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The Backward Problem of Stochastic Convection–Diffusion Equation

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Abstract
In this paper, we consider a backward problem for the stochastic convection–diffusion equation. The source term is driven by the fraction Brownian motion. We illustrate the regularity of the mild solution and prove the instability of this problem. In order to overcome the ill-posedness, we apply a truncated regularization method to obtain a stable numerical approximation to $u(x, t)$. Convergence estimates are presented under the a-priori parameter choice rule. Finally, some numerical experiments are given to show the effectivity of the regularization method.

Keywords Backward problem · Convection–diffusion equation · Existence · Ill-posedness · Truncated regularization method

Mathematics Subject Classification 35R30 · 35R60 · 60G22 · 65M32

1 Introduction

In recent decades, the backward heat conduction problem (BHCP) [15, 16, 23, 40] is very important in environmental science, energy development, fluid mechanics and electronic science. It aims at detecting the previous status of physical field from its

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present information. It is well known that such a problem is severely ill-posed in the Hadamard sense [5, 37]. That is to say, a small perturbation of the measurements may cause a large change in the solution. At present, there are a lot of research achievements on the BHCP of deterministic situation.

On the one hand, the classical convection–diffusion model is the parabolic equation $\frac{\partial u}{\partial t} + \nu \nabla u = D \triangle u$, $x \in D \subset \mathbb{R}^d$, $t > 0$. The backward problem for this equation has been studied extensively. Chen and Liu [3] used a new regularization method to give the regularized solution. The new regularization method is considered with both the number of truncation terms and the approximation accuracy for the final data as multiple regularization parameters. Hào and Nguyen [11] gave a convergence estimate in the sense of L^p-norm by a mollification method for the BHCP, where $p \in (1, \infty)$. Cheng et al. [4] proved the uniqueness of the mild solution for the parabolic equations and applied the discrete Tikhonov regularization method with the generalized cross validation rule to obtain a stable numerical approximation to the initial value. Li et al. [21] used the Carleman estimation to get the conditional stability for such a kind of problem. More related works are available to references [2, 9, 12–14, 17, 22, 26–29, 36, 41].

On the other hand, the time-fractional diffusion equation has attracted the attention of many scholars recently. Wang and Liu [38] considered a backward problem for a time-fractional diffusion process with inhomogeneous media of which the regularization method is same as [3]. Wei and Zhang [39] discussed the backward problem for a time-fractional diffusion-wave equation in a bounded domain and gave the existence, uniqueness and conditional stability. Then, Floridial and Yamamoto [7] improved the theoretical achievements of [39]. Tuan et al. [30] studied a backward problem for a nonlinear fractional diffusion equation and used a filter regularization method to approximate the solution.

With the rapid development of science and technology, more and more scholars realize that the mathematical physical system is inevitably accompanied by random disturbances, which motivates us to add random terms to certain mathematical models. For the direct problem, Foondun [6] considered a stochastic time-fractional diffusion equation (STFDE) over a bounded domain. In [33], random Rayleigh–Stokes equations with Riemann–Liouville fractional derivatives were studied, and the existence and uniqueness of solutions corresponding to two different source terms were discussed. Thach et al. [34, 35] discussed random pseudo-parabolic equations with fractional Caputo derivations for different Hurst parameters and gave the existence, uniqueness and continuity of mild solutions. For the inverse problem, Lv [18] gave a global Carleman estimation for stochastic parabolic equation and studied two kinds of inverse problems. Li and Wang considered an inverse random source problem for the Helmholtz equation driven by a fractional Gaussian field [19] and the time-harmonic Maxwell's equations driven by a centered complex-valued Gaussian vector field with correlated components [20]. Gong et al. [10] considered an inverse random source problem for the STFDE and proved the uniqueness of inverse source problem by the variance of the Fourier transform of the boundary data. About the backward problem, Peng et al. [25] dealt with a nonlocal backward problem for a STFDE and obtained some properties of backward problem with the nonlocally terminal value term. Feng et al. [8] studied a STFDE with the source $f(x)h(t) + g(x)\dot{B}^H(t)$ driven by fractional Brownian motion (fBm) and reconstructed $f(x)$ and $|g(x)|$ from the statistics of the final data $u(x, T, \omega)$. Tuan et al. [32] studied two terminal value problems for bi-parabolic equations driven by Wiener process and fBm ($H \in (\frac{1}{2}, 1)$) and discussed existence and instability of the mild solution. However, to the best of our knowledge, there has not been such work about the stochastic convection–diffusion equation. This is our motivation for this work. In detail, we study the following backward problem for the stochastic convection–diffusion equation driven by fBm:

$$\begin{cases} u_t(x,t) + u_x(x,t) - u_{xx}(x,t) = F(x,t), & (x,t) \in D \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial D \times [0,T], \\ u(x,T) = g(x), & x \in \overline{D}, \end{cases} \tag{1.1}$$

where $D = (0, 1)$ and the random source is assumed as

$$F(x,t) = f(x)h(t) + \sigma(x)\dot{B}^H(t).$$

Here, $f(x)$ and $\sigma(x)$ are deterministic functions with compact supports contained in D, $h(t)$ is also a deterministic function, $B^H(t)$ is the fBm with the Hurst index $H \in (0, 1)$ and $\dot{B}^H(t)$ can be roughly understood as the derivative of $B^H(t)$ with respect to the time t. When $H = \frac{1}{2}$, the fBm reduces to the classical Brownian motion and $\dot{B}^H(t)$ becomes the white noise. Since the source $F(x, t)$ is a random field with low regularity, it is a distribution instead of a function. Moreover, the direct problem has been discussed in [42]. This paper will discuss the regularity and instability of the mild solution of the backward problem (1.1).

It is known that the operator $\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2}$ with the homogeneous Dirichlet boundary condition has an eigensystem $\{\lambda_n, e_n\}_{n=1}^{\infty}$, where $\lambda_n = \frac{1}{4} + n^2\pi^2$, $e_n(x) = \sqrt{2}e^{\frac{x}{2}}\sin(n\pi x)$. The eigenvalues satisfy $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$. And the eigenfunctions $\{e_n\}_{n=1}^{\infty}$ are a complete orthonormal basis weighted by e^{-x} over $L^2(D)$. Here, the corresponding weighted space is called $\widetilde{L}^2(D)$. Specifically,

$$\widetilde{L}^2(D) = \{v \in L^2(D) \big| \int_D e^{-x} v^2(x)dx < \infty\}.$$

In this paper, $(\cdot, \cdot)$ is defined as $(f, g) = \int_D e^{-x} f(x)g(x)dx$ and $\|\cdot\|$ denotes the $\widetilde{L}^2(D)$-norm, i.e., $\|f(\cdot)\| := (\int_D e^{-x} f^2(x)dx)^{\frac{1}{2}} < \infty$, for any functions $f(x), g(x) \in L^2(D)$. Moreover,

$$f(x) = \sum_{n=1}^{\infty} f_n e_n(x), \quad f_n = (f, e_n) = \int_D e^{-x} f(x)e_n(x)dx.$$

The outline of this paper is as follows. In Sect. 2, we firstly provide the mild solution of this problem and some useful lemmas. The regularity of the mild solution and instability of the problem are proved in Sect. 3. In Sect. 4, we propose a truncated regularization method and give a convergence estimation under an a-priori regularization parameter choice rule. At last, some numerical examples are presented in Sect. 5.

2 Preliminaries

In this section, we introduce some useful definitions and lemmas.

Definition 1 $(\Omega, \mathcal{F}, \mathbb{P})$ is called a complete probability space, if Ω is a sample space, $\mathcal{F}$ is a σ-algebra on Ω, and $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure on the measurable space $(\Omega, \mathcal{F})$.

For a random variable X defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the expectation of X is defined by $\mathbb{E}(X) = \int_\Omega X(\omega)d\mathbb{P}(\omega)$. Then, the variance of X is $\mathbb{V}(X) = \mathbb{E}(X - \mathbb{E}(X))^2 = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$. Moreover, for two random variables X and Y, the covariance of X and Y is $\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))]$. In the sequel, the dependence of random variables on the sample $\omega \in \Omega$ will be omitted unless it is necessary to avoid confusion.

Definition 2 [24] A Gaussian process $B^H = \{B^H(t), t \geq 0\}$ is called fractional Brownian motion of Hurst parameter $H \in (0, 1)$ if it has mean zero and the covariance function

$$R(t, s) = \mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right).$$

Particularly, if $H = \frac{1}{2}$, B^H becomes the standard Brownian motion and is usually denoted by W which covariance function turns to be $R(t, s) = t \wedge s$.

The fBm B^H with $H \in (0, 1)$ has a Wiener integral representation

$$B^H(t) = \int_0^t K_H(t, s)dW(s),$$

where K_H is a square integrable kernel and W is the standard Brownian motion.

For a fixed interval $[0, T]$, denote by $\mathcal{E}$ the space of step functions on $[0, T]$ and by $\mathcal{H}$ the closure of $\mathcal{E}$ with respect to the product

$$\langle \chi_{[0,t]}, \chi_{[0,s]} \rangle_{\mathcal{H}} = R(t, s),$$

where $\chi_{[0,t]}$ and $\chi_{[0,s]}$ are the characteristic functions. For $\psi(t), \phi(t) \in \mathcal{H}$, it follows from the Itô isometry that

(1) If $H \in (0, \frac{1}{2})$,

$$\mathbb{E}\left[\int_0^t \psi(s)\mathrm{d}B^H(s) \int_0^t \phi(s)\mathrm{d}B^H(s)\right]$$

$$= \int_0^t \left[K_H(t,\tau)\psi(\tau) + \int_\tau^t (\psi(u) - \psi(\tau)) \frac{\partial K_H(u,\tau)}{\partial u} du \right] \cdot \left[K_H(t,\tau)\phi(\tau) + \int_\tau^t (\phi(u) - \phi(\tau)) \frac{\partial K_H(u,\tau)}{\partial u} du \right] d\tau, \quad (2.1)$$

where

$$K_H(t,\tau) = C_H \left[\left(\frac{t}{\tau}\right)^{H-\frac{1}{2}} (t-\tau)^{H-\frac{1}{2}} - (H - \frac{1}{2})\tau^{\frac{1}{2}-H} \int_\tau^t u^{H-\frac{3}{2}} (u-\tau)^{H-\frac{1}{2}} du \right],$$

and $C_H = \left(\frac{2H}{(1-2H)\beta(1-2H, H+\frac{1}{2})} \right)^{\frac{1}{2}}$ with $\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} \mathrm{d}t$. Since

$$\frac{\partial K_H(u,\tau)}{\partial u} = C_H(H - \frac{1}{2}) \left(\frac{u}{\tau}\right)^{H-\frac{1}{2}} (u-\tau)^{H-\frac{3}{2}},$$

obviously, $K_H(u,\tau) > 0$ and it is decreasing monotonically with respect to u, for $H \in (0, \frac{1}{2})$, $0 < \tau < u$.

(2) If $H = \frac{1}{2}$,

$$\mathbb{E}\left[\int_0^t \psi(s) \mathrm{d}B^H(s) \int_0^t \phi(s) \mathrm{d}B^H(s) \right] = \mathbb{E}\left[\int_0^t \psi(s) dW(s) \int_0^t \phi(s) dW(s) \right] = \int_0^t \psi(s)\phi(s) \mathrm{d}s. \quad (2.2)$$

(3) If $H \in (\frac{1}{2}, 1)$,

$$\mathbb{E}\left[\int_0^t \psi(s) \mathrm{d}B^H(s) \int_0^t \phi(s) \mathrm{d}B^H(s) \right] = \alpha_H \int_0^t \int_0^t \psi(r)\phi(u)|r-u|^{2H-2} du dr, \quad (2.3)$$

where $\alpha_H = H(2H-1)$.

Lemma 2.1 *For $l \geq 0$, $0 < z_1 < z_2 < t$, and $H \in (\frac{1}{2}, 1)$, we have*

$$\int_0^t \int_0^t (t-z_1)^l (t-z_2)^l |z_1 - z_2|^{2H-2} \mathrm{d}z_1 \mathrm{d}z_2 \lesssim t^{2H+2l}.$$

Hereinafter, $a \lesssim b$ or $a \gtrsim b$ stands for $a \leq Cb$ or $a \geq Cb$, where $C > 0$ is a constant and its specific value is not required but should be clear from the context.

Proof Using the binomial expansion, we can obtain that

$$
\begin{aligned}
&\int_0^t \int_0^t (t-z_1)^l (t-z_2)^l |z_1-z_2|^{2H-2} \mathrm{d}z_1 \mathrm{d}z_2 \\
&= 2\int_0^t \int_{z_2}^t (t-z_1)^l (t-z_2)^l (z_1-z_2)^{2H-2} \mathrm{d}z_1 \mathrm{d}z_2 \\
&= 2\int_0^t \int_0^{z_2} z_1^l z_2^l (z_2-z_1)^{2H-2} \mathrm{d}z_1 \mathrm{d}z_2 \\
&= 2\int_0^t \int_0^{z_2} z_1^l z_2^l \sum_{p=0}^{\infty} (-1)^p \binom{2H-2}{p} z_1^p z_2^{2H-2-p} \mathrm{d}z_1 \mathrm{d}z_2 \\
&= 2\sum_{p=0}^{\infty} \int_0^t \int_0^{z_2} (-1)^p \binom{2H-2}{p} z_1^{l+p} z_2^{2H-2-p+l} \mathrm{d}z_1 \mathrm{d}z_2 \\
&= 2\sum_{p=0}^{\infty} (-1)^p \binom{2H-2}{p} \frac{1}{l+1+p} \int_0^t z_2^{2H-1+2l} \mathrm{d}z_2 \\
&= 2\sum_{p=0}^{\infty} (-1)^p \binom{2H-2}{p} \frac{1}{(l+1+p)(2H+2l)} t^{2H+2l} \\
&\lesssim t^{2H+2l}.
\end{aligned}
$$

□

Lemma 2.2 *By the method of separation of variables, the solution of problem (1.1) can be written in the form* $u(x,t) = \sum_{n\in\mathbb{Z}^+} (u(\cdot,t), e_n) e_n(x) =: \sum_{n\in\mathbb{Z}^+} u_n(t) e_n(x)$*, where*

$$
u_n(t) = \Big(g_n \psi_n(T) - \int_t^T f_n h(s) \psi_n(s) \mathrm{d}s - \int_t^T \sigma_n \psi_n(s) \mathrm{d}B^H(s)\Big) \psi_n(-t). \quad (2.4)
$$

Here, $\psi_n(s) := e^{\lambda_n s},\ \lambda_n = \frac{1}{4} + n^2\pi^2,\ g_n = (g, e_n),\ f_n = (f, e_n),\ \sigma_n = (\sigma, e_n)$.

Definition 3 For any $p > 0$, we introduce the following space:

$$
\widetilde{H}^p := \{v \in \widetilde{L}^2(D)\, such\ that\ \|v\|^2_{\widetilde{H}^p} := \sum_{n\in\mathbb{Z}^+} \lambda_n^{2p} (v, e_n)^2 < \infty\}.
$$

Particularly, if $p = 0$, then $\widetilde{H}^p$ turns to $\widetilde{L}^2$.

Definition 4 [31]Let X be a Sobolev space. We introduce the following space: $L^P(\Omega, X) := \{\rho\, is\ X{-}valued\ random\ variable\ on\, \Omega : \|\rho\|^p_{L^p(\Omega,X)} := \mathbb{E}\|\rho\|^p_X < \infty\},\ p \geq 1$.

Definition 5 In order to prove some properties of the solutions, for two nonnegative numbers b and p, let us attempt to introduce a space called the Gevrey-type space [1]:

$$V_p^b := \{f \in \widetilde{L}^2(D), such\ that\ \|f\|^2_{V_p^b} := \sum_{n\in\mathbb{Z}^+} \lambda_n^{2p} e^{2b\lambda_n}(f, e_n)^2 < \infty\}.$$

We note that if $b = 0$, then V_p^b turns to $\widetilde{H}^p$; if $b = p = 0$, then V_p^b turns to $\widetilde{L}^2$. Moreover, we will analyze our problem under the following assumption.

Assumption 1 Let $H \in (0, 1)$ and f, σ and $g \in \widetilde{L}^2(D)$. Moreover, assume that $h \in L^\infty(0, T)$ is a nonnegative function and its support has a positive measure, i.e., $h > C_h > 0$.

3 The Properties of the Problem

3.1 The Regularity of the Solution

In this section, we attempt to find the spaces where we obtain the regularity of the mild solution to the problem. Before obtaining the result, we need the following strong assumptions for the data $\{g, f, \sigma\}$.

Assumption 2 Let $p \geq 0$, we set $g(x),\ f(x),\ \sigma(x) \in V^T_{p+1} \subset V^T_p$.

Before giving the main theorem for the regularity of solution, let us give some important lemmas. From (2.4), if we set $I_1(t, T) := \sum_{n\in\mathbb{Z}^+} \psi_n(T - t)(\cdot, e_n)e_n$, $I_2(t, s) := \sum_{n\in\mathbb{Z}^+} \psi_n(s - t)(\cdot, e_n)e_n$, then

$$u(\cdot, t) = I_1(t, T)g - \int_t^T h(s)I_2(t, s)f ds - \int_t^T I_2(t, s)\sigma \mathrm{d}B^H(s).$$

Lemma 3.1 *For the convenience of subsequent calculation, we first give an estimation formula*

$$\mathbb{E}\Big[\Big(\int_t^T \psi_n(s - t)\mathrm{d}B^H(s)\Big)^2\Big] \lesssim \begin{cases} e^{2\lambda_n(T-t)}(T + T^{2H} + \lambda_n^2 T^{2H+2}), & for\ H \in (0, \frac{1}{2}), \\ e^{2\lambda_n(T-t)}T^{2H}, & for\ H \in [\frac{1}{2}, 1). \end{cases}$$

Proof We discuss the cases $H \in (0, \frac{1}{2})$, $H = \frac{1}{2}$ and $H \in (\frac{1}{2}, 1)$ separately, since the covariance operator of B^H has different forms in these three cases.

(1) For $H \in (0, \frac{1}{2})$, according to (2.1), we have

$$\mathbb{E}\Big[\Big(\int_t^T \psi_n(s - t)\mathrm{d}B^H(s)\Big)^2\Big]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\Big(\int_0^T 1_{[t,T]}(s)\psi_n(s-t)\mathrm{d}B^H(s)\Big)^2\right]\\
&= \int_0^T \Big[K_H(T,s)1_{[t,T]}(s)\psi_n(s-t)\\
&\quad + \int_s^T \big(1_{[t,T]}(u)\psi_n(u-t) - 1_{[t,T]}(s)\psi_n(s-t)\big)\frac{\partial K_H(u,s)}{\partial u}du\Big]^2\mathrm{d}s\\
&= \int_0^t \Big(\int_t^T \psi_n(u-t)\frac{\partial K_H(u,s)}{\partial u}du\Big)^2\mathrm{d}s\\
&\quad + \int_t^T \left[K_H(T,s)\psi_n(s-t) + \int_s^T (\psi_n(u-t)-\psi_n(s-t))\frac{\partial K_H(u,s)}{\partial u}du\right]^2\mathrm{d}s\\
&=: N_1 + N_2. \qquad (3.1)
\end{aligned}
$$

For N_1, using the Hölder inequality, we have

$$
\begin{aligned}
N_1 &= \int_0^t \Big(\int_t^T e^{\lambda_n(u-t)}C_H(H-\frac{1}{2})\left(\frac{u}{s}\right)^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}du\Big)^2\mathrm{d}s\\
&\le C_H^2 e^{2\lambda_n(T-t)}(T-t)\int_0^t\int_t^T \left(\frac{u}{s}\right)^{2H-1}(u-s)^{2H-3}duds\\
&\lesssim e^{2\lambda_n(T-t)}(T-t). \qquad (3.2)
\end{aligned}
$$

Before we talk about N_2, let us give the following estimation for $K_H(T,s)$, $0 < s < T$.

$$
\begin{aligned}
&\int_s^T u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}du\\
&= \int_s^T u^{2H-2}\left[\sum_{p=0}^{\infty}\binom{H-\frac{1}{2}}{p}\left(-\frac{s}{u}\right)^p\right]du\\
&= \sum_{p=0}^{\infty}\binom{H-\frac{1}{2}}{p}(-1)^p s^p\frac{T^{2H-1-p}-s^{2H-1-p}}{2H-1-p}\\
&\le (T^{2H-1}-s^{2H-1})\sum_{p=0}^{\infty}\binom{H-\frac{1}{2}}{p}\frac{(-1)^p}{2H-1-p}\\
&\lesssim s^{2H-1}-T^{2H-1}\\
&\le s^{2H-1}. \qquad (3.3)
\end{aligned}
$$

This implies

$$
K_H^2(T,s) = C_H^2\left[\left(\frac{T}{s}\right)^{H-\frac{1}{2}}(T-s)^{H-\frac{1}{2}} + (\frac{1}{2}-H)s^{\frac{1}{2}-H}\int_s^T u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}du\right]^2
$$

$$\lesssim C_H^2\left[(T-s)^{H-\frac{1}{2}}+s^{H-\frac{1}{2}}\right]^2. \tag{3.4}$$

Therefore, for N_2,

$$\begin{aligned}
N_2 =& \int_t^T \Big[K_H(T,s)\psi_n(s-t) \\
&+ \int_s^T (\psi_n(u-t)-\psi_n(s-t))C_H(H-\frac{1}{2})\left(\frac{u}{s}\right)^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}du\Big]^2 \mathrm{d}s \\
\le& \int_t^T K_H^2(T,s)\psi_n^2(s-t)\mathrm{d}s \\
&+ \int_t^T \Big(\int_s^T (\psi_n(u-t)-\psi_n(s-t))C_H(H-\frac{1}{2})\left(\frac{u}{s}\right)^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}du\Big)^2 \mathrm{d}s \\
=:& N_{21}+N_{22}.
\end{aligned}$$

Here, about N_{21}, using (3.4), it follows that

$$\begin{aligned}
N_{21} &\lesssim \int_t^T ((T-s)^{H-\frac{1}{2}}+s^{H-\frac{1}{2}})^2 e^{2\lambda_n(s-t)}\mathrm{d}s \\
&\lesssim \int_t^T ((T-s)^{2H-1}+s^{2H-1})e^{2\lambda_n(s-t)}\mathrm{d}s \\
&\le e^{2\lambda_n(T-t)}\int_t^T ((T-s)^{2H-1}+s^{2H-1})\mathrm{d}s \\
&\lesssim e^{2\lambda_n(T-t)}((T-t)^{2H}+T^{2H}-t^{2H}) \\
&\lesssim e^{2\lambda_n(T-t)}T^{2H}.
\end{aligned} \tag{3.5}$$

Similarly,

$$\begin{aligned}
N_{22} \le& C_H^2(H-\frac{1}{2})^2\int_t^T\int_s^T (T-s)(\psi_n'(\xi-t))^2(u-s)^2\left(\frac{u}{s}\right)^{2H-1} \\
&(u-s)^{2H-3}duds, \text{ where } \xi\in(s,u) \\
&\le C_H^2(T-t)\lambda_n^2 e^{2\lambda_n(\xi^*-t)}\int_t^T\int_s^T (u-s)^{2H-1}duds, \text{ where } \xi^*\in(t,T) \\
&= \frac{(T-t)^{2H+2}}{2H(2H+1)}C_H^2\lambda_n^2 e^{2\lambda_n(\xi^*-t)} \\
&\lesssim \lambda_n^2(T-t)^{2H+2}e^{2\lambda_n(T-t)}.
\end{aligned} \tag{3.6}$$

Therefore, combining (3.2), (3.5) and (3.6),

$$\begin{aligned}
\mathbb{E}\left[\Big(\int_t^T \psi_n(s-t)\mathrm{d}B^H(s)\Big)^2\right] &\le N_1+N_2 \\
&\lesssim (T-t)e^{2\lambda_n(T-t)}+T^{2H}e^{2\lambda_n(T-t)}
\end{aligned}$$

$$
\begin{aligned}
&+\lambda_n^2(T-t)^{2H+2}e^{2\lambda_n(T-t)}\\
&\leq e^{2\lambda_n(T-t)}(T+T^{2H}+\lambda_n^2T^{2H+2}).
\end{aligned}
$$

(2) For $H=\frac{1}{2}$, according to Itô isometry (2.2), we have

$$
\mathbb{E}\left[\left(\int_t^T\psi_n(s-t)\mathrm{d}B^H(s)\right)^2\right]\leq e^{2\lambda_n(T-t)}T.
$$

(3) For $H\in(\frac{1}{2},1)$, by (2.3) and Lemma 2.1, we have

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_t^T\psi_n(s-t)\mathrm{d}B^H(s)\right)^2\right]\\
&=\mathbb{E}\left[\left(\int_0^T 1_{[t,T]}(s)\psi_n(s-t)\mathrm{d}B^H(s)\right)^2\right]\\
&=\alpha_H\int_0^T\int_0^T 1_{[t,T]}(r)\psi_n(r-t)\cdot 1_{[t,T]}(u)\psi_n(u-t)\mid r-u\mid^{2H-2}dudr\\
&=\alpha_H\int_t^T\int_t^T\psi_n(r-t)\psi_n(u-t)\mid r-u\mid^{2H-2}dudr\\
&\lesssim e^{2\lambda_n(T-t)}T^{2H}.
\end{aligned}
$$

Making an arrangement for the above estimations, we obtain

$$
\mathbb{E}\left[\left(\int_t^T\psi_n(s-t)\mathrm{d}B^H(s)\right)^2\right]\lesssim\begin{cases}e^{2\lambda_n(T-t)}(T+T^{2H}+\lambda_n^2T^{2H+2}), \text{ for } H\in(0,\frac{1}{2}),\\ e^{2\lambda_n(T-t)}T^{2H}, \text{ for } H\in[\frac{1}{2},1).\end{cases}
\tag{3.7}
$$

□

Remark 3.2 Similar to the above calculation methods, for $\delta\geq 0$, we easily get

$$
\mathbb{E}\left[\left(\int_{t+\delta}^T\psi_n(s-t)\mathrm{d}B^H(s)\right)^2\right]\lesssim\begin{cases}e^{2\lambda_n(T-t)}(T+T^{2H}+\lambda_n^2T^{2H+2}), \text{ for } H\in(0,\frac{1}{2}),\\ e^{2\lambda_n(T-t)}T^{2H}, \text{ for } H\in[\frac{1}{2},1).\end{cases}
\tag{3.8}
$$

$$
\mathbb{E}\left[\left(\int_t^{t+\delta}\psi_n(s-t)\mathrm{d}B^H(s)\right)^2\right]\lesssim\begin{cases}e^{2\lambda_n\delta}(\delta+\delta^{2H}+\lambda_n^2\delta^{2H+2}), \text{ for } H\in(0,\frac{1}{2}),\\ e^{2\lambda_n\delta}\delta^{2H}, \text{ for } H\in[\frac{1}{2},1).\end{cases}
\tag{3.9}
$$

Lemma 3.3 *Let $p\geq 0$, $t\in(0,T)$ and assume that g satisfies Assumption 2. Then, if $\delta\geq 0$ is small enough, there hold*

(1) $\|I_1(t,T)g\|_{\widetilde{H}^p} \le \|g\|_{V_p^T}$;
(2) $\|(I_1(t+\delta,T)-I_1(t,T))g\|_{\widetilde{H}^p} \le \delta\|g\|_{V_{p+1}^T}$.

Proof Using Definitions 3 and 5, we can obtain that

(1)

$$\begin{aligned}\|I_1(t,T)g\|_{\widetilde{H}^p}^2 &= \sum_{n\in\mathbb{Z}^+}\lambda_n^{2p}(I_1(t,T)g,e_n)^2\\ &= \sum_{n\in\mathbb{Z}^+}\lambda_n^{2p}e^{2\lambda_n(T-t)}(g,e_n)^2\\ &\le \sum_{n\in\mathbb{Z}^+}\lambda_n^{2p}e^{2\lambda_n T}(g,e_n)^2\\ &= \|g\|_{V_p^T}^2.\end{aligned}$$

(2) Similarly,

$$\begin{aligned}&\|\big(I_1(t+\delta,T)-I_1(t,T)\big)g\|_{\widetilde{H}^p}^2\\ &= \sum_{n\in\mathbb{Z}^+}\lambda_n^{2p}\big((I_1(t+\delta,T)-I_1(t,T))g,e_n\big)^2\\ &= \sum_{n\in\mathbb{Z}^+}\lambda_n^{2p}(g,e_n)^2\big(e^{\lambda_n(T-t-\delta)}-e^{\lambda_n(T-t)}\big)^2.\end{aligned}$$

By mean value theorem of differentiation, we have

$$\|\big(I_1(t+\delta,T)-I_1(t,T)\big)g\|_{\widetilde{H}^p}^2 \le \delta^2\sum_{n\in\mathbb{Z}^+}\lambda_n^{2(p+1)}(g,e_n)^2e^{2\lambda_n T} = \delta^2\|g\|_{V_{p+1}^T}^2.$$

□

Lemma 3.4 *Let $t,s\in(0,T)$, $\delta\ge 0$ and assume that f satisfies Assumption 2, we get*

(1) $\|I_2(t,s)f\|_{\widetilde{H}^p} \le \|f\|_{V_p^T}$;
(2) $\|(I_2(t+\delta,s)-I_2(t,s))f\|_{\widetilde{H}^p} \le \delta\|f\|_{V_{p+1}^T}$.

Proof The proof is similar to Lemma 3.3. □

Remark 3.5 The mild solution and inverse problem may not be stable because $e^{2\lambda_n(T-t)}$ goes to infinity rapidly as $n\to\infty$, so we restrict f and g in Assumption 2.

Lemma 3.6 *Let $\delta\ge 0$ is small enough, and assuming that f and h satisfy Assumptions 1 and 2, we have*

$$\left\|\int_{t+\delta}^T h(s)I_2(t+\delta,s)f\,ds-\int_t^T h(s)I_2(t,s)f\,ds\right\|_{\widetilde{H}^p} \lesssim \delta(T+1)\|h\|_{L^\infty(0,T)}\|f\|_{V_{p+1}^T}.$$

Proof Using the triangle inequality and Lemma 3.4, we have

$$
\begin{aligned}
&\Big\| \int_{t+\delta}^{T} h(s) I_2(t+\delta,s) f ds - \int_{t}^{T} h(s) I_2(t,s) f ds \Big\|_{\widetilde{H}^p} \\
&= \Big\| \int_{t+\delta}^{T} h(s)(I_2(t+\delta,s) - I_2(t,s)) f ds \\
&\quad + \int_{t+\delta}^{T} h(s) I_2(t,s) f ds - \int_{t}^{T} h(s) I_2(t,s) f ds \Big\|_{\widetilde{H}^p} \\
&\le \Big\| \int_{t+\delta}^{T} h(s)(I_2(t+\delta,s) - I_2(t,s)) f ds \Big\|_{\widetilde{H}^p} + \Big\| \int_{t}^{t+\delta} h(s) I_2(t,s) f ds \Big\|_{\widetilde{H}^p} \\
&\le \|h\|_{L^\infty(0,T)} \int_{t+\delta}^{T} \big\| (I_2(t+\delta,s) - I_2(t,s)) f \big\|_{\widetilde{H}^p} \mathrm{d}s \\
&\quad + \|h\|_{L^\infty(0,T)} \int_{t}^{t+\delta} \big\| I_2(t,s) f \big\|_{\widetilde{H}^p} \mathrm{d}s \\
&\le \delta T \|h\|_{L^\infty(0,T)} \|f\|_{V^T_{p+1}} + \delta \|h\|_{L^\infty(0,T)} \|f\|_{V^T_p} \\
&\le \delta (T+1) \|h\|_{L^\infty(0,T)} \|f\|_{V^T_{p+1}}.
\end{aligned}
$$

□

Lemma 3.7 *Let $\delta \ge 0$ is small enough, and assuming that σ satisfies Assumption 2, there holds*

$$
\Big\| \int_{t+\delta}^{T} I_2(t+\delta,s) \sigma \mathrm{d}B^H(s) - \int_{t}^{T} I_2(t,s) \sigma \mathrm{d}B^H(s) \Big\|_{L^2(\Omega,\widetilde{H}^p)} \lesssim \delta^H \|\sigma\|_{V^T_{p+2}}.
$$

Proof According to the triangle inequality of the norm, we have

$$
\begin{aligned}
&\Big\| \int_{t+\delta}^{T} I_2(t+\delta,s) \sigma \mathrm{d}B^H(s) - \int_{t}^{T} I_2(t,s) \sigma \mathrm{d}B^H(s) \Big\|_{L^2(\Omega,\widetilde{H}^p)} \\
&= \Big\| \int_{t+\delta}^{T} (I_2(t+\delta,s) - I_2(t,s)) \sigma \mathrm{d}B^H(s) \\
&\quad + \int_{t+\delta}^{T} I_2(t,s) \sigma \mathrm{d}B^H(s) - \int_{t}^{T} I_2(t,s) \sigma \mathrm{d}B^H(s) \Big\|_{L^2(\Omega,\widetilde{H}^p)} \\
&\le \Big\| \int_{t+\delta}^{T} (I_2(t+\delta,s) - I_2(t,s)) \sigma \mathrm{d}B^H(s) \Big\|_{L^2(\Omega,\widetilde{H}^p)} \\
&\quad + \Big\| \int_{t}^{t+\delta} I_2(t,s) \sigma \mathrm{d}B^H(s) \Big\|_{L^2(\Omega,\widetilde{H}^p)} \\
&=: S_1 + S_2.
\end{aligned}
$$

For S_1, by Definition 4, it follows that

$$\begin{aligned} S_1^2 &= \left\| \int_{t+\delta}^{T} (I_2(t+\delta, s) - I_2(t, s))\sigma \mathrm{d}B^H(s) \right\|^2_{L^2(\Omega, \widetilde{H}^p)} \\ &= \mathbb{E}\left[\left\| \int_{t+\delta}^{T} (I_2(t+\delta, s) - I_2(t, s))\sigma \mathrm{d}B^H(s) \right\|^2_{\widetilde{H}^p} \right] \\ &= \sum_{n\in\mathbb{Z}^+} \mathbb{E}\left[\left\| \int_{t+\delta}^{T} (e^{\lambda_n(s-t-\delta)} - e^{\lambda_n(s-t)})(\sigma, e_n)e_n dB^H(s) \right\|^2_{\widetilde{H}^p} \right] \\ &= \sum_{n\in\mathbb{Z}^+} \mathbb{E}\left[\left(\int_{t+\delta}^{T} (e^{\lambda_n(s-t-\delta)} - e^{\lambda_n(s-t)}) \mathrm{d}B^H(s) \right)^2 \right] \|(\sigma, e_n)e_n\|^2_{\widetilde{H}^p}. \end{aligned}$$

Using mean value theorem of differentiation and (3.8), we have the following estimate

$$\begin{aligned} S_1^2 &= \sum_{n\in\mathbb{Z}^+} \mathbb{E}\left[\left(\int_{t+\delta}^{T} -\delta\lambda_n e^{\lambda_n(\xi-t)} \mathrm{d}B^H(s) \right)^2 \right] \lambda_n^{2p}(\sigma, e_n)^2, \ (\xi \in (s-\delta, s)) \\ &= \sum_{n\in\mathbb{Z}^+} \delta^2\lambda_n^2 \mathbb{E}\left[\left(\int_{t+\delta}^{T} e^{\lambda_n(\xi-t)} \mathrm{d}B^H(s) \right)^2 \right] \lambda_n^{2p}(\sigma, e_n)^2 \\ &\lesssim \begin{cases} \delta^2 \sum_{n\in\mathbb{Z}^+} \lambda_n^{2(p+1)} e^{2\lambda_n T}(T + T^{2H} + \lambda_n^2 T^{2H+2})(\sigma, e_n)^2, \ \text{for } H \in (0, \frac{1}{2}) \\ \delta^2 T^{2H} \sum_{n\in\mathbb{Z}^+} \lambda_n^{2(p+1)} e^{2\lambda_n T}(\sigma, e_n)^2, \ \text{for } H \in [\frac{1}{2}, 1) \end{cases} \\ &\lesssim \delta^2 \|\sigma\|^2_{V^T_{p+2}}. \end{aligned} \tag{3.10}$$

Similarly, for S_2, there holds

$$\begin{aligned} S_2^2 &= \left\| \int_{t}^{t+\delta} I_2(t, s)\sigma \mathrm{d}B^H(s) \right\|^2_{L^2(\Omega, \widetilde{H}^p)} \\ &= \sum_{n\in\mathbb{Z}^+} \left\| \int_{t}^{t+\delta} e^{\lambda_n(s-t)}(\sigma, e_n)e_n \mathrm{d}B^H(s) \right\|^2_{L^2(\Omega, \widetilde{H}^p)} \\ &= \sum_{n\in\mathbb{Z}^+} \mathbb{E}\left[\left(\int_{t}^{t+\delta} e^{\lambda_n(s-t)} \mathrm{d}B^H(s) \right)^2 \right] \|(\sigma, e_n)e_n\|^2_{\widetilde{H}^p} \\ &= \sum_{n\in\mathbb{Z}^+} \mathbb{E}\left[\left(\int_{t}^{t+\delta} e^{\lambda_n(s-t)} \mathrm{d}B^H(s) \right)^2 \right] \lambda_n^{2p}(\sigma, e_n)^2. \end{aligned}$$

By (3.9), we have

$$
\begin{aligned}
S_2^2 &\le \begin{cases} \sum_{n\in\mathbb{Z}^+} e^{2\lambda_n\delta}(\delta+\delta^{2H}+\lambda_n^2\delta^{2H+2})\lambda_n^{2p}(\sigma,e_n)^2, \text{ for } H\in(0,\frac{1}{2}) \\ \sum_{n\in\mathbb{Z}^+} e^{2\lambda_n\delta}\delta^{2H}\lambda_n^{2p}(\sigma,e_n)^2, \text{ for } H\in[\frac{1}{2},1) \end{cases} \\
&\lesssim \delta^{2H}\sum_{n\in\mathbb{Z}^+} e^{2\lambda_n T}\lambda_n^{2p+2}(\sigma,e_n)^2 \\
&\le \delta^{2H}\|\sigma\|^2_{V^T_{p+1}}.
\end{aligned}
\tag{3.11}
$$

Therefore, combining (3.10) and (3.11), we have

$$
\begin{aligned}
&\left\|\int_{t+\delta}^T I_2(t+\delta,s)\sigma\,\mathrm{d}B^H(s)-\int_t^T I_2(t,s)\sigma\,\mathrm{d}B^H(s)\right\|_{L^2(\Omega,\widetilde{H}^p)} \\
&\le \delta\|\sigma\|_{V^T_{p+2}}+\delta^H\|\sigma\|_{V^T_{p+1}} \\
&\lesssim \delta^H\|\sigma\|_{V^T_{p+2}}.
\end{aligned}
$$

□

Theorem 3.8 *Let $\delta\ge 0$. Assuming f, g and σ satisfy Assumption 2, then*

$$
\|u(\cdot,t+\delta)-u(\cdot,t)\|_{L^2(\Omega,\widetilde{H}^p)}\le C_\delta(\|g\|_{V^T_{p+1}}+\|f\|_{V^T_{p+1}}+\|\sigma\|_{V^T_{p+2}}),
$$

where $C_\delta=\max\{\delta,\delta(T+1)\|h\|_{L^\infty(0,T)},\delta^H\}$.

Proof Using the triangle inequality, it follows that

$$
\begin{aligned}
&\|u(\cdot,t+\delta)-u(\cdot,t)\|_{L^2(\Omega,\widetilde{H}^p)} \\
&= \Bigg\| I_1(t+\delta,T)g-\int_{t+\delta}^T h(s)I_2(t+\delta,s)f\,ds-\int_{t+\delta}^T I_2(t+\delta,s)\sigma\,\mathrm{d}B^H(s) \\
&\quad -\left(I_1(t,T)g-\int_t^T h(s)I_2(t,s)f\,ds-\int_t^T I_2(t,s)\sigma\,\mathrm{d}B^H(s)\right)\Bigg\|_{L^2(\Omega,\widetilde{H}^p)} \\
&\le \|(I_1(t+\delta,T)-I_1(t,T))g\|_{\widetilde{H}^p} \\
&\quad +\left\|\int_{t+\delta}^T h(s)I_2(t+\delta,s)f\,ds-\int_t^T h(s)I_2(t,s)f\,ds\right\|_{\widetilde{H}^p} \\
&\quad +\left\|\int_{t+\delta}^T I_2(t+\delta,s)\sigma\,\mathrm{d}B^H(s)-\int_t^T I_2(t,s)\sigma\,\mathrm{d}B^H(s)\right\|_{L^2(\Omega,\widetilde{H}^p)}.
\end{aligned}
$$

Combining Lemmas 3.3(2), 3.6 and 3.7, it is easy to draw the conclusion. □

3.2 The Instability of the Problem

In this section, we will analyze the instability of $u(x,t)$ by the expectation and variance of $u_n(t)$. From (2.4), we have

$$\mathbb{E}(u_n(t)) = g_n\psi_n(T-t) - f_n\int_t^T h(s)\psi_n(s-t)\mathrm{d}s.$$

$$\mathbb{V}ar(u_n(t)) = \sigma_n^2\mathbb{E}\left[\left(\int_t^T \psi_n(s-t)\mathrm{d}B^H(s)\right)^2\right].$$

3.2.1 The Expectation $\mathbb{E}(u_n(t))$

Here, we consider two special cases. On the one hand, if $f \equiv 0$, , then $f_n = 0$. It follows that

$$\psi_n(T-t) \to \infty, \text{ as } n \to \infty, \text{ then } \mathbb{E}(\mathrm{u_n(t)}) \to \infty.$$

On the other hand, if $g \equiv 0$, then $g_n = 0$. Using mean value theorem of integrals and Assumption 1, we have

$$\begin{aligned}\int_t^T h(s)\psi_n(s-t)\mathrm{d}s = \psi_n(\xi)\int_t^T h(s)\mathrm{d}s \geq C_h\psi_n(\xi)(T-t)\\ \to\infty, \text{ as } n \to \infty\, (\xi \in (0, T-t)),\end{aligned}$$

then $\mathbb{E}(u_n(t)) \to \infty$.

3.2.2 The Variance $\mathbb{V}ar(u_n(t))$

We would separately talk about $\mathbb{V}ar(u_n(t))$ for different H.

(1) For the case $H \in (0, \frac{1}{2})$, by (3.1),

$$\begin{aligned}&\mathbb{E}\left[\left(\int_t^T \psi_n(s-t)\mathrm{d}B^H(s)\right)^2\right]\\ &= \int_0^t \left(\int_t^T \psi_n(u-t)\frac{\partial K_H(u,s)}{\partial u}du\right)^2\mathrm{d}s\\ &\quad + \int_t^T \left[K_H(T,s)\psi_n(s-t) + \int_s^T (\psi_n(u-t) - \psi_n(s-t))\frac{\partial K_H(u,s)}{\partial u}du\right]^2\mathrm{d}s\\ &=: N_1 + N_2.\end{aligned}$$

For N_1,

$$N_1 = \int_0^t \left(\int_t^T \psi_n(u-t)C_H(H-\frac{1}{2})(\frac{u}{s})^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}du\right)^2\mathrm{d}s$$

$$= C_H^2(H-\frac{1}{2})^2\int_0^t\Big(\int_t^T e^{(\frac{1}{4}+n^2\pi^2)(u-t)}(\frac{u}{s})^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}du\Big)^2\mathrm{d}s$$
$$\geq C_H^2(H-\frac{1}{2})^2(\frac{1}{4}+n^2\pi^2)\int_0^t\Big(\int_t^T (u-t)(\frac{u}{s})^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}du\Big)^2\mathrm{d}s,$$

where $\int_0^t\Big(\int_t^T(u-t)(\frac{u}{s})^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}du\Big)^2\mathrm{d}s$ is a normal integral; therefore, $N_1\to\infty,$ as $n\to\infty$.
Since $N_2\geq 0$,

$$\mathbb{E}\Bigg[\Big(\int_t^T\psi_n(s-t)\mathrm{d}B^H(s)\Big)^2\Bigg]=N_1+N_2\to\infty,\ \text{as } n\to\infty.$$

(2) For the case $H=\frac{1}{2}$, by (2.2) and mean value theorem, we have

$$\mathbb{E}\Bigg[\Big(\int_t^T\psi_n(s-t)\mathrm{d}B^H(s)\Big)^2\Bigg]$$
$$=\psi_n(\xi)(T-t)\to\infty,\ \text{as } n\to\infty\quad(\xi\in(0,2T-2t)).$$

(3) For the case $H\in(\frac{1}{2},1)$, similarly, by (2.3)

$$\mathbb{E}\Bigg[\Big(\int_t^T\psi_n(s-t)\mathrm{d}B^H(s)\Big)^2\Bigg]$$
$$=\alpha_H\psi_n(\xi_1)\psi_n(\xi_2)\int_t^T\int_t^T|\,r-u\,|^{2H-2}\,\mathrm{d}r\mathrm{d}u\ \ (\xi_1,\xi_2\in(0,2T-2t))$$
$$=2\alpha_H\frac{(T-t)^{2H}}{2H(2H-1)}\psi_n(\xi_1)\psi_n(\xi_2)\to\infty,\ \text{as } n\to\infty.$$

Above all, we can conclude that this problem is instability.

4 The Regularized Solution of the Problem

The instability means the small error in the high-frequency components for g^δ, f^δ and σ^δ will be amplified by the factor $\psi_n(t)$, so we need some regularization methods to compute $u(x,t), t\in(0,1)$ from measured data g^δ, f^δ and σ^δ. Here, we adopt the truncated regularization method to establish the regularized solution for $u(x,t)$ in (1.1). If the noisy data g^δ, f^δ and σ^δ satisfy

$$\|g^\delta-g\|\leq\delta,\quad\|f^\delta-f\|\leq\delta,\quad\|\sigma^\delta-\sigma\|\leq\delta,\tag{4.1}$$

where the constant $\delta \geq 0$ is a noise level, then the regularized solution $u^{M,\delta}$ is defined as

$$u^{M,\delta}(x,t) = \sum_{n=1}^{M} \Big[g_n^\delta \psi_n(T) - \int_t^T f_n^\delta h(s)\psi_n(s)\mathrm{d}s - \int_t^T \sigma_n^\delta \psi_n(s)\mathrm{d}B^H(s) \Big] \psi_n(-t) e_n(x).$$

In addition, we set

$$u^{M}(x,t) = \sum_{n=1}^{M} \Big[g_n \psi_n(T) - \int_t^T f_n h(s)\psi_n(s)\mathrm{d}s - \int_t^T \sigma_n \psi_n(s)\mathrm{d}B^H(s) \Big] \psi_n(-t) e_n(x).$$

To obtain the error estimation between u and $u^{M,\delta}$, the a-priori bound L on the mild solution is needed. In detail, we set

$$\begin{aligned} &\|u(\cdot,0)\|^2_{L^2(\Omega,\widetilde{L}^2(D))} \\ &= \mathbb{E}\Big[\sum_{n=1}^{\infty} \Big(g_n\psi_n(T) - \int_0^T h(s) f_n \psi_n(s)\mathrm{d}s - \int_0^T \sigma_n \psi_n(s)\mathrm{d}B^H(s) \Big)^2 \Big] \\ &\lesssim \sum_{n=1}^{\infty} (g_n\psi_n(T))^2 + \sum_{n=1}^{\infty} \Big(\int_0^T h(s) f_n \psi_n(s)\mathrm{d}s \Big)^2 \\ &\quad + \sum_{n=1}^{\infty} \mathbb{E}\Big[\Big(\int_0^T \sigma_n \psi_n(s)\mathrm{d}B^H(s) \Big)^2 \Big] \\ &\leq L^2. \end{aligned} \tag{4.2}$$

Theorem 4.1 *For $H \in [\frac{1}{2}, 1)$, let M be a positive integer number. Assume that the a-priori bound* (4.2) *and the noise level* (4.1) *hold. Then,*

$$\big\| u^{M,\delta}(\cdot,t) - u(\cdot,t) \big\|^2_{L^2(\Omega,\widetilde{L}^2(D))} \lesssim \psi_M(2T-2t)\delta^2 + \psi_M(-2t)L^2, \quad \textit{for} \quad 0 < t < T.$$

Proof According to the triangle inequality, we get

$$\begin{aligned} &\big\| u^{M,\delta}(\cdot,t) - u(\cdot,t) \big\|^2_{L^2(\Omega,\widetilde{L}^2(D))} \\ &\lesssim \big\| u^{M,\delta}(\cdot,t) - u^M(\cdot,t) \big\|^2_{L^2(\Omega,\widetilde{L}^2(D))} + \big\| u^{M}(\cdot,t) - u(\cdot,t) \big\|^2_{L^2(\Omega,\widetilde{L}^2(D))} \\ &=: I_1 + I_2. \end{aligned} \tag{4.3}$$

For I_1,

$$\begin{aligned} I_1 &= \mathbb{E}\Big\| \sum_{n=1}^{M} \Big[(g_n^\delta - g_n)\psi_n(T) - \int_t^T (f_n^\delta - f_n)h(s)\psi_n(s)\mathrm{d}s \\ &\quad - \int_t^T (\sigma_n^\delta - \sigma_n)\psi_n(s)\mathrm{d}B^H(s) \Big] \psi_n(-t)e_n(\cdot) \Big\|^2 \\ &\lesssim \sum_{n=1}^{M} ((g_n^\delta - g_n)\psi_n(T-t))^2 + \sum_{n=1}^{M} \Big(\int_t^T (f_n^\delta - f_n)h(s)\psi_n(s-t)\mathrm{d}s \Big)^2 \\ &\quad + \sum_{n=1}^{M} \mathbb{E}\Big[\Big(\int_t^T (\sigma_n^\delta - \sigma_n)\psi_n(s-t)\mathrm{d}B^H(s) \Big)^2 \Big] \\ &=: J_1(t,T) + J_2(t,T) + J_3(t,T). \end{aligned}$$

Here,

$$J_1(t,T) \le \psi_M^2(T-t) \sum_{n=1}^{\infty} (g_n^\delta - g_n)^2 \le \delta^2 \psi_M(2T-2t).$$

If Assumption 1 and the noise level (4.1) hold, then

$$\begin{aligned} J_2(t,T) &\le \psi_M^2(T-t)\|h\|_{L^\infty(0,T)}^2 (T-t)^2 \sum_{n=1}^{\infty} (f_n^\delta - f_n)^2 \\ &\le \delta^2 \psi_M(2T-2t)\|h\|_{L^\infty(0,T)}^2 (T-t)^2. \end{aligned}$$

Since

$$J_3(t,T) = \sum_{n=1}^{M} (\sigma_n^\delta - \sigma_n)^2 \mathbb{E}\Big[\Big(\int_t^T \psi_n(s-t)\mathrm{d}B^H(s) \Big)^2 \Big],$$

from (3.7), we have

$$J_3(t,T) \lesssim \delta^2 e^{2\lambda_M(T-t)} T^{2H} \lesssim \delta^2 \psi_M(2T-2t), \text{ for } H \in [\frac{1}{2}, 1).$$

Therefore,

$$I_1 \lesssim \delta^2 \psi_M(2T-2t)(2 + (T-t)^2 \|h\|_{L^\infty(0,T)}^2). \tag{4.4}$$

For I_2 in (4.3), there exists

$$I_2 = \mathbb{E}\Big[\sum_{n=M+1}^{\infty} \Big(g_n\psi_n(T) - \int_t^T h(s)f_n\psi_n(s)\mathrm{d}s - \int_t^T \sigma_n\psi_n(s)\mathrm{d}B^H(s) \Big)^2 \psi_n(-2t) \Big]$$

$$
\begin{aligned}
&\lesssim \sum_{n=M+1}^{\infty} (g_n \psi_n(T-t))^2 \psi_n(-2t) + \sum_{n=M+1}^{\infty} \Big(\int_t^T h(s) f_n \psi_n(s) \mathrm{d}s \Big)^2 \psi_n(-2t) \\
&\quad + \sum_{n=M+1}^{\infty} \mathbb{E}\Big[\Big(\int_t^T \sigma_n \psi_n(s) \mathrm{d}B^H(s) \Big)^2 \Big] \psi_n(-2t) \\
&=: K_1 + K_2 + K_3. \qquad (4.5)
\end{aligned}
$$

For K_1,

$$
K_1 \leq \psi_M(-2t) \sum_{n=1}^{\infty} (g_n \psi_n(T))^2. \qquad (4.6)
$$

For K_2,

$$
K_2 \leq \psi_M(-2t) \sum_{n=1}^{\infty} \Big(\int_0^T h(s) f_n \psi_n(s) \mathrm{d}s \Big)^2. \qquad (4.7)
$$

For K_3,

$$
\begin{aligned}
&\sum_{n=M+1}^{\infty} \mathbb{E}\Big[\Big(\int_t^T \sigma_n \psi_n(s) \mathrm{d}B^H(s) \Big)^2 \Big] \psi_n(-2t) \\
&= \sum_{n=M+1}^{\infty} \sigma_n^2 \psi_n(-2t) \mathbb{E}\Big[\Big(\int_t^T \psi_n(s) \mathrm{d}B^H(s) \Big)^2 \Big].
\end{aligned}
$$

We need to compute $\mathbb{E}\Big[\Big(\int_t^T \sigma_n \psi_n(s) \mathrm{d}B^H(s) \Big)^2 \Big]$ for $H = \frac{1}{2}$ and $H \in (\frac{1}{2}, 1)$, respectively.

(1) For $H = \frac{1}{2}$,

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\int_t^T \psi_n(s) \mathrm{d}B^H(s) \Big)^2 \Big] = \int_t^T \psi_n(2s) \mathrm{d}s \leq \int_0^T \psi_n(2s) \mathrm{d}s \\
&= \mathbb{E}\Big[\Big(\int_0^T \psi_n(s) \mathrm{d}B^H(s) \Big)^2 \Big]. \qquad (4.8)
\end{aligned}
$$

(2) For $H \in (\frac{1}{2}, 1)$,

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\int_t^T \psi_n(s) \mathrm{d}B^H(s) \Big)^2 \Big] \\
&= \int_t^T \int_t^T \psi_n(r) \psi_n(u) \mid r-u \mid^{2H-2} \mathrm{d}r \mathrm{d}u
\end{aligned}
$$

$$\leq \int_0^T \int_0^T \psi_n(r)\psi_n(u) \mid r-u \mid^{2H-2} \mathrm{d}r\mathrm{d}u$$
$$= \mathbb{E}\Big[\Big(\int_0^T \psi_n(s)\mathrm{d}B^H(s)\Big)^2\Big]. \tag{4.9}$$

From (4.8)–(4.9), it concludes that

$$K_3 \leq \psi_M(-2t)\sum_{n=1}^{\infty}\mathbb{E}\Big[\Big(\int_0^T \sigma_n\psi_n(s)\mathrm{d}B^H(s)\Big)^2\Big], \text{ for } H \in [\frac{1}{2}, 1). \tag{4.10}$$

Plunging (4.6), (4.7) and (4.10) into (4.5), we have

$$I_2 \leq \psi_M(-2t)L^2. \tag{4.11}$$

Summing up (4.4) and (4.11), we can conclude that

$$\begin{aligned}
&\big\|u^{M,\delta}(\cdot,t)-u(\cdot,t)\big\|^2_{L^2(\Omega,\widetilde{L}^2(D))}\\
&\quad\lesssim \delta^2\psi_M(2T-2t)\{1+(T-t)^2\|h\|^2_{L^\infty(0,T)}+T^{2H}\}+\psi_M(-2t)L^2\\
&\quad\lesssim \delta^2\psi_M(2T-2t)+\psi_M(-2t)L^2.
\end{aligned}$$

□

Remark 4.2 Particularly, if we choose $M=\lceil(\frac{4\ln(\frac{L}{\delta})-T}{4T\pi^2})^{\frac{1}{2}}\rceil$, then

$$\big\|u^{M,\delta}(\cdot,t)-u(\cdot,t)\big\|^2_{L^2(\Omega,\widetilde{L}^2(D))} \lesssim L^{2(1-\frac{t}{T})}\delta^{2\frac{t}{T}} \to 0, \text{ as } \delta \to 0.$$

Remark 4.3 For $H \in (0, \frac{1}{2})$, since the covariance is negatively correlated, the inequality

$$\mathbb{E}\Big[\Big(\int_t^T \psi_n(s)\mathrm{d}B^H(s)\Big)^2\Big] \lesssim \mathbb{E}\Big[\Big(\int_0^T \psi_n(s)\mathrm{d}B^H(s)\Big)^2\Big]$$

may not be true anymore. In detail,

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^T \psi_n(s)\mathrm{d}B^H(s)\Big)^2\Big]\\
&\quad= \mathbb{E}\Big[\Big(\int_0^t \psi_n(s)\mathrm{d}B^H(s)+\int_t^T \psi_n(s)\mathrm{d}B^H(s)\Big)^2\Big]\\
&\quad= \mathbb{E}\Big[\Big(\int_0^t \psi_n(s)\mathrm{d}B^H(s)\Big)^2\Big]+\mathbb{E}\Big[\Big(\int_t^T \psi_n(s)\mathrm{d}B^H(s)\Big)^2\Big]\\
&\qquad+2\mathbb{E}\Big[\int_0^t \psi_n(s)\mathrm{d}B^H(s)\int_t^T \psi_n(s)\mathrm{d}B^H(s)\Big].
\end{aligned}$$

Because $\mathbb{E}\Big[\int_0^t \psi_n(s)\mathrm{d}B^H(s)\int_t^T \psi_n(s)\mathrm{d}B^H(s)\Big] < 0$ for $H \in (0, \frac{1}{2})$, there is no guarantee that

$$\mathbb{E}\Big[\Big(\int_0^t \psi_n(s)\mathrm{d}B^H(s)\Big)^2\Big] + 2\mathbb{E}\Big[\int_0^t \psi_n(s)\mathrm{d}B^H(s)\int_t^T \psi_n(s)\mathrm{d}B^H(s)\Big] \geq 0.$$

To obtain the error estimation of regularized solution and exact solution for $H \in (0, 1)$, we give another a-prior bound at any time as:

$$\|u(\cdot, t)\|_{L^2(\Omega,\widetilde{H}^p)} \leq L_1. \tag{4.12}$$

Then, using Definition 4, we have

$$\begin{aligned}
&\left\|u^M(\cdot, t) - u(\cdot, t)\right\|^2_{L^2(\Omega,\widetilde{L}^2(D))} \\
&\quad = \sum_{n>M} \lambda_n^{2p}\lambda_n^{-2p}\mathbb{E}|(u(\cdot, t), e_n)|^2 \\
&\quad \leq \lambda_M^{-2p}\|u(\cdot, t)\|^2_{L^2(\Omega,\widetilde{H}^p)} \\
&\quad = \lambda_M^{-2p}L_1^2.
\end{aligned} \tag{4.13}$$

Theorem 4.4 *For $H \in (0, 1)$, $M > 0$ is a integer number. Assume that the a-priori bound (4.12) and the noise level (4.1) hold. We have*

$$\left\|u^{M,\delta}(\cdot, t) - u(\cdot, t)\right\|^2_{L^2(\Omega,\widetilde{L}^2(D))} \lesssim \psi_M(2T - 2t)\delta^2 + \lambda_M^{-2p}L_1^2, \quad \textit{for} \quad 0 < t < T.$$

Proof Combining (4.4) and (4.13), we can easily get the conclusion. □

Particularly, if we choose $M = \lceil(\frac{4\ln(\frac{L_1}{\delta})}{2(T-t)\pi^2})^{\frac{1}{2}}\rceil$, then

$$\left\|u^{M,\delta}(\cdot, t) - u(\cdot, t)\right\|^2_{L^2(\Omega,\widetilde{L}^2(D))} \lesssim L_1\delta + \left(\frac{2(T-t)}{\ln(\frac{L_1}{\delta})}\right)^{2p} L_1^2 \to 0, \text{ as } \delta \to 0.$$

5 Numerical Experiments

In this section, we make some numerical implementations on our inversion scheme. Since the backward problem is ill-posed, we use the truncated regularization method to obtain the regularization solution by the finial value $u(x, T, \omega)$, where the $u(x, T, \omega)$ is given in the direct problem, and we truncate the above series by the first M terms as a regularization. Let N_x and N_t be the number of discrete points in the spatial and temporal directions, respectively, and $x_i = (i-1)h_x, i = 1, 2, \ldots, N_x, t_j = (j-1)h_t, j = 1, 2, \cdots, N_t$, where h_x and h_t are the steps in the spatial direction

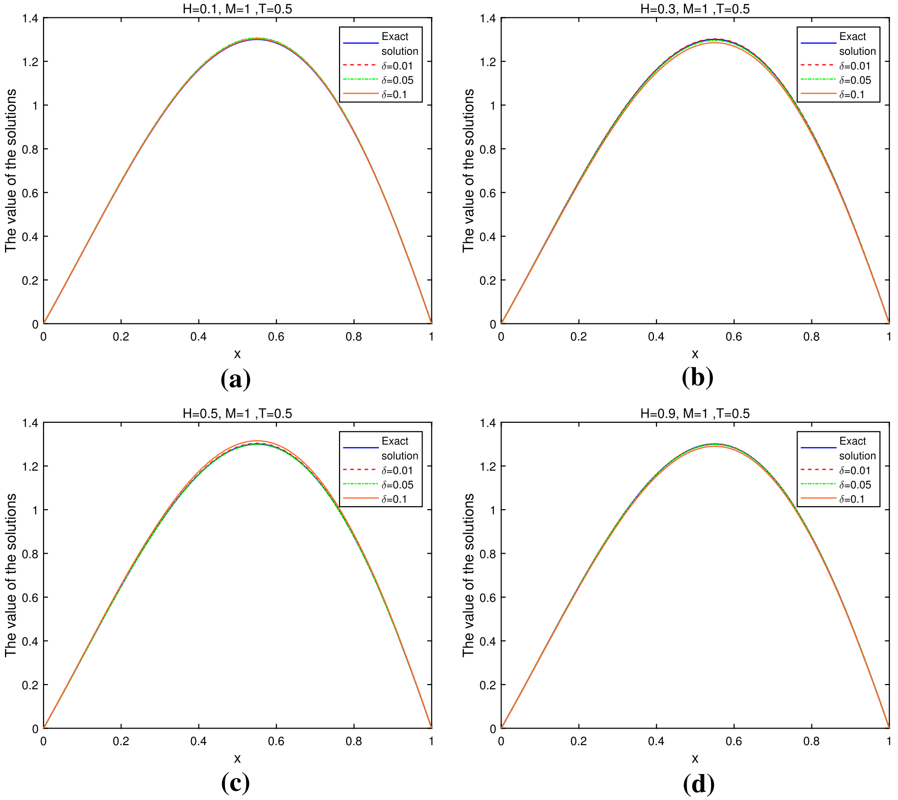

Fig. 1 The reconstruction results for $t = 0$ from noisy measurement data at T=0.5 for the problem with $M = 1$

and time direction, respectively. We solve the direct problem using the following finite difference scheme:

$$\frac{u_i^{j+1} - u_i^j}{h_t} + \frac{u_{i+1}^j - u_{i-1}^j}{2h_x} - \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h_x^2} = F(x_i, t_j),$$

Here, $F(x_i, t_j) = f(x_i)h(t_j) + \sigma(x_i)\frac{B^H(t_{j+1}) - B^H(t_j)}{h_t}$; the initial boundary value condition is discretized as $u_1^j = 0, u_{N_x}^j = 0, u_i^1 = g(x_i)$. In this paper, we choose $N_x = 101$, $N_t = 2^{15} + 1$, $T = 1$ and sample paths $P = 1000$, and some known functions in (2.4) are chosen as

$$f(x) = \sin(x)\cos(\pi x/2), \quad \sigma(x) = x(1 - x), \quad h(t) = 1.$$

Moreover, the data f, σ and g are assumed to be polluted by a uniformly distributed noise with level δ. The parameters M, H and T may vary in different experiments.

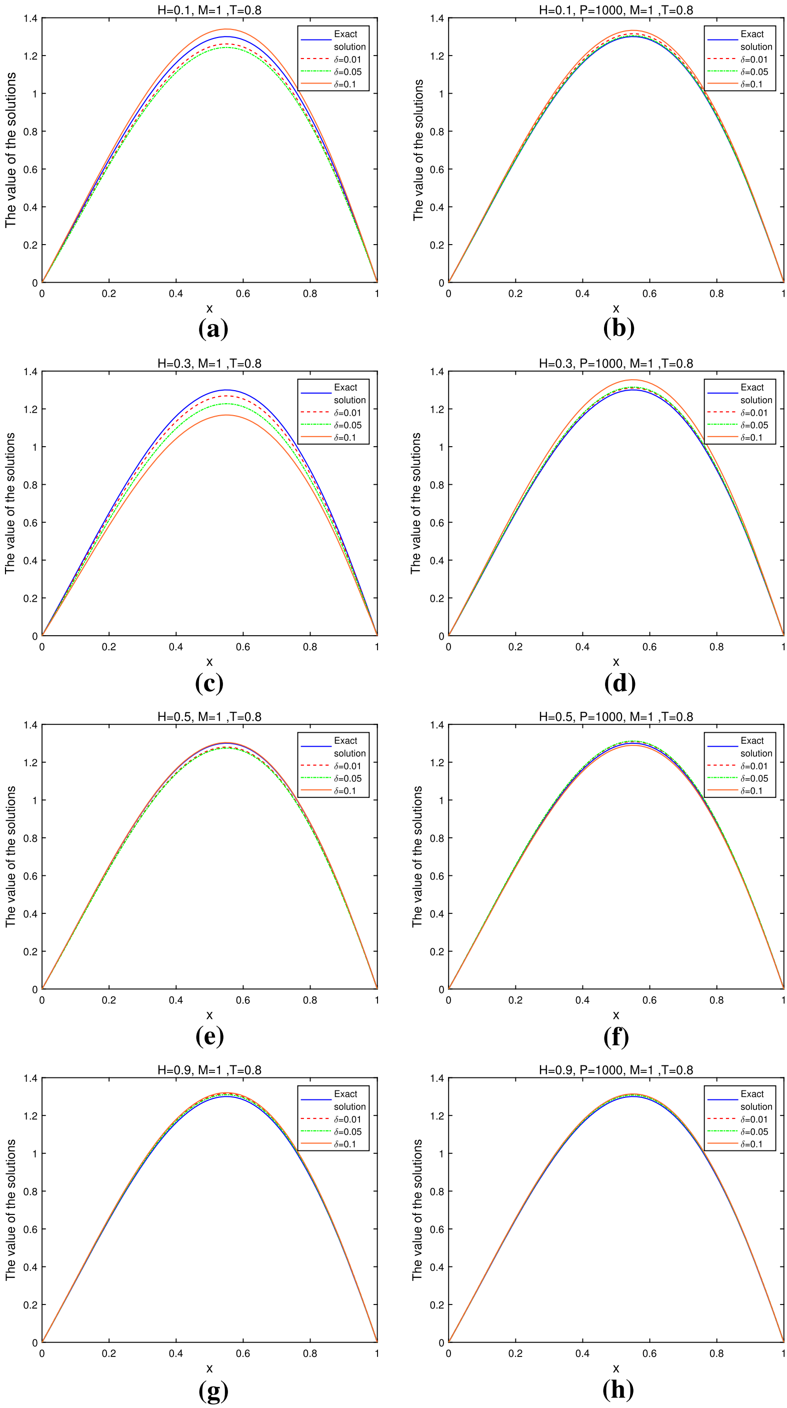

Fig. 2 The exact and approximated solutions for the problem with $T = 0.8$, $M = 1$, (left): one path, (right): the expectation

Figure 1 shows the results of the backward problem with different T, M, $Hand\delta$, which tells us that the recovery would be more accurate if $\delta \geq 0$ is smaller. Based on the results, it can be observed that the regularized results are also acceptable for only one path when $T = 0.5$. Of course, this conclusion can be drawn intuitively from Fig. 2.

6 Conclusion

In this paper, we have studied a backward problem for convection–diffusion equation driven by fBm with Hurst index $H \in (0, 1)$. We obtain the regularity of mild solution and discuss the ill-posedness for the backward problem. The truncated regularization method is introduced to approximate the solution of the problem. Under the a-priori assumption, we obtain the convergence estimate in $\widetilde{L}^2$ norm. Finally, numerical implementations are presented to show the validity of our theorem analysis. However, for this stochastic backward problem, we are not sure if the uniqueness is true. Hope we can do something about it in future.

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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