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A new Brauer-type Z-eigenvalue inclusion set for even-order tensors

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Abstract
A new Brauer-type Z-eigenvalue inclusion set for an even-order real tensor is presented. It is proved that it is tighter than the existing inclusion sets. As an application, a sufficient condition for the positive definiteness of an even-order real symmetric tensor (also a homogeneous polynomial form) and asymptotically stability of time-invariant polynomial systems is given.

Keywords Even-order tensors · Z-eigenvalues · Inclusion sets · Positive definiteness · Asymptotic stability

Mathematics Subject Classification 15A18 · 15A42 · 15A69

1 Introduction

Let m and n be two positive integers with $m \geq 2$ and $n \geq 2$, $[n] = \{1, 2, \ldots, n\}$, $\mathbb{C}$ (resp. $\mathbb{R}$) be the set of all complex (resp. real) numbers, $\mathbb{R}^n$ be the set of all n-dimensional real vectors, $\mathbb{R}^{[m,n]}$ be the set of all order m dimension n real tensors. Let $x = (x_1, x_2, \ldots, x_n)^\top \in \mathbb{R}^n$. Let $\mathscr{A} = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}$, i.e.,

$$a_{i_1 i_2 \ldots i_m} \in \mathbb{R}, \quad i_j \in [n], \quad j \in [m].$$

Let Π_m be the permutation group of m indices.If for any $\pi \in \Pi_m$,

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$$a_{i_1 i_2 \ldots i_m} = a_{i_{\pi(1)} i_{\pi(2)} \ldots i_{\pi(m)}}$$

then $\mathscr{A}$ is called a symmetric tensor [13].

If there are $\lambda \in \mathbb{R}$ and $x = (x_1, x_2, \ldots, x_n)^\top \in \mathbb{R}^n \setminus \{0\}$ such that

$$\mathscr{A} x^{m-1} = \lambda x \quad \text{and} \quad x^\top x = 1,$$

where $\mathscr{A} x^{m-1}$ is an n-dimensional vector, whose i-th component is

$$\left(\mathscr{A} x^{m-1}\right)_i = \sum_{i_2, \ldots, i_m \in [n]} a_{i i_2 \ldots i_m} x_{i_2} \ldots x_{i_m},$$

then λ is called a Z-eigenvalue of $\mathscr{A}$ and x is called a Z-eigenvector associated with λ [10, 13]. Let $\sigma(\mathscr{A})$ be the set of all Z-eigenvalues of $\mathscr{A}$.

The Z-identity tensor is introduced by the authors in [7, 8, 13]. A tensor $\mathscr{E} = (e_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ with m even is called a Z-identity tensor if for any vector $x \in \mathbb{R}^n$,

$$\mathscr{E} x^{m-1} = x \quad \text{and} \quad x^\top x = 1.$$

Note here that an even-order n dimension Z-identity tensor is not unique in general. For instance, the following two tensors are both Z-identity tensors:

Case I. ([8, Definition 2.1]): Let $\mathscr{E}_1 = (e_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}$, where

$$e_{i_1 i_1 i_2 i_2 \ldots i_k i_k} = 1, \quad i_1, i_2, \ldots, i_k \in [n], \quad \text{and} \quad m = 2k;$$

Case II. ([7, Property 2.4]): Let $\mathscr{E}_2 = (e_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}$, where

$$e_{i_1 \ldots i_m} = \frac{1}{m!} \sum_{\pi \in \Pi_m} \delta_{i_{\pi(1)} i_{\pi(2)}} \delta_{i_{\pi(3)} i_{\pi(4)}} \ldots \delta_{i_{\pi(m-1)} i_{\pi(m)}},$$

where δ is the standard Kronecker delta, i,e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

For convenient applications, the Z-identity tensor $\mathscr{E}_2 = (e_{ijkl}) \in \mathbb{R}^{[4,n]}$ is listed as follows:

$$e_{ijkl} = \begin{cases} 1, & \text{if } i = j = k = l, \\ 1/3, & \text{if } i = j \neq k = l, \\ 1/3, & \text{if } i = k \neq j = l, \\ 1/3, & \text{if } i = l \neq j = k, \\ 0, & \text{otherwise.} \end{cases}$$

An even-order m dimension n real symmetric tensor $\mathscr{A}$ defines an m-th degree homogeneous polynomial

$$f(x) = \mathscr{A} x^m = \sum_{i_1, i_2, \ldots, i_m \in [n]} a_{i_1 i_2 \ldots i_m} x_{i_1} x_{i_2} \ldots x_{i_m}. \tag{1}$$

If $f(x) > 0$ for any $x \in \mathbb{R}^n \setminus \{0\}$, then we call that $f(x)$ is positive definite. It is pointed out that $f(x)$ is positive definite if and only if $\mathscr{A}$ is positive definite [13, 14]. On the other hand, if all Z-eigenvalues of the real symmetric tensor $\mathscr{A}$ with order even are positive, then $\mathscr{A}$ is positive definite and therefore $f(x)$ is also positive definite. The positive definiteness of $f(x)$ has extremely important applications in real life. As pointed out in some documents, it is widely used in spectral hypergraph theory [15, 16], automatic control [12] and the stability of nonlinear systems [1, 2].

For judging the positive definiteness of $f(x)$, we must calculate all Z-eigenvalues of an even-order real symmetric tensor $\mathscr{A}$, or calculate the minimum Z-eigenvalue of $\mathscr{A}$. When all Z-eigenvalues of $\mathscr{A}$ are greater than 0, or the minimum Z-eigenvalue of $\mathscr{A}$ is greater than 0, we can judge that $f(x)$ is positive definite. However, if m or n are very large, it is difficult to calculate all Z-eigenvalues of $\mathscr{A}$ and the minimum Z-eigenvalue of $\mathscr{A}$. In order to be able to solve this problem quickly, we can take a very normal and simple method: we only need to judge the signs of all Z-eigenvalues, but not to compute all Z-eigenvalues. In order to achieve this goal, one can construct a set which includes all Z-eigenvalues of $\mathscr{A}$. If this set is just in the right-half complex plane, then he can conclude that all Z-eigenvalues are positive, and consequently, $\mathscr{A}$ is positive definite. The related results are shown in [8, 19–21, 24, 30, 31].

Wang et al. [25] gave the following Z-eigenvalue inclusion set for tensors as follows:

Theorem 1 [25, Theorem 3.2] *Let* $\mathscr{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$. *Then*

$$\sigma(\mathscr{A}) \subseteq \mathscr{L}(\mathscr{A}) = \bigcup_{i \in [n]} \bigcap_{j \in [n], j \neq i} \mathscr{L}_{i,j}(\mathscr{A}).$$

where

$$\mathscr{L}_{i,j}(\mathscr{A}) = \{z \in \mathbb{C} : \big(\mid z \mid -(R_i(\mathscr{A}) - \mid a_{ij\dots j} \mid)\big) \mid z \mid \leq \mid a_{ij\dots j} \mid R_j(\mathscr{A})\}$$

and

$$R_i(\mathscr{A}) = \sum_{i_2, \dots, i_m \in [n]} \mid a_{i i_2 \dots i_m} \mid .$$

From Theorem 1, we can easily see that $0 \in \mathscr{L}(\mathscr{A})$. Therefore, this means that we cannot use the set $\mathscr{L}(\mathscr{A})$ to determine the positive definiteness of a real symmetric tensor $\mathscr{A}$ of even order. However, there are many such similar sets, which can be seen in detail [3–6, 9, 11, 17, 18, 23, 25–29]. Because 0 exists in these inclusion sets, we cannot use such inclusion sets to determine the positive definiteness of even-order real symmetric tensors.

In order to overcome this drawback, Li et al. [8] presented a Z-eigenvalue inclusion interval with n parameters for even-order real tensors as follows:

Theorem 2 [8, Theorem 2.2] *Let* $\mathscr{A} = (a_{i_1\ldots i_m}) \in \mathbb{R}^{[m,n]}$ *with* m *even. Then for any real vector* $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$,

$$\sigma(\mathscr{A}) \subseteq \mathscr{G}(\mathscr{A}, \alpha) := \bigcup_{i\in[n]} \Big(\mathscr{G}_i(\mathscr{A}, \alpha) := \{z \in \mathbb{R} :\mid z - \alpha_i \mid\leq R_i(\mathscr{A}, \alpha_i)\}\Big),$$

where

$$R_i(\mathscr{A}, \alpha_i) = \sum_{\substack{i_2,\ldots,i_m\in[n],\\ e_{ii_2\ldots i_m}\neq 0}} |a_{ii_2\ldots i_m} - \alpha_i e_{ii_2\ldots i_m}| + \sum_{\substack{i_2,\ldots,i_m\in[n],\\ e_{ii_2\ldots i_m}=0}} |a_{ii_2\ldots i_m}|.$$

In order to be able to locate the Z-eigenvalues more accurately, Shen et al. [22] gave the following inclusion set.

Theorem 3 [22, Theorem 1] *Let* $\mathscr{A} = (a_{i_1\ldots i_m}) \in \mathbb{R}^{[m,n]}$ *and* $\mathscr{E} \in \mathbb{R}^{[m,n]}$ *be a* Z*-identity tensor. For any real vector* $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$, *then*

$$\sigma(\mathscr{A}) \subseteq \Upsilon(\mathscr{A}, \alpha) = \bigcup_{i\in[n]} \bigcap_{j\in[n], j\neq i} \Upsilon_{i,j}(\mathscr{A}, \alpha),$$

where

$$\Upsilon_{i,j}(\mathscr{A}, \alpha) = \{z \in \mathbb{R} : (\mid z - \alpha_i \mid - R_i^j(\mathscr{A}, \alpha_i)) \mid z - \alpha_j \mid \\ \leq\mid a_{ij\ldots j} - \alpha_i e_{ij\ldots j} \mid R_j(\mathscr{A}, \alpha_j)\},$$

and

$$R_i^j(\mathscr{A}, \alpha_i) = R_i(\mathscr{A}, \alpha_i) - \mid a_{ij\ldots j} - \alpha_i e_{ij\ldots j} \mid .$$

The remainder of this paper is organized as follows. In Sect. 2, we give a new Z-eigenvalues inclusion set with parameters and prove that it is tighter than that in Theorems 2 and 3. In Sect. 3, we consider two applications of the obtained Z-eigenvalue inclusion sets. The first application is to give a sufficient condition for positive definiteness of an even-order real symmetric tensors (also homogeneous polynomial forms). The second application is to judge the asymptotically stability of time-invariant polynomial systems. Finally, some concluding remarks are given to end this paper in Sect. 4.

2 Main Results

In this section, we give a new inclusion set $\Omega(\mathscr{A}, \alpha)$ and prove that it is tighter than the inclusion set $\mathscr{G}(\mathscr{A}, \alpha)$ in Theorem 2 and the inclusion set $\Upsilon(\mathscr{A}, \alpha)$ in Theorem 3. Before giving the set $\Omega(\mathscr{A}, \alpha)$, we first give some notations and a lemma. Let

$$\Delta = \{(i_2, \ldots, i_m) : i_2 \neq \cdots \neq i_m, \text{ or only two of } i_2, \ldots, i_m \in [n] \text{ are the same}\},$$

$$\overline{\Delta} =\{(i_2,\ldots,i_m) : (i_2,\ldots,i_m) \notin \Delta, i_2,\ldots,i_m \in [n]\},$$
$$N =\{(i_2,\ldots,i_m) : i_2,\ldots,i_m \in [n]\}.$$

Obviously,

$$\Delta \cap \overline{\Delta} = \emptyset, \quad N = \Delta \cup \overline{\Delta}, \text{ and } \overline{\Delta} = N \text{ when } \Delta = \emptyset.$$

Let

$$\Lambda_i =\{(i_2,\ldots,i_m) : e_{ii_2\ldots i_m} \neq 0,\ i_2,\ldots,i_m \in [n]\},$$
$$\overline{\Lambda}_i =\{(i_2,\ldots,i_m) : e_{ii_2\ldots i_m} = 0,\ i_2,\ldots,i_m \in [n]\},$$

and

$$r_i^{\Delta\cap\Lambda_i}(\mathscr{A},\alpha_i) = \frac{1}{(m-2)^{\frac{m-2}{2}}} \sum_{(i_2,\ldots,i_m)\in\Delta\cap\Lambda_i} | a_{ii_2\ldots i_m} - \alpha_i e_{ii_2\ldots i_m} |,$$
$$r_i^{\overline{\Delta}\cap\Lambda_i}(\mathscr{A},\alpha_i) = \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}\cap\Lambda_i} | a_{ii_2\ldots i_m} - \alpha_i e_{ii_2\ldots i_m} |,$$
$$r_i^{\Delta\cap\overline{\Lambda}_i}(\mathscr{A}) = \frac{1}{(m-2)^{\frac{m-2}{2}}} \sum_{(i_2,\ldots,i_m)\in\Delta\cap\overline{\Lambda}_i} | a_{ii_2\ldots i_m} |,$$
$$r_i^{\overline{\Delta}\cap\overline{\Lambda}_i}(\mathscr{A}) = \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}\cap\overline{\Lambda}_i} | a_{ii_2\ldots i_m} |,$$
$$r_i(\mathscr{A},\alpha_i) = r_i^{\Delta\cap\Lambda_i}(\mathscr{A},\alpha_i) + r_i^{\overline{\Delta}\cap\Lambda_i}(\mathscr{A},\alpha_i) + r_i^{\Delta\cap\overline{\Lambda}_i}(\mathscr{A}) + r_i^{\overline{\Delta}\cap\overline{\Lambda}_i}(\mathscr{A}). \quad (2)$$

Then by $\frac{1}{(m-2)^{\frac{m-2}{2}}} \leq 1$ for $m \geq 3$, it can be seen that

$$r_i(\mathscr{A},\alpha_i) \leq R_i(\mathscr{A},\alpha_i), \quad i \in [n]. \quad (3)$$

Lemma 1 [21, Lemma 2.2] *Let* $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, *where* $x_i \in \mathbb{R}$, $i \in [n]$. *If* $y_1, y_2, \ldots, y_k$ *are arbitrary* k *entries of* $x_1, x_2, \ldots, x_n$, *then*

$$| y_1 || y_2 | \cdots | y_k | \leq \frac{1}{k^{\frac{k}{2}}}.$$

Theorem 4 *Let* $\mathscr{A} = (a_{i_1\ldots i_m}) \in \mathbb{R}^{[m,n]}$ *with* m *even. Then for any* $\alpha = (\alpha_1,\ldots,\alpha_n)^\top \in \mathbb{R}^n$,

$$\sigma(\mathscr{A}) \subseteq \Omega(\mathscr{A},\alpha) = \bigcup_{i\in[n]} \bigcap_{j\in[n], j\neq i} \Omega_{i,j}(\mathscr{A},\alpha), \quad (4)$$

where

$$\Omega_{i,j}(\mathscr{A},\alpha)=\{z\in\mathbb{R}:(|z-\alpha_i|-r_i(\mathscr{A},\alpha_i)+|a_{ij\ldots j}|)|z-\alpha_j|\le|a_{ij\ldots j}|r_j(\mathscr{A},\alpha_j)\}.$$

Proof Let λ be any Z-eigenvalue of $\mathscr{A}$ and $x=(x_1,\ldots,x_n)^\top\in\mathbb{R}^n\backslash\{0\}$ be a Z-eigenvector associated with λ. Then

$$\mathscr{A}x^{m-1}=\lambda x=\lambda\mathscr{E}x^{m-1}\quad\text{and}\quad x^\top x=1. \tag{5}$$

Let $|x_t|=\max_{i\in[n]}|x_i|$. Then for any $s\in[n]$ and $s\neq t$, we have

$$\begin{aligned}(\lambda-\alpha_t)x_t &=\lambda x_t-\alpha_t x_t=\lambda x_t-\alpha_t\mathscr{E}x_t^{m-1}\\
&=\sum_{i_2,\ldots,i_m\in[n]}a_{ti_2\ldots i_m}x_{i_2}\ldots x_{i_m}-\alpha_t\sum_{i_2,\ldots,i_m\in[n]}e_{ti_2\ldots i_m}x_{i_2}\ldots x_{i_m}\\
&=\sum_{i_2,\ldots,i_m\in[n]}(a_{ti_2\ldots i_m}-\alpha_t e_{ti_2\ldots i_m})x_{i_2}\ldots x_{i_m}\\
&=\sum_{(i_2,\ldots,i_m)\in\Delta\cap\Lambda_t}(a_{ti_2\ldots i_m}-\alpha_t e_{ti_2\ldots i_m})x_{i_2}\ldots x_{i_m}\\
&\quad+\sum_{(i_2,\ldots,i_m)\in\overline{\Delta}\cap\Lambda_t}(a_{ti_2\ldots i_m}-\alpha_t e_{ti_2\ldots i_m})x_{i_2}\ldots x_{i_m}\\
&\quad+\sum_{(i_2,\ldots,i_m)\in\Delta\cap\overline{\Lambda}_t}a_{ti_2\ldots i_m}x_{i_2}\ldots x_{i_m}\\
&\quad+\sum_{(i_2,\ldots,i_m)\in(\overline{\Delta}\cap\overline{\Lambda}_t)\backslash\{(s,\ldots,s)\}}a_{ti_2\ldots i_m}x_{i_2}\ldots x_{i_m}+a_{ts\ldots s}x_s^{m-1}.\end{aligned}$$

Taking the modulus in above equation and using the triangle inequality and Lemma 1, we have

$$\begin{aligned}|\lambda-\alpha_t||x_t|\le&\sum_{(i_2,\ldots,i_m)\in\Delta\cap\Lambda_t}|a_{ti_2\ldots i_m}-\alpha_t e_{ti_2\ldots i_m}||x_{i_2}|\ldots|x_{i_m}|\\
&+\sum_{(i_2,\ldots,i_m)\in\overline{\Delta}\cap\Lambda_t}|a_{ti_2\ldots i_m}-\alpha_t e_{ti_2\ldots i_m}||x_{i_2}|\ldots|x_{i_m}|\\
&+\sum_{(i_2,\ldots,i_m)\in\Delta\cap\overline{\Lambda}_t}|a_{ti_2\ldots i_m}||x_{i_2}|\ldots|x_{i_m}|\\
&+\sum_{(i_2,\ldots,i_m)\in(\overline{\Delta}\cap\overline{\Lambda}_t)\backslash\{(s,\ldots,s)\}}|a_{ti_2\ldots i_m}||x_{i_2}|\ldots|x_{i_m}|+|a_{ts\ldots s}||x_s|^{m-1}\\
\le&\sum_{(i_2,\ldots,i_m)\in\Delta\cap\Lambda_t}|a_{ti_2\ldots i_m}-\alpha_t e_{ti_2\ldots i_m}||y_1|\ldots|y_{m-2}||x_t|\\
&+\sum_{(i_2,\ldots,i_m)\in\overline{\Delta}\cap\Lambda_t}|a_{ti_2\ldots i_m}-\alpha_t e_{ti_2\ldots i_m}||x_t|^{m-1}\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{(i_2,\ldots,i_m)\in\Delta\cap\overline{\Lambda}_t} | a_{ti_2\ldots i_m} || z_1 | \ldots | z_{m-2} || x_t | \\
&+ \sum_{(i_2,\ldots,i_m)\in(\overline{\Delta}\cap\overline{\Lambda}_t)\backslash\{(s,\ldots,s)\}} | a_{ti_2\ldots i_m} || x_t |^{m-1} + | a_{ts\ldots s} || x_s |^{m-1} \\
\le\ & \frac{1}{(m-2)^{\frac{m-2}{2}}} \sum_{(i_2,\ldots,i_m)\in\Delta\cap\Lambda_t} | a_{ti_2\ldots i_m} - \alpha_t e_{ti_2\ldots i_m} || x_t | \\
&+ \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}\cap\Lambda_t} | a_{ti_2\ldots i_m} - \alpha_t e_{ti_2\ldots i_m} || x_t | \\
&+ \frac{1}{(m-2)^{\frac{m-2}{2}}} \sum_{(i_2,\ldots,i_m)\in\Delta\cap\overline{\Lambda}_t} | a_{ti_2\ldots i_m} || x_t | \\
&+ \sum_{(i_2,\ldots,i_m)\in(\overline{\Delta}\cap\overline{\Lambda}_t)\backslash\{(s,\ldots,s)\}} | a_{ti_2\ldots i_m} || x_t | + | a_{ts\ldots s} || x_s | \\
=\ & r_t^{\Delta\cap\Lambda_t}(\mathscr{A},\alpha_t) | x_t | + r_t^{\overline{\Delta}\cap\Lambda_t}(\mathscr{A},\alpha_t) | x_t | + r_t^{\Delta\cap\overline{\Lambda}_t}(\mathscr{A}) | x_t | \\
&+ \left(r_t^{\overline{\Delta}\cap\overline{\Lambda}_t}(\mathscr{A}) - | a_{ts\ldots s} | \right) | x_t | + | a_{ts\ldots s} || x_s | \\
=\ & (r_t(\mathscr{A},\alpha_t) - | a_{ts\ldots s} |) | x_t | + | a_{ts\ldots s} || x_s |,
\end{aligned} \tag{6}
$$

which implies that

$$
(|\lambda - \alpha_t| - r_t(\mathscr{A},\alpha_t) + | a_{ts\ldots s} |)|x_t| \le |a_{ts\ldots s}||x_s|. \tag{7}
$$

In (6), $|y_1|, \ldots, |y_{m-2}|$ and $|z_1|, \ldots, |z_{m-2}|$ are taken by the following two ways:

(a) If $i_2 \neq \ldots \neq i_m$, then we can enlarge any one of $|x_2|, \ldots, |x_m|$ to $|x_t|$ and keep the others (can be taken as $|y_1|, \ldots, |y_{m-2}|$ and $|z_1|, \ldots, |z_{m-2}|$) unchanged.
(b) If only two of $i_2, \ldots, i_m$ are the same, then we can enlarge one of the two same elements to $|x_t|$ and keep others (can be taken as $|y_1|, \ldots, |y_{m-2}|$ and $|z_1|, \ldots, |z_{m-2}|$) unchanged.

If $|x_s| > 0$ in (7), then from (5), we can get

$$
\begin{aligned}
(\lambda - \alpha_s)x_s &= \lambda x_s - \alpha_s \mathscr{E} x_s^{m-1} \\
&= \sum_{i_2,\ldots,i_m\in[n]} a_{si_2\ldots i_m} x_{i_2} \ldots x_{i_m} - \alpha_s \sum_{i_2,\ldots,i_m\in[n]} e_{si_2\ldots i_m} x_{i_2} \ldots x_{i_m} \\
&= \sum_{(i_2,\ldots,i_m)\in\Delta\cap\Lambda_s} (a_{si_2\ldots i_m} - \alpha_s e_{si_2\ldots i_m}) x_{i_2} \ldots x_{i_m} \\
&\quad + \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}\cap\Lambda_s} (a_{si_2\ldots i_m} - \alpha_s e_{si_2\ldots i_m}) x_{i_2} \ldots x_{i_m} \\
&\quad + \sum_{(i_2,\ldots,i_m)\in\Delta\cap\overline{\Lambda}_s} a_{si_2\ldots i_m} x_{i_2} \ldots x_{i_m}
\end{aligned}
$$

$$+ \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}\cap\overline{\Lambda}_s} a_{si_2\ldots i_m} x_{i_2}\ldots x_{i_m},$$

and

$$\begin{aligned}
|\lambda-\alpha_s||x_s| \le & \sum_{(i_2,\ldots,i_m)\in\Delta\cap\Lambda_s} \mid a_{si_2\ldots i_m}-\alpha_s e_{si_2\ldots i_m} \mid\mid x_{i_2} \mid \ldots \mid x_{i_m} \mid \\
& + \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}\cap\Lambda_s} \mid a_{si_2\ldots i_m}-\alpha_s e_{si_2\ldots i_m} \mid\mid x_{i_2} \mid \ldots \mid x_{i_m} \mid \\
& + \sum_{(i_2,\ldots,i_m)\in\Delta\cap\overline{\Lambda}_s} \mid a_{si_2\ldots i_m} \mid\mid x_{i_2} \mid \ldots \mid x_{i_m} \mid \\
& + \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}\cap\overline{\Lambda}_s} \mid a_{si_2\ldots i_m} \mid\mid x_{i_2} \mid \ldots \mid x_{i_m} \mid \\
\le & \sum_{(i_2,\ldots,i_m)\in\Delta\cap\Lambda_s} \mid a_{si_2\ldots i_m}-\alpha_s e_{si_2\ldots i_m} \mid\mid y_1 \mid \ldots \mid y_{m-2} \mid\mid x_t \mid \\
& + \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}\cap\Lambda_s} \mid a_{si_2\ldots i_m}-\alpha_s e_{si_2\ldots i_m} \mid\mid x_t \mid^{m-1} \\
& + \sum_{(i_2,\ldots,i_m)\in\Delta\cap\overline{\Lambda}_s} \mid a_{si_2\ldots i_m} \mid\mid z_1 \mid \ldots \mid z_{m-2} \mid\mid x_t \mid \\
& + \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}\cap\overline{\Lambda}_s} \mid a_{si_2\ldots i_m} \mid\mid x_t \mid^{m-1} \\
\le & \frac{1}{(m-2)^{\frac{m-2}{2}}} \sum_{(i_2,\ldots,i_m)\in\Delta\cap\Lambda_s} \mid a_{si_2\ldots i_m}-\alpha_s e_{si_2\ldots i_m} \mid\mid x_t \mid \\
& + \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}\cap\Lambda_s} \mid a_{si_2\ldots i_m}-\alpha_s e_{si_2\ldots i_m} \mid\mid x_t \mid \\
& + \frac{1}{(m-2)^{\frac{m-2}{2}}} \sum_{(i_2,\ldots,i_m)\in\Delta\cap\overline{\Lambda}_s} \mid a_{si_2\ldots i_m} \mid\mid x_t \mid \\
& + \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}\cap\overline{\Lambda}_s} \mid a_{si_2\ldots i_m} \mid\mid x_t \mid \\
= & r_s^{\Delta\cap\Lambda_t}(\mathscr{A},\alpha_s) \mid x_t \mid + r_s^{\overline{\Delta}\cap\Lambda_t}(\mathscr{A},\alpha_s) \mid x_t \mid \\
& + r_s^{\Delta\cap\overline{\Lambda}_s}(\mathscr{A}) \mid x_t \mid + r_s^{\overline{\Delta}\cap\overline{\Lambda}_s}(\mathscr{A}) \mid x_t \mid \\
= & r_s(\mathscr{A},\alpha_s) \mid x_t \mid,
\end{aligned} \tag{8}$$

which leads to

$$|\lambda-\alpha_s||x_s| \le r_s(\mathscr{A},\alpha_s)|x_t|. \tag{9}$$

Note here that $|y_1|, \ldots, |y_{m-2}|$ and $|z_1|, \ldots, |z_{m-2}|$ in (8) are taken in the same way as in (6).

Multiplying (7) and (9) yields

$$(|\lambda - \alpha_t| - r_t(\mathscr{A}, \alpha_t) + |\ a_{ts\ldots s}\ |)|\lambda - \alpha_s||x_t||x_s| \le |a_{ts\ldots s}|r_s(\mathscr{A}, \alpha_s)|x_t||x_s|.$$

Furthermore, by $|\ x_t\ ||\ x_s\ | > 0$, we can get

$$(|\lambda - \alpha_t| - r_t(\mathscr{A}, \alpha_t) + |\ a_{ts\ldots s}\ |)|\lambda - \alpha_s| \le |a_{ts\ldots s}|r_s(\mathscr{A}, \alpha_s), \tag{10}$$

which implies that

$$\lambda \in \Omega_{t,s}(\mathscr{A}, \alpha). \tag{11}$$

If $|x_s| = 0$ in (7), by $|x_t| > 0$, we have $|\lambda - \alpha_t| - r_t(\mathscr{A}, \alpha_t) + |\ a_{ts\ldots s}\ | \le 0$, which implies that (10) holds, consequently, (11) holds.

By the arbitrariness of $s \in [n]$, $s \neq t$, we have

$$\lambda \in \bigcap_{s\in[n], s\neq t} \Omega_{t,s}(\mathscr{A}, \alpha).$$

Furthermore, by the uncertainty of choosing $t \in [n]$, we have

$$\lambda \in \bigcup_{t\in[n]} \bigcap_{s\in[n], s\neq t} \Omega_{t,s}(\mathscr{A}, \alpha).$$

Consequently, $\sigma(\mathscr{A}) \subseteq \Omega(\mathscr{A}, \alpha)$. □

The following comparison theorem shows that the Z-eigenvalue inclusion set $\Omega(\mathscr{A}, \alpha)$ in Theorem 4 is tighter (that is, can capture all Z-eigenvalue of $\mathscr{A}$ more accurate) than those in Theorems 2 and 3.

Theorem 5 *Let* $\mathscr{A} = (a_{i_1\ldots i_m}) \in \mathbb{R}^{[m,n]}$. *Then for any* $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$,

$$\Omega(\mathscr{A}, \alpha) \subseteq \Upsilon(\mathscr{A}, \alpha) \subseteq \mathscr{G}(\mathscr{A}, \alpha).$$

Proof From Corollary 1 of [22], it can be seen that $\Upsilon(\mathscr{A}, \alpha) \subseteq \mathscr{G}(\mathscr{A}, \alpha)$. Below we only need to prove $\Omega(\mathscr{A}, \alpha) \subseteq \Upsilon(\mathscr{A}, \alpha)$. Let $z \in \Omega(\mathscr{A}, \alpha)$. Then there are $i, j \in [n]$ and $i \neq j$ such that $z \in \Omega_{i,j}(\mathscr{A}, \alpha)$, i.e.,

$$(|\lambda - \alpha_i| - r_i(\mathscr{A}, \alpha_i) + |\ a_{ij\ldots j}\ |)|\lambda - \alpha_j| \le |a_{ij\ldots j}|r_j(\mathscr{A}, \alpha_j). \tag{12}$$

For Cases I and II of the Z-identity tensor, we have $e_{ij\ldots j} = 0$, that is,

$$|\ a_{ij\ldots j} - \alpha_t e_{ij\ldots j}\ | = |\ a_{ij\ldots j}\ |\ .$$

By (3), we have

$$
\begin{aligned}
R_i^j(\mathscr{A},\alpha_i) &= R_i(\mathscr{A},\alpha_i) - | a_{ij\ldots j} - \alpha_i e_{ij\ldots j} | \\
&= R_i(\mathscr{A},\alpha_i) - | a_{ij\ldots j} | \geq \qquad r_i(\mathscr{A},\alpha_i) - | a_{ij\ldots j} | .
\end{aligned}
$$

By (12), we have

$$
\begin{aligned}
(| z-\alpha_i | - R_i^j(\mathscr{A},\alpha_i)) \, | z-\alpha_j | &\leq (|z-\alpha_i| - r_i(\mathscr{A},\alpha_i) + | a_{ij\ldots j} |)|z-\alpha_j| \\
&\leq |a_{ij\ldots j}| r_j(\mathscr{A},\alpha_j) \\
&\leq | a_{ij\ldots j} - \alpha_i e_{ij\ldots j} | \, R_j(\mathscr{A},\alpha_j),
\end{aligned}
$$

i.e.,

$$
(| z-\alpha_i | - R_i^j(\mathscr{A},\alpha_i)) \, | z-\alpha_j | \leq | a_{ij\ldots j} - \alpha_i e_{ij\ldots j} | \, R_j(\mathscr{A},\alpha_j),
$$

which implies that $z \in \Upsilon_{i,j}(\mathscr{A},\alpha)$. Hence, $\Omega(\mathscr{A},\alpha) \subseteq \Upsilon(\mathscr{A},\alpha)$. □

As $r_i(\mathscr{A},\alpha_i)$, $i \in [n]$, are related to the Z-identity tensor $\mathscr{E}$ and the order and dimension of $\mathscr{A}$, we list the specific form of $\Omega(\mathscr{A},\alpha)$ in Theorem 4 with $m=4$ and $m=6$ by using the similar methods as [21, Corollary 2 and Corollary 3] follows.

Corollary 1 *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[4,n]}$. Then for any $\alpha = (\alpha_1,\ldots,\alpha_n)^\top \in \mathbb{R}^n$,* (4) *holds, where $r_i(\mathscr{A},\alpha_i)$ are taken by the following two cases:*

(i) *If the Z-identify tensor $\mathscr{E}$ is taken as $\mathscr{E}_1$, then*

$$
\begin{aligned}
r_i(\mathscr{A},\alpha_i) &= \frac{1}{2}\sum_{j\neq i} | a_{iijj} - \alpha_i | + | a_{iiii} - \alpha_i | \\
&\quad + \frac{1}{2}\Big(R_i(\mathscr{A}) + \sum_{j\neq i} | a_{ijjj} | - \sum_{j\in[n]} | a_{iijj} | \Big).
\end{aligned}
$$

(ii) *If the Z-identify tensor $\mathscr{E}$ is taken as $\mathscr{E}_2$, then*

$$
\begin{aligned}
r_i(\mathscr{A},\alpha_i) &= \frac{1}{2}\sum_{j\neq i}\Big(| a_{iijj} - \frac{1}{3}\alpha_i | + | a_{ijij} - \frac{1}{3}\alpha_i | + | a_{ijji} - \frac{1}{3}\alpha_i | \Big) \\
&\quad + | a_{iiii} - \alpha_i | + \widetilde{r}_i(\mathscr{A}),
\end{aligned}
$$

where

$$
\widetilde{r}_i(\mathscr{A}) = \frac{1}{2}\Big(R_i(\mathscr{A}) + \sum_{j\neq i} | a_{ijjj} | - \sum_{j\neq i}(| a_{iijj} | + | a_{ijij} | + | a_{ijji} |) - | a_{iiii} | \Big).
$$

Corollary 2 *Let $\mathscr{A} = (a_{i_1\ldots i_6}) \in \mathbb{R}^{[6,n]}$. Then for any $\alpha = (\alpha_1,\ldots,\alpha_n)^\top \in \mathbb{R}^n$,* (4) *holds, $r_i(\mathscr{A},\alpha_i)$ are taken by the following two cases:*

(i) *If the Z-identify tensor $\mathscr{E}$ is taken as $\mathscr{E}_1$, then*

$$r_i(\mathscr{A},\alpha_i) = \sum_{j,k\in[n]} | a_{iijjkk}-\alpha_i | + \begin{cases} R_i(\mathscr{A}) - \sum\limits_{j,k\in[n]} | a_{iijjkk} |, & 2\le n\le 3; \\ R_i(\mathscr{A}) - \sum\limits_{j,k\in[n]} | a_{iijjkk} | - \frac{15}{16}\sum\limits_{(j,k,l,s,t)\in\Delta} | a_{ijklst} |, & n\ge 4. \end{cases}$$

(ii) *If the Z-identify tensor $\mathscr{E}$ is taken as $\mathscr{E}_2$, then* (2) *holds and*

$$r_i^{\Delta\cap\Lambda_i}(\mathscr{A},\alpha_i) = \frac{1}{16}\sum_{j\neq k\neq i}\ \sum_{\{(i_2,\ldots,i_6)\}\in\{(\pi(i,j,j,k,k))\}} \left| a_{ii_2\ldots i_6} - \frac{1}{15}\alpha_i \right|,$$

$$r_i^{\overline{\Delta}\cap\Lambda_i}(\mathscr{A},\alpha_i) = |a_{iiiiii}-\alpha_i| + \sum_{j\neq i}\Bigg(\sum_{\{(i_2,\ldots,i_6)\}\in\{(\pi(i,i,i,j,j))\}} \left| a_{ii_2\ldots i_6} - \frac{1}{5}\alpha_i \right| + \sum_{\{(i_2,\ldots,i_6)\}\in\{(\pi(i,j,j,j,j))\}} \left| a_{ii_2\ldots i_6} - \frac{1}{5}\alpha_i \right| \Bigg),$$

$$\begin{aligned} r_i^{\Delta\cap\overline{\Lambda}_i}(\mathscr{A}) = \frac{1}{16}\Bigg\{ & \sum_{j\neq k\neq l\neq i}\Bigg(\sum_{\{(i_2,\ldots,i_6)\}\in\{(\pi(i,j,j,k,l))\}} \left| a_{ii_2\ldots i_6} \right| \\ & + \sum_{\{(i_2,\ldots,i_6)\}\in\{(\pi(i,i,j,k,l))\}} \left| a_{ii_2\ldots i_6} \right| \\ & + \sum_{\{(i_2,\ldots,i_6)\}\in\{(\pi(j,j,k,k,l))\}} \left| a_{ii_2\ldots i_6} \right| \\ & + \sum_{\{(i_2,\ldots,i_6)\}\in\{(\pi(j,j,j,k,l))\}} \left| a_{ii_2\ldots i_6} \right| \Bigg) \\ & + \sum_{j\neq k\neq l\neq s\neq i}\Bigg(\sum_{\{(i_2,\ldots,i_6)\}\in\{(\pi(i,j,k,l,s))\}} \left| a_{ii_2\ldots i_6} \right| \\ & + \sum_{\{(i_2,\ldots,i_6)\}\in\{(\pi(j,j,k,l,s))\}} \left| a_{ii_2\ldots i_6} \right| \Bigg) \\ & + \sum_{j\neq k\neq l\neq s\neq p\neq i}\ \sum_{\{(i_2,\ldots,i_6)\}\in\{(\pi(j,k,l,s,p))\}} \left| a_{ii_2\ldots i_6} \right| \Bigg\}, \end{aligned}$$

$$\begin{aligned} r_i^{\overline{\Delta}\cap\overline{\Lambda}_i}(\mathscr{A}) = {} & R_i(\mathscr{A}) - |a_{iiiiii}| - 16 r_i^{\Delta\cap\overline{\Lambda}_i}(\mathscr{A}) \\ & - \sum_{j\neq k\neq i}\ \sum_{\{(i_2,\ldots,i_6)\}\in\{(\pi(i,j,j,k,k))\}} \left| a_{ii_2\ldots i_6} \right| \\ & - \sum_{j\neq i}\Bigg(\sum_{\{(i_2,\ldots,i_6)\}\in\{(\pi(i,i,i,j,j))\}} \left| a_{ii_2\ldots i_6} \right| \end{aligned}$$

$$+\sum_{\{(i_2,\ldots,i_6)\}\in\{(\pi(i,j,j,j,j))\}} \left|a_{ii_2\ldots i_6}\right|\Big),$$

where $\{(\pi(i,j,k,l,s))\}$ *represents the set of all permutations of indexes* i,j,k,l,s.

3 Applications

In this section, two applications are considered. By using the inclusion set $\Omega(\mathscr{A},\alpha)$ in Theorem 4, we give sufficient conditions for the positive definiteness of an even-order real symmetric tensor (also the homogeneous polynomial forms) and the asymptotically stability of time-invariant polynomial systems.

3.1 Positive Definiteness of Homogeneous Polynomial Forms

Based on the inclusion interval $\mathscr{G}(\mathscr{A},\alpha)$ in Theorem 2, Li et al. in [8] obtained a sufficient condition of the positive definiteness of an even-order tensor as follows.

Definition 1 [8, Definition 3.1] Let $\mathscr{A}\in\mathbb{R}^{[m,n]}$ with m even and $\alpha=(\alpha_1,\ldots,\alpha_n)^\top\in\mathbb{R}^n$. We call $\mathscr{A}$ an α-strictly diagonally dominant tensor of even order if

$$\alpha_i > R_i(\mathscr{A},\alpha_i),\quad i\in[n]. \tag{13}$$

Theorem 6 [8, Theorem 3.2] *Let* $\mathscr{A}=(a_{i_1\ldots i_m})\in\mathbb{R}^{[m,n]}$ *with* m *even, and* λ *be a* Z*-eigenvalue of* $\mathscr{A}$. *If* $\mathscr{A}$ *is an* α*-strictly diagonally dominant tensor with all* $\alpha_i>0$ *for each* $i\in[n]$, *then* $\lambda>0$. *Furthermore, if* $\mathscr{A}$ *is also symmetric, then* $\mathscr{A}$ *is positive definite, consequently,* $f(x)$ *defined in* (1) *is positive definite.*

Based on the inclusion interval $\Upsilon(\mathscr{A},\alpha)$ in Theorem 3, Shen et al. in [22] obtained a sufficient condition of the positive definiteness of an even-order weakly symmetric tensor as follows.

Theorem 7 [22, Theorem 3] *Let* λ *be a* Z*-eigenvalue of* $\mathscr{A}=(a_{i_1\ldots i_m})\in\mathbb{R}^{[m,n]}$ *and* $\mathscr{E}\in\mathbb{R}^{[m,n]}$ *be a* Z*-identity tensor. If there exists a positive real vector* $\alpha=(\alpha_1,\ldots,\alpha_n)^\top$ *and* $i,j\in[n]$ *with* $j\neq i$ *such that*

$$\Big(\alpha_i - R_i^j(\mathscr{A},\alpha_i)\Big)\alpha_j >\mid a_{ij\ldots j}-\alpha_i e_{ij\ldots j}\mid R_j(\mathscr{A},\alpha_j), \tag{14}$$

then $\lambda>0$. *Further, if* $\mathscr{A}$ *weakly symmetric, then* $\mathscr{A}$ *is positive definite and* $f(x)$ *defined in* (1) *is positive definite.*

Based on the inclusion interval $\Omega(\mathscr{A},\alpha)$ in Theorem 4, a sufficient condition for the positive definiteness of an even-order tensor can be obtained.

Definition 2 Let $\mathscr{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$ with m even. If there is $\alpha = (\alpha_1, \dots, \alpha_n)^\top \in \mathbb{R}^n$ such that for any $i, j \in [n]$ and $j \neq i$,

$$(|\alpha_i| - r_i(\mathscr{A}, \alpha_i) + | a_{ij\dots j} |)|\alpha_j| > |a_{ij\dots j}| r_j(\mathscr{A}, \alpha_j), \tag{15}$$

then we call $\mathscr{A}$ an even-order double α-strictly diagonally dominant tensor.

Now, the relationship between α-strictly diagonally dominant tensors and double α-strictly diagonally dominant tensors is discussed.

Theorem 8 *Let $\mathscr{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$. If $\mathscr{A}$ is an α-strictly diagonally dominant tensor, then $\mathscr{A}$ is a double α-strictly diagonally dominant tensor.*

Proof Let $\mathscr{A}$ be an α-strictly diagonally dominant tensor. Then for each $i \in [n]$, we have

$$|\alpha_i| > R_i(\mathscr{A}, \alpha_i) \geq r_i(\mathscr{A}, \alpha_i),$$

that is,

$$|\alpha_i| > r_i(\mathscr{A}, \alpha_i),$$

which implies that

$$|\alpha_i| - r_i(\mathscr{A}, \alpha_i) + |a_{ij\dots j}| > |a_{ij\dots j}|, \quad j \in [n], \quad j \neq i. \tag{16}$$

For this index $j \in [n]$, we have

$$| \alpha_j | > r_j(\mathscr{A}, \alpha_j). \tag{17}$$

Multiplying (16) and (17) yields (15), which implies that $\mathscr{A}$ is a double α-strictly diagonally dominant tensor. □

Theorem 9 *Let $\mathscr{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$ with m even and λ be any Z-eigenvalue of $\mathscr{A}$. If there is positive vector $\alpha = (\alpha_1, \dots, \alpha_n)^\top \in \mathbb{R}^n$ such that $\mathscr{A}$ is a double α-strictly diagonally dominant tensor, then $\lambda > 0$. Furthermore, if $\mathscr{A}$ is symmetric, then $\mathscr{A}$ is positive definite and, consequently, $f(x)$ is positive definite.*

Proof Suppose on the contrary that $\lambda \leq 0$. According to Theorem 4, we have $\lambda \in \Omega(\mathscr{A}, \alpha)$, which implies that there are $i, j \in [n]$ and $i \neq j$ such that $\lambda \in \Omega_{i,j}(\mathscr{A}, \alpha)$, i.e.,

$$(|\lambda - \alpha_i| - r_i(\mathscr{A}, \alpha_i) + | a_{ij\dots j} |)|\lambda - \alpha_j| \leq |a_{ij\dots j}| r_j(\mathscr{A}, \alpha_j). \tag{18}$$

On the other hand, by $\alpha_i > 0$, $\alpha_j > 0$, $\lambda \leq 0$ and (15), it follows that

$$\begin{aligned}(|\lambda - \alpha_i| - r_i(\mathscr{A}, \alpha_i) + |a_{ij\dots j}|)|\lambda - \alpha_j| &\geq (|\alpha_i| - r_i(\mathscr{A}, \alpha_i) + | a_{ij\dots j} |)|\alpha_j| \\ &> |a_{ij\dots j}| r_j(\mathscr{A}, \alpha_j). \end{aligned} \tag{19}$$

It is easy to see that (18) and (19) contradict each other. Consequently, $\lambda > 0$. Furthermore, if $\mathscr{A}$ is symmetric tensor, then all Z-eigenvalues of $\mathscr{A}$ are positive, which implies that $\mathscr{A}$ is positive definite and, consequently, $f(x)$ is positive definite. □

Finally, an example is given to verify the effectiveness of Theorem 9. Before that, a lemma is recalled.

Lemma 2 [21, Lemma 4.2] *Let*

$$g(x) = x - \frac{1}{a}\sum_{i\in[n]}|x - b_i| - c$$

be a real-valued function about x, where a is a positive integer, $b_i \in \mathbb{R}$ and $b_1 \le b_2 \le \ldots \le b_n$ with $n \ge a$, and $c \in \mathbb{R}$.

(I) *Assume that a is odd.*

(I.i) *If n is odd, then*

$$\max_{x\in\mathbb{R}} g(x) = \frac{1}{a}\left(\sum_{i=1}^{\frac{n+a}{2}} b_i - \sum_{i=\frac{n+a}{2}+1}^{n} b_i\right) - c, \tag{20}$$

and this takes place for every $x \in [b_{\frac{n+a}{2}}, b_{\frac{n+a}{2}+1}]$ if $b_{\frac{n+a}{2}} \ne b_{\frac{n+a}{2}+1}$, and only for $x = b_{\frac{n+a}{2}}$ if $b_{\frac{n+a}{2}} = b_{\frac{n+a}{2}+1}$. Note that let $[b_{\frac{n+a}{2}}, b_{\frac{n+a}{2}+1}]$ be $[b_{\frac{n+a}{2}}, +\infty)$ if $b_{\frac{n+a}{2}+1}$ does not exist.

(I.ii) *If n is even, then*

$$\max_{x\in\mathbb{R}} g(x) = \frac{1}{a}\left(\sum_{i=1}^{\frac{n+a-1}{2}} b_i - \sum_{i=\frac{n+a+3}{2}}^{n} b_i\right) - c, \tag{21}$$

and this maximum is reached when $x = b_{\frac{n+a+1}{2}}$.

(II) *Assume that a is even. If n is odd, then (21) holds. And if n is even, then (20) holds.*

Example 1 Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[4,3]}$ with entries defined as follows:

$$\begin{aligned}
&a_{1111} = 2.6, \quad a_{2222} = 3.2, \quad a_{3333} = 2, \quad a_{1112} = a_{1121} = a_{1211} = a_{2111} = 0.4,\\
&a_{1122} = a_{1212} = a_{1221} = a_{2112} = a_{2121} = a_{2211} = 0.9,\\
&a_{1133} = a_{1313} = a_{1331} = a_{3113} = a_{3131} = a_{3311} = 1.1,\\
&a_{1233} = a_{1323} = a_{1332} = a_{2133} = a_{2313} = a_{2331} = 0.4,\\
&a_{3123} = a_{3132} = a_{3213} = a_{3231} = a_{3312} = a_{3321} = 0.3,\\
&a_{2223} = a_{2232} = a_{2322} = a_{3222} = 0.4,\\
&a_{2233} = a_{2323} = a_{2332} = a_{3223} = a_{3232} = a_{3322} = 1,
\end{aligned}$$

Table 1 Numerical results of (15) with the Z-identify tensor $\mathscr{E}_1$

	$(\lvert\alpha_i\rvert - r_i(\mathscr{A},\alpha_i)+\mid a_{ij\ldots j}\mid)\lvert\alpha_j\rvert$	$\lvert a_{ij\ldots j}\rvert r_j(\mathscr{A},\alpha_j)$
$i=1$ and $j=2$	0.400	0
$i=1$ and $j=3$	1.200	0
$i=2$ and $j=1$	1.875	0.96
$i=2$ and $j=3$	0.200	0
$i=3$ and $j=1$	1.125	0
$i=3$ and $j=2$	2.000	1.56

and $a_{ijkl}=0$ for otherwise.

Our goal is to judge the positive definiteness of $\mathscr{A}$. Because of the form of $R_i(\mathscr{A},\alpha_i)$ and $r_i(\mathscr{A},\alpha_i)$ being related to the Z-identify tensor $\mathscr{E}$, we now divide two cases to consider the positive definiteness of $\mathscr{A}$.

Case I. Let the Z-identify tensor $\mathscr{E}$ be $\mathscr{E}_1=(e_{ijkl})\in\mathbb{R}^{[4,3]}$, i.e.,

$$e_{1111}=e_{1122}=e_{1133}=e_{2222}=e_{2211}=e_{2233}=e_{3333}=e_{3311}=e_{3322}=1,$$

and $e_{ijkl}=0$ for otherwise.

Proposition 1 of [21] shows that Theorem 6 cannot be used to judge the positive definiteness of $\mathscr{A}$ when the Z-identify tensor $\mathscr{E}$ in $R_i(\mathscr{A},\alpha_i)$ is taken as $\mathscr{E}_1$. Now, we consider using Theorem 7 to judge the positive definiteness of $\mathscr{A}$. Let $\alpha=(2.5,4,2)^\top$. By

$$(\alpha_2-R_2^1(\mathscr{A},\alpha_2))\alpha_1=-4.25<1.88=\mid a_{2111}-\alpha_2 e_{2111}\mid R_1(\mathscr{A},\alpha_1),$$

it can be seen that (14) does not hold for $i=2$ and $j=1$, which implies that we cannot use Theorem 7 to judge the positive definiteness of $\mathscr{A}$ for this α. However, for this α, we can use Theorem 9 to judge the positive definiteness of $\mathscr{A}$. In fact, by the numerical results of (15) listed in Table 1, it can be seen that (15) holds for all $i,j\in[n]$ and $j\neq i$, which implies that we can use Theorem 9 to judge the positive definiteness of $\mathscr{A}$.

We also use the Z-eigenvalue inclusion sets to judge the positive definiteness of $\mathscr{A}$. By Theorem 1, we have

$$\mathscr{L}(\mathscr{A})=\left\{z\in\mathbb{C}:\mid z\mid\leq\frac{5.9+\sqrt{45.21}}{2}\right\}.$$

By Theorem 2, we have

$$\mathscr{G}(\mathscr{A},\alpha)=\{z\in\mathbb{R}:\mid z-4\mid\leq 13.5\}.$$

By Theorem 3, we have

$$\Upsilon(\mathscr{A},\alpha)=\{z\in\mathbb{R}:(\mid z-4\mid-5.7)\mid z-2.5\mid\leq 1.88\}.$$

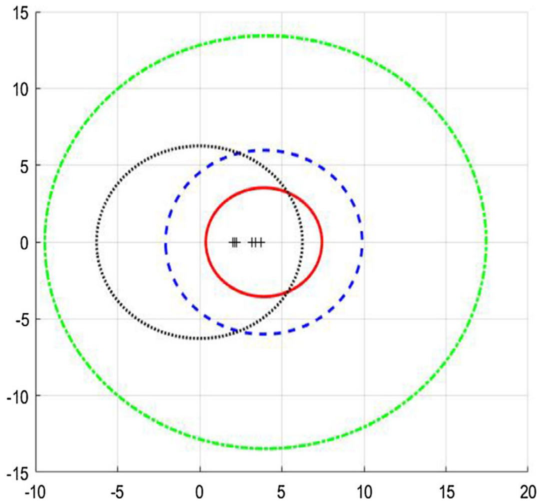


Fig. 1 Comparisons of $\mathscr{L}(\mathscr{A})$, $\mathscr{G}(\mathscr{A}, \alpha)$, $\Upsilon(\mathscr{A}, \alpha)$ and $\Omega(\mathscr{A}, \alpha)$ with $\mathscr{E}_1$ and $\alpha = (2.5, 4, 2)^\top$

By Theorem 4, we have

$$\Omega(\mathscr{A}, \alpha) = \{z \in \mathbb{R} : (\mid z - 4 \mid -3.25) \mid z - 2.5 \mid \leq 0.96\}.$$

The Z-eigenvalue inclusion sets $\mathscr{L}(\mathscr{A})$, $\mathscr{G}(\mathscr{A}, \alpha)$, $\Upsilon(\mathscr{A}, \alpha)$, $\Omega(\mathscr{A}, \alpha)$ and the exact Z-eigenvalues are drawn in Fig. 1, where they are represented by black dotted boundary, green stippled boundary, blue dotted boundary, red solid boundary and black "+," respectively.

Case II. Let the Z-identify tensor $\mathscr{E}$ be $\mathscr{E}_2 = (e_{ijkl}) \in \mathbb{R}^{[4,3]}$, i.e.,

$$e_{1111} = e_{2222} = e_{3333} = 1, \quad e_{1122} = e_{1212} = e_{1221} = \frac{1}{3}, \quad e_{1133} = e_{1313} = e_{1331} = \frac{1}{3},$$

and $e_{ijkl} = 0$ for otherwise.

Then

$$\begin{aligned} &R_i(\mathscr{A}, \alpha_i) \\ &\quad =\mid a_{iiii} - \alpha_i \mid + \sum_{j \neq i} \left(\mid a_{iijj} - \frac{1}{3}\alpha_i \mid + \mid a_{ijij} - \frac{1}{3}\alpha_i \mid + \mid a_{ijji} - \frac{1}{3}\alpha_i \mid \right) + \gamma_i, \end{aligned}$$

where

$$\gamma_i = R_i(\mathscr{A}) - \mid a_{iiii} \mid - \sum_{j \neq i} (\mid a_{iijj} \mid + \mid a_{ijij} \mid + \mid a_{ijji} \mid), \quad i \in [3].$$

Table 2 Numerical results of (15) with the Z-identify tensor $\mathscr{E}_2$

	$(\lvert\alpha_i\rvert - r_i(\mathscr{A}, \alpha_i) + \lvert a_{ij\ldots j}\rvert)\lvert\alpha_j\rvert$	$\lvert a_{ij\ldots j}\rvert r_j(\mathscr{A}, \alpha_j)$
$i = 1$ and $j = 2$	5.100	0
$i = 1$ and $j = 3$	12.000	0
$i = 2$ and $j = 1$	6.625	0.32
$i = 2$ and $j = 3$	13.200	0
$i = 3$ and $j = 1$	1.625	0
$i = 3$ and $j = 2$	1.500	0.32

Firstly, we use Theorem 6 to judge the positive definiteness of $\mathscr{A}$. Suppose that there is $\alpha = (\alpha_1, \alpha_2, \alpha_3)^\top \in \mathbb{R}^3$ such that (13) holds, which implies that

$$\begin{aligned} f(\alpha_i) :=& \alpha_i - |a_{iiii} - \alpha_i| - \sum_{j \neq i} \left(|a_{iijj} - \frac{1}{3}\alpha_i| + |a_{ijij} - \frac{1}{3}\alpha_i| + |a_{ijji} - \frac{1}{3}\alpha_i| \right) \\ =& \alpha_i - \frac{1}{3}\left[3|\alpha_i - a_{iiii}| + \sum_{j \neq i}(|\alpha_i - 3a_{iijj}| + |\alpha_i - 3a_{ijij}| + |\alpha_i - 3a_{ijji}|) \right] \\ >& \gamma_i. \end{aligned}$$

By Lemma 2, we have

$$\begin{aligned} \max_{\alpha_1 \in \mathbb{R}} f(\alpha_1) &= 2 < 2.4 = \gamma_1, \\ \max_{\alpha_2 \in \mathbb{R}} f(\alpha_2) &= 2.5 < 2.8 = \gamma_2, \\ \max_{\alpha_3 \in \mathbb{R}} f(\alpha_3) &= 1.7 < 2.2 = \gamma_3, \end{aligned}$$

which shows that there is not α_1, α_2 and α_3 such that (13) holds and implies that we cannot use Theorem 6 to judge the positive definiteness of $\mathscr{A}$.

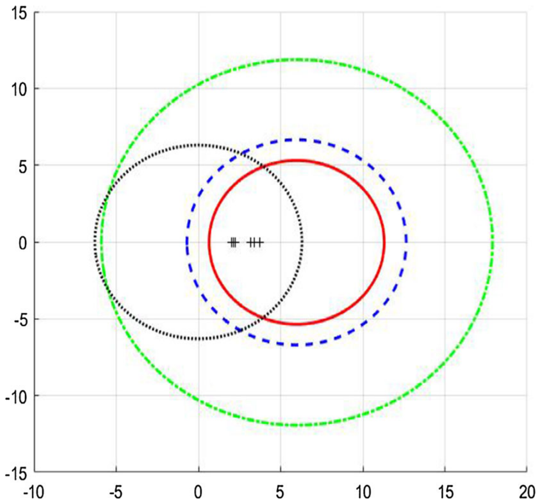
Secondly, we use Theorem 7 to judge the positive definiteness of $\mathscr{A}$. Let $\alpha = (2.5, 3, 6)^\top$. By

$$(\alpha_3 - R_3^2(\mathscr{A}, \alpha_3))\alpha_2 = -3 < 0.56 = | a_{3222} - \alpha_3 e_{3222} | R_2(\mathscr{A}, \alpha_2),$$

it can be seen that (14) does not hold for $i = 3$ and $j = 1$, which implies that we cannot use Theorem 7 to judge the positive definiteness of $\mathscr{A}$.

However, for this $\alpha = (2.5, 3, 6)^\top$, we can use Theorem 9 to judge the positive definiteness of $\mathscr{A}$. The numerical results of (15) are listed in Table 2. From Table 2, it can be seen that (15) holds for all $i, j \in [n]$ and $j \neq i$, which implies that we can use Theorem 9 to judge the positive definiteness of $\mathscr{A}$. In fact, all Z-eigenvalues of $\mathscr{A}$ are 2.0000, 2.0035, 2.0224, 2.1335, 2.2539, 3.2022, 3.4147 and 3.7271.

Fig. 2 Comparisons of $\mathscr{L}(\mathscr{A})$, $\mathscr{G}(\mathscr{A}, \alpha)$, $\Upsilon(\mathscr{A}, \alpha)$ and $\Omega(\mathscr{A}, \alpha)$ with $\mathscr{E}_2$ and $\alpha = (2.5, 3, 6)^\top$



We also use the Z-eigenvalue inclusion sets to judge the positive definiteness of $\mathscr{A}$. Let $\alpha = (2.5, 3, 6)^\top$. By Theorem 2, we have

$$\mathscr{G}(\mathscr{A}, \alpha) = \{z \in \mathbb{R} :\mid z - 6 \mid\leq 11.9\}.$$

By Theorem 3, we have

$$\Upsilon(\mathscr{A}, \alpha) = \{z \in \mathbb{R} : (\mid z - 6 \mid -6.7) \mid z - 2.5 \mid\leq 0\}.$$

By Theorem 4, we have

$$\Omega(\mathscr{A}, \alpha) = \{z \in \mathbb{R} : (\mid z - 6 \mid -5.35) \mid z - 2.5 \mid\leq 0\}.$$

The Z-eigenvalue inclusion sets $\mathscr{L}(\mathscr{A})$, $\mathscr{G}(\mathscr{A}, \alpha)$, $\Upsilon(\mathscr{A}, \alpha)$, $\Omega(\mathscr{A}, \alpha)$ and the exact Z-eigenvalues are drawn in Fig. 2, where they are represented by black dotted boundary, green stippled boundary, blue dotted boundary, red solid boundary and black "+," respectively.

From Figs. 1 and 2, it is easy to see that:

(i) $0 \in \mathscr{L}(\mathscr{A})$, $0 \in \mathscr{G}(\mathscr{A}, \alpha)$ and $0 \in \Upsilon(\mathscr{A}, \alpha)$; consequently, the sets $\mathscr{L}(\mathscr{A})$, $\mathscr{G}(\mathscr{A}, \alpha)$ and $\Upsilon(\mathscr{A}, \alpha)$ cannot be used to judge the positive definiteness of $\mathscr{A}$.
(ii) $\sigma(\mathscr{A}) \subseteq \Omega(\mathscr{A}, \alpha) \subset \mathbb{C}^+$, where $\mathbb{C}^+$ denotes the set of all complex numbers with positive real part, which implies that all Z-eigenvalues of $\mathscr{A}$ are positive and hence $\mathscr{A}$ is positive definite.
(iii) $\sigma(\mathscr{A}) \subseteq \Omega(\mathscr{A}, \alpha) \subseteq \Upsilon(\mathscr{A}, \alpha) \subseteq \mathscr{G}(\mathscr{A}, \alpha)$, and $\mathscr{L}(\mathscr{A})$ and $\Omega(\mathscr{A}, \alpha)$ do not contain each other.

This example shows that no matter we take the Z-identify tensor $\mathscr{E}$ as $\mathscr{E}_1$ or $\mathscr{E}_2$, we can use Theorems 4 and 9 to judge the positive definiteness of the even-order tensors.

3.2 Asymptotically Stability of Time-Invariant Polynomial Systems

Consider the asymptotically stability of the time-invariant polynomial system

$$\Sigma : \dot{x} = \mathscr{A}^{(2)}x + \mathscr{A}^{(4)}x^3 + \cdots + \mathscr{A}^{(2k)}x^{2k-1}, \tag{22}$$

where $\mathscr{A}^{(t)} = (a_{i_1 \dots i_t}) \in \mathbb{R}^{[t,n]}$, $t = 2, 4, \dots, 2k$, and $x = (x_1, \dots, x_n)^\top$; see [1, 2]. A sufficient condition such that the nonlinear system (22) above is asymptotically stable is gave by Deng et al. in [1] as follows.

Theorem 10 [1, Theorem 3.3] *For the nonlinear system Σ in (22), if $-\mathscr{A}^{(t)}$ is positive definite, where $t = 2, 4, \dots, 2k$, then the equilibrium point of Σ is asymptotically stable.*

By Theorems 9 and 10, a sufficient condition for the asymptotically stability can be given.

Theorem 11 *For the nonlinear system Σ in (22), if $-\mathscr{A}^{(t)}$ satisfies all conditions of Theorem 9, where $t = 2, 4, \dots, 2k$, then the equilibrium point of Σ is asymptotically stable.*

Example 2 Consider the following polynomial system

$$\begin{aligned}
\Sigma : \dot{x}_1 &= -5x_1 + 2x_2 + 2x_3 - 2.6x_1^3 - 0.9x_1^2x_2 - 3.3x_1x_3^2 - 2.7x_1x_2^2 - 1.8x_2x_3^2, \\
\dot{x}_2 &= 2x_1 - 5x_2 + 2x_3 - 0.5x_1^3 - 3.2x_2^3 - 0.6x_2^2x_3 \\
&\quad -3.0x_2x_3^2 - 2.7x_1^2x_2 - 1.8x_1x_3^2, \\
\dot{x}_3 &= 2x_1 + 2x_2 - 5x_3 - 0.2x_2^3 - 2.4x_3^3 - 3.3x_1^2x_3 - 3.0x_2^2x_3 - 3.6x_1x_2x_3.
\end{aligned}$$

Apparently, Σ can be written as $\dot{x} = \mathscr{A}^{(2)}x + \mathscr{A}^{(4)}x^3$, where $x = (x_1, x_2, x_3)^\top$,

$$\mathscr{A}^{(2)} = \begin{pmatrix} -5 & 2 & 2 \\ 2 & -5 & 2 \\ 2 & 2 & -5 \end{pmatrix}$$

and $\mathscr{A}^{(4)} = (a_{ijkl}) \in \mathbb{R}^{[4,3]}$ whose entries are as follows:

$$\begin{aligned}
&a_{1111} = -2.6, \quad a_{2222} = -3.2, \quad a_{3333} = -2.4, \\
&a_{1112} = a_{1121} = a_{1211} = a_{2111} = -0.3, \\
&a_{1122} = a_{1212} = a_{1221} = a_{2112} = a_{2121} = a_{2211} = -0.9, \\
&a_{1133} = a_{1313} = a_{1331} = a_{3113} = a_{3131} = a_{3311} = -1.1, \\
&a_{1233} = a_{1323} = a_{1332} = a_{2133} = a_{2313} = a_{2331} = -0.6, \\
&a_{3123} = a_{3132} = a_{3213} = a_{3231} = a_{3312} = a_{3321} = -0.6, \\
&a_{2223} = a_{2232} = a_{2322} = a_{3222} = -0.2, \\
&a_{2233} = a_{2323} = a_{2332} = a_{3223} = a_{3232} = a_{3322} = -1.0,
\end{aligned}$$

Table 3 Numerical results of (15) with the Z-identify tensor $\mathscr{E}_1$ for $-\mathscr{A}^{(4)}$

	$(\|\alpha_i\| - r_i(-\mathscr{A}^{(4)}, \alpha_i) + \| a_{ijjj} \|)\|\alpha_j\|$	$\|a_{ijjj}\| r_j(-\mathscr{A}^{(4)}, \alpha_j)$
$i = 1$ and $j = 2$	0.80	0
$i = 1$ and $j = 3$	1.54	0
$i = 2$ and $j = 1$	2.25	0.84
$i = 2$ and $j = 3$	1.12	0
$i = 3$ and $j = 1$	1.35	0
$i = 3$ and $j = 2$	2.00	0.72

Table 4 Numerical results of (15) with the Z-identify tensor $\mathscr{E}_2$ for $-\mathscr{A}^{(4)}$

	$(\|\alpha_i\| - r_i(-\mathscr{A}^{(4)}, \alpha_i) + \| a_{ijjj} \|)\|\alpha_j\|$	$\|a_{ijjj}\| r_j(-\mathscr{A}^{(4)}, \alpha_j)$
$i = 1$ and $j = 2$	3.00	0
$i = 1$ and $j = 3$	3.50	0
$i = 2$ and $j = 1$	13.25	1.2
$i = 2$ and $j = 3$	5.00	0
$i = 3$ and $j = 1$	4.75	0
$i = 3$ and $j = 2$	3.30	0.1

and $a_{ijkl} = 0$ for otherwise.

It is easy to see that $-\mathscr{A}^{(2)}$ is positive definite. Now, we judge the positive definiteness of $-\mathscr{A}^{(4)}$ by taking the Z-identify tensor $\mathscr{E}$ as $\mathscr{E}_1$ and $\mathscr{E}_2$.

Case I. Let the Z-identify tensor $\mathscr{E}$ be $\mathscr{E}_1 = (e_{ijkl}) \in \mathbb{R}^{[4,3]}$.

Taking $\alpha = (3, 4, 2.8)^\top$, the numerical results of (15) are listed in Table 3. From Table 3, it can be seen that (15) holds for all $i, j \in [3]$ and $i \neq j$. Hence, $-\mathscr{A}^{(4)}$ is positive definite by Theorem 9.

Case II. Let the Z-identify tensor $\mathscr{E}$ be $\mathscr{E}_2 = (e_{ijkl}) \in \mathbb{R}^{[4,3]}$.

Taking $\alpha = (5, 3, 2)^\top$, the numerical results of (15) are listed in Table 4. From Table 4, it can be seen that (15) holds for all $i, j \in [3]$ and $i \neq j$, which implies that $-\mathscr{A}^{(4)}$ is positive definite by Theorem 9.

All in all, no matter the Z-identify tensor $\mathscr{E}$ is taken as $\mathscr{E}_1$ or $\mathscr{E}_2$, we can both judge the positive definiteness of $-\mathscr{A}^{(4)}$ by Theorem 9. In fact, all Z-eigenvalues of $-\mathscr{A}^{(4)}$ are 1.8287, 1.9538, 2.2721, 2.4000, 3.0829, 3.0838, 3.0954, 3.8162 and 3.9610. Furthermore, by Theorem 11, the equilibrium point of Σ is asymptotically stable.

4 Conclusions

In this paper, we firstly presented a new Z-eigenvalue inclusion set $\Omega(\mathscr{A}, \alpha)$ in Theorem 4. Subsequently, we in Theorem 5 proved that it is tighter than the inclusion set $\mathscr{G}(\mathscr{A}, \alpha)$ in Theorem 2.2 of [8] and the inclusion set $\Upsilon(\mathscr{A}, \alpha)$ in Theorem 1 of

[22]. As an application of the new set $\Omega(\mathscr{A}, \alpha)$, we obtained a sufficient condition for the positive definiteness of an even-order real symmetric tensor (also homogeneous polynomial forms) in Theorem 9 and obtained a sufficient condition for the asymptotically stability of time-invariant polynomial systems in Theorem 11. Finally, we used Examples 1 and 2 to verify the validity of Theorems 9 and 11.

However, how to choose appropriate parameter vector α to minimize the Z-eigenvalue inclusion set $\Omega(\mathscr{A}, \alpha)$ in Theorem 4 is still an unsolved problem. We will continue to study this problem in the future.

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Declarations

Conflict of interest The author declares that he has no competing interests.

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