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Approximation Solutions of Some Nonlocal Dispersal Problems

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Abstract

This paper is concerned with a class of nonlocal dispersal problem with Dirichlet boundary conditions. We analyze the limit of solutions when the dispersal kernel is rescaled. Our main results reveal that the solutions of Dirichlet heat equation can be approximated by the nonlocal dispersal equation. The investigation also shows that the nonlocal dispersal equation is similar to the convection–diffusion equation by taking another special kernel function.

Keywords Nonlocal dispersal · Evolution equation · Approximation

Mathematics Subject Classification 35B40 · 35K57 · 92D25

1 Introduction

Let $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, continuous function such that $\int_{\mathbb{R}^N} K(y,x)\,dy = 1$ for all $x \in \mathbb{R}^N$. Nonlocal dispersal equation of the form

$$u_t(x,t) = \int_{\mathbb{R}^N} K(x,y)u(y,t)\,dy - u(x,t), \tag{1.1}$$

and variations of it have been widely used to model diffusion process [1, 3]. As stated in [9, 11], if $u(x,t)$ is thought as a density at position x at time t and the probability distribution that individuals jump from y to x is given by $K(x,y)$, then

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$\int_{\mathbb{R}^N} K(x, y)u(y, t)\, dy$ denotes the rate at which individuals are arriving to position x from all other places and $u(x, t) = \int_{\mathbb{R}^N} K(y, x)u(x, t)\, dy$ is the rate at which they are leaving position x to all other places. This consideration, in the absence of external sources, leads immediately to that $u(x, t)$ satisfies (1.1). For recent references on nonlocal dispersal equations, see [2–4, 17, 19] and references therein.

It is known from [1, 9] that nonlocal dispersal equation shares many properties with the classical heat equation. Moreover, Cortazar et al. [6] proved that a suitable rescaled nonlocal equation with convolution kernel function can approximate the classical heat equation with Dirichlet boundary condition. We refer to [5, 13–16, 18] for the recent study of nonlocal rescaled problems. In the present paper, we study nonlocal dispersal problem with non-homogeneous kernel functions. We then analyze the approximation solutions when the dispersal kernel is rescaled. To do this, let us first consider the nonlocal dispersal equation

$$\begin{cases} u_t(x, t) = \int_{\mathbb{R}^N} J(x, x-y)[u(y, t) - u(x, t)]\, dy & \text{in } \Omega \times (0, \infty), \\ u(x, t) = g(x, t) & \text{in } \mathbb{R}^N \backslash \Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $J : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a nonnegative dispersal kernel such that $\int_{\mathbb{R}^N} J(x, y)dy = 1$ for any $x \in \mathbb{R}^N$. The function $g(x, t)$ is defined for $x \in \mathbb{R}^N \backslash \Omega$, $t > 0$ and $u_0(x)$ is defined for $x \in \Omega$. In (1.2), the values of $u(x, t)$ are prescribed outside Ω, which is analogous to the Dirichlet boundary condition for heat equation [1, 4]. Throughout this paper, we make the following assumptions.

$(A1)$ $J : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is nonnegative, smooth, $J(0, 0) > 0$, $J(x, y) = J(|x|, |y|)$ for $x, y \in \mathbb{R}^N$. Moreover, $\int_{\mathbb{R}^N} J(x, y)dy = 1$ and $\int_{\mathbb{R}^N} J(x, y)|y|^2 dy < \infty$ for any $x \in \mathbb{R}^N$.

$(A2)$ The function $g(x, t)$ and $u_0(x)$ are smooth functions.

It follows from the assumption $(A1)$ that the dispersal kernel function J may not be a symmetric function. However, we shall prove that the rescaled nonlocal dispersal equation of (1.2) is analogous to heat equation

$$\begin{cases} u_t(x, t) = \Delta u(x, t) & \text{in } \Omega \times (0, \infty), \\ u(x, t) = g(x, t) & \text{in } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1.3}$$

Note that the regularity of solution $u(x, t)$ to (1.3) is related to the properties of Ω, $u_0(x)$ and $g(x, t)$, see [7, 8]. So in this paper, we assume that $u(x, t)$ is the unique solution to (1.3) and

$$u \in C^{2+\alpha, 1+\alpha/2}(\bar{\Omega} \times [0, T])$$

for some $0 < \alpha < 1$. Take $\varepsilon > 0$, we consider the rescaled kernel function of $J(x)$ as follows

$$J_\varepsilon(x, \xi) = \frac{1}{\varepsilon^N d(x)} J\left(x, \frac{\xi}{\varepsilon}\right),$$

here $d(x)$ is given by

$$d(x) = \frac{1}{2N} \int_{\mathbb{R}^N} J(x, y)|y|^2 dy. \tag{1.4}$$

By $(A1)$, we know that $d(x)$ is positive and finite for $x \in \Omega$. We then consider the nonlocal dispersal equation

$$\begin{cases} (u_\varepsilon)_t(x,t) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} J_\varepsilon(x, x-y)[u_\varepsilon(y,t) - u_\varepsilon(x,t)]\,dy & \text{in } \Omega \times (0,\infty), \\ u_\varepsilon(x,t) = g(x,t) & \text{in } \mathbb{R}^N \backslash \Omega \times (0,\infty), \\ u_\varepsilon(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1.5}$$

Existence and uniqueness of solutions to (1.5) will be established in Sect. 2. We show that there exists a unique solution $u_\varepsilon(x,t)$ to (1.5) such that

$$u_\varepsilon \in C^1([0,\infty); L^1(\Omega)).$$

Now, we are ready to state the main result.

Theorem 1.1 *Assume that* $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T])$ *is the solution of (1.3) and* $u_\varepsilon(x,t)$ *is the solution of (1.5), respectively. Then, there exists* $C = C(T)$ *such that*

$$\sup_{t\in[0,T]} \|u_\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^\infty(\Omega)} \le C\varepsilon^\alpha \to 0 \text{ as } \varepsilon \to 0.$$

From Theorem 1.1, we can see that the nonlocal dispersal Eq. (1.5) is similar to the Dirichlet heat equation (1.3). It follows from the classical works of Ignat and Rossi [10] that the asymmetric nonlocal dispersal equation may be similar to the convection–diffusion equation. However, our result shows that the nonlocal dispersal Eq. (1.5) may also be similar to the diffusion equation without convection.

In the second part of this paper, let us consider the nonlocal dispersal equation

$$\begin{cases} u_t(x,t) = \int_{\mathbb{R}^N} J(y, x-y)[u(y,t) - u(x,t)]\,dy & \text{in } \Omega \times (0,\infty), \\ u(x,t) = g(x,t) & \text{in } \mathbb{R}^N \backslash \Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1.6}$$

For $\varepsilon > 0$, we use the rescaled kernel function

$$J^\varepsilon(y,\xi) = \frac{1}{\varepsilon^N d(x)} J\left(y, \frac{\xi}{\varepsilon}\right)$$

and study the rescaled nonlocal dispersal equation

$$\begin{cases} u^\varepsilon_t(x,t) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} J^\varepsilon(y, x-y)[u^\varepsilon(y,t) - u^\varepsilon(x,t)]\,dy & \text{in } \Omega \times (0,\infty), \\ u^\varepsilon(x,t) = g(x,t) & \text{in } \mathbb{R}^N \backslash \Omega \times (0,\infty), \\ u^\varepsilon(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.7}$$

here, $d(x)$ is given in (1.4). Existence and uniqueness of solutions to (1.6) and (1.7) will be established in Sect. 2. We show that there exists a unique solution $u^\varepsilon(x,t)$ to (1.5) such that

$$u^\varepsilon \in C^1([0,\infty); L^1(\Omega)).$$

In order to get a simple statement, we assume further that J satisfies the following condition.

$(A3)$ There exists $c > 0$ such that $J(x, y) = 0$ for $|x| > c$ and $|y| > c$.

We shall prove that the rescaled nonlocal dispersal Eq. (1.7) will approximate the convection–diffusion equation

$$\begin{cases} u_t = \Delta u + q(x)\cdot\nabla u & \text{in } \Omega\times(0,\infty),\\ u(x,t) = g(x,t) & \text{in } \partial\Omega\times(0,\infty),\\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.8}$$

where $q(x) = (q_1(x), q_2(x), \cdots, q_N(x))$ and $q_i(x)$ is given by

$$q_i(x) = \frac{1}{d(x)}\int_{\mathbb{R}^N} \frac{\partial J(x,y)}{\partial x} y y_i dy$$

for $i = 1, 2, \cdots, N$ and $y = (y_1, y_2, \cdots, y_N)$.

Theorem 1.2 *Assume that* $(A1) - (A3)$ *hold. Let* $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$ *be the solution of (1.8) and* $u^\varepsilon(x,t)$ *be the solution of (1.7), respectively. Then, we have*

$$\sup_{t\in[0,T]} \|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \textit{ as } \varepsilon\to 0.$$

By Theorem 1.2, we know that the nonlocal Dirichlet Eq. (1.7) is similar to the classical convection–diffusion equation. Our main results reveal that nonlocal dispersal equation with non-homogeneous kernel function may also be similar to convection–diffusion equation.

The rest of the paper is organized as follows. In Sect. 2, we prove existence and uniqueness of solutions to our nonlocal models. The main results are proved in Sect. 3.

2 Existence and Uniqueness

In this section, we establish the existence and uniqueness of solutions to our main models. Here by a solution of (1.2), it is understood in an integral sense.

Definition 2.1 A solution of (1.2) is a function $u \in C([0,\infty); L^1(\Omega))$ such that

$$u(x,t) = u_0(x) + \int_0^t \int_{\mathbb{R}^N} J(x, x-y)[u(y,s) - u(x,s)]\,dyds,\ \ (x,t)\in\Omega\times(0,\infty),$$

and

$$u(x,t) = g(x,t),\ \ (x,t) \in \mathbb{R}^N \backslash \Omega \times (0,\infty).$$

The solution of (1.6) can be defined by a similar way. Since the argument for (1.6) is analogous, we first study the model (1.2). Existence and uniqueness will be obtained by Banach's fixed point theorem. Fix $t_0 > 0$ and consider the space

$$X_{t_0} = \{w \in C([0,t_0]; L^1(\Omega))\}.$$

We then know that X_{t_0} is a Banach space with norm

$$|||w||| = \max_{0 \le t \le t_0} ||w||_{L^1(\Omega)}.$$

Define $\mathcal{T} : X_{t_0} \to X_{t_0}$ by

$$\mathcal{T}_{u_0}(w)(x,t) = u_0(x) + \int_0^t \int_{\mathbb{R}^N} J(x, x-y)[u(y,s) - u(x,s)]\, dyds,\ \ (x,t) \in \Omega \times (0,\infty),$$

where it is assumed that

$$u(x,t) = g(x,t),\ \ (x,t) \in \mathbb{R}^N \backslash \Omega \times (0,\infty).$$

Lemma 2.2 *Assume that $u_0, v_0 \in L^1(\Omega)$. Then, there exists $C > 0$ depending only on J and Ω such that*

$$|||\mathcal{T}_{u_0}(w) - \mathcal{T}_{v_0}(z)||| \le Ct_0|||u - v||| + ||u_0 - v_0||_{L^1(\Omega)}$$

for $u, v \in \mathrm{X}_{t_0}$.

Proof Note that

$$\begin{aligned} &||\mathcal{T}_{u_0}(u) - \mathcal{T}_{v_0}(z)||_{L^1(\Omega)} \\ &\quad \le \int_\Omega \left| \int_0^t \int_{\mathbb{R}^N} J(x, x-y)([u(y,s) - v(y,s)] - [u(x,s) - v(x,s]) \, dyds \right| \\ &\qquad \times dx + ||u_0 - v_0||_{L^1(\Omega)} \\ &\quad \le \int_0^t \int_\Omega \int_\Omega J(x, x-y)|u(y,s) - v(y,s)|\, dydxds \\ &\qquad + \int_0^t \int_\Omega \int_\Omega J(x, x-y)|u(y,s) - v(y,s)|\, dydxds + ||u_0 - v_0||_{L^1(\Omega)}, \end{aligned}$$

we obtain that

$$|||\mathcal{T}_{u_0}(w) - \mathcal{T}_{v_0}(z)||| \le Ct_0|||u - v||| + ||u_0 - v_0||_{L^1(\Omega)}$$

for some $C > 0$. □

Theorem 2.3 *For every $u_0 \in L^1(\Omega)$, there exists a unique solution $u(x,t)$ to (1.2) and*

$$u \in C^1([0,\infty); L^1(\Omega)). \tag{2.1}$$

Proof By the definition of $\mathcal{T}_{u_0}$, we know that $\mathcal{T}_{u_0}$ maps X_{t_0} into X_{t_0}. Then, it follows from Lemma 2.2 that $\mathcal{T}_{u_0}$ is a contraction map if we choose t_0 small enough such that $Ct_0 < 1$. By Banach's fixed point theorem, we get the existence and uniqueness of solutions in the interval $[0, t_0]$. Then, we take $u(x, t_0) \in L^1(\Omega)$ as initial value and we can obtain a solution up to $[0, 2t_0]$. Iterating this procedure, we get a solution $u(x, t)$ such that

$$u \in C([0,\infty); L^1(\Omega)). \tag{2.2}$$

But for any $\delta \neq 0$ and $t > 0$, we have

$$\begin{aligned} &u(x, t+\delta) - u(x,t) \\ &= \int_t^{t+\delta} \int_{\mathbb{R}^N} J(x, x-y)[u(y,s) - u(x,s)]\, dyds. \end{aligned}$$

It follows from (2.2) and Lebesgue theorem that

$$\lim_{\delta \to 0+} \frac{u(x,t+\delta) - u(x,t)}{\delta} = \int_{\mathbb{R}^N} J(x, x-y)[u(y,t) - u(x,t)]\, dyds.$$

Hence, we know that (2.1) holds. □

Analogously, we obtain the following result.

Theorem 2.4 *For every $u_0 \in L^1(\Omega)$, there exists a unique solution $u(x,t)$ to (1.6) and*

$$u \in C^1([0,\infty); L^1(\Omega)).$$

At the end of this section, we give the comparison principle. The sub-super solutions are defined as follows.

Definition 2.5 A function $u \in C((0, T); L^1(\Omega))$ is a super-solution to (1.2) if

$$\begin{cases} u_t(x,t) \geq \int_{\mathbb{R}^N} J(x, x-y)[u(y,t) - u(x,t)]\, dy, & x \in \Omega, t > 0, \\ u(x,t) \geq g(x,t), & x \in \mathbb{R}^N \backslash \Omega, t > 0, \\ u(x, 0) \geq u_0(x), & x \in \Omega. \end{cases}$$

The sub-solution is defined analogously by reversing the inequalities.

We have the following results on sub-super solutions. One can see [6] for a similar proof.

Theorem 2.6 *Assume that $u(x,t)$, $v(x,t)$ are a pair of super-sub solutions to (1.2). Then, $u(x,t) \geq v(x,t)$ for $(x,t) \in \Omega \times (0,\infty)$.*

Theorem 2.7 *Assume that $u(x,t)$, $v(x,t)$ are a pair of super-sub solutions to (1.6). Then, $u(x,t) \geq v(x,t)$ for $(x,t) \in \Omega \times (0,\infty)$.*

3 Proof of Main Results

In this section, we shall prove the main results of this paper. We first consider the case of nonlocal dispersal Eq. (1.2).

Proof of Theorem 1.1 In (1.3), the functions $g(x,t)$ and $u_0(x)$ are smooth, and we then can extend $u(x,t)$ to the whole space, see [8, 12]. Let $\widetilde{v}(x,t)$ be a $C^{2+\alpha,1+\alpha/2}$ extension of $u(x,t)$ to $\mathbb{R}^N \times [0,T]$ and consider the operator

$$L_\varepsilon(\omega) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} J_\varepsilon(x,x-y)[\omega(y,t)-\omega(x,t)]\,dy.$$

We can see that $\widetilde{v}(x,t)$ satisfies

$$\begin{cases} \widetilde{v}_t(x,t) = L_\varepsilon(\widetilde{v})(x,t) + F_\varepsilon(x,t), & x\in\Omega, t\in(0,T],\\ \widetilde{v}(x,t) = g(x,t)+G(x,t), & x\in\mathbb{R}^N\backslash\Omega, t\in(0,T],\\ \widetilde{v}(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{3.1}$$

where

$$F_\varepsilon(x,t) = -L_\varepsilon(\widetilde{v})(x,t) + \Delta\widetilde{v}(x,t).$$

Since $G(x,t)=\widetilde{v}(x,t)-g(x,t)$ is smooth and $G(x,t)=0$ for $x\in\partial\Omega$, we can find $M_1>0$, such that

$$|G(x,t)| \le M_1\varepsilon \tag{3.2}$$

for x satisfying $\operatorname{dist}(x,\partial\Omega)\le\varepsilon$.

The existence and uniqueness of solution $u_\varepsilon(x,t)$ to (1.5) are followed by Theorem 2.3. Define $\omega_\varepsilon(x,t)=\widetilde{v}(x,t)-u_\varepsilon(x,t)$, then we get

$$\begin{cases} (\omega_\varepsilon)_t(x,t) = L_\varepsilon(\omega_\varepsilon)(x,t) + F_\varepsilon(x,t), & x\in\Omega, t\in(0,T],\\ \omega_\varepsilon(x,t) = G(x,t), & x\in\mathbb{R}^N\backslash\Omega, t\in(0,T],\\ \omega_\varepsilon(x,0) = 0, & x\in\Omega. \end{cases}$$

Note that $\widetilde{v}\in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$, we claim that there exists $M_2>0$ such that

$$\sup_{t\in[0,T]}\|F_\varepsilon\|_{L^\infty(\Omega)} = \sup_{t\in[0,T]}\|\Delta\widetilde{v}-L_\varepsilon(\widetilde{v})\|_{L^\infty(\Omega)} \le M_2\varepsilon^\alpha. \tag{3.3}$$

In fact, we know that

$$\begin{aligned} &\Delta\widetilde{v}(x,t) - L_\varepsilon(\widetilde{v})(x,t)\\ &= \Delta\widetilde{v}(x,t) - \frac{1}{\varepsilon^2 d(x)}\int_{\mathbb{R}^N} J_\varepsilon(x,x-y)[\widetilde{v}(y,t)-\widetilde{v}(x,t)]\,dy\\ &= \Delta\widetilde{v}(x,t) - \frac{1}{\varepsilon^{N+2} d(x)}\int_{\mathbb{R}^N} J\left(x,\frac{x-y}{\varepsilon}\right)[\widetilde{v}(y,t)-\widetilde{v}(x,t)]\,dy\\ &= \Delta\widetilde{v}(x,t) - \frac{1}{\varepsilon^2 d(x)}\int_{\mathbb{R}^N} J(x,z)[\widetilde{v}(x-\varepsilon z,t)-\widetilde{v}(x,t)]\,dz. \end{aligned}$$

But $\widetilde{v} \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T])$, a simple argument from Taylor's theorem shows that

$$\begin{aligned}
&\Delta\widetilde{v}(x,t) - L_\varepsilon(\widetilde{v})(x,t) \\
&\quad = \Delta\widetilde{v}(x,t) + \frac{1}{\varepsilon d(x)} \sum_{i=1}^{N} \frac{\partial \widetilde{v}(x,t)}{\partial x_i} \int_{\mathbb{R}^N} J(x,z) z_i \, dz \\
&\qquad - \frac{1}{2d(x)} \sum_{i,j=1}^{N} \frac{\partial^2 \widetilde{v}(x,t)}{\partial x_i \partial x_j} \int_{\mathbb{R}^N} J(x,z) z_i z_j \, dz + O(\varepsilon^\alpha).
\end{aligned}$$

By the assumption $(A1)$, we have

$$\int_{\mathbb{R}^N} J(x,z) z_i \, dz = 0$$

for $i = 1, 2 \cdots, N$ and

$$\int_{\mathbb{R}^N} J(x,z) z_i z_j \, dz = 0$$

for $i, j = 1, 2 \cdots, N$ and $i \neq j$. Accordingly,

$$\begin{aligned}
&\Delta\widetilde{v}(x,t) - L_\varepsilon(\widetilde{v})(x,t) \\
&\quad = \Delta\widetilde{v}(x,t) - \frac{1}{2d(x)} \sum_{i=1}^{N} \frac{\partial^2 \widetilde{v}(x,t)}{\partial x_i^2} \int_{\mathbb{R}^N} J(x,z) z_i^2 \, dz + O(\varepsilon^\alpha) \\
&\quad = O(\varepsilon^\alpha).
\end{aligned}$$

Hence,

$$\sup_{t\in[0,T]} \|F_\varepsilon\|_{L^\infty(\Omega)} = \sup_{t\in[0,T]} \|\Delta\widetilde{v} - L_\varepsilon(\widetilde{v})\|_{L^\infty(\Omega)} \leq M_2 \varepsilon^\alpha.$$

This gives that (3.3) holds.

Now, denote

$$\overline{w}(x,t) = M_1 \varepsilon^\alpha t + M_2 \varepsilon.$$

For $x \in \Omega$, we have

$$\overline{w}_t(x,t) - L_\varepsilon(\overline{w})(x,t) = M_1 \varepsilon^\alpha \geq F_\varepsilon(x,t) = (w_\varepsilon)_t(x,t) - L_\varepsilon(w_\varepsilon)(x,t). \quad (3.4)$$

In view of (3.1–3.2), by choosing M_2 large, we obtain

$$\overline{w}(x,t) \geq w_\varepsilon(x,t)$$

for $x \in \mathbb{R}^N \backslash \Omega$ such that $dist(x, \partial\Omega) \leq \varepsilon$ and $t \in [0,T]$. Moreover, it is clear that

$$\overline{w}(x,0) = K_2 \varepsilon \geq w_\varepsilon(x,0) = 0. \quad (3.5)$$

Thanks to (3.4–3.5), we have $\overline{w}(x,t)$ is the super-solution of (3.1). This yields

$$w_\varepsilon(x,t) \le \overline{w}(x,t) = K_1\varepsilon^\alpha t + K_2\varepsilon.$$

By a similar way, we can show that

$$\underline{w} = -K_1\varepsilon^\alpha t - K_2\varepsilon$$

is a sub-solution and

$$w_\varepsilon(x,t) \ge \underline{w}(x,t) = -K_1\varepsilon^\alpha t - K_2\varepsilon.$$

Hence,

$$\sup_{t\in[0,T]} \|u_\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^\infty(\Omega)} \le C\varepsilon^\alpha \to 0 \quad as \quad \varepsilon \to 0$$

and we end the proof. □

In order to prove Theorem 1.2, we consider the elliptic equation

$$\begin{cases} v_t(x,t) = \sum_{i,j=1}^N a_{ij}^\varepsilon(x)\frac{\partial^2 v(x,t)}{\partial x_i \partial x_j} + \sum_{i=1}^N q_i^\varepsilon(x)\frac{\partial v(x,t)}{\partial x_i} & \text{in } \Omega\times(0,\infty), \\ v(x,t) = g(x,t) & \text{in } \partial\Omega\times(0,\infty), \\ v(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.6}$$

where $\varepsilon > 0$, the coefficients

$$a_{ij}^\varepsilon(x) = \frac{1}{2d(x)}\int_{\mathbb{R}^N} J(x-\varepsilon z, z) z_i z_j \, dz$$

and

$$q_i^\varepsilon(x) = -\frac{1}{\varepsilon d(x)}\int_{\mathbb{R}^N} J(x-\varepsilon z, z) z_i \, dz$$

for $i, j = 1, 2\cdots, N$. We know from [8, 12] that (3.6) admits a unique solution

$$v^\varepsilon \in C^{2+\alpha, 1+\alpha/2}(\bar{\Omega}\times[0,T]).$$

We then have the following results.

Lemma 3.1 *Assume that* $(A1)-(A3)$ *hold. Let* $v(x,t)$ *and* $v^\varepsilon(x,t)$ *be the solutions of (1.8) and (3.6), respectively. Then, we have*

$$\sup_{t\in[0,T]} \|v^\varepsilon(\cdot,t) - v(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \text{ as } \varepsilon \to 0. \tag{3.7}$$

Proof Since

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} J(x-\varepsilon z, z) z_i z_j \, dz = \int_{\mathbb{R}^N} J(x,z) z_i z_j \, dz \text{ uniformly in } \bar{\Omega}$$

for $i, j = 1, 2, \cdots, N$ and

$$\int_{\mathbb{R}^N} J(x,z) z_i z_j \, dz = 0$$

for $i \neq j$, we get

$$\lim_{\varepsilon \to 0} a_{ij}^{\varepsilon}(x) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

On the other hand, we have

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \frac{1}{\varepsilon} J(x - \varepsilon z, z) z_i \, dz = -\int_{\mathbb{R}^N} \frac{\partial J(x,z)}{\partial x} z z_i \, dz.$$

and so

$$\lim_{\varepsilon \to 0} q_i^{\varepsilon}(x) = q_i(x) \text{ uniformly in } \bar{\Omega}$$

for $i = 1, 2, \cdots, N$. Hence, we know that (3.7) holds. □

Lemma 3.2 *Assume that $(A1) - (A3)$ hold. Let $v^{\varepsilon}(x,t)$ be the solution of (3.6) and $u^{\varepsilon}(x,t)$ be the solution of (1.7), respectively. Then, we have*

$$\sup_{t \in [0,T]} \|u^{\varepsilon}(\cdot,t) - v^{\varepsilon}(\cdot,t)\|_{L^{\infty}(\Omega)} \to 0 \text{ as } \varepsilon \to 0.$$

Proof Let $\widetilde{v}^{\varepsilon}(x,t) \in \mathbb{R}^N \times [0,T]$ be the extension of $v^{\varepsilon}(x,t)$ to $\mathbb{R}^N \times [0,T]$, where $v^{\varepsilon}(x,t)$ is the unique solution to (3.6). Define the operator

$$L^{\varepsilon}(\omega) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N} J^{\varepsilon}(y, x-y)[\omega(y,t) - \omega(x,t)] \, dy.$$

Then, $\widetilde{v}^{\varepsilon}(x,t)$ satisfies

$$\begin{cases} \widetilde{v}_t^{\varepsilon}(x,t) = L^{\varepsilon}(\widetilde{v}^{\varepsilon})(x,t) + F^{\varepsilon}(x,t), & x \in \Omega, \ t \in (0,T], \\ \widetilde{v}^{\varepsilon}(x,t) = g(x,t) + G(x,t), & x \in \mathbb{R}^N \backslash \Omega, \ t \in (0,T], \\ \widetilde{v}^{\varepsilon}(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where

$$F^{\varepsilon}(x,t) = -L^{\varepsilon}(\widetilde{v}^{\varepsilon})(x,t) + \sum_{i,j=1}^{N} a_{ij}^{\varepsilon}(x) \frac{\partial^2 \widetilde{v}^{\varepsilon}(x,t)}{\partial x_i \partial x_j} + \sum_{i=1}^{N} q_i^{\varepsilon}(x) \frac{\partial \widetilde{v}^{\varepsilon}(x,t)}{\partial x_i}.$$

Besides, since $G(x,t) = \widetilde{v}^{\varepsilon}(x,t) - g(x,t)$ is smooth and $G(x,t) = 0$ if $x \in \partial\Omega$, then there exists $M_1 > 0$ such that

$$|G(x,t)| \leq M_1 \varepsilon$$

for x such that $\mathrm{dist}(x, \partial\Omega) \leq \varepsilon$.

Denote $w^\varepsilon(x,t) = \widetilde{v}^\varepsilon(x,t) - u^\varepsilon(x,t)$, then we have

$$\begin{cases} w_t^\varepsilon(x,t) = L^\varepsilon(w^\varepsilon)(x,t) + F^\varepsilon(x,t), & x \in \Omega, t \in (0,T], \\ w^\varepsilon(x,t) = G(x,t), & x \in \mathbb{R}^N \backslash \Omega, t \in (0,T], \\ w^\varepsilon(x,0) = 0, & x \in \Omega. \end{cases}$$

We claim that for $\varepsilon > 0$ is small, there exists $M_2 > 0$ such that

$$\sup_{t\in[0,T]} ||F_\varepsilon||_{L^\infty(\Omega)} \le M_2 \varepsilon^\alpha. \tag{3.8}$$

In fact, we always have

$$\begin{aligned} &\sum_{i,j=1}^N a_{ij}^\varepsilon(x) \frac{\partial^2 \widetilde{v}^\varepsilon(x,t)}{\partial x_i \partial x_j} + \sum_{i=1}^N q_i^\varepsilon(x) \frac{\partial \widetilde{v}^\varepsilon(x,t)}{\partial x_i} - L^\varepsilon(\widetilde{v}^\varepsilon)(x,t) \\ &= \sum_{i,j=1}^N a_{ij}^\varepsilon(x) \frac{\partial^2 \widetilde{v}^\varepsilon(x,t)}{\partial x_i \partial x_j} \\ &\quad + \sum_{i=1}^N q_i^\varepsilon(x) \frac{\partial \widetilde{v}^\varepsilon(x,t)}{\partial x_i} - \frac{1}{\varepsilon^{N+2} d(x)} \int_{\mathbb{R}^N} J\left(y, \frac{x-y}{\varepsilon}\right) [\widetilde{v}(y,t) - \widetilde{v}(x,t)]\, dy. \end{aligned}$$

Set $z = (x-y)/\varepsilon$, then we get

$$\begin{aligned} &\frac{1}{\varepsilon^{N+2}} \int_{\mathbb{R}^N} J\left(y, \frac{x-y}{\varepsilon}\right) [\widetilde{v}(y,t) - \widetilde{v}(x,t)]\, dy \\ &= \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N} J(x-\varepsilon z, z) [\widetilde{v}(x-\varepsilon z, t) - \widetilde{v}(x,t)]\, dz \\ &= \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N} J(x-\varepsilon z, z) \left[\sum_{i=1}^N \frac{\partial \widetilde{v}(x,t)}{\partial x_i} (-\varepsilon z_i) \right. \\ &\quad \left. + \frac{1}{2} \sum_{i,j=1}^N \frac{\partial^2 \widetilde{v}(x,t)}{\partial x_i \partial x_j} (-\varepsilon z_i)(-\varepsilon z_j) + O(\varepsilon^{2+\alpha}) \right] dz \\ &= -\int_{\mathbb{R}^N} \sum_{i=1}^N \frac{1}{\varepsilon} J(x-\varepsilon z, z) z_i \frac{\partial \widetilde{v}(x,t)}{\partial x_i} dz \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^N J(x-\varepsilon z, z) z_i z_j \frac{\partial^2 \widetilde{v}(x,t)}{\partial x_i \partial x_j} dz + O(\varepsilon^\alpha), \end{aligned}$$

this also shows that (3.8) holds,

Denote

$$\overline{w}(x,t) = K_1 \varepsilon^\alpha t + K_2 \varepsilon.$$

For $x \in \Omega$, by the claim above, we have

$$\overline{w}(x,t) - \widetilde{L}(\bar{w})(x,t) = K_1\varepsilon^{\alpha} \geq F_{\varepsilon}(x,t) = (w_{\varepsilon})_t(x,t) - L^{\varepsilon}(w_{\varepsilon})(x,t). \qquad (3.9)$$

We can take K_2 large enough such that

$$\overline{w}(x,t) \geq M_1\varepsilon \geq |G(x,t)|$$

for $x \in \mathbb{R}^N \backslash \Omega$ satisfying $dist(x, \partial\Omega) \leq \varepsilon$ and $t \in [0, T]$. Moreover, we have

$$\overline{w}(x,0) = K_2\varepsilon > w_{\varepsilon}(x,0) = 0. \qquad (3.10)$$

Thanks to (3.9–3.10), we use the comparison principle to obtain

$$w_{\varepsilon}(x,t) \leq \overline{w}(x,t) = K_1\varepsilon^{\alpha}t + K_2\varepsilon.$$

Similarly, we can show that

$$w_{\varepsilon}(x,t) \geq \underline{w}(x,t) = -K_1\varepsilon^{\alpha}t - K_2\varepsilon.$$

Hence,

$$\sup_{t\in[0,T]} ||u_{\varepsilon}(\cdot,t) - v(\cdot,t)||_{L^{\infty}(\Omega)} \leq C\varepsilon^{\alpha}.$$

We end the proof. □

At last, let $v \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0, T])$ be the solution of (1.8). Then, we have

$$\|u^{\varepsilon}(\cdot,t) - v(\cdot,t)\|_{L^{\infty}(\Omega)} \leq \|v^{\varepsilon}(\cdot,t) - v(\cdot,t)\|_{L^{\infty}(\Omega)} + \|u^{\varepsilon}(\cdot,t) - v^{\varepsilon}(\cdot,t)\|_{L^{\infty}(\Omega)}.$$

It follows from Lemmas 3.1–3.2 that

$$\sup_{t\in[0,T]} \|u^{\varepsilon}(\cdot,t) - v(\cdot,t)\|_{L^{\infty}(\Omega)} \to 0 \text{ as } \varepsilon \to 0$$

and we end the proof of Theorem 1.2.

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