Bull. Malays. Math. Sci. Soc. (2023) 46:11
https://doi.org/10.1007/s40840-022-01395-w

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Quantum State Transfer on Neighborhood Corona of Two Graphs

Xiao-Qin Zhang[1] · Qi Xiong[1] · Gui-Xian Tian[1] · Shu-Yu Cui[2]

Received: 4 July 2022 / Revised: 14 October 2022 / Accepted: 4 November 2022 /
Published online: 22 November 2022

Abstract
Given two graphs G_1 of order n_1 and G_2, the neighborhood corona of G_1 and G_2, denoted by $G_1 \bigstar G_2$, is the graph obtained by taking one copy of G_1 and taking n_1 copies of G_2, in the meanwhile, linking all the neighbors of the i-th vertex of G_1 with all vertices of the i-th copy of G_2. In our work, we give some conditions that $G_1 \bigstar G_2$ is not periodic. Furthermore, we demonstrate some sufficient conditions for $G_1 \bigstar G_2$ having no perfect state transfer. Some examples are provided to explain our results. In addition, for the reason that the graph admitting perfect state transfer is rare, we also consider pretty good state transfer on neighborhood corona of two graphs. We show some sufficient conditions for $G_1 \bigstar G_2$ admitting pretty good state transfer.

Keywords Quantum state transfer · Neighborhood Corona · Spectrum · Perfect state transfer · Pretty good state transfer

Mathematics Subject Classification 05C50 · 81P68

Communicated by Rosihan M. Ali.

✉ Gui-Xian Tian
gxtian@zjnu.cn

Xiao-Qin Zhang
1136081490@qq.com

Qi Xiong
252321425@qq.com

Shu-Yu Cui
cuishuyu@zjnu.cn

[1] Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China

[2] Xingzhi College, Zhejiang Normal University, Jinhua 321004, China

1 Introduction

Quantum walk is a natural generalization of classical random walk on graphs. In 1998, Farhi and Gutmann [19] firstly put forward the concept of continuous-time quantum walk. Given a graph G, let $A(G)$, $L(G)$ and $Q(G)$ be its adjacency matrix, Laplacian matrix and signless Laplacian matrix, respectively. Suppose that $X(G)$ is a Hermitian matrix associated with G, then the unitary matrix $H_G(t) = \exp(-\mathbf{i}tX(G))$ is the transition matrix of continuous-time quantum walk corresponding to $X(G)$ for $\mathbf{i}^2 = -1$ and $t > 0$, where $X(G)$ may be $A(G)$, $L(G)$, $Q(G)$ and so on. In 2003, Bose [6] studied the task of information transition in a quantum spin system. Christandl et al. [13] showed that this task can be lessened to the question of perfect state transfer. Let e_x^n be a column vector of order n whose element of x-th position is 1 and 0, otherwise. Sometimes, e_x^n can be recorded briefly by e_x. If

$$\exp(-\mathbf{i}tX(G))e_x = \lambda e_y \tag{1}$$

with $|\lambda| = 1$ for two vertices x, y of G, then we say that G admits *perfect state transfer* (PST for short) relative to matrix $X(G)$ between x and y at time t. Particularly, if $X(G) = A(G)$ (resp. $L(G)$ or $Q(G)$), then G admits *perfect state transfer* (resp. *Laplacian perfect state transfer* (LPST for short) or *signless Laplacian perfect state transfer* (SLPST for short)).

Given that graphs admitting PST are rare, Godsil [22] proposed a new concept called pretty good state transfer, whose restriction is more relaxing than that of PST. The transpose of x is denoted as x^T. If

$$|e_y^T \exp(-\mathbf{i}\tau X(G))e_x| > 1 - \epsilon, \tag{2}$$

for any $\epsilon > 0$, then we say that G admits *pretty good state transfer* (PGST for short) between two vertices x and y at time τ. Similarly, if $X(G) = A(G)$ (resp. $L(G)$ or $Q(G)$), then G admits *pretty good state transfer* (resp. *Laplacian pretty good state transfer*(LPGST for short) or *signless Laplacian pretty good state transfer* (SLPGST for short)).

Recently, many articles focus on PST and PGST of composite graphs which are obtained by some graph operations. For example, Li et al. [25] considered LPST and LPGST of $\mathcal{Q}$-graph. They showed that, for an r-regular graph G, if $r + 1$ is a prime, then the $\mathcal{Q}$-graph of G has no LPST but admits LPGST. Recently, Zhang et al. [36] studied SLPST and SLPGST in $\mathcal{Q}$-graph. Ackelsberg et al. [1] considered LPST and LPGST of corona graphs. They showed that corona graph $G \circ H$ has no LPST, but it occurs LPGST under some special conditions. In 2017, Ackelsberg et al. [2] considered PST and PGST of corona graphs. They showed that $G \circ K_n$ has no PST and $G \circ K_1$ admits PGST under some suitable conditions. In 2021, Tian et al. [34] considered SLPST and SLPGST of corona graphs. They demonstrated that $K_2 \circ H$ has no SLPST between the two vertices of K_2. They also showed that $G \circ \overline{K_m}$ admits SLPGST under some suitable condition. In 2021, Wang and Liu [35] considered LPST and LPGST of edge complemented coronas. They gave some sufficient conditions such that edge complemented corona $G \diamond H$ has no LPST. They also showed that $G \diamond H$ admits LPGST under some suitable conditions. Recently, Li et al. [26] gave some sufficient conditions for extended neighborhood coronas to have Laplacian perfect state transfer. What is noteworthy is that, due to some nice algebraic structures of Cayley graphs over finite abelian groups, their PST and PGST have also received widely attention. For example, Bašić [4, 5], Pal [30], Cheung and Godsil [12] gave some characterizations on circulant graphs and cubelike graphs admitting PST. Tan et al. [33] presented a characterization of Cayley graphs over abelian groups admitting PST, which generalized some known results on circulant graphs and cubelike graphs. Along this line, PST on weight Cayley graphs and PEST on Cayley graphs were studied in [8] and [11], respectively. Recently, PGST on Cayley graphs are also investigated in [10, 31]. For more details about these directions, readers may refer to [3, 7, 9, 14–16, 18, 22, 28, 37] and the cited references therein.

Motivated by the aforementioned results, we mainly focus on PST and PGST of neighborhood corona of two graphs. Given two graphs G_1 of order n_1 and G_2, the *neighborhood corona* of G_1 and G_2, denoted by $G_1\bigstar G_2$, is the graph obtained by taking one copy of G_1 and n_1 copies of G_2, in the meanwhile, linking all the neighbors of the i-th vertex of G_1 with all vertices of the i-th copy of G_2. In our work, we first give some conditions that $G_1\bigstar G_2$ is not periodic. With the help of these conditions, we demonstrate some sufficient conditions for $G_1\bigstar G_2$ having no PST. Furthermore, some examples are provided to explain our results. Finally, for the reason that the graph admitting PST is rare, we also consider PGST on neighborhood corona of two graphs. We show some sufficient conditions for $G_1\bigstar G_2$ admitting PGST. It turns out that $G_1\bigstar \overline{K_{n_2}}$ with G_1 admitting PST, $C_4\bigstar \overline{K_{n_2}}$ and $C_4\bigstar G$ have PGST under some special conditions.

2 Preliminaries

Throughout this paper, we only consider undirected simple graphs. Let $\mathbf{j}_n$ and J_n be the all-one column vector of order n and all-one square matrix of order n, respectively. Let $[n]$ be the set of $\{1, 2, \ldots, n\}$.

Suppose that G is a graph of order n and λ_j is the eigenvalue of $A(G)$ with multiplicity l_j for $j \in [p]$, where $l_1 + \cdots + l_p = n$. The spectrum of $A(G)$ is denoted by Sp(G), then Sp$(G) = \{\lambda_j^{l_j} : j \in [p]\}$. Let $\{x_1^{(j)}, x_2^{(j)}, \ldots, x_{l_j}^{(j)}\}$ be an orthonomal basis of the eigenvalue space V_{λ_j} of λ_j. The eigenprojector of λ_j, denoted by $f_{\lambda_j}(G)$, is $f_{\lambda_j}(G) = \sum_{i=1}^{l_j} x_i^{(j)}(x_i^{(j)})^T$ and $\sum_{j=1}^{p} f_{\lambda_j}(G) = I$. Obviously, $f_{\lambda_j}(G)f_{\lambda_k}(G) = 0$ for $j \neq k$ and $f_{\lambda_j}(G)$ is an idempotent matrix. According to the eigenprojectors, we get the spectral decomposition of $A(G)$, i.e.,

$$A(G) = A(G)\sum_{j=1}^{p} f_{\lambda_j}(G) = \sum_{j=1}^{p} \lambda_j f_{\lambda_j}(G). \tag{3}$$

Hence,

$$H_G(t) = \exp(-\mathbf{i}tA(G)) = \sum_{j=1}^{p} e^{-\mathbf{i}t\lambda_j} f_{\lambda_j}(G). \tag{4}$$

The eigenvalue support of vertex x, denoted by $\mathrm{supp}_G(x)$, is the set of eigenvalues λ_j such that $f_{\lambda_j}(G)e_x \neq 0$. Two vertices x and y are *strongly cospectral* whenever $f_{\lambda_j}(G)e_x = \pm f_{\lambda_j}(G)e_y$ for any $j \in [p]$. Let $S^+ = \{\lambda : f_\lambda(G)e_x = f_\lambda(G)e_y\}$ and $S^- = \{\lambda : f_\lambda(G)e_x = -f_\lambda(G)e_y\}$.

Theorem 2.1 *(*Coutinho [14]*) Assume that G is a graph with vertex set satisfying $|V(G)| \geq 2$, and $u, v \in V(G)$. If λ_0 is the maximum eigenvalue of G, then G admits PST between the vertices u and v if and only if the following conditions hold.*

(i) *Two vertices, u and v, are strongly cospectral.*
(ii) *Nonzero elements in $supp_G(u)$ are either all integers or all quadratic integers. Moreover, for each eigenvalue $\lambda \in supp_G(u)$, there exists a square-free integer Δ and integers a, b_λ such that,*

$$\lambda = \frac{1}{2}(a + b_\lambda\sqrt{\Delta}).$$

Here, we allow $\Delta = 1$ if all eigenvalues in $supp_G(u)$ are integers, and $a = 0$ if all eigenvalues in $supp_G(u)$ are multiples of $\sqrt{\Delta}$.
(iii) *$\lambda \in S^+$ if and only if $\frac{\lambda_0 - \lambda}{g\sqrt{\Delta}}$ is even and $\lambda \in S^-$ if and only if $\frac{\lambda_0 - \lambda}{g\sqrt{\Delta}}$ is odd, where*

$$g = \gcd\left(\left\{\frac{\lambda_0 - \lambda}{\sqrt{\Delta}} : \lambda \in supp_G(u)\right\}\right).$$

Moreover, if the conditions above hold, then the following also hold.

(1) *There exists a minimum time $\tau_0 > 0$ at which PST occurs between u and v, and*

$$\tau_0 = \frac{1}{g}\frac{\pi}{\sqrt{\Delta}}.$$

(2) *The time of PST, τ is an odd multiple of τ_0.*
(3) *The phase of PST is given by $\lambda = e^{-\mathbf{i}\tau\lambda_0}$.*

In order to characterize graphs admitting PST (or PGST), the following two lemmas play a crucial role in our study process.

Lemma 2.2 *(*Godsil [20]*) If a graph G admits PST between two vertices u and v at time t, then G is periodic at vertex u (or v) at time 2t.*

Lemma 2.3 *(*Godsil [21]*) A graph G at vertex v is periodic if and only if one of the following conditions holds:*

(i) *all elements of* $supp_G(v)$ *are integers;*
(ii) *for each eigenvalue of* $supp_G(v)$*, there is a square-free integer* Δ*, integer* a *and corresponding some integer* b_λ *so that* $\lambda = \frac{1}{2}(a + b_\lambda\sqrt{\Delta})$.

Theorem 2.4 *(*Hardy and Wright [24]*) Assume that* $1, \lambda_1, \ldots, \lambda_m$ *are linearly independent over* $\mathbb{Q}$*. Then, for any real numbers* $\alpha_1, \ldots, \alpha_m$ *and* $N > 0$*,* $\epsilon > 0$*, there exist integers* $\alpha > N$ *and* $\gamma_1, \ldots, \gamma_m$ *such that*

$$|\alpha\lambda_k - \gamma_k - \alpha_k| < \epsilon, \tag{5}$$

for each $k \in [m]$*. Equivalently, (5) can be restated by* $\alpha\lambda_k - \gamma_k \approx \alpha_k$ *for omitting the dependence on* ϵ*.*

Lemma 2.5 *(*Richards [32]*) The set* $\{\sqrt{\Delta} : \Delta$ *is a square-free integer*$\}$ *is linearly independent over* $\mathbb{Q}$*.*

The following characterization of quadratic integer was originally given by Dedekind in his supplements of lectures by Dirichlet (see [17]). For ease of reading, Coutinho reproduced the proof of this result in [14].

Lemma 2.6 *(*Dirichlet and Dedekind [17]*) A real number* λ *is a quadratic integer if and only if there exist integers* a*,* b *and* Δ *such that* Δ *is square-free and one of the following cases holds:*

(i) $\lambda = a + b\sqrt{\Delta}$ *and* $\Delta \equiv 2, 3 \ (mod\ 4)$*;*
(ii) $\lambda = \frac{1}{2}(a + b\sqrt{\Delta})$*,* $\Delta \equiv 1 \ (mod\ 4)$*, and* a *and* b *have the same parity.*

3 Neighborhood Corona of Graphs

We denote the neighborhood corona of graph G_1 and G_2 by $G_1 \bigstar G_2$. Let the ordered pair $V(G_1 \bigstar G_2) = V(G_1) \times (0 \cup V(G_2))$ be the vertex set of $G_1 \bigstar G_2$ and $|V(G_1)| = n_1$, $|V(G_2)| = n_2$. According to the definition of $G_1 \bigstar G_2$, the adjacency relation is given by:

$$((x, y), (x', y')) \in E(G_1 \bigstar G_2) \Longleftrightarrow \begin{cases} y = y' = 0 \quad \text{and} \quad (x, x') \in E(G_1) \qquad \text{or} \\ x = x' \quad \text{and} \quad (y, y') \in E(G_2) \qquad \text{or} \\ (x, x') \in E(G_1) \quad \text{and exactly one of } y \text{ and } y' \text{ is } 0. \end{cases} \tag{6}$$

According to the adjacency relation above, we easily get the adjacency matrix of $G_1 \bigstar G_2$ as follows:

$$A(G_1 \bigstar G_2) = \begin{pmatrix} A(G_1) & A(G_1) \otimes \mathbf{j}_{n_2}^T \\ A(G_1) \otimes \mathbf{j}_{n_2} & I_{n_1} \otimes A(G_2) \end{pmatrix}, \tag{7}$$

where "$\otimes$" denotes the Kronecker product of two matrices.

At first, we recall the adjacency spectrum of $G_1 \bigstar G_2$, which is attributed to Gopalapillai in [23].

Theorem 3.1 *(Gopalapillai [23]) Let G_1 be any graph of order n_1 and G_2 be any k-regular connected graph with n_2 vertices. Let $Sp(G_1) = \{\lambda_i^{l_i} | i \in [p]\}$ and $Sp(G_2) = \{\eta_j^{l'_j} | j \in [q]\}$, where the power represents the multiplicity. Then, the eigenvalues of $A(G_1 \bigstar G_2)$ are*

(1)

$$\lambda_{i+} = \frac{\lambda_i + k + \sqrt{(\lambda_i - k)^2 + 4n_2\lambda_i^2}}{2} \tag{8}$$

and

$$\lambda_{i-} = \frac{\lambda_i + k - \sqrt{(\lambda_i - k)^2 + 4n_2\lambda_i^2}}{2} \tag{9}$$

with multiplicity l_i for $i \in [p]$.

(2) η_j with multiplicity n_1 for $j \in [q] \setminus \{1\}$.

In particular, if there is some $i_0 \in [p]$ such that $\lambda_{i_0} = 0$, then

$$\lambda_{i_0+} = \frac{\lambda_{i_0} + k + \sqrt{(\lambda_{i_0} - k)^2 + 4n_2\lambda_{i_0}^2}}{2} = k \tag{10}$$

and

$$\lambda_{i_0-} = \frac{\lambda_{i_0} + k - \sqrt{(\lambda_{i_0} - k)^2 + 4n_2\lambda_{i_0}^2}}{2} = 0 \tag{11}$$

with multiplicity l_{i_0}.

Proposition 3.2 *Let G_1 be any graph with n_1 vertices and G_2 be a k-regular graph with n_2 vertices. Let $Sp(G_1) = \{\lambda_i{}^{l_i} | i \in [p]\}$ and $Sp(G_2) = \{\eta_j{}^{l'_j} | j \in [q]\}$, where the respective multiplicities of λ_i and η_j are l_i and l'_j for $i \in [p]$, $j \in [q]$.*

(i) If $\lambda_i \neq 0$ for any $i \in [p]$, then

(1) the eigenprojector of eigenvalue $\lambda_{i\pm}$ of $A(G_1 \bigstar G_2)$ is

$$F_{\lambda_{i\pm}}(G_1 \bigstar G_2) = \frac{\lambda_i{}^2}{(\lambda_{i\pm} - k)^2 + n_2\lambda_i{}^2} \begin{pmatrix} \frac{(\lambda_{i\pm}-k)^2}{\lambda_i{}^2} f_{\lambda_i}(G_1) & \frac{(\lambda_{i\pm}-k)}{\lambda_i} f_{\lambda_i}(G_1) \otimes \mathbf{j}_{n_2}^T \\ \frac{(\lambda_{i\pm}-k)}{\lambda_i} f_{\lambda_i}(G_1) \otimes \mathbf{j}_{n_2} & f_{\lambda_i}(G_1) \otimes J_{n_2} \end{pmatrix} \tag{12}$$

for $i \in [p]$;

(2) *the eigenprojector of eigenvalue η_j of $A(G_1 \bigstar G_2)$ is*

$$F_{\eta_j}(G_1 \bigstar G_2) = \begin{pmatrix} 0 & 0 \\ 0 & I_{n_1} \otimes (f_{\eta_j}(G_2) - \delta_{\eta_j,k}\frac{1}{n_2}J_{n_2}) \end{pmatrix} \tag{13}$$

for $j \in [q]$.

Hence, the spectrum decomposition of $A(G_1\bigstar G_2)$ is given by:

$$A(G_1\bigstar G_2)=\sum_{i=1}^{p}\sum_{\pm}\lambda_{i\pm}F_{\lambda_{i\pm}}(G_1\bigstar G_2)+\sum_{j=1}^{q}\eta_j F_{\eta_j}(G_1\bigstar G_2). \tag{14}$$

(i) If there is some $i_0\in[p]$ such that $\lambda_{i_0}=0$, then

(1) the eigenprojector of eigenvalue $\lambda_{i\pm}$ of $A(G_1\bigstar G_2)$ is

$$F_{\lambda_{i\pm}}(G_1\bigstar G_2)=\frac{{\lambda_i}^2}{(\lambda_{i\pm}-k)^2+n_2{\lambda_i}^2}\begin{pmatrix}\frac{(\lambda_{i\pm}-k)^2}{{\lambda_i}^2}f_{\lambda_i}(G_1) & \frac{(\lambda_{i\pm}-k)}{\lambda_i}f_{\lambda_i}(G_1)\otimes\mathbf{j}_{n_2}^T\\ \frac{(\lambda_{i\pm}-k)}{\lambda_i}f_{\lambda_i}(G_1)\otimes\mathbf{j}_{n_2} & f_{\lambda_i}(G_1)\otimes J_{n_2}\end{pmatrix} \tag{15}$$

for $i\in[p]$ and $i\neq i_0$;

(2) the eigenprojector of eigenvalue η_j of $A(G_1\bigstar G_2)$ is

$$F_{\eta_j}(G_1\bigstar G_2)=\begin{pmatrix}0 & 0\\ 0 & I_{n_1}\otimes f_{\eta_j}(G_2)\end{pmatrix} \tag{16}$$

for $j\in[q]$ and $j\neq 1$;

(3) the eigenprojector of eigenvalue k of $A(G_1\bigstar G_2)$ is

$$F_k(G_1\bigstar G_2)=\frac{1}{n_2}\begin{pmatrix}0 & 0\\ 0 & f_0(G_1)\otimes J_{n_2}\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & I_{n_1}\otimes(f_k(G_2)-\frac{1}{n_2}J_{n_2})\end{pmatrix}. \tag{17}$$

(4) the eigenprojector of eigenvalue 0 of $A(G_1\bigstar G_2)$ is

$$F_0(G_1\bigstar G_2)=\begin{pmatrix}f_0(G_1) & 0\\ 0 & 0\end{pmatrix}. \tag{18}$$

Hence, the spectrum decomposition of $A(G_1\bigstar G_2)$ is given by:

$$\begin{aligned}A(G_1\bigstar G_2)=&\sum_{i\neq i_0}\sum_{\pm}\lambda_{i\pm}F_{\lambda_{i\pm}}(G_1\bigstar G_2)+\sum_{j=2}^{q}\eta_j F_{\eta_j}(G_1\bigstar G_2)\\ &+kF_k(G_1\bigstar G_2)+0F_0(G_1\bigstar G_2).\end{aligned} \tag{19}$$

Proof (i) Suppose that $\{x_j^{(i)}|j\in[l_i]\}$ is the set of all orthonormal eigenvectors of eigenvalue λ_i of G_1 and $\{y_i^{(j)}|i\in[l'_j]\}$ is the set of all orthonormal eigenvectors of eigenvalue η_j of G_2 for $i\in[p]$, $j\in[q]$. The eigenvectors of $A(G_1\bigstar G_2)$ can be easily obtained from the proof of Theorem 2.1 in [23]. For the convenience of readers, we give the detailed proof. Let

$$X_{i\pm}^{j} = \frac{|\lambda_i|}{\sqrt{(\lambda_{i\pm} - k)^2 + n_2\lambda_i^2}} \begin{pmatrix} \frac{\lambda_{i\pm}-k}{\lambda_i} x_j^{(i)} \\ x_j^{(i)} \otimes \mathbf{j}_{n_2} \end{pmatrix}$$

for $j \in [l_i]$, $i \in [p]$. According to (7), then

$$\begin{aligned} A(G_1 \bigstar G_2) X_{i\pm}^{j} &= \frac{|\lambda_i|}{\sqrt{(\lambda_{i\pm}-k)^2+n_2\lambda_i^2}} \begin{pmatrix} A(G_1) & A(G_1) \otimes \mathbf{j}_{n_2}^T \\ A(G_1) \otimes \mathbf{j}_{n_2} & I_{n_1} \otimes A(G_2) \end{pmatrix} \begin{pmatrix} \frac{\lambda_{i\pm}-k}{\lambda_i} x_j^{(i)} \\ x_j^{(i)} \otimes \mathbf{j}_{n_2} \end{pmatrix} \\ &= \frac{|\lambda_i|}{\sqrt{(\lambda_{i\pm}-k)^2+n_2\lambda_i^2}} \begin{pmatrix} (\lambda_{i\pm} - k + n_2\lambda_i) x_j^{(i)} \\ \lambda_{i\pm} x_j^{(i)} \otimes \mathbf{j}_{n_2} \end{pmatrix} \\ &= \lambda_{i\pm} \frac{|\lambda_i|}{\sqrt{(\lambda_{i\pm}-k)^2+n_2\lambda_i^2}} \begin{pmatrix} \frac{\lambda_{i\pm}-k}{\lambda_i} x_j^{(i)} \\ x_j^{(i)} \otimes \mathbf{j}_{n_2} \end{pmatrix} \\ &= \lambda_{i\pm} X_{i\pm}^{j} \end{aligned}$$

for $i \in [p]$ and $j \in [l_i]$, where the last equality holds because $(\lambda_{i\pm} - k + n_2\lambda_i)\lambda_i = (\lambda_{i\pm} - k)\lambda_{i\pm}$. Hence, according to the definition of eigenprojector, one gets

$$\begin{aligned} F_{\lambda_{i\pm}}(G_1 \bigstar G_2) &= \sum_{j=1}^{l_i} X_{i\pm}^{j} (X_{i\pm}^{j})^T \\ &= \sum_{j=1}^{l_i} \frac{\lambda_i{}^2}{(\lambda_{i\pm} - k)^2 + n_2\lambda_i{}^2} \begin{pmatrix} \frac{\lambda_{i\pm}-k}{\lambda_i} x_j^{(i)} \\ x_j^{(i)} \otimes \mathbf{j}_{n_2} \end{pmatrix} \begin{pmatrix} \frac{\lambda_{i\pm}-k}{\lambda_i} x_j^{(i)} \\ x_j^{(i)} \otimes \mathbf{j}_{n_2} \end{pmatrix}^T \\ &= \frac{\lambda_i{}^2}{(\lambda_{i\pm} - k)^2 + n_2\lambda_i{}^2} \begin{pmatrix} \frac{(\lambda_{i\pm}-k)^2}{\lambda_i{}^2} f_{\lambda_i}(G_1) & \frac{(\lambda_{i\pm}-k)}{\lambda_i} f_{\lambda_i}(G_1) \otimes \mathbf{j}_{n_2}{}^T \\ \frac{(\lambda_{i\pm}-k)}{\lambda_i} f_{\lambda_i}(G_1) \otimes \mathbf{j}_{n_2} & f_{\lambda_i}(G_1) \otimes J_{n_2} \end{pmatrix}. \end{aligned} \tag{20}$$

Let $Y_j^{ii'} = \begin{pmatrix} 0 \\ e_{i'} \otimes y_i^{(j)} \end{pmatrix}$ for $j \in [q]$ and $i \in [l'_j]$, $i' \in [n_1]$, where $e_{i'}$ is the characteristic vector of order n_1 and $y_i^{(j)}$ is a unit vector orthogonal to $\mathbf{j}_{n_2}$. Then,

$$\begin{aligned} A(G_1 \bigstar G_2) Y_j^{ii'} &= \begin{pmatrix} A(G_1) & A(G_1) \otimes \mathbf{j}_{n_2}^T \\ A(G_1) \otimes \mathbf{j}_{n_2} & I_{n_1} \otimes A(G_2) \end{pmatrix} \begin{pmatrix} 0 \\ e_{i'} \otimes y_i^{(j)} \end{pmatrix} \\ &= \eta_j \begin{pmatrix} 0 \\ e_{i'} \otimes y_i^{(j)} \end{pmatrix} \\ &= \eta_j Y_j^{ii'}. \end{aligned}$$

Thus,

$$
\begin{aligned}
F_{\eta_j}(G_1 \bigstar G_2) &= \sum_{i'=1}^{n_1} \sum_{i=1}^{l_j{}'} Y_j^{ii'} (Y_j^{ii'})^T \\
&= \sum_{i'=1}^{n_1} \sum_{i=1}^{l_j{}'} \begin{pmatrix} 0 \\ e_{i'} \otimes y_i^{(j)} \end{pmatrix} \begin{pmatrix} 0 \\ e_{i'} \otimes y_i^{(j)} \end{pmatrix}^T \\
&= \sum_{i'=1}^{n_1} \begin{pmatrix} 0 & 0 \\ 0 & e_{i'}(e_{i'})^T \otimes \sum_{i=1}^{l_j{}'} y_i^{(j)} (y_i^{(j)})^T \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & I_{n_1} \otimes f_{\eta_j}(G_2) \end{pmatrix},
\end{aligned}
$$

where $j \in [q] \setminus \{1\}$. If $j = 1$,

$$
F_{\eta_1}(G_1 \bigstar G_2) = F_k(G_1 \bigstar G_2) = \begin{pmatrix} 0 & 0 \\ 0 & I_{n_1} \otimes (f_k(G_2) - \frac{1}{n_2} J_{n_2}) \end{pmatrix}.
$$

Therefore, the spectral decomposition of $A(G_1 \bigstar G_2)$ is given by:

$$
A(G_1 \bigstar G_2) = \sum_{i=1}^{p} \sum_{\pm} \lambda_{i\pm} F_{\lambda_{i\pm}}(G_1 \bigstar G_2) + \sum_{j=1}^{q} \eta_j F_{\eta_j}(G_1 \bigstar G_2).
$$

(ii) If there is some $i_0 \in [p]$ such that $\lambda_{i_0} = 0$, then $\lambda_{i_0+} = k$ and $\lambda_{i_0-} = 0$. Similar to the proof of (i), we only need to compute $F_k(G_1 \bigstar G_2)$ and $F_0(G_1 \bigstar G_2)$. Let

$$
X_{i_0+}^j = \frac{1}{\sqrt{n_2}} \begin{pmatrix} 0 \\ x_j^{(i_0)} \otimes \mathbf{j}_{n_2} \end{pmatrix} \text{ and } X_{i_0-}^j = \begin{pmatrix} x_j^{(i_0)} \\ 0 \end{pmatrix}
$$

for $j \in [l_{i_0}]$. Then,

$$
A(G_1 \bigstar G_2) X_{i_0+}^j = \frac{1}{\sqrt{n_2}} \begin{pmatrix} A(G_1) & A(G_1) \otimes \mathbf{j}_{n_2}^T \\ A(G_1) \otimes \mathbf{j}_{n_2} & I_{n_1} \otimes A(G_2) \end{pmatrix} \begin{pmatrix} 0 \\ x_j^{(i_0)} \otimes \mathbf{j}_{n_2} \end{pmatrix} = \frac{k}{\sqrt{n_2}} \begin{pmatrix} 0 \\ x_j^{(i_0)} \otimes \mathbf{j}_{n_2} \end{pmatrix}
$$

and

$$
A(G_1 \bigstar G_2) X_{i_0-}^j = \begin{pmatrix} A(G_1) & A(G_1) \otimes \mathbf{j}_{n_2}^T \\ A(G_1) \otimes \mathbf{j}_{n_2} & I_{n_1} \otimes A(G_2) \end{pmatrix} \begin{pmatrix} x_j^{(i_0)} \\ 0 \end{pmatrix} = 0 \begin{pmatrix} x_j^{(i_0)} \\ 0 \end{pmatrix}
$$

for $j \in [l_{i_0}]$. Thus, we have

$$\begin{aligned}
F_k(G_1\bigstar G_2) &= \sum_{j=1}^{l_{i_0}} X_{i_0+}^j (X_{i_0+}^j)^T + \begin{pmatrix} 0 & 0 \\ 0 & I_{n_1} \otimes (f_k(G_2) - \frac{1}{n_2} J_{n_2}) \end{pmatrix} \\
&= \sum_{j=1}^{l_{i_0}} \frac{1}{n_2} \begin{pmatrix} 0 \\ x_j^{(i_0)} \otimes \mathbf{j}_{n_2} \end{pmatrix} \begin{pmatrix} 0 \\ x_j^{(i_0)} \otimes \mathbf{j}_{n_2} \end{pmatrix}^T + \begin{pmatrix} 0 & 0 \\ 0 & I_{n_1} \otimes (f_k(G_2) - \frac{1}{n_2} J_{n_2}) \end{pmatrix} \\
&= \frac{1}{n_2} \begin{pmatrix} 0 & 0 \\ 0 & \sum_{j=1}^{l_{i_0}} x_j^{(i_0)} (x_j^{(i_0)})^T \otimes J_{n_2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I_{n_1} \otimes (f_k(G_2) - \frac{1}{n_2} J_{n_2}) \end{pmatrix} \\
&= \frac{1}{n_2} \begin{pmatrix} 0 & 0 \\ 0 & f_0(G_1) \otimes J_{n_2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I_{n_1} \otimes (f_k(G_2) - \frac{1}{n_2} J_{n_2}) \end{pmatrix}
\end{aligned}$$

and

$$F_0(G_1\bigstar G_2) = \sum_{j=1}^{l_{i_0}} X_{i_0-}^j (X_{i_0-}^j)^T = \begin{pmatrix} \sum_{j=1}^{l_{i_0}} x_j^{(i_0)} (x_j^{(i_0)})^T & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} f_0(G_1) & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, the spectral decomposition of $A(G_1\bigstar G_2)$ is given by:

$$\begin{aligned}
A(G_1\bigstar G_2) = &\sum_{i \neq i_0} \sum_{\pm} \lambda_{i\pm} F_{\lambda_{i\pm}}(G_1\bigstar G_2) + \sum_{j=2}^{q} \eta_j F_{\eta_j}(G_1\bigstar G_2) \\
&+ k F_k(G_1\bigstar G_2) + 0 F_0(G_1\bigstar G_2).
\end{aligned}$$

□

Proposition 3.3 *Let G_1 be any connected graph with n_1 vertices and G_2 be any k-regular graph with n_2 vertices. For $u, v \in V(G_1)$,*

(i) if $\lambda_j \neq 0$ for any $j \in [p]$, then

$$\begin{aligned}
e_{(u,0)} \exp(-\mathbf{i}t A(G_1\bigstar G_2)) e_{(v,0)} = &\sum_{j=1}^{p} e^{-\mathbf{i}t \frac{\lambda_j + k}{2}} e_u f_{\lambda_j}(G_1) e_v \Bigg(\cos \frac{\Delta_{\lambda_j} t}{2} \\
&+ \mathbf{i} \frac{k - \lambda_j}{\Delta_{\lambda_j}} \sin \frac{\Delta_{\lambda_j} t}{2} \Bigg),
\end{aligned} \tag{21}$$

where $\Delta_{\lambda_j} = \sqrt{(\lambda_j - k)^2 + 4 n_2 \lambda_j^2}$ for $j \in [p]$.

(ii) if there is some $j_0 \in [p]$ such that $\lambda_{j_0} = 0$, then

$$\begin{aligned}
&e_{(u,0)} \exp(-\mathbf{i}t A(G_1\bigstar G_2)) e_{(v,0)} \\
&= \sum_{j \neq j_0} e^{-\mathbf{i}t \frac{\lambda_j + k}{2}} e_u f_{\lambda_j}(G_1) e_v (\cos \frac{\Delta_{\lambda_j} t}{2} + \mathbf{i} \frac{k - \lambda_j}{\Delta_{\lambda_j}} \sin \frac{\Delta_{\lambda_j} t}{2}) + e_u f_0(G_1) e_v,
\end{aligned} \tag{22}$$

where $\Delta_{\lambda_j} = \sqrt{(\lambda_j - k)^2 + 4 n_2 \lambda_j^2}$ for $j \in [p]$ and $j \neq j_0$.

Proof (i) Since $\lambda_j \neq 0$ for any $j \in [p]$, according to (14), then the transition matrix of $A(G_1 \bigstar G_2)$ is given by:

$$\exp(-\mathbf{i}tA(G_1 \bigstar G_2)) = \sum_{j=1}^{p} \sum_{\pm} e^{-\mathbf{i}t\lambda_{j\pm}} F_{\lambda_{j\pm}}(G_1 \bigstar G_2) + \sum_{j'=1}^{q} e^{-\mathbf{i}t\eta_{j'}} F_{\eta_{j'}}(G_1 \bigstar G_2). \tag{23}$$

From (23), (12) and (13), then the element of $\exp(-\mathbf{i}tA(G_1 \bigstar G_2))$ relevant to vertices $(u, 0)$ and $(v, 0)$ is given by:

$$\begin{aligned}
&e_{(u,0)} \exp(-\mathbf{i}tA(G_1 \bigstar G_2))e_{(v,0)} \\
&\quad = e_{(u,0)} \sum_{j=1}^{p} \sum_{\pm} e^{-\mathbf{i}t\lambda_{j\pm}} F_{\lambda_{j\pm}}(G_1 \bigstar G_2)e_{(v,0)} \\
&\qquad + e_{(u,0)} \sum_{j'=1}^{q} e^{-\mathbf{i}t\eta_{j'}} F_{\eta_{j'}}(G_1 \bigstar G_2)e_{(v,0)} \\
&\quad = \sum_{j=1}^{p} e^{-\mathbf{i}t\frac{\lambda_j+k}{2}} \sum_{\pm} e^{\mp\mathbf{i}t\frac{\Delta_{\lambda_j}}{2}} \frac{{\lambda_j}^2}{(\lambda_{j\pm}-k)^2+n_2{\lambda_j}^2} \frac{(\lambda_{j\pm}-k)^2}{{\lambda_j}^2} e_u f_{\lambda_j}(G_1)e_v \\
&\quad = \sum_{j=1}^{p} e^{-\mathbf{i}t\frac{\lambda_j+k}{2}} e_u f_{\lambda_j}(G_1)e_v \sum_{\pm} e^{\mp\mathbf{i}t\frac{\Delta_{\lambda_j}}{2}} \frac{(\lambda_{j\pm}-k)^2}{(\lambda_{j\pm}-k)^2+n_2{\lambda_j}^2} \\
&\quad = \sum_{j=1}^{p} e^{-\mathbf{i}t\frac{\lambda_j+k}{2}} e_u f_{\lambda_j}(G_1)e_v \left(\frac{(\lambda_{j+}-k)^2}{(\lambda_{j+}-k)^2+n_2{\lambda_j}^2} + \frac{(\lambda_{j-}-k)^2}{(\lambda_{j-}-k)^2+n_2{\lambda_j}^2} \right) \cos\frac{\Delta_{\lambda_j}t}{2} \\
&\qquad + \left(\frac{(\lambda_{j-}-k)^2}{(\lambda_{j-}-k)^2+n_2{\lambda_j}^2} - \frac{(\lambda_{j+}-k)^2}{(\lambda_{j+}-k)^2+n_2{\lambda_j}^2} \right) \mathbf{i}\sin\frac{\Delta_{\lambda_j}t}{2}.
\end{aligned}$$

In the light of (8) and (9), we obtain the following three equalities,

$$(\lambda_{j+} - k)(\lambda_{j-} - k) = -n_2\lambda_j^2, \tag{24}$$

$$(\lambda_{j+} - k)^2 + (\lambda_{j-} - k)^2 = (\lambda_j - k)^2 + 2n_2\lambda_j^2, \tag{25}$$

and

$$(\lambda_{j-} - k)^2 - (\lambda_{j+} - k)^2 = \Delta_{\lambda_j}(k - \lambda_j). \tag{26}$$

Thus,

$$\frac{(\lambda_{j+} - k)^2}{(\lambda_{j+} - k)^2 + n_2{\lambda_j}^2} + \frac{(\lambda_{j-} - k)^2}{(\lambda_{j-} - k)^2 + n_2{\lambda_j}^2} = 1$$

and

$$\frac{(\lambda_{j-}-k)^2}{(\lambda_{j-}-k)^2+n_2\lambda_j{}^2}-\frac{(\lambda_{j+}-k)^2}{(\lambda_{j+}-k)^2+n_2\lambda_j{}^2}=\frac{k-\lambda_j}{\Delta_{\lambda_j}}.$$

Therefore,

$$e_{(u,0)}\exp(-\mathbf{i}tA(G_1\bigstar G_2))e_{(v,0)}=\sum_{j=1}^{p}e^{-\mathbf{i}t\frac{\lambda_j+k}{2}}e_u f_{\lambda_j}(G_1)e_v\Bigg(\cos\frac{\Delta_{\lambda_j}t}{2}+\mathbf{i}\frac{k-\lambda_j}{\Delta_{\lambda_j}}\sin\frac{\Delta_{\lambda_j}t}{2}\Bigg).$$

(ii) If there is some $j_0\in[p]$ such that $\lambda_{j_0}=0$, then the proof is totally similar to that of (i), here we omit it. □

4 Perfect State Transfer

In this section, we mainly focus on PST of neighborhood corona of two graphs. First we recall the following lemma.

Lemma 4.1 *(*Li, Liu and Zhang [27]*) Let G be a k-regular connected graph with n vertices, for any vertex v of G.*

(i) If G is not a complete graph, then $|supp_G(v)|\geq 3$*;*
(ii) If G is a complete graph, then $|supp_G(v)|=2$.

Proof Suppose that $\{\lambda_i|i\in[p]\}$ is the set of all distinct eigenvalues of $A(G)$ and $\lambda_1>\lambda_2>\cdots>\lambda_p$. Denote the eigenprojector of eigenvalue λ_i by $f_{\lambda_i}(G)$. Since G is k-regular, then $L(G)=D(G)-A(G)=kI_n-A(G)$. Let $\{\theta_i|i\in[p]\}$ be the set of all distinct eigenvalues of $L(G)$ and $\theta_1<\theta_2<\cdots<\theta_p$, then $\theta_i=k-\lambda_i$. It is clear that $f_{\lambda_i}(G)=f_{\theta_i}(G)$ for the regular graph G. Since, for any $\theta_i\in\mathrm{supp}_{L(G)}(v)$, $f_{\theta_i}(G)e_v\neq 0$, then $\lambda_i\in\mathrm{supp}_G(v)$ for $f_{\lambda_i}(G)e_v=f_{\theta_i}(G)e_v\neq 0$. Conversely, since, for any $\lambda_i\in\mathrm{supp}_G(v)$, $f_{\lambda_i}(G)e_v\neq 0$, then $\theta_i\in\mathrm{supp}_{L(G)}(v)$ for $f_{\theta_i}(G)e_v=f_{\lambda_i}(G)e_v\neq 0$. Therefore, we obtain the desired results from Lemma 4.2 in [27]. □

Lemma 4.2 *If* 0 *is not an eigenvalue of* $A(G_1)$ *and* G_2 *is a k-regular graph. Then,* $(v,0)$ *is periodic whenever* (v,w) *is periodic in* $G_1\bigstar G_2$ *for* $v\in V(G_1)$, $w\in V(G_2)$.

Proof According to (12) and (13), it is obvious that the element of $\mathrm{supp}_{G_1\bigstar G_2}(v,0)$ contains in $\mathrm{supp}_{G_1\bigstar G_2}(v,w)$. If (v,w) is periodic in $G_1\bigstar G_2$, then $(v,0)$ is periodic in $G_1\bigstar G_2$ according to Lemma 2.3. □

Lemma 4.3 *Let* G_1 *be a connected graph and* G_2 *be a k-regular graph. Suppose that there is some* $i\in[p]$ *such that the eigenvalue* $\lambda_i=0$ *for* $A(G_1)$*. For any* $v\in V(G_1)$ *and* $w\in V(G_2)$,

(i) if $0 \notin \mathrm{supp}_{G_1}(v)$*, then* $(v, 0)$ *is periodic in* $G_1 \bigstar G_2$ *whenever* (v, w) *is periodic in* $G_1 \bigstar G_2$*;*
(ii) if $0 \in \mathrm{supp}_{G_1}(v)$*, then* $\mathrm{supp}_{G_1 \bigstar G_2}(v, 0) \setminus \{0\} \subseteq \mathrm{supp}_{G_1 \bigstar G_2}(v, w)$*.*

Proof (i) Assume that $0 \notin \mathrm{supp}_{G_1}(v)$. According to the (ii) of Proposition 3.2, one has $\mathrm{supp}_{G_1 \bigstar G_2}(v, 0) \subseteq \mathrm{supp}_{G_1 \bigstar G_2}(v, w)$. Now, we obtain the desired results from Lemma 2.3.

(ii) If $0 \in \mathrm{supp}_{G_1}(v)$, then $f_0(G_1)e_v \neq 0$. According to (18), we easily obtain that $F_0(G_1 \bigstar G_2)e_{(v,0)} \neq 0$ and $F_0(G_1 \bigstar G_2)e_{(v,w)} = 0$. Hence, $0 \in \mathrm{supp}_{G_1 \bigstar G_2}(v, 0)$, but $0 \notin \mathrm{supp}_{G_1 \bigstar G_2}(v, w)$. □

Theorem 4.4 *Assume that* G_1 *is an* r*-regular connected integral graph with* n_1 *vertices. Let* G_2 *be a* k*-regular graph with* n_2 *vertices. If* $\sqrt{(r-k)^2 + 4n_2r^2}$ *is not an integer, then* (v, w) *is not periodic for any* $v \in V(G_1)$*,* $w \in V(G_2) \cup \{0\}$ *in* $G_1 \bigstar G_2$*. Moreover, there is no PST in* $G_1 \bigstar G_2$*.*

Proof Since G_1 is r-regular, then $A(G_1)\mathbf{j}_{n_1} = r\mathbf{j}_{n_1}$. According to the definition of eigenprojector, the eigenprojector of eigenvalue r in G_1 is given by $f_r(G_1) = \frac{1}{n_1}J_{n_1}$ as G_1 is a connected graph. Therefore, $f_r(G_1)e_v \neq 0$ for any $v \in V(G_1)$. In other words, $r \in \mathrm{supp}_{G_1}(v)$ for any $v \in V(G_1)$.

Suppose that $G_1 \bigstar G_2$ admits PST between vertex $(v, 0)$ and another vertex. Then, $(v, 0)$ is periodic in $G_1 \bigstar G_2$. According to Lemma 2.3 and Theorem 2.1, nonzero elements of $\mathrm{supp}_{G_1 \bigstar G_2}(v, 0)$ are all integers or the form of $\frac{a+b\sqrt{\Delta}}{2}$ for integer a, square-free integer Δ and some integer b.

Case 1: Nonzero elements of $\mathrm{supp}_{G_1 \bigstar G_2}(v, 0)$ are all integers. Since $r \in \mathrm{supp}_{G_1}(v)$, then $F_{r\pm}(G_1 \bigstar G_2)e_{(v,0)} \neq 0$ for $f_r(G_1)e_v \neq 0$. Hence $\lambda_{r\pm} \in \mathrm{supp}_{G_1 \bigstar G_2}(v, 0)$. Since all the elements of $\mathrm{supp}_{G_1 \bigstar G_2}(v, 0)$ are integers, then λ_{r+}, λ_{r-}, and $\lambda_{r+} - \lambda_{r-}$ are integers. According to $\lambda_{r+} = \frac{r+k+\sqrt{(r-k)^2+4n_2r^2}}{2}$ and $\lambda_{r-} = \frac{r+k-\sqrt{(r-k)^2+4n_2r^2}}{2}$, one has

$$\lambda_{r+} - \lambda_{r-} = \sqrt{(r-k)^2 + 4n_2r^2}. \tag{27}$$

The left of (27) is an integer but the right of (27) is not an integer. This contradicts that nonzero elements of $\mathrm{supp}_{G_1 \bigstar G_2}(v, 0)$ are integers.

Case 2: All the elements of $\mathrm{supp}_{G_1 \bigstar G_2}(v, 0)$ are the form of $\frac{a+b\sqrt{\Delta}}{2}$ for integer a, square-free integer Δ and some integer b. According to Lemma 4.1, there is $\lambda \in \mathrm{supp}_{G_1}(v)$ and $\lambda \neq r$, such that $\lambda_{\pm} \in \mathrm{supp}_{G_1 \bigstar G_2}(v, 0)$ for $f_\lambda(G_1)e_v \neq 0$. Notice that $r_{\pm} \in \mathrm{supp}_{G_1 \bigstar G_2}(v, 0)$ and $\sqrt{(r-k)^2 + 4n_2r^2}$ is not an integer. Let $\lambda_{\pm} = \frac{a+b_{\pm}\sqrt{\Delta}}{2}$ for integer a, square-free integer $\Delta > 1$ and some integer $b_{\pm}$. In the light of (24), one has

$$(\lambda_+ - k)(\lambda_- - k) = -n_2\lambda^2. \tag{28}$$

According to the form of $\lambda_\pm$, we obtain the following equality:

$$\frac{1}{4}[(a-2k)^2 + b_+b_-\Delta] + \frac{1}{4}(a-2k)(b_+ + b_-)\sqrt{\Delta} = -n_2\lambda^2. \tag{29}$$

Since $\sqrt{\Delta}$ is irrational, (29) holds if and only if either $a - 2k = 0$, or $b_+ + b_- = 0$. If $b_+ + b_- = 0$, then $a = \lambda_+ + \lambda_- = \lambda + k$, which implies that $|\mathrm{supp}_{G_1}(v)| = 1$. At that time, it contradicts with Lemma 4.1. If $a - 2k = 0$, then $\lambda_\pm = k + \frac{b_\pm\sqrt{\Delta}}{2}$. Therefore,

$$(\lambda_+ - k) + (\lambda_- - k) = \lambda - k. \tag{30}$$

The left of (30) is a rational multiple of $\sqrt{\Delta}$, but the right of (30) is an integer. Thus, (30) can not hold, which means that $\lambda_\pm$ can not be the form of quadratic integer. Therefore, not all the elements of $\mathrm{supp}_{G_1\bigstar G_2}(v, 0)$ are the form of quadratic integer.

According to Case 1 and Case 2, $(v, 0)$ is not periodic in $G_1\bigstar G_2$ for any $v \in V(G_1)$ by Lemma 2.3. Furthermore, (v, w) is not periodic in $G_1\bigstar G_2$ for any $v \in V(G_1)$ and any $w \in V(G_2) \cup \{0\}$ by Lemma 4.2. Finally, according to Lemma 2.2, there is no PST in $G_1\bigstar G_2$. □

In Theorem 4.4, we proved that there is no PST in $G_1\bigstar G_2$ when $\sqrt{(r-k)^2 + 4n_2r^2}$ is a non-integer for two regular graphs G_1 and G_2. Thus, a natural problem arises: *if $\sqrt{(r-k)^2 + 4n_2r^2}$ is an integer, must $G_1\bigstar G_2$ have PST?* The answer is negative. First observe that the neighborhood corona $K_2\bigstar G_2$ is just the usual corona $K_2 \circ G_2$ for the complete graph K_2. Let $K_2\bigstar\overline{K_{n_2}}$, if $n_2 = l(l+1)$ for any positive integer l, then $\sqrt{1+4n_2}$ is an integer. However, Fan and Godsil [18] proved that $K_2\bigstar\overline{K_{n_2}}$ has no PST for all n_2. In addition, as stated by the reviewer, we usually want to know whether there exists a neighborhood corona graph admitting PST. In fact, if we admit G_1 to be a multigraph, then the neighborhood corona graphs admitting PST really do exist. For example, Ackelsberg et al. [2] proved that, for any positive integer l, $K_2(2)\bigstar\overline{K_{4l^2-1}}$ admits PST between the vertices of $K_2(2)$ at time $\frac{\pi}{2}$, where $K_2(2)$ is the digon (a multigraph on two vertices which are connected by two parallel edges). This fact also shows that it is interesting to study PST of multigraphs or weighted graphs.

Remark that, in Theorem 4.4, we consider the existence of PST for the neighborhood corona $G_1\bigstar G_2$ in the case where both graphs are regular. In what follows, we shall consider the case when the graph G_1 is non-regular.

Theorem 4.5 *Assume that G_1 is any connected graph. Let G_2 be any connected k-regular graph with n_2 vertices. If there exists nonzero element $\lambda_i \in \mathrm{supp}_{G_1}(v)$ such that $\lambda_i \in \mathbb{Z}\sqrt{\Delta'}$ for some $i \in [p]$ and some square-free integer $\Delta' > 1$, where $\mathbb{Z}$ is the set of all integers. Then, (v, w) is not periodic in $G_1\bigstar G_2$ for $v \in V(G_1)$, and any $w \in V(G_2) \cup \{0\}$.*

Proof Suppose that there is PST in $G_1\bigstar G_2$ between vertex $(v, 0)$ and another vertex, then $(v, 0)$ is periodic in $G_1\bigstar G_2$. Thus, according to Lemma 2.3 and Theorem 2.1, nonzero elements of $\mathrm{supp}_{G_1\bigstar G_2}(v, 0)$ are either all integers, or all quadratic integers.

Case 1: Nonzero elements of $\mathrm{supp}_{G_1\bigstar G_2}(v,0)$ are all integers.

Since $\lambda_i \in \mathrm{supp}_{G_1}(v)$, then $f_{\lambda_i}(G_1)e_v \neq 0$. According to (12), then $\lambda_{i\pm} \in \mathrm{supp}_{G_1\bigstar G_2}(v,0)$ for $F_{\lambda_{i\pm}}(G_1\bigstar G_2)e_{(v,0)} \neq 0$. Hence λ_{i+} and λ_{i-} are both integers. Furthermore, $\lambda_{i+}+\lambda_{i-}$ is also an integer. According to (8) and (9),

$$\lambda_{i+}+\lambda_{i-} = \lambda_i + k. \tag{31}$$

Since $\lambda_i \in \mathbb{Z}\sqrt{\Delta'}$ is an irrational number, then the right of (31) is an irrational number. However, the left of (31) is an integer, it is impossible. Hence, not all $\mathrm{supp}_{G_1\bigstar G_2}(v,0)$ are integers.

Case 2: There is some integer a and square-free integer Δ such that all the elements of $\mathrm{supp}_{G_1\bigstar G_2}(v,0)$ are the form of $\frac{a+b_{\lambda_{i\pm}}\sqrt{\Delta}}{2}$ for some integer $b_{\lambda_{i\pm}}$ relevant to $\lambda_{i\pm}$. Notice that here Δ is different from the previous Δ' in the statement of the theorem.

According to Case 1, $\lambda_{i\pm} \in \mathrm{supp}_{G_1\bigstar G_2}(v,0)$. Since $(\lambda_i-k)^2+4n_2\lambda_i^2$ is an irrational number according to $\lambda_i \in \mathbb{Z}\sqrt{\Delta'}$, then there is an integer a and square-free integer $\Delta > 1$ such that $\lambda_{i\pm} = \frac{a+b_{\lambda_{i\pm}}\sqrt{\Delta}}{2}$ for some integer $b_{\lambda_{i\pm}}$ corresponding to $\lambda_{i\pm}$. In the light of (24), then $(\lambda_{i+}-k)(\lambda_{i-}-k) = -n_2\lambda_i^2$. According to the form of $\lambda_{i\pm}$, one has, by a simple calculation,

$$\frac{1}{4}[(a-2k)^2+b_{\lambda_{i+}}b_{\lambda_{i-}}\Delta]+\frac{1}{4}(a-2k)(b_{\lambda_{i+}}+b_{\lambda_{i-}})\sqrt{\Delta} = -n_2\lambda_i^2. \tag{32}$$

Since $\lambda_i \in \mathbb{Z}\sqrt{\Delta'}$, then $-n_2\lambda_i^2$ is an integer. Since $\Delta > 1$, then $\sqrt{\Delta}$ is an irrational number. In the light of (32), we obtain that either $a-2k=0$, or $b_{\lambda_{i+}}+b_{\lambda_{i-}}=0$. If $b_{\lambda_{i+}}+b_{\lambda_{i-}}=0$, then $a=\lambda_{i+}+\lambda_{i-}=\lambda_i+k$. It contradicts that a is an integer beacause $\lambda_i \in \mathbb{Z}\sqrt{\Delta'}$ is an irrational number. If $a-2k=0$, then $a=2k$. Hence $\lambda_{i+} = k+\frac{b_{\lambda_{i+}}\sqrt{\Delta}}{2}$ and $\lambda_{i-} = k+\frac{b_{\lambda_{i-}}\sqrt{\Delta}}{2}$. According to the equalities above, then

$$(\lambda_{i+}-k)+(\lambda_{i-}-k) = \lambda_i - k \tag{33}$$

After taking square both sides of (33), we easily obtain that the left side is a rational number, but the right side is irrational, a contradiction. Therefore, neither $a-2k=0$ nor $b_{\lambda_{i+}}+b_{\lambda_{i-}}=0$. Furthermore, nonzero elements of $\mathrm{supp}_{G_1\bigstar G_2}(v,0)$ are not all quadratic integers.

Now, it follows from Case 1, Case 2 and Lemma 2.3 that $(v,0)$ is not periodic in $G_1\bigstar G_2$. Therefore, (v,w) is not periodic in $G_1\bigstar G_2$ from Lemma 4.3. □

Theorem 4.6 *The neighborhood corona $P_3\bigstar G$ has no PST for a connected k-regular graph G.*

Proof It is known that $\mathrm{Sp}(P_3) = \{\sqrt{2},0,-\sqrt{2}\}$. By a simple calculation, then the eigenvalue support of the middle vertex is the set $\{\sqrt{2},-\sqrt{2}\}$ and the eigenvalue support of the endpoints of P_3 are the set $\{\sqrt{2},0,-\sqrt{2}\}$. Based on Theorem 4.5, $(v,0)$ is not a periodic vertex in $P_3\bigstar G$ for any $v \in V(P_3)$, then (v,w) is not a periodic vertex in $P_3\bigstar G$ for any $v \in V(P_3)$, $w \in V(G)\cup\{0\}$. According to Lemma 2.2, $P_3\bigstar G$ has no PST. □

5 Pretty Good State Transfer

Given that the neighborhood corona graphs having PST are rare, it is meaningful for us to search some neighborhood corona graphs admitting PGST. In this section, we mainly study PGST in neighborhood corona of two graphs.

Theorem 5.1 *Assume that G_1 has PST at time $\frac{\pi}{g}$ between vertices x and y, where g is defined in Theorem 2.1. If $\sqrt{4n_2+1}$ is not an integer, then there is PGST in $G_1\bigstar\overline{K_{n_2}}$ between vertices $(x,0)$ and $(y,0)$.*

Proof Since G_1 has PST at time $\frac{\pi}{g}$, then all eigenvalues λ_j are integers by Theorem 2.1 for $j\in[p]$. The following proof is divided into two cases.

Case 1: $0\notin \mathrm{supp}_{G_1}(y)$.

According to (i) of Proposition 3.3, we have

$$e_{(x,0)}\exp(-\mathbf{i}tA(G_1\bigstar\overline{K_{n_2}}))e_{(y,0)}=\sum_{j=1}^{p}e^{-\mathbf{i}t\frac{\lambda_j}{2}}e_xf_{\lambda_j}(G_1)e_y\Bigg(\cos\frac{\Delta_{\lambda_j}t}{2}+\mathbf{i}\frac{-\lambda_j}{\Delta_{\lambda_j}}\sin\frac{\Delta_{\lambda_j}t}{2}\Bigg),\tag{34}$$

where $\Delta_{\lambda_j}=\sqrt{(4n_2+1)\lambda_j^2}$ for $j\in[p]$. Since $f_{\lambda_j}(G_1)e_y=0$ for $\lambda_j\notin\mathrm{supp}_{G_1}(y)$, then

$$e_{(x,0)}\exp(-\mathbf{i}tA(G_1\bigstar\overline{K_{n_2}}))e_{(y,0)}=\sum_{\lambda_j\in\mathrm{supp}_{G_1}(y)}e^{-\mathbf{i}t\frac{\lambda_j}{2}}e_xf_{\lambda_j}(G_1)e_y\Bigg(\cos\frac{\Delta_{\lambda_j}t}{2}+\mathbf{i}\frac{-\lambda_j}{\Delta_{\lambda_j}}\sin\frac{\Delta_{\lambda_j}t}{2}\Bigg).\tag{35}$$

In order to prove that $G_1\bigstar\overline{K_{n_2}}$ has PGST between $(x,0)$ and $(y,0)$, we need to find a time t_0 such that

$$\left|\sum_{\lambda_j\in\mathrm{supp}_{G_1}(y)}e^{-\mathbf{i}t_0\frac{\lambda_j}{2}}e_xf_{\lambda_j}(G_1)e_y(\cos\frac{\Delta_{\lambda_j}t_0}{2}+\mathbf{i}\frac{-\lambda_j}{\Delta_{\lambda_j}}\sin\frac{\Delta_{\lambda_j}t_0}{2})\right|\approx 1.$$

Since $\sqrt{4n_2+1}$ is not an integer, then $\Delta_{\lambda_j}=\sqrt{(4n_2+1)\lambda_j^2}$ is not an integer for integer $\lambda_j\in\mathrm{supp}_{G_1}(y)$. Let $\Delta_{\lambda_j}=a_j\sqrt{b_j}$ for each $\lambda_j\in\mathrm{supp}_{G_1}(y)$, where a_j, $b_j\in\mathbb{Z}^+$ and b_j is the square-free part of $\Delta_{\lambda_j}^2$. Then, the disjoint union $\{1\}\cup\{\sqrt{b_j}:\lambda_j\in\mathrm{supp}_{G_1}(y)\}$ is linearly independent over $\mathbb{Q}$ by Lemma 2.5. According to Theorem 2.4, there are integers α and c_j for each $\lambda_j\in\mathrm{supp}_{G_1}(y)$ such that

$$\alpha\sqrt{b_j}-c_j\approx-\frac{1}{2g}\sqrt{b_j}.\tag{36}$$

If $b_j = b_{j'}$ for two different eigenvalues $\lambda_j, \lambda_{j'} \in \mathrm{supp}_{G_1}(y)$, then $c_j = c_{j'}$. Multiplying $4a_j$ to two sides of (36), we have $\Delta_{\lambda_j} \approx \frac{4a_jc_j}{4\alpha+\frac{2}{g}}$. Let $t_0 = (4\alpha + \frac{2}{g})\pi$. Then, $\cos\frac{\Delta_{\lambda_j}t_0}{2} \approx \cos 2a_jc_j\pi = 1$. Hence,

$$\begin{aligned}
&\left|e_{(x,0)}\exp(-\mathbf{i}t_0A(G_1\bigstar\overline{K_{n_2}}))e_{(y,0)}\right| \\
&= \left|\sum_{\lambda_j\in\mathrm{supp}_{G_1}(y)} e^{-\mathbf{i}t_0\frac{\lambda_j}{2}} e_x f_{\lambda_j}(G_1)e_y(\cos\frac{\Delta_{\lambda_j}t_0}{2} + \mathbf{i}\frac{-\lambda_j}{\Delta_{\lambda_j}}\sin\frac{\Delta_{\lambda_j}t_0}{2})\right| \\
&\approx \left|\sum_{\lambda_j\in\mathrm{supp}_{G_1}(y)} e^{-\mathbf{i}t_0\frac{\lambda_j}{2}} e_x f_{\lambda_j}(G_1)e_y\right| \\
&= \left|\sum_{\lambda_j\in\mathrm{supp}_{G_1}(y)} e^{-\mathbf{i}\frac{\pi}{g}\lambda_j} e_x f_{\lambda_j}(G_1)e_y\right| \\
&= 1,
\end{aligned}$$

where the last equality holds by Theorem 2.1.

Case 2: $0 \in \mathrm{supp}_{G_1}(y)$.

According to (ii) of Proposition 3.3, if there exists $0 = \lambda_{j_0} \in \mathrm{supp}_{G_1}(y)$, then

$$\begin{aligned}
e_{(x,0)}\exp(-\mathbf{i}tA(G_1\bigstar\overline{K_{n_2}}))e_{(y,0)} = \sum_{j\neq j_0} e^{-\mathbf{i}t\frac{\lambda_j}{2}} e_x f_{\lambda_j}(G_1)e_y\Big(\cos\frac{\Delta_{\lambda_j}t}{2} \\
+\mathbf{i}\frac{-\lambda_j}{\Delta_{\lambda_j}}\sin\frac{\Delta_{\lambda_j}t}{2}\Big) + e_x f_0(G_1)e_y, \qquad (37)
\end{aligned}$$

where $\Delta_{\lambda_j} = \sqrt{(4n_2+1)\lambda_j^2}$ for $j \neq j_0$. Similar to the discussion in Case 1, we can obtain desired result based on the equalities $1 = e^{-\mathbf{i}\frac{\pi}{g}0}$ and $e^{-\mathbf{i}t_0\frac{\lambda_j}{2}} = e^{-\mathbf{i}\frac{\pi}{g}\lambda_j}$.

Therefore, there exists PGST on $G_1\bigstar\overline{K_{n_2}}$ between $(x, 0)$ and $(y, 0)$.

Theorem 5.1 gives a sufficient condition such that $G_1\bigstar\overline{K_{n_2}}$ admits PGST when G_1 has PST at $\frac{\pi}{g}$. It is well known that C_4 has PST between its two antipodal vertices at $\frac{\pi}{2}$. Therefore, we immediately obtain the following corollary.

Corollary 5.2 *If $\sqrt{1+4n_2}$ is not an integer, then $C_4\bigstar\overline{K_{n_2}}$ admits PGST.*

Proof Recall that $\mathrm{Sp}(C_4) = \{2, 0^2, -2\}$ and C_4 has PST at time $\frac{\pi}{2}$ between two antipodal vertices (see [14]). Without loss of generality, suppose that C_4 admits PST between vertices v_1 and v_3. According to the definition of eigenvalue support, then $\mathrm{supp}_{C_4}(v_1) = \{2, 0, -2\} = \mathrm{supp}_{C_4}(v_3)$. Since $\sqrt{1+4n_2}$ is not an integer and $e^{-\mathbf{i}\frac{\pi}{2}0} = 1$, then $C_4\bigstar\overline{K_{n_2}}$ admits PGST at vertices $(v_1, 0)$ and $(v_3, 0)$ by Theorem 5.1. □

Theorem 5.2 implies that $C_4\bigstar\overline{K_{n_2}}$ admits PGST when $\sqrt{1+4n_2}$ is not an integer and $G_2=\overline{K_{n_2}}$ is not a connected graph. Next, we consider PGST on $C_4\bigstar G_2$ whenever G_2 is a k-regular connected graph.

Theorem 5.3 *Assume that G_2 is a k-regular connected graph with vertices n_2. If neither of $\sqrt{(2+k)^2+16n_2}$ and $\sqrt{(2-k)^2+16n_2}$ is an integer and $k=0(mod4)$, then $C_4\bigstar G_2$ admits PGST.*

Proof Suppose that $V(C_4)=\{v_1,v_2,v_3,v_4\}$. According to the proof of Theorem 5.2, $\mathrm{Sp}(C_4)=\{2,0^2,-2\}$ and $\mathrm{supp}_{C_4}(v_1)=\mathrm{supp}_{C_4}(v_3)=\{2,0,-2\}$. Let $S=\mathrm{supp}_{C_4}(v_3)$. Since $f_{\lambda_j}(C_4)e_{v_3}=0$ for any $\lambda_j\notin S$, then Proposition 3.3 implies that

$$e_{(v_1,0)}\exp(-\mathbf{i}tA(C_4\bigstar G_2))e_{(v_3,0)} \\ =\sum_{\lambda_j\neq 0} e^{-\mathbf{i}t\frac{\lambda_j+k}{2}}e_{v_1}f_{\lambda_j}(C_4)e_{v_3}\left(\cos\frac{\Delta_{\lambda_j}t}{2}+\mathbf{i}\frac{k-\lambda_j}{\Delta_{\lambda_j}}\sin\frac{\Delta_{\lambda_j}t}{2}\right)+e_{v_1}f_0(C_4)e_{v_3}, \tag{38}$$

where $\Delta_{\lambda_j}=\sqrt{(\lambda_j-k)^2+4n_2\lambda_j^2}$ for $j\in\{1,3\}$. Hence, in order to prove that $C_4\bigstar G_2$ occurs PGST between vertices $(v_1,0)$ and $(v_3,0)$, we only need to find some t_0 such that

$$\left|e_{(v_1,0)}\exp(-\mathbf{i}t_0A(C_4\bigstar G_2))e_{(v_3,0)}\right| \\ =\left|\sum_{\lambda_j\neq 0} e^{-\mathbf{i}t_0\frac{\lambda_j+k}{2}}e_{v_1}f_{\lambda_j}(C_4)e_{v_3}(\cos\frac{\Delta_{\lambda_j}t_0}{2}+\mathbf{i}\frac{k-\lambda_j}{\Delta_{\lambda_j}}\sin\frac{\Delta_{\lambda_j}t_0}{2})+e_{v_1}f_0(C_4)e_{v_3}\right| \\ \approx 1. \tag{39}$$

Since $\lambda_1=2$ and $\lambda_3=-2$, obviously, $\Delta_{\lambda_3}=\sqrt{(2+k)^2+16n_2}$ and $\Delta_{\lambda_1}=\sqrt{(2-k)^2+16n_2}$. Based on the given condition, neither of $\sqrt{(2+k)^2+16n_2}$ and $\sqrt{(2-k)^2+16n_2}$ is an integer, then $\Delta_{\lambda_j}=a_j\sqrt{b_j}$ for $j\in\{1,3\}$, where $a_j,b_j\in\mathbb{Z}^+$ and b_j is square-free part of $\Delta_{\lambda_j}^2$. It is not difficult to see that $\{1\}\cup\{\sqrt{b_j}:j\in\{1,3\}\}$ is linearly independent over $\mathbb{Q}$. According to Theorem 2.4, there are integers α and d_j such that

$$\alpha\sqrt{b_j}-d_j\approx-\frac{1}{4}\sqrt{b_j}. \tag{40}$$

If $\sqrt{b_1}=\sqrt{b_3}$, then $d_1=d_3$. Multiplying $4a_j$ on both sides of (40), then $(4\alpha+1)a_j\sqrt{b_j}\approx 4d_ja_j$ implies $\Delta_{\lambda_j}\approx\frac{4d_ja_j}{(4\alpha+1)}$. Take $t_0=(4\alpha+1)\pi$. By a simple calculation,

$$\cos\frac{\Delta_{\lambda_j}t_0}{2}\approx\cos\frac{\frac{4d_ja_j}{(4\alpha+1)}(4\alpha+1)\pi}{2}=\cos 2a_jd_j\pi=1.$$

Thus,

$$\begin{aligned}
&\left|e_{(v_1,0)} \exp(-\mathbf{i}t_0 A(C_4 \bigstar G_2)) e_{(v_3,0)}\right| \\
&\quad = \left|\sum_{\lambda_j \neq 0} e^{-\mathbf{i}t_0 \frac{\lambda_j+k}{2}} e_{v_1} f_{\lambda_j}(C_4) e_{v_3} (\cos \frac{\Delta_{\lambda_j} t_0}{2} + \mathbf{i} \frac{k-\lambda_j}{\Delta_{\lambda_j}} \sin \frac{\Delta_{\lambda_j} t_0}{2}) + e_{v_1} f_0(C_4) e_{v_3}\right| \\
&\quad \approx \left|\sum_{\lambda_j \neq 0} e^{-\mathbf{i}t_0 \frac{k}{2}} e^{-\mathbf{i}t_0 \frac{\lambda_j}{2}} e_{v_1} f_{\lambda_j}(C_4) e_{v_3} + e_{v_1} f_0(C_4) e_{v_3}\right|.
\end{aligned} \tag{41}$$

Since $e^{-\mathbf{i}t_0 \frac{k}{2}} = 1$ whenever $k = 0(\text{mod}4)$, $e^{-\mathbf{i}t_0 \frac{\lambda_j}{2}} = e^{-\mathbf{i}\frac{\pi}{2}\lambda_j}$ and $e^{-\mathbf{i}\frac{\pi}{2}0} = 1$, then

$$\begin{aligned}
&\left|\sum_{\lambda_j \neq 0} e^{-\mathbf{i}t_0 \frac{k}{2}} e^{-\mathbf{i}t_0 \frac{\lambda_j}{2}} e_{v_1} f_{\lambda_j}(C_4) e_{v_3} + e_{v_1} f_0(C_4) e_{v_3}\right| \\
&= \left|\sum_{\lambda_j \neq 0} e^{-\mathbf{i}\frac{\pi}{2}\lambda_j} e_{v_1} f_{\lambda_j}(C_4) e_{v_3} + e^{-\mathbf{i}\frac{\pi}{2}0} e_{v_1} f_0(C_4) e_{v_3}\right| \\
&= \left|\sum_{\lambda_j \in S} e^{-\mathbf{i}\frac{\pi}{2}\lambda_j} e_{v_1} f_{\lambda_j}(C_4) e_{v_3}\right| \\
&= 1,
\end{aligned} \tag{42}$$

where the last equality holds as C_4 admits PST at time $\frac{\pi}{2}$ between vertices v_1 and v_3. □

Example The complete graph K_5 is a 4-regular connected graph, then $k = 4$ and $n_2 = 5$. By a simple calculation, $\sqrt{(2+k)^2 + 16n_2} = \sqrt{116}$ and $\sqrt{(2-k)^2 + 16n_2} = \sqrt{84}$. Neither of $\sqrt{(2+k)^2 + 16n_2}$ and $\sqrt{(2-k)^2 + 16n_2}$ is an integer, then $C_4 \bigstar K_5$ admits PGST.

Acknowledgements We are very grateful to the reviewers for their valuable, detailed comments and thoughtful suggestions, which led to a substantial improvement on the presentation and contents of this paper. This work is supported in part by the National Natural Science Foundation of China (Nos. 11801521, 12071048).

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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