is subordinate to ϕ. Then $f * \psi$ is subordinate to $\phi * \psi$ in $\mathbb{D}$.*

□

3 Main Results

Cesàro means not only play an important role in areas like approximation theory, summability, Fourier analysis etc., but also find many applications in geometric function theory. One of the striking properties of the polynomial approximations $\sigma_n^{(\alpha)}(z, f)$ is that they converge to f in the sense of compact convergence as $n \to \infty$. In the following two theorems we establish analogues of Theorem 1.1 for some transformations of $\sigma_n^{(\alpha)}(z, f), \alpha = 2, 3$.

Theorem 3.1 *For all elements $f \in \mathscr{K}$ and for all positive integers n, we have*

$$\frac{n}{n+1}\sigma_n^{(2)}(z, f) \prec f(z) \tag{1}$$

in $\mathbb{D}$. This result is sharp for every n.

□

Proof Let $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ be any member of the class $\mathscr{K}$. Then

$$\frac{n}{n+1}\sigma_n^{(2)}(z, f) = \frac{n}{n+1}z + \frac{n(n-1)}{(n+1)^2}a_2 z^2 + \frac{(n-1)(n-2)}{(n+1)^2}a_3 z^3 + \cdots + \frac{2.1}{(n+1)^2}a_n z^n.$$

Thus, in view of the Definition 2.1, the assertion (1) holds if and only if the sequence

$$\left\langle \frac{n}{n+1}, \frac{n(n-1)}{(n+1)^2}, \frac{(n-1)(n-2)}{(n+1)^2}, \cdots, \frac{2.1}{(n+1)^2}, 0, 0, \cdots \right\rangle$$

is a subordinating factor sequence. By Lemma 2.2, this is equivalent to

$$\Re F(z) > 0, \quad z \in \mathbb{D}.$$

where

$$F(z) = 1 + \frac{2}{(n+1)^2}\left\{(n+1)nz + n(n-1)z^2 + (n-1)(n-2)z^3 + \cdots + 2.1z^n\right\}.$$
$$= 1 + \frac{2}{(n+1)^2}F_n(z)\ (say). \tag{2}$$

Here

$$F_n(z) = (n+1)nz + n(n-1)z^2 + (n-1)(n-2)z^3 + \cdots + 2.1z^n, \tag{3}$$

from which we get,

$$(1-z)F_n(z) = (n+1)nz - 2nz^2 - 2(n-1)z^3 - \cdots - 2z^{n+1}.$$
$$= (n+1)nz - 2z\sum_{k=1}^{n}(n-k+1)z^k.$$

It compiles to

$$F_n(z) = n(n+1)\frac{z}{1-z} - 2\frac{z}{1-z}\left[\sum_{k=1}^{n}(n-k+1)z^k\right].$$

Setting $z = e^{i\theta}, 0 \le \theta < 2\pi$ and making use of Lemma 2.3, we get

$$F_n(e^{i\theta}) = n(n+1)\left(\frac{e^{i\theta}}{1-e^{i\theta}}\right) - 2\left(\frac{e^{i\theta}}{1-e^{i\theta}}\right)\cdot\left[-\frac{n+1}{2} + \frac{\sin^2(n+1)\theta/2}{2\sin^2\theta/2} + i\frac{(n+1)\sin\theta - \sin(n+1)\theta}{4\sin^2\theta/2}\right]. \tag{4}$$

On substituting $\frac{e^{i\theta}}{1-e^{i\theta}} = \left(\frac{-1}{2} + \frac{i}{2}\cot\theta/2\right)$, we get

$$F_n(e^{i\theta}) = n(n+1)\left(\frac{-1}{2} + \frac{i}{2}\cot\theta/2\right) - 2\left(\frac{-1}{2} + \frac{i}{2}\cot\theta/2\right)\cdot\left[-\frac{n+1}{2} + \frac{\sin^2(n+1)\theta/2}{2\sin^2\theta/2} + i\frac{(n+1)\sin\theta - \sin(n+1)\theta}{4\sin^2\theta/2}\right]. \tag{5}$$

From (2), (3) and (5), we obtain

$$\Re(F(e^{i\theta})) = 1 + \frac{2}{(n+1)^2}\left\{\frac{-(n+1)^2}{2} + \left[\frac{\sin^2(n+1)\theta/2}{2\sin^2\theta/2} + \frac{(n+1)\sin\theta - \sin(n+1)\theta}{4\sin^2\theta/2}\cot\theta/2\right]\right\}.$$

or

$$\Re(F(e^{i\theta})) = \frac{2}{(n+1)^2}\left[\frac{\sin^2(n+1)\theta/2}{2\sin^2\theta/2} + \frac{(n+1)\sin\theta - \sin(n+1)\theta}{4\sin^2\theta/2}\cot\theta/2\right].$$

We note that $(n+1)\sin\theta - \sin(n+1)\theta$ and $\cot\theta/2$ are both non-negative for $\theta \in [0,\pi]$ and are both negative for $\theta \in (\pi, 2\pi)$. Hence,

$$\Re(F(z)) > 0 \ \text{ for all } \ \theta \in [0, 2\pi).$$

In order to prove the sharpness of our result, we consider the function $h(z) = z/(1-z)$ which is a member of the class $\mathscr{K}$. We have

$$\begin{aligned}\sigma_n^{(2)}(z,h) &= z + \frac{n(n-1)}{n(n+1)}z^2 + \frac{(n-1)(n-2)}{n(n+1)}z^3 + \cdots + \frac{2.1}{n(n+1)}z^n.\\ &= \frac{1}{n(n+1)}F_n(z).\end{aligned}$$

In view of (5), taking $z = e^{i\theta}$, $0 \le \theta < 2\pi$, we get

$$\begin{aligned}\sigma_n^{(2)}(e^{i\theta},h) = \frac{1}{n(n+1)}&\left\{n(n+1)\left(\frac{-1}{2} + \frac{i}{2}\cot\theta/2\right) - 2\left(\frac{-1}{2} + \frac{i}{2}\cot\theta/2\right)\cdot\right.\\ &\left.\cdot\left[-\frac{n+1}{2} + \frac{\sin^2(n+1)\theta/2}{2\sin^2\theta/2} + i\frac{(n+1)\sin\theta - \sin(n+1)\theta}{4\sin^2\theta/2}\right]\right\}.\end{aligned}$$

Taking $\theta = 2\pi/(n+1)$, for any positive real number ρ, we have

$$\begin{aligned}\Re\rho\,\sigma_n^{(2)}(e^{i\theta},h) &= -\frac{\rho}{2n}\left\{(n+1) + \left(1 - \frac{1}{\sin^2(\frac{\pi}{n+1})}\right)\right\}\\ &\ge -\frac{\rho(n+1)}{2n},\end{aligned}$$

because for all $n \in \mathbb{N}$,

$$1 - \frac{1}{\sin^2(\pi/(n+1))} \le 0.$$

It follows that if $\rho > n/(n+1)$, then $\Re\rho\sigma_n^{(2)}(z,h) < -1/2$ and hence $\rho\sigma_n^{(2)}(z,h)$ will not be subordinate to h in $\mathbb{D}$ as h maps $\mathbb{D}$ onto the right half plane $\Re w > -1/2$. This completes the proof.

For the sake of illustration, graphs of $h(\partial\mathbb{D})$ and $(n/(n+1))\sigma_n^{(2)}(\partial\mathbb{D},h)$, $n = 2, 3, 4$ are plotted in Fig. 1, and Fig. 2 is an enlargement of the critical portion of Fig. 1. □

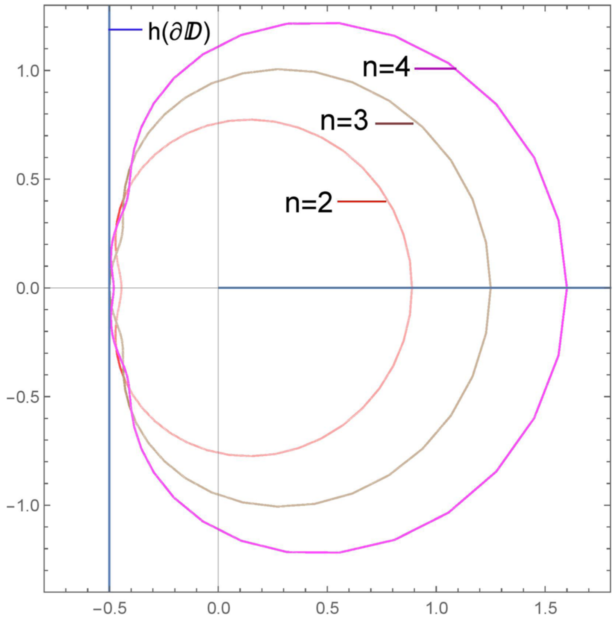

Fig. 1 Graphs of $h(\partial\mathbb{D})$ and $(n/(n+1))\sigma_n^{(2)}(\partial\mathbb{D}, h)$, for $n = 2, 3, 4$

Theorem 3.2 *For all $f \in \mathcal{K}$ and for all positive integers n, we have*

$$\frac{2n(n+2)}{(2n+1)(n+3)}\sigma_n^{(3)}(z, f) \prec f(z) \tag{6}$$

in $\mathbb{D}$. This result is sharp for every n.

□

Proof For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we have

$$\frac{2n(n+2)}{(2n+1)(n+3)}\sigma_n^{(3)}(z, f) = \frac{2}{(2n+1)(n+3)(n+1)} \Big[(n+2)(n+1)nz + (n+1)n(n-1)z^2 + n(n-1)(n-2)z^3 + \ldots + 3.2.1z^n\Big].$$

The assertion (6) will hold, if

$$\left\langle \frac{2n(n+1)(n+2)}{(2n+1)(n+3)(n+1)}, \frac{2n(n+1)(n-1)}{(2n+1)(n+3)(n+1)}, \frac{2n(n-1)(n-2)}{(2n+1)(n+1)(n+3)}, \cdots, \frac{3.2.1}{(2n+1)(n+1)(n+3)}, 0, 0, \cdots \right\rangle$$

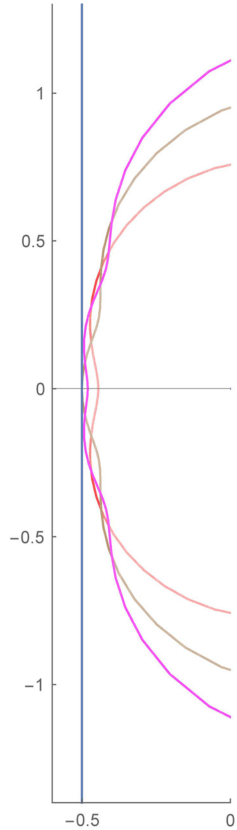

Fig. 2 Enlargement of the critical portion of Fig.1.

is a subordinating factor sequence. By Lemma 2.2, it is equivalent to show that

$$\Re G(z) > 0$$

for all z in $\mathbb{D}$, where

$$G(z) = 1 + \frac{4}{(2n+1)(n+3)(n+1)} \Big\{ (n+2)(n+1)nz + (n+1)n(n-1)z^2 \\ +(n+1)n(n-1)z^2 + n(n-1)(n-2)z^3 + \ldots + 3.2.1z^n \Big\}. \tag{7}$$

If we write

$$G_n(z) = (n+2)(n+1)nz + (n+1)n(n-1)z^2 + n(n-1)(n-2)z^3 + \cdots + 3.2.1z^n,$$

then

$$(1-z)G_n(z) = (n+2)(n+1)nz - 3(n+1)nz^2 - 3n(n-1)z^3 - \ldots - 3.2.1z^{n+1}$$
$$= (n+2)(n+1)nz - 3z\left[(n+1)nz + n(n-1)z^2 + \ldots + 2.1z^n\right].$$

Thus, we obtain

$$G(z) = 1 + \frac{4}{(n+1)(n+3)(2n+1)}\{n(n+1)(n+2)\frac{z}{1-z} - 3\frac{z}{1-z}F_n(z)\},$$

where F_n is as in (3). Setting $z = e^{i\theta},\ 0 \le \theta < 2\pi$ and using (4), we get

$$G(e^{i\theta}) = 1 + \frac{4}{(n+1)(n+3)(2n+1)}\left\{n(n+1)(n+2)\frac{e^{i\theta}}{1-e^{i\theta}}\right.$$
$$-3\frac{e^{i\theta}}{1-e^{i\theta}}\left\{n(n+1)\left(\frac{e^{i\theta}}{1-e^{i\theta}}\right)\right.$$
$$\left.\left.-2\left(\frac{e^{i\theta}}{1-e^{i\theta}}\right)\left[-\frac{n+1}{2} + \frac{\sin^2(n+1)\theta/2}{2\sin^2\theta/2} + i\frac{(n+1)\sin\theta - \sin(n+1)\theta}{4\sin^2\theta/2}\right]\right\}\right\}.$$

On substituting $\frac{e^{i\theta}}{1-e^{i\theta}} = \left(-\frac{1}{2} + \frac{i}{2}\cot\theta/2\right)$ and $(\frac{e^{i\theta}}{1-e^{i\theta}})^2 = \frac{-e^{i\theta}}{4\sin^2\theta/2}$, we get

$$G(e^{i\theta}) = 1 + \frac{4}{(n+1)(n+3)(2n+1)}\left\{n(n+1)(n+2)\left(\frac{-1}{2} + \frac{i}{2}\cot\theta/2\right)\right.$$
$$+3(n+1)^2\left(\frac{e^{i\theta}}{4\sin^2\theta/2}\right)$$
$$\left.-6\left(\frac{e^{i\theta}}{4\sin^2\theta/2}\right)\frac{\sin^2(n+1)\theta/2}{2\sin^2\theta/2} - 6\left(i\frac{e^{i\theta}}{4\sin^2\theta/2}\right)\frac{(n+1)\sin\theta - \sin(n+1)\theta}{4\sin^2\theta/2}\right\}.$$

Writing the real part, we have

$$\Re G(e^{i\theta}) = 1 + \frac{4}{(n+1)(n+3)(2n+1)}\left\{\frac{-n(n+1)(n+2)}{2} + \frac{3}{4}(n+1)^2\frac{\cos\theta}{\sin^2\theta/2}\right.$$
$$\left.-\frac{6}{8}\frac{\cos\theta\sin^2(n+1)\theta/2}{\sin^4\theta/2} + \frac{6}{16}\frac{\sin\theta[(n+1)\sin\theta - \sin(n+1)\theta]}{\sin^4\theta/2}\right\}.$$

Writing $\sin^2(n+1)\theta/2 = (1-\cos(n+1)\theta)/2$ and regrouping the terms, we have

$$\Re G(e^{i\theta}) = 1 + \frac{4}{(n+1)(n+3)(2n+1)}\left\{\frac{-n(n+1)(n+2)}{2} + \frac{3}{4}(n+1)^2(-1+\cot^2\theta/2)\right.$$
$$\left.+\frac{6}{16}\frac{\cos(n+2)\theta}{\sin^4\theta/2} - \frac{6}{16}\frac{\cos\theta}{\sin^4\theta/2} + \frac{6}{4}(n+1)\cot^2\theta/2\right\},$$

i.e.,

$$\Re G(e^{i\theta}) = 1 + \frac{4}{(n+1)(n+3)(2n+1)} \left\{ \frac{-(n+1)(n+3)(2n+1)}{4} + \frac{3}{4}(n+1)(n+3)\cot^2\frac{\theta}{2} + \frac{3}{8}\frac{\cos(n+2)\theta - \cos\theta}{\sin^4\theta/2} \right\},$$

or

$$\Re G(e^{i\theta}) = 1 + \frac{4}{(n+1)(n+3)(2n+1)} \left\{ \frac{-(n+1)(2n+1)(n+3)}{4} + \frac{3}{2}\cot^2\frac{\theta}{2}\left(\frac{(n+3)(n+1)}{2} + \frac{\cos(n+2)\theta - \cos\theta}{\sin^2\theta}\right) \right\},$$

which further simplifies to

$$\Re G(e^{i\theta}) = \frac{4}{(n+1)(n+3)(2n+1)} \left\{ \frac{3}{2}\cot^2\frac{\theta}{2}\left(\frac{(n+3)(n+1)}{2} + 2\frac{\cos^2(n+2)\theta/2 + \sin^2\theta/2}{\sin^2\theta}\right) \right\}.$$

All terms on the right-hand side of above expression are positive. So, we have

$$\Re G(z) > 0$$

for all z in $\mathbb{D}$.

Thus, the desired result holds.

To prove the sharpness of our result, we consider the function $h(z) = z/(1-z)$ which is a member of the class $\mathscr{K}$. We have

$$\sigma_n^{(3)}(z,h) = z + \frac{(n+1)n(n-1)}{n(n+1)(n+2)}z^2 + \frac{n(n-1)(n-2)}{n(n+1)(n+2)}z^3 + \cdots \frac{3.2.1}{n(n+1)(n+2)}z^n.$$

Setting $z = e^{i\theta}$, we get

$$\sigma_n^{(3)}(e^{i\theta},h) = \frac{1}{n(n+1)(n+2)}[(n+2)(n+1)ne^{i\theta} + (n+1)n(n-1)e^{2i\theta} + n(n-1)(n-2)e^{3i\theta} + \cdots 3.2.1e^{ni\theta}].$$

For a positive real number ρ, we have

$$\rho\sigma_n^{(3)}(e^{i\theta},h) = \frac{\rho}{n(n+1)(n+2)} \left\{ n(n+1)(n+2)\frac{e^{i\theta}}{1-e^{i\theta}} - 3\frac{e^{i\theta}}{1-e^{i\theta}} \left\{ (n+1)^2\frac{e^{i\theta}}{1-e^{i\theta}} - 2\frac{e^{i\theta}}{1-e^{i\theta}} \left(\frac{\sin^2(n+1)\theta/2}{2\sin^2\theta/2} + i\frac{(n+1)\sin\theta - \sin(n+1)\theta}{4\sin^2\theta/2} \right) \right\} \right\}.$$

Simplifying, we obtain

$$\Re\rho\sigma_n^{(3)}(e^{i\theta},h) = \frac{\rho}{n(n+1)(n+2)}\left\{\frac{-(n+1)(n+3)(2n+1)}{4} + \frac{3}{4}(n+1)(n+3)\cot^2\frac{\theta}{2} + \frac{3}{8}\frac{\cos(n+2)\theta - \cos\theta}{\sin^4\theta/2}\right\}.$$

For $\theta = \pi$, we get

$$\Re\rho\,\sigma_n^{(3)}(e^{i\theta},h) = \frac{\rho}{n(n+1)(n+2)}\left\{\frac{-(n+1)(n+3)(2n+1)}{4} + \frac{3}{8}(1+\cos(n+2)\pi)\right\} \tag{8}$$

$$\geq -\frac{\rho(n+3)(2n+1)}{n(n+2)}. \tag{9}$$

It follows that if $\rho > 2n(n+2)/((n+3)(2n+1))$, then $\Re\rho\sigma_n^{(3)}(z,h) < -1/2$ and hence $\rho\sigma_n^{(3)}(z,h)$ will not be subordinate to h in $\mathbb{D}$ as h maps $\mathbb{D}$ onto the right half plane $\Re w > -1/2$.

This completes the proof.

For visual illustration, graphs of $h(\partial\mathbb{D})$ and $2n(n+2)/((2n+1)(n+3))\sigma_n^{(3)}(\partial\mathbb{D},h)$, $n = 2, 3, 4$, are plotted in Fig. 3, and Fig. 4 is an enlargement of the critical portion of Fig. 3. □

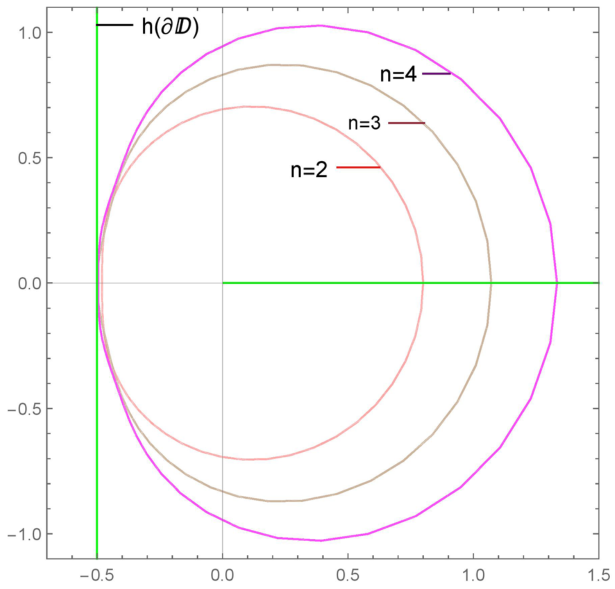

Fig. 3 Graphs of $h(\partial\mathbb{D})$ and $2n(n+2)/((2n+1)(n+3))\sigma_n^{(3)}(\partial\mathbb{D},h)$, $for\ n = 2, 3, 4$

Fig. 4 Enlargement of the critical portion of Fig. 3.

Ruscheweyh [11] proved that $\sigma_n^{(3)}(z, z/(1-z)) \in \mathcal{K} \subset \mathcal{S}^*(1/2)$. Since the class $\mathcal{S}^*(1/2)$ is closed under convolution (see [10], Theorem 3.1), so $\sigma_n^{(3)}(z, f) = \sigma_n^{(3)}(z, z/(1-z)) * f \in \mathcal{S}^*(1/2)$ for every $f \in \mathcal{S}^*(1/2)$. In the next theorem we establish that $\sigma_n^{(2)}(z, f)$ in Theorem 1.3 can be replaced with $\sigma_n^{(3)}(z, f)$.

Theorem 3.3 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be a member of the class* $\mathcal{S}^*(1/2)$. *Then for every positive integer n and each* $z \in \mathbb{D}$, *we have*

$$\Re \frac{V_n(z, f)}{\sigma_n^{(3)}(z, f)} > 0.$$

□

Proof Consider the function

$$\begin{aligned} F_n(z) = (1-z)\Bigg[&\frac{n}{(n+1)} + \frac{n}{(n+1)}z + \frac{n}{(n+3)}z^2 \\ &+\frac{n^2}{(n+3)(n+4)}z^3 + \frac{n^2(n-1)}{(n+3)(n+4)(n+5)}z^4 \\ &+ \ldots + \frac{n^2(n-1)(n-2)(n-3)\ldots 4}{(n+3)(n+4)(n+5)\ldots(2n)}z^{n-1}\Bigg]. \end{aligned} \tag{10}$$

Then

$$\begin{aligned}\sigma_n^{(3)}(z,f)*\frac{z}{1-z}F_n(z) &= \left\{z+\frac{n-1}{n+2}z^2+\frac{(n-1)(n-2)}{(n+2)(n+1)}z^3+\ldots\right.\\ &\quad\left.+\frac{3.2.1}{(n+2)(n+1)n}z^n\right\}\\ &\quad *\left\{\frac{n}{(n+1)}z+\frac{n}{(n+1)}z^2+\frac{n}{(n+3)}z^3\right.\\ &\quad\left.+\frac{n^2}{(n+3)(n+4)}z^4+\ldots+\frac{n^2(n-1)(n-2)(n-3)...4}{(n+3)(n+4)(n+5)...(2n)}z^n\right\}\\ &= \frac{n}{n+1}z+\frac{n(n-1)}{(n+1)(n+2)}z^2+\frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)}z^3\\ &\quad+\ldots\frac{n(n-1)(n-2)\ldots4.3.2.1}{(n+1)(n+2)\ldots(2n)}z^n.\\ &= V_n(z,f).\end{aligned}$$

Also, the function F_n defined above is regular (in fact, an entire function) in $\mathbb{D}$ and can be written in the form

$$\begin{aligned}F_n(z) &= \left[\frac{n}{(n+1)}-\frac{n}{(n+1)}\left(1-\frac{n+1}{n+3}\right)z^2-\frac{n}{(n+3)}\left(1-\frac{n}{n+4}\right)z^3\right.\\ &\quad-\frac{n^2}{(n+3)(n+4)}\left(1-\frac{n-1}{n+5}\right)z^4-\\ &\quad\left....-\frac{n^2(n-1)(n-2)...5}{(n+3)(n+4)...(2n-1)}\left(1-\frac{4}{2n}\right)z^{n-1}-\frac{n^2(n-1)(n-2)...4}{(n+3)(n+4)...(2n)}z^n\right].\end{aligned} \tag{11}$$

In view of (10) and (11), it is clear that in $\mathbb{D}$, we have

$$\Re F_n(z)\geq F_n(|z|)>F_n(1)=0.$$

Taking $f(z)=\sigma_n^{(3)}(z,f)$, $g(z)=z/(1-z)$ and $F(z)=F_n(z)$ in Lemma 2.4, we immediately get

$$\Re\frac{\sigma_n^{(3)}(z,f)*z/(1-z)F_n(z)}{\sigma_n^{(3)}(z,f)*z/(1-z)}=\Re\frac{V_n(z,f)}{\sigma_n^{(3)}(z,f)}>0,\ z\in\mathbb{D}, \tag{12}$$

as $\Re F_n(z)>0$ in $\mathbb{D}$.
This completes the proof. □

It is known (see [6]) that for $\alpha\geq 1$ and $n\in\mathbb{N}$, $\sigma_n^{(\alpha)}(z,z/(1-z))\in\mathscr{C}$. Then, using the fact that the class $\mathscr{C}$ is closed under convolution with convex functions (see [10], Theorem 2.2), we immediately get that for $\alpha\geq 1$ and $n\in\mathbb{N}$, $\sigma_n^{(\alpha)}(z,f)\in\mathscr{C}$ for all $f\in\mathscr{K}$. Singh and Singh [13] proved that if $f\in\mathscr{K}$, then $z/2\prec V_2(z,f)\prec\sigma_2^{(1)}(z,f)$ in $\mathbb{D}$. The theorem below generalizes this result of Singh and Singh [13] in the sense that the superordinate function $\sigma_2^{(1)}(z, f)$ can be replaced with $\sigma_2^{(\alpha)}(z, f)$, where α, $\alpha \geq 1$, is any real number.

Theorem 3.4 *If $f \in \mathcal{K}$, then for all real numbers α, $\alpha \geq 1$,*

$$z/2 \prec V_2(z, f) \prec \sigma_2^{(\alpha)}(z, f)$$

in $\mathbb{D}$.

□

Proof We note that for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$,

$$V_2(z, f) = \frac{2}{3}z + \frac{1}{6}a_2 z^2$$

and

$$\sigma_2^{(\alpha)}(z, f) = z + \frac{1}{1+\alpha}a_2 z^2.$$

For every $f \in \mathcal{K}$, the relation, $z/2 \prec V_2(z, f)$, is well known. As $\sigma_2^{(\alpha)}(z, f)$ is univalent in $\mathbb{D}$ for $\alpha \geq 1$ and $V_2(0, f) = \sigma_2^{(\alpha)}(0, f)$, we need to show only that for all $f \in \mathcal{K}$,

$$V_2(\mathbb{D}, f) \subset \sigma_2^{(\alpha)}(\mathbb{D}, f).$$

But this is equivalent to showing that for each real θ, the polynomial

$$R(z) = z + \frac{1}{1+\alpha}a_2 z^2 - \frac{2}{3}e^{i\theta} - \frac{1}{6}a_2 e^{2i\theta}$$

has a zero on $\overline{\mathbb{D}}$. Suppose that for some θ, $R(z)$ has no zero in $\mathbb{D}$. Then the polynomial

$$T(z) = \left(\frac{2}{3}e^{i\theta} + \frac{1}{6}a_2 e^{2i\theta}\right) z^2 - z - \frac{a_2}{1+\alpha}$$

has both zeros on $\overline{\mathbb{D}}$. Hence by Lemma 2.5, condition (i), we must have

$$\left|\frac{a_2}{1+\alpha}\right| \leq \frac{2}{3}\left|1 + \frac{a_2 e^{i\theta}}{4}\right|$$

Writing $a_2 = \rho e^{i\phi} (\rho \leq 1)$ and $\phi + \theta = \psi$, this is equivalent to

$$\frac{\rho}{1+\alpha} \leq \frac{2}{3} \left| 1 + \frac{\rho e^{i\psi}}{4} \right| \tag{13}$$

From condition (ii) of Lemma 2.5, we must have

$$\left| \frac{a_2}{1+\alpha} + \frac{2}{3} e^{-i\theta} + \frac{\bar{a}_2}{6} e^{-2i\theta} \right| \leq \left| \frac{2}{3} e^{i\theta} + \frac{\bar{a}_2}{6} e^{2i\theta} \right|^2 - \left| \frac{a_2}{1+\alpha} \right|^2.$$

Again, writing $a_2 = \rho e^{i\phi}$, $\rho \leq 1$, and $\phi + \theta = \psi$, this is equivalent to

$$6 \left| \frac{6\rho}{1+\alpha} e^{i\psi} + 4 + \rho e^{-i\psi} \right| \leq \left| 4 + \rho e^{i\psi} \right|^2 - \frac{36\rho^2}{(1+\alpha)^2}$$

or,

$$6 \left| \frac{6\rho}{1+\alpha} e^{i\psi} + 4 + \rho e^{-i\psi} \right| \leq \left(\left| 4 + \rho e^{i\psi} \right| - \frac{6\rho}{1+\alpha} \right) \left(\left| 4 + \rho e^{i\psi} \right| + \frac{6\rho}{1+\alpha} \right). \tag{14}$$

As

$$\left| \frac{6\rho}{1+\alpha} e^{i\psi} + 4 + \rho e^{-i\psi} \right| \geq \left| 4 + \rho e^{-i\psi} \right| - \left| \frac{6\rho}{1+\alpha} e^{i\psi} \right| = \left| 4 + \rho e^{i\psi} \right| - \frac{6\rho}{1+\alpha},$$

therefore, if (14) holds, we must have

$$\left(\left| 4 + \rho e^{i\psi} \right| - \frac{6\rho}{1+\alpha} \right) \left(\left| 4 + \rho e^{i\psi} \right| + \frac{6\rho}{1+\alpha} - 6 \right) \geq 0. \tag{15}$$

Obviously, minimum of left-hand side of (15) occurs at $\psi = \pi$; so, we must have

$$\left(\rho - \frac{4(1+\alpha)}{7+\alpha} \right) \left(\rho - \frac{2(1+\alpha)}{5-\alpha} \right) \leq 0. \tag{16}$$

Now, at $\psi = \pi$, (13) gives: $\rho \leq 4(1+\alpha)/(7+\alpha)$. But $4(1+\alpha)/(7+\alpha) \geq 1$ for $\alpha \geq 1$ and also for $1 \leq \alpha \leq 5$, $2(1+\alpha)/(5-\alpha) \geq 1$. As $\rho \leq 1$, (16) gives a contradiction (and therefore, $R(z)$ has a zero in $\mathbb{D}$) except if $\rho = 1$ ($\alpha = 1$), $\psi = \pi$ and $\alpha > 5$. When $\rho = 1$ and $\psi = \pi$, we easily verify that $-e^{-i\phi}$ is a zero of $R(z)$ on $\overline{\mathbb{D}}$. For $\alpha > 5$, we proceed as follows. If $I(z) = z/(1-z)$, then

$$\max_{\theta} \left| V_2(e^{i\theta}, I) \right| = \max_{\theta} \sqrt{\frac{4}{9} + \frac{1}{36} + \frac{2\cos\theta}{9}} = \frac{5}{6},$$

and

$$\min_{\theta} |\sigma_2^{(\alpha)}(e^{i\theta}, I)| = \min_{\theta} \sqrt{1 + \frac{1}{(1+\alpha)^2} + \frac{2\cos\theta}{1+\alpha}} = 1 - \frac{1}{(1+\alpha)} > \frac{5}{6}$$

as $\alpha > 5$. Thus, $V_2(\mathbb{D}, I)$ is contained in the disc $\{z : |z| \leq 5/6\}$ and for $\alpha > 5$, $\sigma_2^{(\alpha)}(\mathbb{D}, I)$ contains the disc $\{z : |z| \leq 5/6\}$. Therefore, $V_2(z, I) \prec \sigma_2^{(\alpha)}(z, I), \quad \alpha > 5$. As, $\sigma_2^{(\alpha)}(z, I)$ is convex for $\alpha > 5$ (infact, for $\alpha \geq 3$, see [11]), using Lemma 2.6, we immediately conclude that $V_2(z, f) \prec \sigma_2^{(\alpha)}(z, f)$ in $\mathbb{D}$, for all $\alpha > 5$ and $f \in \mathscr{K}$. This completes the proof. □

Author Contributions Each author contributed equally to the research and preparation of manuscript.

Funding The first author, Manju Yadav, acknowledges financial support from CSIR-UGC, Govt. of India, in the form of JRF vide Award Letter No. 211610111581.

Data Availability Statement Not Applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest with respect to the research.

References

1. Basgöze, T., Frank, J. L., Keogh, F. R.: On Convex Univalent Functions. Canad. J. Math. 22(1): 123-127 (1970)
2. Bustoz, J.: Jacobi polynomial sums and univalent Cesáro means. Proc. Amer. Math. Soc. **50**, 259–264 (1975)
3. Egerváry, E.: Abbildungseigenschaften der arithmetischen Mittel der geometrischen Reihe. Math. Z. **42**, 221–230 (1937)
4. Fezér, L.: On new properties of arithmetical means of the partial sums of fourier series. J. Math. Phys. **13**, 1–17 (1934)
5. Goodman, A.W.: Univalent Functions, vol. I. Mariner Publishing Co., Tampa, Florida (1983)
6. Lewis, J.: Applications of a convolution theorem to Jacobi polynomials. SIAM J. Math. Anal. **19**, 1110–1120 (1979)
7. Pólya, G., Schoenberg, I. J.: Remarks on de la Vallée Poussin means and convex conformal maps of the circle, Pacific J. Math., 8, 295–334 (1958)
8. Robertson, M.S.: Applications of subordination principle to univalent functions. Pacific J. Math. **11**(1), 315–324 (1961)
9. Robertson, M.S.: Power series with multiply monotonic coefficients. Michigan Math. J. **16**, 27–31 (1969)
10. Ruscheweyh, S., Sheil-Small, T.: Hadamard products of Schlicht functions and the Pólya-Schoenberg conjecture, Comment. Math. Helv., 48, 119–135 (1973)
11. Ruscheweyh, S.: Geometric properties of the Cesáro means. Res. Math. **22**, 739–748 (1992)
12. Ruscheweyh, S. and Salinas, L.: Subordination by Cesáro means, Complex Variables Theory Appl. **21**, 279–285 (1993)
13. Singh, S., Singh, R.: Subordination by univalent functions. Proc. Amer. Math. Soc. **82**, 39–47 (1981)
14. Singh, R., Singh, S.: On the partial sums, Cesáro and de la Vallée Poussin means of convex and starlike functions of order 1/2. Proc. Indian Acad. Sci. Math. Sci. **105**(4), 399–404 (1995)

15. Wilf, H.S.: Subordinating factor sequence for convex maps of the unit circle. Proc. Amer. Math. Soc. **12**, 689–693 (1961)