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Asymptotic Radial Solution of Parabolic Tempered Fractional Laplacian Problem

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Abstract

We study parabolic equation with the tempered fractional Laplacian and logarithmic nonlinearity by the direct method of moving planes. We first prove several important theorems, such as asymptotic maximum principle, asymptotic narrow region principle and asymptotic strong maximum principle for antisymmetric functions, which are critical factors in the process of moving planes. Then, we further derive some properties of asymptotic radial solution to parabolic equation with the tempered fractional Laplacian and logarithmic nonlinearity in a unit ball. These consequences can be applied to investigate more nonlinear nonlocal parabolic equations.

Keywords Fractional parabolic equation · Logarithmic nonlinearity · Asymptotic maximum principle · Tempered fractional Laplacian · Asymptotic symmetry and monotonicity

Mathematics Subject Classification 35R11

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1 Introduction

In recent years, fractional parabolic equations have captured more and more scholars' attention. In [1], Li and Chen presented some parabolic systems, which are extensively used in the simulation of elementary chemical reactions. In [2], Wang and Chen proved some Hopf's lemmas and provide several examples concerning the application of Hopf's lemmas in deriving the properties of solutions for the following parabolic equations:

$$\frac{\partial z(x,t)}{\partial t} + (-\Delta)_p^s z(x,t) = f(t, z(x,t)),$$

here $0 < s < 1$ and $2 \leq p < \infty$. In [3], Chen, Wu and Wang considered fractional parabolic equations with indefinite nonlinearities and proved the nonexistence of its solution. In [4], Poláčik and Quittner introduced indefinite parabolic problem with the regular Laplacian as follows,

$$\frac{\partial z(x,t)}{\partial t} - \Delta z(x,t) = a(x_1) z^p(x,t), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

where $x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^n$, $t \in \mathbb{R}$, a is nondecreasing continuous function, and they received the nonexistence of bounded positive solutions of the above equation. In [5], in view of nonlocal parabolic problems, Chen, Wang and Niu developed the asymptotic method of moving planes and applied it on bounded or unbounded domains. More details can be seen in [6], [7].

To the best of authors' knowledge, not much is known about parabolic equation with the tempered fractional Laplacian and logarithmic nonlinearity. Here, we mainly focus on the following equation:

$$\frac{\partial z(x,t)}{\partial t} - (\Delta + \lambda)^{\frac{\beta}{2}} z(x,t) = a z(x,t) \ln \mid z(x,t) + 1 \mid^p, \tag{1.1}$$

where $a, p > 0$ and the tempered fractional Laplacian operator is defined as

$$(\Delta + \lambda)^{\frac{\beta}{2}} z(x,t) = -C_{n,\beta,\lambda} P.V. \int_{R^n} \frac{z(x,t) - z(y,t)}{e^{\lambda|x-y|} \mid x - y \mid^{n+\beta}} dy,$$

where $\beta \in (0, 2)$, λ is a sufficient small positive constant and $C_{n,\beta,\lambda} = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}} |\Gamma(-\beta)|}$. $P.V.$ presents the Cauchy principle value and $\Gamma(t) = \int_0^\infty s^{t-1} e^{-s} ds$ is the Gamma function. Moreover, let $z \in C^{1,1}_{loc} \bigcap \mathcal{L}_{2s}$, $\mathcal{L}_{2s} = \{z(\cdot, t) \in L^1_{loc}(R^n) \mid \int_{R^n} \frac{|z(x,t)|}{1+|x|^{n+\beta}} dx < +\infty\}$.

The fractional Laplacian $\Delta^{\frac{2}{\beta}}$ is the generator of the β-stable Lévy process, in which the second and all higher order moments diverge. It sometimes is referred to as a shortcoming when applied to physical processes. So a parameter λ is introduced to temper the temper Lévy process. The tempered Lévy flight has finite second moment, which solves the problem about the divergence of second moment of the jump length of the Lévy flight. For a short time, the tempered Lévy flight exhibits the dynamics of the Lévy flight, while after a sufficiently long time it turns to normal diffusion. Moreover, the tempered fractional Laplacian equation governs the probability function of position of the particles and some works on the tempered fractional Laplacian have been done by scholars. For example, in [8], Zhang, Deng and Karniadakis established numerical methods in the Riesz basis Galerkin framework with respect to the tempered fractional Laplacian. In [9], Zhang, Hou, Ahmad and Wang studied the Choquard equation involving a generalized nonlinear tempered fractional p-Laplacian. In [10], considering the two-dimensional tempered fractional Laplacian $(\Delta+\lambda)^{\frac{\beta}{2}}$, Sun, Nie and Deng derived the finite difference discretization and applied it to work out the tempered fractional Poisson equation with Dirichlet boundary conditions. Moreover, they also derived the error estimates. In addition, more results on tempered fractional Laplacian operator can be found in [11, 12, 23].

The nonlocal property of the fractional Laplacian operator creates some difficulties to study it. To overcome this difficulty, an extension method was introduced by Caffarelli and Slivestre [13], which converts the nonlocal problem into a high-dimensional local one. It has been used by many authors (see [14] and the references therein). In addition, the method of moving planes in integral forms also has been widely used to study the nonlocal problems. However, some nonlocal operators cannot be solved by the above method. In [15], Chen, Li and Li put forward a novel approach: a direct method of moving planes method, which is a new idea to solve the fractional Laplacian problems. In [16], Wang and Ren employed the direct method of moving planes method to devote to a nonlinear Schrödinger equation with the fractional Laplacian and Hardy potential. In addition, the radial symmetry result of standing waves have been established. In [17], Zhang and Nie studied two nonlinear equations by the direct method of moving planes concerning Logarithmic Laplacian. Numerous results can be seen in [18]–[21].

In this article, we study parabolic equation involving the tempered fractional Laplacian and logarithmic nonlinearity with the aid of the direct method of moving planes. A great diversity of elliptic equations involving fractional Laplacian operator have been studied by many authors. Here, *we make a new attempt to study parabolic equation with the tempered fractional Laplacian and logarithmic nonlinearity to obtain asymptotic symmetry and monotonicity of its radial solution.* We believe that the basic theory of such nonlinear temporary parallel problems established in this article will provide theoretical support for many mathematical models coming from the real world and will also help the development of numerical solutions of this type of problems.

2 Results

Before proving important theorems, we first define

$$T_\alpha=\{x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n \mid x_1=\alpha,\ \text{for}\ \in\mathbb{R}\}$$

being the moving planes and

$$\Sigma_\alpha = \{x \in \mathbb{R}^n \mid x_1 < \alpha\}$$

being the region to the left of Σ_α.

Also,

$$x^\alpha = (2\alpha - x_1, x_2, \cdots, x_n)$$

is the reflection of x about T_α. Meanwhile, we denote

$$z_\alpha(x,t) = z(x^\alpha, t) \; and \; M_\alpha(x,t) = z_\alpha(x,t) - z(x,t).$$

From the above notation, $M_\alpha(x,t)$ is antisymmetric about T_α, a.e.,

$$M_\alpha(x_1, x_2, \cdots, x_n, t) = -M_\alpha(2\alpha - x_1, (x_1)', t).$$

In order to state properties of solutions to parabolic equation, we need to define the ρ-limit set of z :

$$\rho(z) := \{\, \psi \mid \psi = \lim z(\cdot, t_k) \text{ as } t_k \to \infty\}.$$

It follows from [22] that $\rho(z) \subset C_0(\mathbb{R}^n)$ and $\rho(z)$ is compact. Moreover, it satisfies

$$\lim_{t\to\infty} dist_{C_0(\mathbb{R}^n)}(z(x,t), \rho(z)) = 0.$$

where $C_0(\mathbb{R}^n)$ is the class of all continuous functions which vanishes to zero at infinity, relative to the supremum norm defined finely. For each $\psi(x) \in \rho(z)$, we define an ρ-limit of $\psi_\alpha(x)$ as follows:

$$\Psi_\alpha(x) = \psi(x^\alpha) - \psi(x) = \psi_\alpha(x) - \psi(x).$$

For this part, we state important theorems as follows. The three theorems are asymptotic narrow region principle (Lemma 2.1), asymptotic maximum principle (Lemma 2.2) and asymptotic strong maximum principle (Lemma 2.3) concerning antisymmetric functions, respectively.

Lemma 2.1 *Suppose that* Λ *is a region in* Σ_α*, which is contained in*

$$\{\, x \mid \alpha - q < x_1 < \alpha \,\}$$

for q *small. Moreover,* $M_\alpha(x,t) \in (C^{1,1}_{loc}(\Lambda) \cap \mathcal{L}_{2s}) \times \mathrm{C}^1([\bar{t}, \infty))$ *is lower semi-continuous and uniformly bounded with respect to* x *on* $\overline{\Lambda}$*, where* $\bar{t}$ *is large enough,* *and*

$$\begin{cases} \frac{\partial M_\alpha(x,t)}{\partial t} - (\Delta+\lambda)^{\frac{\beta}{2}} M_\alpha(x,t) = [ap(\ln \mid \xi(x,t)+1 \mid + \frac{\xi(x,t)}{\xi(x,t)+1})]M_\alpha(x,t),\ (x,t) \in \\ \Lambda \times [\bar{t},\infty), \\ M_\alpha(x,t) \geq 0, \qquad (x,t) \in (\Sigma_\alpha \backslash \Lambda) \times [\bar{t},\infty), \\ M_\alpha(x,t) = -M_\alpha(x^\alpha,t), \qquad (x,t) \in \Lambda \times [\bar{t},\infty), \end{cases} \tag{2.1}$$

where $\xi(x,t)$ falls in between $z_\alpha(x,t)$ and $z(x,t)$.

Case 1: when Λ is bounded narrow region, for q being small enough , then

$$\varliminf_{t\to\infty} \mathrm{M}_\alpha(x,t) \geq 0, \quad \forall x \in \Lambda; \tag{2.2}$$

Case 2: when Λ is unbounded region, if

$$\varliminf_{|x|\to\infty} \mathrm{M}_\alpha(x,t) \geq 0, \text{ uniformly for } t \geq \bar{t}. \tag{2.3}$$

Then, (2.2) still holds.

Proof **Case 1**: let m be a constant. Set

$$\widehat{M}_\alpha(x,t) = e^{mt} M_\alpha(x,t),$$

then $\hat{M}_\alpha(x,t)$ satisfies

$$\begin{aligned} &\frac{\partial \widehat{M}_\alpha(x,t)}{\partial t} - (\Delta+\lambda)^{\frac{\beta}{2}} \widehat{M}_\alpha(x,t) \\ &= [m + ap(\ln \mid \xi(x,t)+1 \mid + \frac{\xi(x,t)}{\xi(x,t)+1})]\widehat{M}_\alpha(x,t),\ (x,t) \\ &\in \Lambda \times [\bar{t},\infty). \end{aligned} \tag{2.4}$$

Nevertheless, proving (2.2) is equivalent to prove the following (2.5),

$$\widehat{M}_\alpha(x,t) \geq \min\{0, \inf_{\Lambda} \widehat{M}_\alpha(x,\bar{t})\}, \quad (x,t) \in \Lambda \times [\bar{t},T),\ \forall\, T > \bar{t}. \tag{2.5}$$

Suppose that (2.5) is not false, under the condition that M_α on $\overline{\Lambda} \times [\bar{t},T]$ is lower semi-continuous, there is $(x^{'},t^{'}) \in \Lambda \times (\bar{t},T]$, then

$$\widehat{M}_\alpha(x^{'},t^{'}) = \min_{\Sigma_\alpha \times (\bar{t},T]} \widehat{M}_\alpha(x,t) < \min\{0, \inf_{\Lambda} \widehat{M}_\alpha(x,\bar{t})\}, \tag{2.6}$$

then we have

$$\frac{\partial \widehat{M}_\alpha(x^{'},t^{'})}{\partial t} \leq 0 \tag{2.7}$$

and

$$\begin{aligned}
&-(\Delta+\lambda)^{\frac{\beta}{2}}\widehat{M}_\alpha(x^{'},t^{'})\\
=&C_{n,\beta,\lambda}P.V.\int_{\mathbb{R}^n}\frac{\widehat{M}_\alpha(x^{'},t^{'})-\widehat{M}_\alpha(y,t^{'})}{e^{\lambda|x^{'}-y|}\mid x^{'}-y\mid^{n+\beta}}dy\\
=&C_{n,\beta,\lambda}P.V.\int_{\Sigma_\alpha}\frac{\widehat{M}_\alpha(x^{'},t^{'})-\widehat{M}_\alpha(y,t^{'})}{e^{\lambda|x^{'}-y|}\mid x^{'}-y\mid^{n+\beta}}dy+\\
&C_{n,\beta,\lambda}P.V.\int_{\mathbb{R}^n\setminus\Sigma_\alpha}\frac{\widehat{M}_\alpha(x^{'},t^{'})-\widehat{M}_\alpha(y,t^{'})}{e^{\lambda|x^{'}-y|}\mid x^{'}-y\mid^{n+\beta}}dy\\
=&C_{n,\beta,\lambda}P.V.\int_{\Sigma_\alpha}\frac{\widehat{M}_\alpha(x^{'},t^{'})-\widehat{M}_\alpha(y,t^{'})}{e^{\lambda|x^{'}-y|}\mid x^{'}-y\mid^{n+\beta}}dy+\\
&C_{n,\beta,\lambda}P.V.\int_{\Sigma_\alpha}\frac{\widehat{M}_\alpha(x^{'},t^{'})-\widehat{M}_\alpha(y,t^{'})}{e^{\lambda|x^{'}-y^\alpha|}\mid x^{'}-y^{\alpha|^{n+\beta}}}dy\\
\leq&C_{n,\beta,\lambda}P.V.\int_{\Sigma_\alpha}\frac{2\widehat{M}_\alpha(x^{'},t^{'})}{e^{\lambda|x^{'}-y^\alpha|}\mid x^{'}-y^{\alpha|^{n+\beta}}}dy\\
=&\frac{C}{e^{\lambda q}\cdot q^\beta}\widehat{M}_\alpha(x^{'},t^{'}).
\end{aligned}\tag{2.8}$$

Here, we suppose

$$D=\{y\mid\ q<y_1-(x^{'})_1<2q,\ \ \mid y'-(x^{'})^{'}\mid<q\},$$

in consequence

$$\begin{aligned}
&\int_{\Sigma_\alpha}\frac{1}{e^{\lambda|x^{'}-y^\alpha|}\mid x^{'}-y^{\alpha|^{n+\beta}}}dy\geq\int_D\frac{1}{e^{\lambda|x^{'}-y^\alpha|}\mid x^{'}-y^{\alpha|^{n+\beta}}}dy\\
&\geq\frac{m_1}{e^{\lambda q}\cdot q^{n+\beta}}\cdot\mid D\mid\\
&=\frac{m_1}{e^{\lambda q}\cdot q^\beta}.
\end{aligned}$$

Hence, by (2.4), we obtain

$$\begin{aligned}
\frac{\partial\widehat{M}_\alpha(x^{'},t^{'})}{\partial t}&=(\Delta+\lambda)^{\frac{\beta}{2}}\widehat{M}_\alpha(x^{'},t^{'})+\left[m+ap(\ln\mid\xi(x^{'},t^{'})+1\mid+\frac{\xi(x^{'},t^{'})}{\xi(x^{'},t^{'})+1})\right]\\
\widehat{M}_\alpha(x^{'},t^{'})&\geq\left[-\frac{m_1}{e^{\lambda q}\cdot q^\beta}+m+ap(\ln\mid\xi(x^{'},t^{'})+1\mid+\frac{\xi(x^{'},t^{'})}{\xi(x^{'},t^{'})+1})\right]\widehat{M}_\alpha(x^{'},t^{'}).
\end{aligned}\tag{2.9}$$

We choose small enough q such that in (2.9),

$$[-\frac{m_1}{e^{\lambda q}\cdot q^{\beta}}+m+ap(\ln|\xi(x^{'},t^{'})+1|+\frac{\xi(x^{'},t^{'})}{\xi(x^{'},t^{'})+1})]\widehat{M}_{\alpha}(x^{'},t^{'})> 0,$$

which contradicts (2.7).

Since M_α is bounded, by the definition of boundedness, we have $m_2 > 0$ and for $(x,t)\in\Lambda\times[\bar{t},T), \forall\, T>\bar{t}$

$$\widehat{M}_{\alpha}(x,t)\geq\min\{0,\inf_{\Lambda}\widehat{M}_{\alpha}(x,\bar{t})\}\geq -m_2. \tag{2.10}$$

As a result,

$$M_{\alpha}(x,t)\geq e^{-mt}(-m_2), \forall\, t>\bar{t}.$$

Taking the limit as $t\to\infty$, we get

$$\varliminf_{t\to\infty} M_{\alpha}(x,t)\geq 0,\quad x\in\Lambda.$$

Case 2: While Λ is unbounded domain, we know that the minimum value point of $\widehat{M}_{\alpha}(x,t)$ can be obtained by assuming condition (2.3). Then, a final conclusion is reached through a process equivalent to **Case 1**.
This completes the proof. □

Lemma 2.2 *Suppose that $\Lambda\subset\Sigma_\alpha$ is bounded and $N(x,t)\in(C^{1,1}_{loc}(\Lambda)\cap\mathcal{L}_{2s})\times C^1([0,\infty))$ is lower semi-continuous with respect to x on $\overline{\Lambda}$, if*

$$\begin{cases}\frac{\partial N(x,t)}{\partial t}-(\Delta+\lambda)^{\frac{\beta}{2}}N(x,t)\geq[ap(\ln|\xi(x,t)+1|+\frac{\xi(x,t)}{\xi(x,t)+1})]N(x,t), (x,t)\in\\ \Lambda\times[0,\infty),\\ N(x^{\alpha},t)=-N(x,t), & (x,t)\in\Sigma_\alpha\times[0,\infty),\\ N(x,t)\geq 0, & (x,t)\in(\Sigma_\alpha\backslash\Lambda)\times[0,\infty),\\ N(x,0)\geq 0, & x\in\Lambda,\end{cases}$$

where $\xi(x,t)$ falls in between $z_\alpha(x,t)$ and $z(x,t)$, then for $(x,t)\in\Lambda\times[0,T]$, $\forall\, T>0$,

$$N(x,t)\geq 0. \tag{2.11}$$

Proof Set

$$\hat{N}(x,t)=e^{mt}N(x,t),$$

then $\hat{N}(x,t)$ satisfies

$$\frac{\hat{N}(x,t)}{\partial t} - (\Delta+\lambda)^{\frac{\beta}{2}}\hat{N}(x,t) \geq [m+2a(\ln \mid \xi(x,t)+1 \mid + \frac{\xi(x,t)}{\xi(x,t)+1})]\hat{N}(x,t), \ (x,t) \in \Lambda \times [0,\infty). \tag{2.12}$$

We assert that

$$\hat{N}(x,t) \geq \inf_{\Lambda} \hat{N}(x,0), \quad (x,t) \in \Lambda \times [\bar{t}, T].$$

Suppose it is violated, in view of the continuity of $N(x,t)$, we can obtain that there is $(x^{'}, t^{'}) \in \Lambda \times (0, T]$, then

$$\hat{N}(x^{'}, t^{'}) = \min_{\Sigma_\alpha \times (0,T]} \hat{N}(x,t) < \ 0,$$

We choose $m < 0$ such that

$$m + ap(\ln \mid \xi(x^{'}, t^{'}) + 1 \mid + \frac{\xi(x^{'}, t^{'})}{\xi(x^{'}, t^{'}) + 1}) < 0,$$

then

$$[m + ap(\ln \mid \xi(x^{'}, t^{'}) + 1 \mid + \frac{\xi(x^{'}, t^{'})}{\xi(x^{'}, t^{'}) + 1})]\hat{N}(x^{'}, t^{'}) > 0.$$

However,

$$\frac{\partial \hat{N}(x^{'}, t^{'})}{\partial t} \leq 0$$

and

$$\begin{aligned}
&-(\Delta+\lambda)^{\frac{\beta}{2}}\hat{N}(x^{'}, t^{'}) \\
&= C_{n,\beta,\lambda} P.V. \int_{\mathbb{R}^n} \frac{\hat{N}(x^{'}, t^{'}) - \hat{N}(y, t^{'})}{e^{\lambda|x^{'}-y|} \mid x^{'} - y \mid^{n+\beta}} dy \\
&= C_{n,\beta,\lambda} P.V. \int_{\Sigma_\alpha} \frac{\hat{N}(x^{'}, t^{'}) - \hat{N}(y, t^{'})}{e^{\lambda|x^{'}-y|} \mid x^{'} - y \mid^{n+\beta}} dy + \\
&C_{n,\beta,\lambda} P.V. \int_{\mathbb{R}^n \setminus \Sigma_\alpha} \frac{\hat{N}(x^{'}, t^{'}) - \hat{N}(y, t^{'})}{e^{\lambda|x^{'}-y|} \mid x^{'} - y \mid^{n+\beta}} dy \\
&= C_{n,\beta,\lambda} P.V. \int_{\Sigma_\alpha} \frac{\hat{N}(x^{'}, t^{'}) - \hat{N}(y, t^{'})}{e^{\lambda|x^{'}-y|} \mid x^{'} - y \mid^{n+\beta}} dy + \\
&C_{n,\beta,\lambda} P.V. \int_{\Sigma_\alpha} \frac{\hat{N}(x^{'}, t^{'}) - \hat{N}(y, t^{'})}{e^{\lambda|x^{'}-y^\alpha|} \mid x^{'} - y^{\alpha|^{n+\beta}}} dy \\
&\leq C_{n,\beta,\lambda} P.V. \int_{\Sigma_\alpha} \frac{2\hat{N}(x^{'}, t^{'})}{e^{\lambda|x^{'}-y^\alpha|} \mid x^{'} - y^{\alpha|^{n+\beta}}} dy \\
&< \ 0.
\end{aligned}$$

This yields a contradiction with (2.12). Therefore,

$$N(x,t) \geq 0.$$

□

Lemma 2.3 *Suppose that* $M_\alpha(x,t) \in (C^{1,1}_{loc}(\Lambda) \cap \mathcal{L}_{2s}) \times C^1([\bar{t},\infty))$ *is bounded, where* $\bar{t}$ *is large enough, and*

$$\begin{cases} \frac{\partial M_\alpha(x,t)}{\partial t} - (\Delta+\lambda)^{\frac{\beta}{2}} M_\alpha(x,t) = [ap(\ln \mid \xi(x,t)+1 \mid + \frac{\xi(x,t)}{\xi(x,t)+1})] M_\alpha(x,t), (x,t) \\ \in \Sigma_\alpha \times [\bar{t},\infty), \\ M_\alpha(x,t) = -M_\alpha(x^\alpha,t), & (x,t) \in \Sigma_\alpha \times [\bar{t},\infty), \\ \varliminf_{t\to\infty} M_\alpha(x,t) \geq 0, & x \in \Sigma_\alpha, \end{cases} \tag{2.13}$$

where $\xi(x,t)$ *falls in between* $z_\alpha(x,t)$ *and* $z(x,t)$*. Assume that* Ψ_α *is greater than* 0 *somewhere in* Σ_α*.*

Then $\Psi_\alpha(x)$ *is greater than* 0 *in* Σ_α*.*

Proof For every $\psi \in \rho(z)$, notice that the definition of $\rho(z)$, we infer that there exists t_k and $t_k \to \infty$, $M_\alpha(x,t_k) \to \Psi_\alpha(x)$. Let

$$M_k(x,t) = M_\alpha(x, t+t_k-1).$$

It follows that

$$\frac{\partial M_k(x,t)}{\partial t} - (\Delta+\lambda)^{\frac{\beta}{2}} M_k(x,t) = [ap(\ln \mid \xi_k(x,t)+1 \mid + \frac{\xi_k(x,t)}{\xi_k(x,t)+1})] M_k(x,t), \quad (x,t) \in \Sigma_\alpha \times [\bar{t},\infty),$$

with $\xi_k(x,t) = \xi(x,t+t_k-1)$. In view of standard parabolic regularity estimates [22], leads to

$$\frac{\partial M_k(x,t)}{\partial t} - (\Delta+\lambda)^{\frac{\beta}{2}} M_k(x,t) \to \frac{\partial M_\infty(x,t)}{\partial t} - (\Delta+\lambda)^{\frac{\beta}{2}} M_\infty(x,t),$$
$$\xi_k(x,t) \to \xi_\infty(x,t), \qquad k \to \infty,$$

where $(x,t) \in \Sigma_\alpha \times [0,2]$ and $M_\infty(x,t)$ is Hölder continuous with respect to x and t.

Let $t=1$, we have

$$M_\alpha(x,t_k) = M_k(x,1) \to M_\infty(x,1) = \Psi_\alpha(x), \quad k \to \infty.$$

Choose $m > 0$, then

$$m + 2a(\ln \mid \xi_\infty(x,t)+1 \mid + \frac{\xi_\infty(x,t)}{\xi_\infty(x,t)+1}) > 0.$$

Let

$$\widehat{M}(x,t) = e^{mt} M_\infty(x,t).$$

By $\varliminf\limits_{t\to\infty} M_\alpha(x,t) \geq 0$, we get

$$\widehat{M}(x,t) \geq 0, \quad in \ \ \Sigma_\alpha \times [0,2].$$

Base on above, we can see that

$$\begin{aligned} \frac{\partial \widehat{M}(x,t)}{\partial t} - (\Delta+\lambda)^{\frac{\beta}{2}} \widehat{M}(x,t) = [m + ap(\ln \mid \xi_\infty(x,t) \\ +1 \mid + \tfrac{\xi_\infty(x,t)}{\xi_\infty(x,t)+1})]\widehat{M}(x,t) \geq 0, \\ (x,t) \in \Sigma_\alpha \times [0,2]. \end{aligned} \tag{2.14}$$

Considering the assumption that Ψ_α is greater than 0 somewhere in Σ_α, through continuity, we derive that there is a set $E \subset\subset \Sigma_\alpha$, then

$$\Psi_\alpha(x) > m_3 > 0, \quad x \in E. \tag{2.15}$$

here m_3 is a constant.

Because $M_\infty(x,t)$ is continuous, by the definition of continuity, we have $0 < \varepsilon_0 < 1$, then

$$M_\infty(x,t) > \frac{m_4}{2}, \ \ (x,t) \in E \times [1-\varepsilon_0, 1+\varepsilon_0].$$

To simplify, we transform $E \times [1-\varepsilon_0, 1+\varepsilon_0]$ into $E \times [0,2]$. For $(x,t) \in E \times [0,2]$, we attain

$$M_\infty(x,t) > \frac{m_4}{2}. \tag{2.16}$$

Considering arbitrary $\hat{x} \in \Sigma_\alpha \backslash E$, define $\gamma = min\{dist(\hat{x}, E), dist(\hat{x}, T_\alpha)\}$, thus $B_\gamma(\hat{x}) \subset \Sigma_\alpha \backslash E$.

Then, we set up a lower solution in $B_\gamma(\hat{x}) \times [0,2]$. Let

$$\overline{M}(x,t) = \zeta_{E\cup E_\alpha}(x)\widehat{M}(x,t) + \varepsilon\tau(t)h(x),$$

with

$$\zeta_{E\cup E_\alpha}(x) = \begin{cases} 1 & x \in E \cup E_\alpha, \\ 0 & other, \end{cases}$$

$$\tau(t) = \begin{cases} 1 & t \in [\frac{1}{2}, 1], \\ 0 & t \notin [0, 2], \end{cases}$$
$$h(x) = (\gamma^2 - | x - \hat{x} |^2)_+^s - (\gamma^2 - | x - \hat{x}^\alpha |^2)_+^s.$$

Observing that $h(\hat{x}) = \gamma^{2s}$, $h(\hat{x}^\alpha) = -h(\hat{x})$. In the meantime,

$$-(\Delta + \lambda)^{\frac{\beta}{2}} h(x) \leq m_5, \tag{2.17}$$

where m_5 is an invariable.

Considering $(x, t) \in B_\gamma(\hat{x}) \times [0, 2]$, we deduce

$$\begin{aligned}
&-(\Delta + \lambda)^{\frac{\beta}{2}} (\zeta_{E\cup E_\alpha}(x)\widehat{M}(x, t)) \\
&= C_{n,\beta,\lambda} P.V. \int_{\mathbb{R}^n} \frac{\zeta_{E\cup E_\alpha}(x)\widehat{M}(x, t) - \zeta_{E\cup E_\alpha}(y)\widehat{M}(y, t)}{e^{\lambda|x-y|} | x - y |^{n+\beta}} dy \\
&= C_{n,\beta,\lambda} P.V. \int_{\mathbb{R}^n} \frac{-\zeta_{E\cup E_\alpha}(y)\widehat{M}(y, t)}{e^{\lambda|x-y|} | x - y |^{n+\beta}} dy \\
&= C_{n,\beta,\lambda} P.V. \int_E \frac{-\widehat{M}(y, t)}{e^{\lambda|x-y|} | x - y |^{n+\beta}} + C_{n,\beta,\lambda} P.V. \int_E \frac{-\widehat{M}(y^\alpha, t)}{e^{\lambda|x-y^\alpha|} | x - y^\alpha |^{n+\beta}} dy \\
&= C_{n,\beta,\lambda} P.V. \int_E \left(\frac{1}{e^{\lambda|x-y^\alpha|} | x - y^\alpha |^{n+\beta}} - \frac{1}{e^{\lambda|x-y|} | x - y |^{n+\beta}}\right)\widehat{M}(y, t) dy \\
&\leq -m_6,
\end{aligned} \tag{2.18}$$

where $m_6 > 0$. In the light of (2.17) with (2.18), then

$$\frac{\partial \overline{M}(x, t)}{\partial t} - (\Delta + \lambda)^{\frac{\beta}{2}} \overline{M}(x, t) = \varepsilon\tau'(t)h(x) - (\Delta + \lambda)^{\frac{\beta}{2}} (\zeta_{E\cup E_\alpha}(x)\widehat{M}(x, t)) - \varepsilon\tau(t)$$
$$(\Delta + \lambda)^{\frac{\beta}{2}} h(x) \leq \varepsilon\tau'(t)h(x) - m_6 + \varepsilon\tau(t)m_5.$$

Next, take $\varepsilon > 0$ small enough, we have

$$\frac{\partial \overline{M}(x, t)}{\partial t} - (\Delta + \lambda)^{\frac{\beta}{2}} \overline{M}(x, t) \leq 0. \tag{2.19}$$

Let $N(x, t) = \widehat{M}(x, t) - \overline{M}(x, t)$, by (2.14) and (2.19), we can get

$$\frac{\partial N(x,t)}{\partial t} - (\Delta + \lambda)^{\frac{\beta}{2}} N(x, t) \geq [ap(\ln | \xi_k(x, t) + 1 | + \frac{\xi_k(x,t)}{\xi_k(x,t)+1})]N(x, t),\ (x, t) \in B_\gamma(\hat{x}) \times [0, 2].$$

Also,

$$N(x, t) = \widehat{M}(x, t) - \overline{M}(x, t) \geq 0, \quad (x, t) \in (\Sigma_\alpha \backslash B_\gamma(\hat{x})) \times [0, 2],$$

and

$$N(x,0)\geq 0,\quad x\in\Sigma_\alpha.$$

As a result, by Theorem 2.2, we have

$$N(x,t)\geq 0,\quad (x,t)\in B_\gamma(\hat{x})\times[0,2].$$

which means

$$N(x,t)=e^{mt}M_\infty(x,t)-\varepsilon\tau(t)h(x)\geq 0,\quad (x,t)\in B_\gamma(\hat{x})\times[0,2].$$

Let $t=1$, thus $\tau(t)=1$. So

$$M_\infty(x,1)\geq e^{-m}\varepsilon h(x),\quad x\in B_\gamma(\hat{x}).$$

In view of $h(\hat{x})=\gamma^{2s}$, we can get

$$\Psi_\alpha(\hat{x})=e^{-m}\varepsilon\gamma^{2s}>0. \tag{2.20}$$

Since $\tilde{x}$ is arbitrary in $\Sigma_\alpha\backslash E$, by (2.15) and (2.20), we obtain

$$\Psi_\alpha(x)>0,\quad x\in\Sigma_\alpha.$$

The proof is completed. □

3 Example

In this section, we will apply above theorems to study the properties of asymptotic radial solution for the parabolic problem in $B_1(0)$ by using the direct moving planes.

Lemma 3.1 *Let* $z(x,t)\in(C^{1,1}_{loc}(B_1(0))\cap C(\overline{B_1(0)}))\times C^1((0,\infty))$ *be a positive uniformly bounded solution of the following equation*

$$\begin{cases}\frac{\partial z(x,t)}{\partial t}-(\Delta+\lambda)^{\frac{\beta}{2}}z(x,t)=az(x,t)\ln|z(x,t)+1|^p, & (x,t)\in B_1(0)\times(0,\infty),\\ z(x,t)=0, & (x,t)\in B_1^c(0)\times(0,\infty),\end{cases} \tag{3.1}$$

where $a>0$. *Then two conclusions can be derived as follows:*

(H_1) $\psi(x)=0$.

(H_2) $\psi(x)$ *is necessarily radical symmetric. Moreover, it is strictly decreasing with respect to the origin.*

Proof Choose arbitrary direction from the origin as the positive direction of x_1, the following T_α, Σ_α, x^α, z_α, M_α, ψ, Ψ_α have be introduced in Section 2. Set

$$\Lambda_\alpha = \{x \in B_1(0) \mid x_1 < \alpha\}.$$

By (3.1), we can get

$$\begin{cases} \frac{\partial M_\alpha(x,t)}{\partial t} - (\Delta+\lambda)^{\frac{\beta}{2}} M_\alpha(x,t) = [ap(\ln \mid \xi(x,t)+1 \mid + \frac{\xi(x,t)}{\xi(x,t)+1})]M_\alpha(x,t),\ (x,t) \in \\ \Lambda_\alpha \times (0,\infty), \\ M_\alpha(x,t) = -M_\alpha(x^\alpha,t), \qquad (x,t) \in \Lambda_\alpha \times (0,\infty), \end{cases} \tag{3.2}$$

where $\xi(x,t)$ falls in between $z_\alpha(x,t)$ and $z(x,t)$.

For case (H_1), given that $\psi(x) = 0$, the above conclusion can be derived easily. So here we consider case (H_2), that is $\psi \neq 0$ in $B_1(0)$. The proof of the above theorem can be accomplished by two steps.

Step 1. We prove the following inequality,

$$\Psi_\alpha \geq 0,\ \ x \in \Lambda_\alpha,\ \text{ for all } \ \in æ(z),\ \text{ when } \ > -1 \text{ and close enough to } -1. \tag{3.3}$$

Since when $(x,t) \in B_1^c(0) \times (0,\infty)$, $z(x,t) = 0$, consequently

$$M_\alpha(x,t) \geq 0,\ x \in \Sigma_\alpha \backslash \Lambda_\alpha,\ t \in (0,\infty).$$

Together with (3.2), we use Theorem 2.1 to derive (3.3).

Step 2. On account of (3.3), we will move the plane T_α as long as the inequality holds. Let

$$\alpha_0 = \sup\{ \leq 0 \mid 9\text{-}(x) \geq 0,\ \forall \ \in æ(z), x \in 3\text{-}, {}^{-} \leq \}, \tag{3.4}$$

then we explain

$$\alpha_0 = 0.$$

If not, in view of (3.4), then

$$\Psi_{\alpha_0}(x) \geq 0,\ x \in \Lambda_{\alpha_0}.$$

We Firstly show that for every $\psi \in \rho(z)$, there is $x^1 \in \Sigma_{\alpha_0}$, then

$$\Psi_{\alpha_0}(x^1) > 0. \tag{3.5}$$

If (3.5) is not true, in case that there is $\bar{\psi} \in \rho(z)$,

$$\bar{\Psi}_{\alpha_0}(x) = \bar{\psi}_{\alpha_0}(x) - \psi(x) \equiv 0\ in\ \Sigma_{\alpha_0}.$$

Otherwise, $z(x,t)=0, (x,t)\in B_1^c(0)\times(0,\infty)$. As a result, $\bar{\psi}(x)\equiv 0$ in $B_1^c(0)\cap\Sigma_{\alpha_0}$. Morewhile, by continuity there is $x^0\in B_1(0)$, then $\bar{\psi}(x^0)=0$.

For above $\bar{\psi}$, we know that there is t_k and it satisfies $t_k\to\infty$, $z(x,t_k)\to\bar{\psi}(x)$. Then in view of [22], we have

$$\frac{\partial z_\infty(x,t)}{\partial t}-(\Delta+\lambda)^{\frac{\beta}{2}}z_\infty(x,t)=az_\infty(x,t)\ln\mid z_\infty(x,t)+1\mid^p$$

and $z_\infty(x,1)=\bar{\psi}(x)$. Observing that $z_\infty(x,t)\geq 0$, we get $\frac{\partial z_\infty(x^0,1)}{\partial t}\leq 0$ and

$$-(\Delta+\lambda)^{\frac{\beta}{2}}z_\infty(x^0,1)=C_{n,\beta,\lambda}P.V.\int_{B_1(0)}\frac{-z_\infty(y,1)}{e^{\lambda|x^0-y|}\mid x^0-y\mid^{n+\beta}}dy<0.$$

Since $\bar{\psi}\neq 0$ in $B_1(0)$, there ensures $z_\infty(y,1)\neq 0$. Therefore, $az_\infty(x^0,1)\ln\mid z_\infty(x^0,1)+1\mid^2<0$. This is a contradiction. So (3.5) is established.

Now taking account of Theorem 2.3, it is clear that for every $\psi\in\rho(z)$,

$$\Psi_{\alpha_0}(x)>0, x\in\Lambda_{\alpha_0}. \tag{3.6}$$

It follows that for any $\delta>0$, there is a general constant m_7, then

$$\Psi_{\alpha_0}(x)\geq m_7>0, x\in\overline{\Lambda_{\alpha_0-\delta}}. \tag{3.7}$$

Because Ψ_α is continuous, we deduce that $\varepsilon>0$, satisfies

$$\Psi_\alpha(x)\geq\frac{C_0}{2}>0, x\in\overline{\Lambda_{\alpha_0-\delta}}, \forall\alpha\in(\alpha_0,\alpha_0+\varepsilon), \tag{3.8}$$

which implies

$$M_\alpha(x,t)\geq 0, x\in\overline{\Lambda_{\alpha_0-\delta}}, \forall\alpha\in(\alpha_0,\alpha_0+\varepsilon).$$

For $\delta,\varepsilon>0$ being small enough, it is clear that for $\alpha\in(\alpha_0,\alpha_0+\varepsilon)$, $\Lambda_\alpha\backslash\Lambda_{\alpha_0-\delta}$ is a narrow region. Then taking account of Theorem 2.1, one can get

$$\Psi_\alpha(x)\geq 0,\ \forall x\in\Lambda_\alpha\backslash\Lambda_{\alpha_0-\delta}. \tag{3.9}$$

Together with (3.8), then

$$\Psi_\alpha(x)\geq 0,\ \forall x\in\Lambda_\alpha,\ \forall\alpha\in(\alpha_0,\alpha_0+\varepsilon),\ \forall\psi\in\rho(z),$$

which yields a contradiction. Consequently, $\alpha_0=0$. Thus,

$$\Psi_0(x)\geq 0,\ x\in\Lambda_0$$

or

$$\psi(x_1, \cdots, x_n) \geq \psi(-x_1, \cdots, x_n),\ 0 < x_1 < 1. \tag{3.10}$$

Because the direct of x_1 is arbitrary, we derive the radial symmetry of $\psi(x)$.

The monotone property follows from

$$\Psi_\alpha > 0,\ x \in \Lambda_\alpha,\ \forall -1 < \alpha < 0,$$

which is similar to the process of (3.6). □

4 Conclusion

In this paper, we deal with parabolic equation with the tempered fractional Laplacian and logarithmic nonlinearity. First, we prove asymptotic maximum principle, asymptotic narrow region principle and asymptotic strong maximum principle for antisymmetric functions. Then, we prove some properties of parabolic equation in a unit ball.

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Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

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