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Some Estimations for the Generalized Relative Operator Entropy

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Abstract
In this paper, we investigate a notion of the generalized relative operator entropy, which develops the theory of the relative operator entropy introduced by Fujii and Kamei, and a notion of the Csiszar operator f-divergence mapping. We estimate some upper and lower bounds of the generalized relative operator entropy and generalized operator Shannon entropy. In particular, we reach some new bounds for the relative operator entropy, the operator q-geometric mean, and the χ^2-divergence. Mainly, our results extend some known operator inequalities.

Keywords Operator inequality · Operator Shannon-type inequality · Relative operator entropy · Generalized relative operator entropy · q-geometric mean

Mathematics Subject Classification 47A63 · 46L05 · 46L60.

1 Introduction

The quantum relative entropy is a very important quantity in quantum information theory [32]. It satisfies many significant relations such as monotonicity property under quantum channels [20]. Relative entropies or generalized divergences are used to derive the second-order asymptotic expansions and strong converse theorems.

Generalized entropic functions are in an active area of research. Hence, several lower and upper bounds on these functions are of interest. These quantities are defined on pairs of positive operators and usually required to be nonnegative on pairs of

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states. Relative entropies serve at least two purposes in both classical and quantum information theory:

(1) a notion of distance on the set of probability distributions or quantum states,
(2) acting as parent quantities for entropic quantities such as the Shannon entropy or the von Neumann entropy.

The relative operator entropy is one of the most important concepts in both the quantum information theory and statistical physics. Operator entropic quantities are also interesting mathematical subjects with many attractive properties. Operator entropy inequalities have been investigated by some mathematicians [1–3, 13, 15, 22, 26, 30].

The relative operator entropy of strictly positive operators A and B on a Hilbert space was introduced in the noncommutative information theory by Fujii and Kamei [11] via the quantity

$$S(A|B) := A^{\frac{1}{2}}(\ln A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

The generalized relative operator entropy for strictly positive operators A, B and $q \in \mathbb{R}$ was defined in [14] by setting

$$S_q(A|B) := A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^q(\ln A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

In particular, when $q = 0$, it leads to the relative operator entropy $S(A|B)$.

We introduced in [29] a generalized notion of the relative operator entropy and the Tsallis relative operator entropy and called them the relative operator (α, β)-entropy and the Tsallis relative operator (α, β)-entropy. For the strictly positive operators A, B and the real numbers $\alpha \neq 0, \beta$, we defined

$$S_{\alpha,\beta}(A|B) := A^{\frac{\beta}{2}}(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^\alpha(\ln A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})A^{\frac{\beta}{2}},$$
$$T_{\alpha,\beta}(A|B) := A^{\frac{\beta}{2}}\ln_\lambda(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})A^{\frac{\beta}{2}},$$

where $\ln_\lambda X = \frac{X^\lambda - 1}{\lambda}$ and proved the joint convexity or concavity of these concepts under certain conditions concerning α and β. We clarified in [26] the upper and lower bounds for the relative operator (α, β)-entropy and Tsallis relative operator (α, β)-entropy according to the operator (α, β)-geometric mean introduced in [24].

Drogomir found in [8] some bounds for the difference

$$\frac{m \ln m}{M - m}(MA - B) + \frac{M \ln M}{M - m}(B - mA) - S_1(A|B) \tag{1}$$

and he specified in [7] some bounds for the following difference

$$S(A|B) - \frac{\ln m}{M - m}(MA - B) - \frac{\ln M}{M - m}(B - mA), \tag{2}$$

where A, B are two strictly positive operators such that $mA \leq B \leq MA$ for some $m, M > 0$ with $m < M$.

We generalized Dragomir's results in [23] and identified some upper and lower bounds for the difference

$$\frac{m^q \ln m}{M-m}(MA-B)+\frac{M^q \ln M}{M-m}(B-mA)-S_q(A|B), \tag{3}$$

where A and B are two strictly positive operators such that $mA \le B \le MA$ for some $m, M \in (0, e^{\frac{2q-1}{q(1-q)}}]$ with $m < M$ and $0 < q \le 1$. In particular, when $q \to 1^-$ in (3), we get (1). We also identified several bounds for the difference

$$T_\lambda(A|B)-\frac{m^\lambda-1}{\lambda(M-m)}(MA-B)-\frac{M^\lambda-1}{\lambda(M-m)}(B-mA), \tag{4}$$

where A and B are as above and $0 < \lambda \le 1$. When $\lambda \to 0^+$ in (4), we get (2).

We denote by $B(\mathcal{H})$ the C^*-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$. For self-adjoint operators A and B in $B(\mathcal{H})$, we write $A \ge B$ (resp. $A > B$) if $A - B$ is positive (resp. strictly positive). A self-adjoint operator A in $B(\mathcal{H})$ is positive if $\langle Ah, h\rangle \ge 0$ for $h \in \mathcal{H}$ and strictly positive if $\langle Ah, h\rangle > 0$ for $h \in \mathcal{H}$.

In this paper, we investigate a notion of the generalized relative operator entropy and call it the generalized f-relative operator entropy, which develops the theory of the relative operator entropy introduced by Fujii and Kamei. We also investigate a notion of the Csiszar operator f-divergence mapping. We estimate some new upper and lower bounds of the generalized f-relative operator entropy, generalized operator Shannon entropy, operator q-geometric mean, and χ^2-divergence. Our results recover some known operator inequalities.

2 Bounds of the Generalized *f*-Relative Operator Entropy

The classical perspective function associated with f, defined on a convex set $\mathcal{C} \subseteq \mathbb{R}^n$, is a function of two variables on the subset

$$K := \{(s,t) : s > 0, \ \ \frac{t}{s} \in \mathcal{C}\} \subseteq \mathbb{R}^{n+1}$$

considered by $P_f(s,t) := sf(\frac{t}{s})$ (cf. [17]).

Given a convex function $f : [0,\infty) \to \mathbb{R}$, the f-divergence functional

$$I_f(P,Q)=\sum_{i=1}^{n} q_i f(\frac{p_i}{q_i}),$$

was introduced by Csiszar [5] as a generalized measure of information, a distance function between two probability distributions $P = \{p_1, ..., p_n\}$, $Q = \{q_1, ..., q_n\}$. Note that the term $q_i f(\frac{p_i}{q_i})$ in the definition of the f-divergence functional $I_f(P,Q)$

is the classical perspective of f in (q_i, p_i) in the sense that $q_i f(\frac{p_i}{q_i}) = P_f(q_i, p_i)$ and thus

$$I_f(P, Q) = \sum_{i=1}^{n} P_f(q_i, p_i).$$

The relative entropy or Kullback–Leibler distance [19] between two probability distributions $P = \{p_1, ..., p_n\}$, $Q = \{q_1, ..., q_n\}$ was defined by

$$D(P||Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}.$$

The χ^2-divergence was proposed by Pearson [33] via the formula

$$\chi^2(P, Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i},$$

and the Hellinger distance [4] was defined by

$$H(P, Q) = \frac{1}{2} \sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2.$$

Note that we can state them as the f-divergence functional with a suitable representing function as follows.

$$\begin{aligned} D(P||Q) &= I_{-\log t}(Q, P), \\ \chi^2(P, Q) &= I_{\frac{(t-1)^2}{t}}(Q, P), \\ H(P, Q) &= I_{\frac{1}{2}(\sqrt{t}-1)^2}(Q, P). \end{aligned}$$

A fully noncommutative perspective of the one variable function f defined in [9] by setting

$$P_f(A, B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$

where A is a strictly positive operator and B is a self-adjoint operator on a Hilbert space $\mathcal{H}$ such that the spectrum of the operator $A^{-1/2} B A^{-1/2}$ lies in the domain of the function f. Effros considered in [10] the case where each pair in the argument of the perspective consists of commuting operators. The necessary and sufficient conditions for joint convexity (resp. concavity) of the noncommutative perspective and generalized perspective functions are established in [9]. For the other applications concerning this concept, see [24, 28, 29, 31]. The axiomatic theory for connections has been discussed by Kubo and Ando [18]. They proved the existence of an affine-order isomorphism between the class of connections and the class of positive operator monotone functions. Albeit this affine order isomorphism has a perspective form, the axiomatic theory for connections was only considered for the class of positive operator monotone functions.

Taking ideas from these facts, we motivate to introduce the notion of the Csiszar operator f-divergence mapping. Let $\mathbf{A} = \{A_1, ..., A_n\}, \mathbf{B} = \{B_1, ..., B_n\}$ be two finite sequences of strictly positive operators such that the spectrum of the operators $A_i^{-1/2} B_i A_i^{-1/2}$ lies in the closed interval $\mathbb{I}$ and let $f : \mathbb{I} \to \mathbb{R}$ be a continuous function. We consider the Csiszar operator f-divergence mapping by setting

$$I_f(\mathbf{B}, \mathbf{A}) = \sum_{i=1}^{n} P_f(A_i, B_i).$$

The first part of the following lemma was proved in [26, Theorem 2.1] for the noncommutative generalized perspective functions.

Lemma 1 *Let r, s, k be real-valued and continuous functions on the closed interval $\mathbb{I}$. If $r(t) \le s(t) \le k(t)$ for $t \in \mathbb{I}$, then*

(i) for every strictly positive operator A and every self-adjoint operator B such that the spectrum of the operator $A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ lies in $\mathbb{I}$, we have

$$P_r(A, B) \le P_s(A, B) \le P_k(A, B),$$

(ii) for two finite sequences of strictly positive operators $\boldsymbol{A} = \{A_1, ..., A_n\}, \boldsymbol{B} = \{B_1, ..., B_n\}$ such that the spectrum of the operators $A_i^{-1/2} B_i A_i^{-1/2}$ lies in $\mathbb{I}$, we have

$$I_r(\boldsymbol{B}, \boldsymbol{A}) \le I_s(\boldsymbol{B}, \boldsymbol{A}) \le I_k(\boldsymbol{B}, \boldsymbol{A}).$$

Proof (i) In view of the assumption, we get

$$r(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \le s(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \le k(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})$$

for the strictly positive operator A and the self-adjoint operator B. By multiplying $A^{\frac{1}{2}}$ from both sides, we obtain the desired inequalities.

(ii) It is a simple consequence of part (i).

Definition 1 Let $f : [0, \infty) \to \mathbb{R}$ be a twice differentiable function and $q \in \mathbb{R}$. For two strictly positive operators A and B, we consider the generalized f-relative operator entropy by setting

$$S_q^f(A|B) := A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^q f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

We can recognize the significance of this definition when one considers the function f various known functions. For instance, when one considers $f(t) = 1, q \in$

$[0, 1]$, $f(t) = \ln(t)$, $q \in \mathbb{R}$, $f(t) = \frac{t^{k\lambda} - t^{(k-1)\lambda}}{\lambda}$, $\lambda, q \in \mathbb{R}$, $\lambda \neq 0$, $k \in \mathbb{Z}$, we reach the operator q-geometric mean [14]

$$A \natural_q B = A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^q A^{\frac{1}{2}},$$

the generalized relative operator entropy [14]

$$S_q(A|B) = A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^q \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}},$$

the generalized Tsallis relative operator entropy [25, 34]

$$\widetilde{T}_{q,k,\lambda}(A, B) = \frac{A \natural_{q+k\lambda} B - A \natural_{q+(k-1)\lambda} B}{\lambda},$$

respectively.

In particular, when $\mathbf{A} = \{A_1, ..., A_n\}$, $\mathbf{B} = \{B_1, ..., B_n\}$ are two finite sequences of strictly positive operators, we reach the generalized operator Shannon entropy introduced in [21] for a positive operator monotone function f and $q \in \mathbb{R}$ by

$$S_q^f(\mathbf{A}|\mathbf{B}) = \sum_{i=1}^{n} S_q^f(A_i|B_i).$$

Under certain conditions, the upper and lower bounds for $S_q^f(\mathbf{A}|\mathbf{B})$ were given in [21] as an extension of an inequality due to Furuta [14].

Define $\omega(t) := t^q f(t)$ for $q \in \mathbb{R}$, where $f : [0, \infty) \to \mathbb{R}$ is a twice differentiable function and consider

$$\mathbb{J}_q := \{t \geq 0 : \omega''(t) \geq 0\}.$$

Hence, the function $\omega(t)$ is convex on $\mathbb{J}_q$ for $q \in \mathbb{R}$. In particular, when we consider $f(t) = \ln t$, the natural logarithm function, a simple calculation indicates that $\mathbb{J}_q := (0, e^{\frac{2q-1}{q(1-q)}}]$ for $0 < q \leq 1$. Consequently, $\mathbb{J}_{1^-} = (0, \infty)$ and $\mathbb{J}_{0^+} = \emptyset$. Moreover, for $q > 1$ or $q < 0$, a routine verification shows that $\mathbb{J}_q := [e^{\frac{2q-1}{q(1-q)}}, \infty)$ and so $\mathbb{J}_{1^+} = (0, \infty)$, $\mathbb{J}_{0^-} = \emptyset$. Note that the function $t^q \ln t$ is convex on $\mathbb{J}_q$ for $0 < q \leq 1$ and on $\mathbb{J}_q$ for $q > 1$ or $q < 0$.

Throughout the paper and for the sake of simplified writing, we consider

$$r(u) := \min\left\{\frac{u-m}{M-m}, \frac{M-u}{M-m}\right\} = \frac{1}{2} - \left|\frac{u - \frac{M+m}{2}}{M-m}\right|,$$

$$R(u) := \max\left\{\frac{u-m}{M-m}, \frac{M-u}{M-m}\right\} = \frac{1}{2} + \left|\frac{u - \frac{M+m}{2}}{M-m}\right|,$$

$$K_q^f(m, M) := \frac{m^q f(m) + M^q f(M)}{2} - \left(\frac{M+m}{2}\right)^q f\left(\frac{M+m}{2}\right),$$

$$\begin{aligned}\Psi(t) &:= (t-m)(M-t),\\ L_f(t) &:= \frac{m^q f(m)}{M-m}(M-t) + \frac{M^q f(M)}{M-m}(t-m),\\ \Gamma(t) &:= t^{q-1}(tf'(t)+qf(t)).\end{aligned}$$

where $0 < m < M$ and $q \in \mathbb{R}$. The function L_f is the line joining the points $(m, m^q f(m))$ and $(M, M^q f(M))$. Note that convexity of the function $\omega(t) = t^q f(t)$ on $\mathbb{J}_q$ shows that

$$K_q^f(m, M) \geq 0 \tag{5}$$

for $m, M \in [\alpha, \beta] \subseteq \mathbb{J}_q$ with $0 < m < M$ and $q \in \mathbb{R}$.

In what follows, we provide the upper and lower bounds for $S_q^f(A|B)$ and $S_q^f(\mathbf{A}|\mathbf{B})$. In particular, we reach [23, Theorems 2, 3, 4] by considering $f(t) = \ln t$ and $0 < q \leq 1$ in Theorems 1, 2, and 3, respectively. Moreover, by letting $q \to 1^+$ in Corollary 3 we obtain [8, Theorem 3], and by putting $q \to 1^-$ in Theorem 2 we reach [8, Theorem 2]. We generalize [23, Theorems 2, 3, 4] for the case where $q \notin (0, 1)$.

Theorem 1 *Let A and B be two strictly positive operators such that $mA \leq B \leq MA$ for some $m, M \in [\alpha, \beta] \subseteq \mathbb{J}_q$ with $0 < m < M$ and $q \in \mathbb{R}$. Then,*

$$\begin{aligned}0 \leq P_{L_f}(A, B) - S_q^f(A|B) &\leq \frac{\Gamma(M)-\Gamma(m)}{M-m} P_\Psi(A, B)\\ &\leq \frac{1}{4}(M-m)\Big(\Gamma(M)-\Gamma(m)\Big)A.\end{aligned}$$

Proof We apply [8, Lemma 1] for the function $\omega(t) = t^q f(t)$, $t \in [\alpha, \beta] \subseteq \mathbb{J}_q$. Then,

$$\begin{aligned}0 &\leq (1-c)\omega(x) + c\omega(y) - \omega((1-c)x + cy)\\ &\leq c(1-c)(y-x)(\omega'_-(y) - \omega'_+(x)),\end{aligned} \tag{6}$$

where $c \in [0, 1]$ and $x, y \in [\alpha, \beta]$. Substitute $x = m$, $y = M$, and $c = \frac{u-m}{M-m}$ in (6) to get

$$\begin{aligned}0 &\leq \frac{m^q f(m)}{M-m}(M-u) + \frac{M^q f(M)}{M-m}(u-m) - u^q f(u)\\ &\leq \frac{M^{q-1}(Mf'(M)+qf(M)) - m^{q-1}(mf'(m)+qf(m))}{M-m}\Psi(u)\\ &= \frac{\Gamma(M)-\Gamma(m)}{M-m}\Psi(u).\end{aligned} \tag{7}$$

The function $\Psi(u)$ attains its maximum value at $u = \frac{M+m}{2}$, and the maximum value is $\frac{1}{4}(M-m)^2$. So,

$$\frac{\Gamma(M)-\Gamma(m)}{M-m}\Psi(u) \leq \frac{1}{4}(M-m)\Big(\Gamma(M)-\Gamma(m)\Big). \tag{8}$$

Combining inequalities (7), (8) and regarding Lemma 1 and taking the perspective, we conclude the result.

The following corollaries are straightforward consequences of Theorem 1 and provide the upper and lower bounds for $S_q^f(A|B)$ and $S_q^f(\mathbf{A}|\mathbf{B})$, the generalized operator Shannon entropy, introduced in [21].

Corollary 1 *Let A and B be two strictly positive operators such that $mA \leq B \leq MA$ for some $m, M \in [\alpha, \beta] \subseteq \mathbb{J}_q$ with $0 < m < M$ and $q \in \mathbb{R}$. Then,*

$$\begin{aligned} & P_{L_f}(A, B) - \frac{1}{4}(M-m)\Big(\Gamma(M) - \Gamma(m)\Big)A \\ & \leq P_{L_f}(A, B) - \frac{\Gamma(M) - \Gamma(m)}{M-m} P_\Psi(A, B) \\ & \leq S_q^f(A|B) \\ & \leq P_{L_f}(A, B). \end{aligned}$$

Corollary 2 *Let $\boldsymbol{A} = \{A_1, ..., A_n\}, \boldsymbol{B} = \{B_1, ..., B_n\}$ be two finite sequences of strictly positive operators such that $mA_i \leq B_i \leq MA_i$ for some $m, M \in [\alpha, \beta] \subseteq \mathbb{J}_q$ with $0 < m < M$ and $q \in \mathbb{R}$. Then,*

$$I_{L_f}(\boldsymbol{B}, \boldsymbol{A}) - \frac{\Gamma(M) - \Gamma(m)}{M-m} I_\Psi(\boldsymbol{B}, \boldsymbol{A}) \leq S_q^f(\boldsymbol{A}|\boldsymbol{B}) \leq I_{L_f}(\boldsymbol{B}, \boldsymbol{A}).$$

In particular, considering $f(t) = \ln t$ and $0 < q \leq 1$ in Theorem 1 we get $\mathbb{J}_q = (0, e^{\frac{2q-1}{q(1-q)}}]$ where we reach [23, Theorem 2]. Moreover, we generalize [23, Theorem 2] for the cases $q > 1$ or $q < 0$ as follows.

Remark 1 We note that there is no result for the case where $q \to 0$, since in this case the convex domain of the function $\ln t$ is empty. In the last section, we will solve this problem.

Corollary 3 *Let A and B be two strictly positive operators such that $mA \leq B \leq MA$ for some $m, M \in \mathbb{J}_q = [e^{\frac{2q-1}{q(1-q)}}, \infty)$ with $0 < m < M$ and $q > 1$ or $q < 0$. Then,*

$$\begin{aligned} 0 & \leq \frac{m^q \ln m}{M-m}(MA - B) + \frac{M^q \ln M}{M-m}(B - mA) - S_q(A|B) \\ & \leq \frac{M^{q-1}(1 + q\ln M) - m^{q-1}(1 + q\ln m)}{M-m} P_\Psi(A, B) \\ & \leq \frac{1}{4}(M-m)\Big(M^{q-1}(1 + q\ln M) - m^{q-1}(1 + q\ln m)\Big)A. \qquad (9) \end{aligned}$$

By letting $q \to 1^+$ in Corollary 3, we get $\mathbb{J}_{1^+} = (0, \infty)$ and we deduce [8, Theorem 3] via the fact that $S_1(A|B) = -S(B|A)$.

We obtain the lower and upper bounds for the operator q-geometric mean in the case $q < 0$ or $q > 1$.

Corollary 4 *Let A and B be two strictly positive operators such that $mA \le B \le MA$ for some $0 < m < M$. If $q < 0$ or $q > 1$, then*

$$\alpha_1 A + \beta B \le A \natural_q B \le \alpha_2 A + \beta B,$$

where

$$\alpha_1 = \frac{m^q M - mM^q}{M-m} - \frac{q}{4}(M-m)(M^{q-1} - m^{q-1}),$$
$$\alpha_2 = \frac{m^q M - mM^q}{M-m},$$
$$\beta = \frac{M^q - m^q}{M-m}.$$

Proof Consider $f(t) = 1$ in Corollary 1 and note that in this situation $\mathbb{J}_q = (0, \infty)$ and

$$P_{L_f}(A, B) = \frac{m^q}{M-m}(MA - B) + \frac{M^q}{M-m}(B - mA),$$
$$S_q^f(A|B) = A \natural_q B,$$
$$\Gamma(M) - \Gamma(m) = q(M^{q-1} - m^{q-1}).$$

A simplification gets the desired result.

Remark 2 Dragomir in [6] proved that if $\phi : D \to \mathbb{R}$ is a convex function defined on a convex subset $D \subset \mathbb{R}$, then

$$2r\Big[\frac{\phi(x)+\phi(y)}{2} - \phi(\frac{x+y}{2})\Big] \le (1-c)\phi(x) + c\phi(y) - \phi((1-c)x + cy)$$
$$\le 2R\Big[\frac{\phi(x)+\phi(y)}{2} - \phi(\frac{x+y}{2})\Big]$$

for any $x, y \in D$ and $c \in [0, 1]$, where $r = \min\{c, 1-c\}$ and $R = \max\{c, 1-c\}$.

By the notations as in Remark 2, we have the following results.

Theorem 2 *Let A and B be two strictly positive operators such that $mA \le B \le MA$ for some $m, M \in [\alpha, \beta] \subseteq \mathbb{J}_q$ with $0 < m < M$ and $q \in \mathbb{R}$. Then,*

$$2K_q^f(m, M)P_r(A, B) \le P_{L_f}(A, B) - S_q^f(A|B) \le 2K_q^f(m, M)P_R(A, B).$$

Proof Apply Remark 2 for the convex function $\phi(t) = t^q f(t)$, $t \in [\alpha, \beta] \subseteq \mathbb{J}_q$ to obtain

$$2r\Big[\frac{x^q f(x)+y^q f(y)}{2}-(\frac{x+y}{2})^q f(\frac{x+y}{2})\Big]$$
$$\leq (1-c)x^q f(x)+cy^q f(y)-((1-c)x+cy)^q f((1-c)x+cy)$$
$$\leq 2R\Big[\frac{x^q f(x)+y^q f(y)}{2}-(\frac{x+y}{2})^q f(\frac{x+y}{2})\Big] \tag{10}$$

for any $x, y \in [\alpha, \beta]$ and $c \in [0, 1]$, where $r = \min\{c, 1-c\}$ and $R = \max\{c, 1-c\}$. Substitute $x = m$, $y = M$, and $c = \frac{u-m}{M-m}$ with $u \in [m, M]$ in (10) to get

$$2K_q^f(m, M)r(u) \leq m^q f(m)\frac{M-u}{M-m}+M^q f(M)\frac{u-m}{M-m}-u^q f(u)$$
$$\leq 2K_q^f(m, M)R(u). \tag{11}$$

By using Lemma 1 and taking the perspective, we get the desired inequalities.

As a consequence of Theorem 2, we give the upper and lower bounds for the generalized f-relative operator entropy and generalized operator Shannon entropy.

Corollary 5 *Let A and B be two strictly positive operators such that $mA \leq B \leq MA$ for some $m, M \in [\alpha, \beta] \subseteq \mathbb{J}_q$ with $0 < m < M$ and $q \in \mathbb{R}$. Then,*

$$P_{L_f}(A, B)-2K_q^f(m, M)P_R(A, B) \leq S_q^f(A|B)$$
$$\leq P_{L_f}(A, B)-2K_q^f(m, M)P_r(A, B). \tag{12}$$

Corollary 6 *Let $\boldsymbol{A} = \{A_1, ..., A_n\}$, $\boldsymbol{B} = \{B_1, ..., B_n\}$ be two finite sequences of strictly positive operators such that $mA_i \leq B_i \leq MA_i$ for some $m, M \in [\alpha, \beta] \subseteq \mathbb{J}_q$ with $0 < m < M$ and $q \in \mathbb{R}$. Then,*

$$I_{L_f}(\boldsymbol{B}, \boldsymbol{A})-2K_q^f(m, M)I_R(\boldsymbol{B}, \boldsymbol{A}) \leq S_q^f(\boldsymbol{A}|\boldsymbol{B})$$
$$\leq I_{L_f}(\boldsymbol{B}, \boldsymbol{A})-2K_q^f(m, M)I_r(\boldsymbol{B}, \boldsymbol{A}).$$

In light of our results and by considering $f(t) = \ln t$ and $0 < q \leq 1$ in Theorem 2, we obtain [23, Theorem 3], and by putting $q \to 1^-$ we reach [8, Theorem 2]. Furthermore, we generalize [23, Theorem 3] for the cases $q > 1$ or $q < 0$ as follows, since in this case the convex domain of the function $t^q \ln t$ is $\mathbb{J}_q$:

Corollary 7 *Let A and B be two strictly positive operators such that $mA \leq B \leq MA$ for some $m, M \in \mathbb{J}_q = [e^{\frac{2q-1}{q(1-q)}}, \infty)$ with $0 < m < M$ and $q > 1$ or $q < 0$. Then,*

$$2K_q^{\ln}(m, M)P_r(A, B) \leq \frac{m^q \ln m}{M-m}(MA-B)+\frac{M^q \ln M}{M-m}(B-mA)-S_q(A|B)$$
$$\leq 2K_q^{\ln}(m, M)P_R(A, B). \tag{13}$$

We obtain the lower and upper bounds for the operator q-geometric mean in the case $q < 0$ or $q > 1$.

Corollary 8 *Let A and B be two strictly positive operators such that $mA \le B \le MA$ for some $0 < m < M$. If $q < 0$ or $q > 1$, then*

$$\alpha_2 A + \beta B - \kappa P_R(A, B) \le A\natural_q B \le \alpha_2 A + \beta B - \kappa P_r(A, B),$$

where

$$\kappa = m^q + M^q - 2(\frac{m+M}{2})^q$$

and α_2, β are the same as Corollary 4.

Proof Consider $f(t) = 1$ in Corollary 5 and note that

$$P_{L_f}(A, B) = \frac{m^q}{M-m}(MA - B) + \frac{M^q}{M-m}(B - mA),$$
$$S_q^f(A|B) = A\natural_q B,$$
$$\kappa = 2K_q^f(m, M) = m^q + M^q - 2(\frac{m+M}{2})^q.$$

A simple verification shows the result.

Theorem 3 *Let A and B be two strictly positive operators such that $mA \le B \le MA$ for some $m, M \in [\alpha, \beta] \subseteq \mathbb{J}_q$ with $0 < m < M$ and $q \in \mathbb{R}$. If there exist the constants γ_1, γ_2 such that $\gamma_1 \le \omega''(t) \le \gamma_2$ for every $t \in [m, M]$, then*

$$\frac{1}{2}\gamma_1 P_\Psi(A, B) \le P_{L_f}(A|B) - S_q^f(A|B) \le \frac{1}{2}\gamma_2 P_\Psi(A, B).$$

Proof By applying [8, Lemma 2] for the function $\omega(t) = t^q f(t)$, $t \in [m, M]$, we get

$$\begin{aligned}\frac{1}{2}c(1-c)\gamma_1(y-x)^2 &\le (1-c)\omega(x) + c\omega(y) - \omega((1-c)x + cy) \\ &\le \frac{1}{2}c(1-c)\gamma_2(y-x)^2, \end{aligned} \tag{14}$$

where $c \in [0, 1]$ and $x, y \in [m, M]$. Substitute $x = m$, $y = M$, and $c = \frac{u-m}{M-m}$, in (14), to find

$$\begin{aligned}\frac{1}{2}\gamma_1(u-m)(M-u) &\le \frac{M-u}{M-m}m^q f(m) + \frac{u-m}{M-m}M^q f(M) - u^q f(u) \\ &\le \frac{1}{2}\gamma_2(u-m)(M-u). \end{aligned} \tag{15}$$

By applying Lemma 1, we reach the desired inequalities.

In view of Theorem 3, we find the other bounds for the generalized f-relative operator entropy and generalized operator Shannon entropy.

Corollary 9 *Let A and B be two strictly positive operators such that $mA \le B \le MA$ for some $m, M \in [\alpha, \beta] \subseteq \mathbb{J}_q$ with $0 < m < M$ and $q \in \mathbb{R}$. If there exist the constants γ_1, γ_2 such that $\gamma_1 \le \omega''(t) \le \gamma_2$ for every $t \in [m, M]$, then*

$$P_{L_f}(A, B) - \frac{\gamma_2}{2} P_\Psi(A, B) \le S_q^f(A|B) \le P_{L_f}(A|B) - \frac{\gamma_1}{2} P_\Psi(A, B).$$

Corollary 10 *Let $\boldsymbol{A} = \{A_1, ..., A_n\}$, $\boldsymbol{B} = \{B_1, ..., B_n\}$ be two finite sequences of strictly positive operators such that $mA_i \le B_i \le MA_i$ for some $m, M \in [\alpha, \beta] \subseteq \mathbb{J}_q$ with $0 < m < M$ and $q \in \mathbb{R}$. If there exist the constants γ_1, γ_2 such that $\gamma_1 \le \omega''(t) \le \gamma_2$ for every $t \in [m, M]$, then*

$$I_{L_f}(\boldsymbol{B}, \boldsymbol{A}) - \frac{\gamma_2}{2} I_\Psi(\boldsymbol{B}, \boldsymbol{A}) \le S_q^f(\boldsymbol{A}|\boldsymbol{B}) \le I_{L_f}(A|B) - \frac{\gamma_1}{2} I_\Psi(\boldsymbol{B}, \boldsymbol{A}).$$

According to Theorem 3, if we consider $f(t) = \ln t$ and $0 < q \le 1$, then we obtain [23, Theorem 4] with

$$\gamma_1 = M^{q-2}(2q - 1 + q(q-1)\ln M) \ge 0,$$
$$\gamma_2 = m^{q-2}(2q - 1 + q(q-1)\ln m) \ge 0.$$

Hence, $0 \le \frac{1}{2}\gamma_1 P_\Psi(A, B)$.

Corollary 11 *[23, Theorem 4] Let A and B be two strictly positive operators such that $mA \le B \le MA$ for some $m, M \in \mathbb{J}_q = (0, e^{\frac{2q-1}{q(1-q)}}]$ with $0 < m < M$ and $0 < q \le 1$. Then,*

$$\begin{aligned} 0 &\le \frac{1}{2}\gamma_1 P_\Psi(A, B) \\ &\le \frac{m^q \ln m}{M - m}(MA - B) + \frac{M^q \ln M}{M - m}(B - mA) - S_q(A|B) \\ &\le \frac{1}{2}\gamma_2 P_\Psi(A, B). \end{aligned} \tag{16}$$

Since the convex domain of the function $t^q \ln t$ is $\mathbb{J}_q$ for $q < 0$ or $q > 1$, we may generalize Corollary 11 for the case where $q \in (-\infty, 0) \cup (1, \infty)$.

Corollary 12 *Let A and B be two strictly positive operators such that $mA \le B \le MA$ for some $m, M \in \mathbb{J}_q = [e^{\frac{2q-1}{q(1-q)}}, \infty)$ with $0 < m < M$ and $q < 0$ or $q > 1$. Then, the inequalities (16) hold.*

The following corollary is a consequence of Theorem 3 when we consider $f(t) = \ln t$ and $q \to 1^-$.

Corollary 13 *[8, Theorem 4] Let A and B be two strictly positive operators such that $mA \le B \le MA$ for some $m, M \in (0, \infty)$ with $0 < m < M$. Then,*

$$
\begin{aligned}
0 &\le \frac{1}{2M} P_{\Psi}(A, B) \le \frac{m \ln m}{M - m}(MA - B) + \frac{M \ln M}{M - m}(B - mA) + S(B|A) \\
&\le \frac{1}{2m} P_{\Psi}(A, B).
\end{aligned}
$$

3 The Lower and Upper Bounds for $S(A|B)$

Having in mind to establish the bounds for the difference (2), one can let $q \to 0$ in the inequalities (9), (13), and (16). But, in this situation, we know that the convex domain $\mathbb{J}_0$ is empty. So, when q tends to 0, the established results in the previous section for the difference (2) do not mean. Hence, to achieve the goal, we change our attention to the concave domain of the function ω. Let us consider $\mathbb{C}_q := \{t \ge 0 : \omega''(t) \le 0\}$ for $q \in \mathbb{R}$. Then, clearly, $\mathbb{C}_q = \overline{\mathbb{J}_q^c}$ and the function ω is concave on $\mathbb{C}_q$. Note that in this case $\mathbb{C}_0 = (0, \infty)$ and the domain $\mathbb{C}_0$ is not empty and as a result the problem mentioned in Remark 1 is solved. So, one can discuss on the lower and upper bounds for the relative operator entropy $S(A|B)$. Indeed, we recognize the bounds of $S(A|B)$.

We now conclude the following result for the concave function ω on $\mathbb{C}_q$. The proof of the previous section can be repeated for the concave function ω on $\mathbb{C}_q$, and we omit the proof, since the function $-\omega$ is convex on $\mathbb{J}_q = \overline{\mathbb{C}_q^c}$. Note that concavity of the function $\omega(t) = t^q f(t)$ on $\mathbb{C}_q$ shows that

$$
K_q^f(m, M) \le 0 \tag{17}
$$

for $m, M \in [\alpha, \beta] \subseteq \mathbb{C}_q$ with $0 < m < M$ and $q \in \mathbb{R}$.

Theorem 4 *Let A and B be two strictly positive operators such that $mA \le B \le MA$ for some $m, M \in [\alpha, \beta] \subseteq \mathbb{C}_q$ with $0 < m < M$ and $q \in \mathbb{R}$. Then,*

(i) $0 \le S_q^f(A|B) - P_{L_f}(A, B) \le \frac{\Gamma(m) - \Gamma(M)}{M - m} P_{\Psi}(A, B)$,

(ii) $-2K_q^f(m, M) P_r(A, B) \le S_q^f(A|B) - P_{L_f}(A, B) \le -2K_q^f(m, M) P_R(A, B)$.

By considering $f(t) = \ln t$ and $q \to 0$ in parts (i) and (ii) of Theorem 4, we get the bounds for the difference (2) as follows.

Corollary 14 *Let A and B be two strictly positive operators such that $mA \le B \le MA$ for some $m, M > 0$ with $m < M$. Then,*

$$
\begin{aligned}
0 &\le S(A|B) - \frac{\ln m}{M - m}(MA - B) - \frac{\ln M}{M - m}(B - mA) \\
&\le \frac{1}{Mm} P_{\Psi}(A, B).
\end{aligned} \tag{18}
$$

Corollary 15 *Let A and B be two strictly positive operators such that $mA \le B \le MA$ for some $m, M > 0$ with $m < M$. Then,*

$$-2K_0^{\ln}(m, M)P_r(A, B) \le S(A|B) - \frac{\ln m}{M-m}(MA - B) - \frac{\ln M}{M-m}(B - mA)$$
$$\le -2K_0^{\ln}(m, M)P_R(A, B). \tag{19}$$

Note that $-K_0^{\ln}(m, M) = \ln \frac{m+M}{2\sqrt{mM}}$.

By applying Theorem 4, one can get the bounds for the generalized operator Shannon entropy based on the notion of the Csiszar operator f-divergence mapping.

Corollary 16 *Let $\mathbf{A} = \{A_1, ..., A_n\}$, $\mathbf{B} = \{B_1, ..., B_n\}$ be two finite sequences of strictly positive operators such that $mA_i \le B_i \le MA_i$ for some $m, M \in [\alpha, \beta] \subseteq \mathbb{C}_q$ with $0 < m < M$ and $q \in \mathbb{R}$. Then,*

(i) $0 \le S_q^f(\mathbf{A}|\mathbf{B}) - I_{L_f}(\mathbf{B}, \mathbf{A}) \le \frac{\Gamma(m)-\Gamma(M)}{M-m} I_\Psi(\mathbf{B}, \mathbf{A})$,

(ii) $-2K_q^f(m, M)I_r(\mathbf{B}, \mathbf{A}) \le S_q^f(\mathbf{A}|\mathbf{B}) - I_{L_f}(\mathbf{B}, \mathbf{A}) \le -2K_q^f(m, M)I_R(\mathbf{B}, \mathbf{A})$.

As a numerical consequence, one can realize the bounds of the χ^2-divergence as follows.

Example 1 Let $P = \{p_1, ..., p_n\}$ and $Q = \{q_1, ..., q_n\}$ be two probability distributions with $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$ and $0 < mp_i \le q_i \le Mp_i$. Then,

$$\frac{(M-m)(M-1)}{mM(m+M)} \le -\chi^2(P, Q) + \frac{(M-1)(1-m)}{mM} \le \frac{(M-m)(M+1)}{mM(m+M)}.$$

Proof Consider $A_i = p_i I$, $B_i = q_i I$ in Corollary 16 (ii), where I is the identity operator. So, the bounds of the Csiszar operator f-divergence mapping for the concave function $f(t) = -\frac{(t-1)^2}{t}$ with $q = 0$ can be obtained as follows:

$$\frac{(M-m)^2}{mM(m+M)} I_r(P, Q) \le -\chi^2(P, Q) + \frac{(M-1)(1-m)}{mM}$$
$$\le \frac{(M-m)^2}{mM(m+M)} I_R(P, Q), \tag{20}$$

where

$$I_r(P, Q) = \frac{1}{2} - \frac{1}{2(M-m)} \sum_{i=1}^n |2q_i - (m+M)p_i|,$$
$$I_R(P, Q) = \frac{1}{2} + \frac{1}{2(M-m)} \sum_{i=1}^n |2q_i - (m+M)p_i|.$$

A simple verification and using the fact that the absolute value for the real numbers satisfies the triangle inequality, we reach

$$I_R(P,Q) = \frac{1}{2} + \frac{1}{2(M-m)} \sum_{i=1}^{n} |2q_i - (m+M)p_i| \leq \frac{1}{2} + \frac{2+m+M}{2(M-m)},$$

$$\frac{1}{2} - \frac{2-m-M}{2(M-m)} \leq \frac{1}{2} - \frac{1}{2(M-m)} \sum_{i=1}^{n} |2q_i - (m+M)p_i| = I_r(P,Q).$$

Therefore, by replacing the lower and upper bounds of I_r and I_R in (20), respectively, we conclude the result.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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