Bull. Malays. Math. Sci. Soc. (2022) 45:3335–3365
https://doi.org/10.1007/s40840-022-01383-0

BULLETIN *of the* MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

The Solutions of Critical Nonlinear Dirac Equations with Degenerate Potential

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Received: 9 May 2022 / Revised: 25 August 2022 / Accepted: 27 August 2022 /
Published online: 15 September 2022

Abstract

The main purpose of this paper is to look for solutions of the following critical nonlinear Dirac equation

$$-i\varepsilon\alpha \cdot \nabla u + a\beta u + V(x)u = K(x)|u|^{p-2}u + Q(x)|u|u \quad x \in \mathbb{R}^3,$$

where $\varepsilon > 0$ is a small parameter, $a > 0$ is a constant, $p \in (5/2, 3)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is triplets of matrices, $\alpha_1, \alpha_2, \alpha_3$ and β are 4×4 Pauli-Dirac matrices. The potential $V(x)$ may attain $\pm a$ at somewhere or at infinity, $K, Q \in C^1(\mathbb{R}^3, \mathbb{R}^+)$ are two functions. When $\varepsilon > 0$ small, we will prove the existence and concentration of the solutions by using variational methods under some mild assumptions on the potentials V, K and Q.

Keywords Nonlinear Dirac equation · Variational methods · Concentration · Semiclassical states · Critical exponent

Mathematics Subject Classification Primary 35A15; Secondary 35Q40 · 49J35

Communicated by Rosihan M. Ali.

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1 Introduction and Main Results

In this paper, we concerned with the following nonlinear Dirac equation with critical nonlinearities

$$-i\varepsilon\alpha\cdot\nabla u+a\beta u+V(x)u=K(x)|u|^{p-2}u+Q(x)|u|u, \tag{1.1}$$

where $u:\mathbb{R}^3\to\mathbb{C}^4$ is a spinor field, $\nabla=(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3})$, $a>0$ is a constant. α_1, α_2, α_3 and β are 4×4 Pauli-Dirac matrices:

$$\alpha_k=\begin{pmatrix}0&\sigma_k^*\\ \sigma_k&0\end{pmatrix}\quad 1\le k\le 3,\quad \beta=\begin{pmatrix}I_2&0\\0&-I_2\end{pmatrix}$$

with

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

where σ_k^* is the conjugate transpose of σ_k. It is well known that the most general form of Eq. (1.1) is

$$-i\hbar\partial_t\psi=ic\hbar\Sigma_{k=1}^3\alpha_k\partial_k\psi-mc^2\beta\psi-M(x)\psi+F_\psi(x,\psi), \tag{1.2}$$

where $\hbar$ stands for Planck constant, $m>0$ denotes the mass of particle, c is speed of light. Equation (1.2) plays an important role in quantum electrodynamics [21]. In mathematics, under the assumptions $F(x,e^{i\theta}\psi)=F(x,\psi)$ for any $\theta\in[0,2\pi]$ and $\psi(t,x)=e^{\frac{i\mu}{\hbar}t}w(x)$, then the Eq. (1.2) is equivalent to the following stationary equation

$$i\hbar\Sigma_{k=1}^3\alpha_k\partial_k w+a\beta w+V(x)w=F_w(x,w), \tag{1.3}$$

where $a=mc$, $V(x)=(\frac{M(x)}{c}+\mu)I_4$ and $F_w(x,w)=\frac{1}{c}F_\psi(x,\psi)$. Especially, Eq. (1.1) can be regarded as a generalized stationary equation of (1.2) in the case that $F=\frac{1}{p}K(x)|\psi|^p+\frac{1}{3}Q(x)|\psi|^3$ and $\varepsilon=\hbar$. The external fields in (1.3) arise in models of mathematical models of particle physics for many years [22, 24]. The most common examples of nonlinear Dirac equation are the massive Thirring model [29] (vector self-interaction) and the Soler model [27] (scalar self-interaction). Various nonlinearities appear in models for unified field theories. For more physical background one can refer to [28].

For the Soler model $F(w)=\frac{1}{2}H(w\overline{w})$, $H\in C^2(\mathbb{R},\mathbb{R})$, by using variational methods, Esteban and Séré [19] obtained infinitely many solutions under the following assumptions:

$$V(x)\equiv\omega,\ H'(s)\cdot s\ge\theta H(s),\ F(-w)=F(w)\text{ and }\omega\in(-a,0)$$

for all $s \in \mathbb{R}$ and some $\theta > 1$. This may be the first literature to study the nonlinear Dirac equation by using variational theory. After that, Bartsch and Ding [2] obtained the standarding wave solution of Eq. (1.3) under $V(x)$ and $F(x, w)$ are period depend on x. This is a change in the study of nonlinear Dirac equations from autonomous systems to non-autonomous systems. Their work benefits from the critical point theory of strongly indefinite functional developed in [1]. Further, Ding and Ruf [16] considered the Coulomb-type potential and obtained the existence and multiplicity of solutions for asymptotically quadratic nonlinearities. For more results on the existence and multiplicity of solutions of (1.3), we refer to the literature [12, 20] and their references.

According to [13], when the Plank constant $\hbar > 0$ is small enough and tends to zero, the solution of (1.3) is called semiclassical states. From physical point of view, this is related to the correspondence principle proposed by Niels. Bohr in the early development of quantum mechanics. This principle describes a corresponding relationship between quantum mechanics and classical mechanics, it provides a new view of physics. To the best of our knowledge, there have been many literatures seeking the existence and concentration phenomenon of the semiclassical states for nonlinear Dirac equations. Under the condition $V(x) = 0$ and $F_w(x, w) = P(x)|w|^{p-2}w$, $2 < p < 3$, Ding [13] obtained ground state solutions of (1.3) which concentrate the maximum points of $P(x)$ as $\hbar \to 0$, it is the first result about semiclassical state of the nonlinear Dirac equation. This results was later generalized to the case

$$V(x) \not\equiv 0, \quad \min_{x\in\mathbb{R}^3} V(x) < \liminf_{|x|\to\infty} V(x) \tag{1.4}$$

and the nonlinearity with the form $F_w(x, w) = f(|w|)w$ in [15], where nonlinearity is subcritical. When the potential V satisfies (1.4), Ding and Ruf [17] also considered Eq. (1.3) with the nonlinearity $F_w(x, w) = P(x)(g(|w|) + |w|)w$. In [18], Ding and Xu proposed the following local condition of the potential $V(x)$: there is a bounded domain $\Lambda \subset \mathbb{R}^3$ such that

$$\min_{x\in\overline{\Lambda}} V(x) < \min_{x\in\partial\Lambda} V(x) \tag{1.5}$$

and they established the same conclusion as [15]. It is worth mentioning that this local condition (1.5) weakens (1.4). In fact, (1.5) is similar to the classical global condition proposed by Rabinowitz [26] in nonlinear Schödinger equation. For more semiclassical results, we refer the reader to the surveys [5–7, 14, 31, 32, 34] for reference to the literature.

In this paper, we first construct the semiclassical states of the critical Dirac equation with degenerate potential(the potential V may attain $\pm a$ or approach $\pm a$ at ∞), and then discuss the concentration phenomenon of the semiclassical state as $\hbar = \varepsilon \to 0$. To state our main results, we need the following assumptions.

(V) $V \in C^1(\mathbb{R}^3, \mathbb{R})$ satisfies $\sup_{\mathbb{R}^3} |V(x)| \leq a$, there exist the constants $\tau \in (0, 2)$ and $\nu \in (0, +\infty)$, such that

$$a - |V(x)| \geq \frac{\nu}{1 + |x|^\tau}.$$

(K) $K \in C^1(\mathbb{R}^3, \mathbb{R})$ and $0 < k_1 \le K(x) \le k_2(1+|x|)^{\tau'}$ for any $x \in \mathbb{R}^3$ with constants $k_1 > 0$, $k_2 > 0$ and $\tau' > 0$.

(Q) $Q \in C^1(\mathbb{R}^3, \mathbb{R})$ and $0 < q_1 \le Q(x) \le q_2 < \infty$ for any $x \in \mathbb{R}^3$ with constants $q_1 > 0$ and $q_2 > 0$.

(S) There is a bounded domain $\Lambda \subset \mathbb{R}^3$ with smooth boundary $\partial\Lambda$ such that

$$\vec{n}(x) * \nabla V(x) > 0, \; \nabla K(x) * \nabla V(x) < 0 \text{ for any } x \in \partial\Lambda,$$
$$\nabla Q(x) * \nabla V(x) < 0, \; \nabla Q(x) * \nabla K(x) > 0 \text{ for any } x \in \partial\Lambda,$$

where $\vec{n}(x)$ denotes the unit outward normal vector to $\partial\Lambda$ at x.

Without loss of generality, we assume $0 \in \Lambda$. For any set $\Omega \subset \mathbb{R}^3$, $\delta > 0$, $\varepsilon > 0$, we define

$$\Omega^\delta = \left\{ x \in \mathbb{R}^3 : \text{dist}(x, \Omega) := \inf_{y \in \Omega} |x - y| < \delta \right\},$$
$$\Omega_\varepsilon = \left\{ x \in \mathbb{R}^3 : \varepsilon x \in \Omega \right\}.$$

Denote for $\delta > 0$ small $\mathcal{O}(\delta) = \{x \in \Lambda : \text{dist}(x, \partial\Lambda) > \delta\}$. Then there is $\delta_0 > 0$ such that $\sup_{\Lambda^{\delta_0} \setminus \mathcal{O}(\delta_0)} \nabla K(x) * \nabla V(x) < 0$ and $\sup_{\Lambda^{\delta_0} \setminus \mathcal{O}(\delta_0)} \nabla Q(x) * \nabla V(x) < 0$. The main results of this paper are as follows.

Theorem 1.1 *Suppose that assumptions* (V),(K),(Q) *and* (S) *hold. Then, for* $p \in (5/2, 3)$*, there exists* $\varepsilon_0 > 0$ *such that if* $0 < \varepsilon < \varepsilon_0$*, Eq.* (1.1) *has a nontrivial solution* u_ε*, satisfying that for any* $\delta > 0$*, there exist* $C_1 = C_1(\delta) > 0$ *and* $C_2 = C_2(\delta) > 0$ *such that*

$$|u_\varepsilon| \le C_2 \exp\left(-C_1 \left(\frac{dist(x, \mathcal{O}(\delta))}{\varepsilon} \right)^{\frac{2-\tau}{2}} \right).$$

Our problem concerns the Sobolev critical situations, so it is difficult to deal with compactness in order to get semiclassical state. As we will see, the energy functional associated to Eq. (1.1) is strongly indefinite. Thus, we cannot use the standard critical point theory [33] to solve it. On the other hand, we allow the potential $V(x)$ can be reach a or tends to a at infinite. This potential V destroys the linking structure of the energy functional. In order to overcome these difficulties, we follow the methods in references [6] and [32]. We first introduce a truncation function and adjust the nonlinear term appropriately. Secondly, we make use of an idea of the penalization approach similar to that used in [4, 8, 9] in the energy functional by subtracting a penalized functional term P_ε, which ensures the linking structure of the energy functional. Combining truncation techniques and the penalization functional P_ε, it makes the Palais-Smale sequences bounded and relatively compact, so we can deal with the modified problem. Finally, by some regularity and L^∞ estimate of solutions which solves modified problem, we can get the semiclassical state of Eq. (1.1).

The paper is organised as follows. In the next section we present some preliminary notions on the Dirac operator, introduce the modified functional and give some basic lemmas. In Sect. 3, by using an abstract linking theorem, we prove the existence of nontrivial solutions of the modified problems when ε is small. In Sect. 4, we give a profile decomposition with respect to a family of solution $\{u_\varepsilon\}$ which obtained in Sect. 3 and get some regularity estimates on the $\{u_\varepsilon\}$. Finally, in Sect. 5, we finish the proof of the main theorem.

2 Preliminaries

Firstly, using the scaling $w(x) = u(\varepsilon x)$, we can rewrite the Eq. (1.1) as the following equivalent equation

$$-i\alpha \cdot \nabla w + a\beta w + V(\varepsilon x)w = K(\varepsilon x)|w|^{p-2}w + Q(\varepsilon x)|w|w \quad x \in \mathbb{R}^3. \tag{2.1}$$

If w is a solution of Eq. (2.1), then $u(x) := w(x/\varepsilon)$ is a solution of the Eq. (1.1). Therefore, we will mainly focus on this equivalent equation in the remaining part of the paper.

For convenience, let $H_0 := -i\alpha \cdot \nabla + a\beta$ denotes the Dirac operator, it is a self-adjoint operator on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $\mathcal{D}(H_0) = H^1(\mathbb{R}^3, \mathbb{C}^4)$. According to [19], we know that

$$\sigma(H_0) = \sigma_c(H_0) = \mathbb{R}\backslash(-a, a),$$

where $\sigma(H_0)$ and $\sigma_c(H_0)$ denote the spectrum and the continuous spectrum of H_0, respectively. Consequently, the space $L^2(\mathbb{R}^3, \mathbb{C}^4)$ possesses the orthogonal decomposition:

$$L^2(\mathbb{R}^3, \mathbb{C}^4) = L^+ \oplus L^-, \quad u = u^+ + u^-$$

such that H_0 is positive definite in L^+ and negative in L^-. Let $|H_0|$ denote the absolute value of H_0 and $|H_0|^{\frac{1}{2}}$ denote its square root. We define $E := \mathcal{D}(|H_0|^{\frac{1}{2}})$, then by [19], we know that E is a Hilbert space if endowed with the inner product

$$(u, v) = \mathrm{Re}\left(|H_0|^{\frac{1}{2}}u, |H_0|^{\frac{1}{2}}v\right)_{L_2},$$

and the induced norm $\|u\|^2 = (u, u)$, where Re stands for the real part of a complex number. By [19], this norm is equivalent to the usual $H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$-norm, therefore, E embeds continuously into $L^q(\mathbb{R}^3, \mathbb{C}^4)$ for all $q \in [2, 3]$ and compactly into $L^q_{loc}(\mathbb{R}^3, \mathbb{C}^4)$ for all $q \in [1, 3)$. Moreover, since $\sigma(H_0) = \mathbb{R}\backslash(-a, a)$, we have

$$a|u|_2^2 \le \|u\|^2, \quad \text{for all } u \in E. \tag{2.2}$$

Furthermore, E can be decomposed as follows

$$E = E^+ \oplus E^-,$$

where $E^+ = E \cap L^+$ and $E^- = E \cap L^-$ and the sum is orthogonal with respect to inner product $(\cdot,\cdot)$ and $(\cdot,\cdot)_{L_2}$. In addition, it follows from [18, Proposition 2.1] that

$$c_q\|u^\pm\|_q^q \le \|u\|_q^q \text{ for all } u \in E,$$

where $c_q > 0$ is a constant.

The energy functional of (2.1) is

$$\begin{aligned}J_\varepsilon(w) =& \frac{1}{2}\int_{\mathbb{R}^3}(-i\alpha\cdot\nabla w, w) + (a\beta w, w)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}(V(\varepsilon x)w, w)\mathrm{d}x\\ &- \frac{1}{p}\int_{\mathbb{R}^3}K(\varepsilon x)|w|^p\mathrm{d}x - \frac{1}{3}\int_{\mathbb{R}^3}Q(\varepsilon x)|w|^3\mathrm{d}x.\end{aligned}$$

By the decomposition $E = E^+ \oplus E^-$, we can rewrite J_ε as follows

$$\begin{aligned}J_\varepsilon(w) =& \frac{1}{2}(\|w^+\|^2 - \|w^-\|^2) + \frac{1}{2}\int_{\mathbb{R}^3}(V(\varepsilon x)w, w)\mathrm{d}x\\ &- \frac{1}{p}\int_{\mathbb{R}^3}K(\varepsilon x)|w|^p dx - \frac{1}{3}\int_{\mathbb{R}^3}Q(\varepsilon x)|w|^3\mathrm{d}x.\end{aligned}$$

According to standard arguments, we know that $J_\varepsilon : E \to \mathbb{R}$ is of class C^1. For $w, v \in E$, there holds

$$J_\varepsilon'(w)v = \mathrm{Re}\int_{\mathbb{R}^3}(H_0 w + V(\varepsilon x)w - K(\varepsilon x)|w|^{p-2}w - Q(\varepsilon x)|w|w)\cdot v\mathrm{d}x,$$

where $w \cdot v$ express the usual inner product in $\mathbb{C}^4$. Moreover, in [15, Lemma 2.1] it is proved that critical points of J_ε are weak solutions of nonlinear Dirac Eq. (2.1).

From now on, we will construct a penalized functional P_ε as that used in [4, 10, 11] and a truncation function as that used in [6] and so that our modified functional have nontrivial critical points.

Let $\varphi \in C^\infty(\mathbb{R}^+, [0, 1])$ be a cut-off function such that $\varphi(t) = 1$ if $0 \le t \le 1$, $\varphi(t) = 0$ if $t \ge 2$ and for any $t \ge 0$. Set $b_\varepsilon(t) = \varphi(\varepsilon t)$ and $m_\varepsilon(t) = \int_0^t b_\varepsilon(s)\mathrm{d}s$ for any $t \ge 0$.

Let $\zeta \in C^\infty(\mathbb{R}^+, [0, 1])$ be a cut-off function such that $\zeta(t) = 0$ if $t \ge \delta_0$, and $\zeta(t) = 1$ if $0 \le t \le \delta_0/2$, and $\zeta'(t) \le 0$ for any $t \ge 0$. Define $\chi(x) = \zeta(\mathrm{dist}(x, \Lambda))$ and

$$g_\varepsilon(x, t) = \min\{h_\varepsilon(x, t), \phi(x)\} \text{ for any } t \ge 0, x \in \mathbb{R}^3,$$

where $\phi(x) = \frac{\kappa}{1+|x|^{4+\tau'}}$ and

$$h_\varepsilon(x,t)=K(\varepsilon x)t^{p-2}+\frac{p}{3}Q(\varepsilon x)t^{p-2}\left(m_\varepsilon(t^2)\right)^{\frac{3-p}{2}}$$
$$+\frac{3-p}{3}Q(\varepsilon x)t^{p}\left(m_\varepsilon(t^2)\right)^{\frac{3-p}{2}-1}b_\varepsilon(t^2).$$

Let us define

$$f_\varepsilon(x,t)=\chi(\varepsilon x)h_\varepsilon(x,t)+(1-\chi(\varepsilon x))g_\varepsilon(x,t),$$

then for $(x,t)\in\mathbb{R}^3\times\mathbb{R}^+$ and $G_\varepsilon(x,t)=\int_0^t g_\varepsilon(x,s)s\mathrm{d}s$

$$\begin{aligned}F_\varepsilon(x,t)&=\int_0^t f_\varepsilon(x,s)s\mathrm{d}s\\&=\chi(\varepsilon x)\left(\frac{1}{p}K(\varepsilon x)t^p+\frac{1}{3}Q(\varepsilon x)\left(m_\varepsilon(t^2)\right)^{\frac{3-p}{2}}\right)+(1-\chi(\varepsilon x))\,G_\varepsilon(x,t).\end{aligned}$$

We denote the sets $\mathcal{V}_\pm:=\{x\in\mathbb{R}^3: V(x)=\pm a\}$ and $\mathcal{V}:=\mathcal{V}_+\cup\mathcal{V}_-$. By (V), we can choose l_0 large enough, such that $(\mathcal{V})^{2\delta}\subset B(0,l_{0/2})$. Setting χ_+ and χ_- be the characteristic function of the sets

$$\mathcal{B}_+:=(\mathcal{V}_+)^\delta\cup\left\{|x|\geq l_0: V(x)\geq\frac{3a}{4}\right\},\ \mathcal{B}_-:=(\mathcal{V}_-)^\delta\cup\left\{|x|\geq l_0: V(x)\leq-\frac{3a}{4}\right\}.$$

Without loss of generality, assume that δ is small enough, there exists a $\theta\in(0,1)$ satisfying

$$\pm V(x)\geq\frac{3a}{4}\ \text{ for } x\in\mathcal{B}_\pm\ \text{ and } V(x)\in[-\theta a,\theta a] \text{ for } x\notin\mathcal{B}=\mathcal{B}_+\cup\mathcal{B}_-.$$

For $\phi(x)=\frac{\kappa}{1+|x|^{4+\tau'}}$, we define $\xi,\widehat{\xi}:\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$ by

$$\xi(x,t)=\begin{cases}0, & t\leq\phi(x);\\ \frac{1}{\phi(x)}(t-\phi(x))^2, & \phi(x)<t<2\phi(x);\\ 2t-3\phi(x), & t\geq 2\phi(x),\end{cases}\quad \widehat{\xi}(x,t)=\int_{-\infty}^t\xi(x,s)\mathrm{d}s,$$

and define the penalized functional $P_\varepsilon: E\to\mathbb{R}$ by

$$P_\varepsilon(w)=\frac{a}{8}\int_{\mathbb{R}^3}\widetilde{\chi}(\varepsilon x)\widehat{\xi}(x,|w|)\mathrm{d}x,$$

where $\widetilde{\chi}(x)=\chi_+(x)-\chi_-(x)$. It is clear that $P_\varepsilon: E\to\mathbb{R}$ is of class C^1 and

$$P'_\varepsilon(w)v=\frac{a}{8}\mathrm{Re}\int_{\mathbb{R}^3}\widetilde{\chi}(\varepsilon x)\widetilde{\xi}(x,|w|)w\cdot v\mathrm{d}x\ \text{ for any } v\in C_0^\infty(\mathbb{R}^3,\mathbb{C}^4),\tag{2.3}$$

where $\widetilde{\xi}(x,t) \in C^1(\mathbb{R}^3 \times \mathbb{R}, [0,2])$,

$$\widetilde{\xi}(x,t) := \frac{1}{t}\xi(x,t) \begin{cases} 0, & t \le \phi(x); \\ \frac{(t-\phi(x))^2}{t\phi(x)}, & \phi(x) < t < 2\phi(x); \\ \frac{2t-3\phi(x)}{t}, & t \ge 2\phi(x). \end{cases}$$

Moreover, for $w_n \rightharpoonup w$ weakly in E, there holds

$$P'_\varepsilon(w_n)v \to P'_\varepsilon(w)v \ \text{ for any } v \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4).$$

Now we define the modified functional $\Phi_\varepsilon : E \to \mathbb{R}$

$$\begin{aligned}\Phi_\varepsilon(w) &= \frac{1}{2}\int_{\mathbb{R}^3} (-i\alpha\cdot\nabla + a\beta)\, w \cdot w \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} (V(\varepsilon x)w, w)\,\mathrm{d}x \\ &\quad - P_\varepsilon(w) - \int_{\mathbb{R}^3} F_\varepsilon(x,|w|)\mathrm{d}x \\ &= \frac{1}{2}\left(\|w^+\|^2 - \|w^-\|^2\right) + \frac{1}{2}\int_{\mathbb{R}^3} (V(\varepsilon x)w, w)\,\mathrm{d}x - P_\varepsilon(w) - \int_{\mathbb{R}^3} F_\varepsilon(x,|w|)\mathrm{d}x.\end{aligned}$$

By (V), (K), (Q) and (2.3), we know that Φ_ε is of class C^1, and for $w, v \in E$, there holds

$$\Phi'_\varepsilon(w)v = \mathrm{Re}\int_{\mathbb{R}^3} (H_0 w + V(\varepsilon x)w - \frac{a}{8}\widetilde{\chi}(\varepsilon x)\widetilde{\xi}(x,|w|)w - f_\varepsilon(x,|w|)w)\cdot v\mathrm{d}x,$$

and the critical points correspond to weak solutions of

$$-i\alpha\cdot\nabla w + a\beta w + V(\varepsilon x)w - \frac{a}{8}\widetilde{\chi}(\varepsilon x)\widetilde{\xi}(x,|w|)w = f_\varepsilon(x,|w|)w.$$

Lemma 2.1 *For small $\varepsilon_0 > 0$ and $\varepsilon \in (0,\varepsilon_0)$, the energy functional Φ_ε satisfies the Palais-Smale condition.*

Proof Assuming $\{w_n\} \subset E$ is a Palais-Smale sequence for Φ_ε, i.e., $\{\Phi_\varepsilon(w_n)\} \subset \mathbb{R}$ is bounded and $\Phi'_\varepsilon(w_n) \to 0$ in E^*, we shall show that $\{w_n\}$ has a convergent subsequence in E. We first verify the bounded-ness of $\{w_n\}$ in E. Observing that

$$\begin{aligned}o_n(1)\|w_n\| &= \Phi'_\varepsilon(w_n)(w_n^+ - w_n^-) \\ &= \|w_n\|^2 + \mathrm{Re}\int_{\mathbb{R}^3} V(\varepsilon x)w_n\cdot(w_n^+ - w_n^-)\mathrm{d}x \\ &\quad - \mathrm{Re}\int_{\mathbb{R}^3} f_\varepsilon(x,|w_n|)w_n\cdot(w_n^+ - w_n^-)\mathrm{d}x \\ &\quad - \frac{a}{8}\mathrm{Re}\int_{\mathbb{R}^3} \widetilde{\chi}(\varepsilon x)\widetilde{\xi}(x,|w_n|)w_n\cdot(w_n^+ - w_n^-)\mathrm{d}x\end{aligned}$$

$$= \|w_n\|^2 + \int_{\mathbb{R}^3} (V(\varepsilon) - \frac{a}{8}\widetilde{\chi}(\varepsilon x)\widetilde{\xi}(x, |w_n|))(|w_n^+|^2 - |w_n^-|^2)\mathrm{d}x$$
$$- \mathrm{Re} \int_{\mathbb{R}^3} f_\varepsilon(x, |w_n|)w_n \cdot (w_n^+ - w_n^-)\mathrm{d}x. \tag{2.4}$$

By the similar argument as [32, Lemma 2.2], we get

$$\int_{\mathbb{R}^3} \left(V(\varepsilon x) - \frac{a}{8}\widetilde{\chi}(\varepsilon x)\widetilde{\xi}(x, |w_n|)\right)\left(|w_n^+|^2 - |w_n^-|^2\right)\mathrm{d}x$$
$$\geq -\max\{\theta, \frac{7}{8}\}\|w_n\|^2 - C\varepsilon^{2\tau'+5}. \tag{2.5}$$

On the other hand, it may be assumed that $\Phi_\varepsilon(w_n) \to c$, then

$$c + \|w_n\| \geq \Phi_\varepsilon(w_n) - \frac{1}{2}\Phi_\varepsilon'(w_n)w_n$$
$$= \int_{\mathbb{R}^3} \left(\frac{1}{2} f_\varepsilon(x, |w_n|)|w_n|^2 - F_\varepsilon(x, |w_n|)\right)\mathrm{d}x + \frac{1}{2}P_\varepsilon'(w_n)w_n - P_\varepsilon(w_n). \tag{2.6}$$

By the definition of P_ε, (2.2) and the fact $\|\chi_-(\varepsilon x)\phi\|_2 \leq C\varepsilon^{\tau'+5/2}$, we deduce

$$\frac{1}{2}P_\varepsilon'(w_n)w_n - P_\varepsilon(w_n) \geq -\frac{3a}{2}\int_{\mathbb{R}^3} \chi_-(\varepsilon x)|w_n|\phi\mathrm{d}x$$
$$\geq -C\|\chi_-(\varepsilon x)\phi\|_2\|w_n\|_2 \geq -C\varepsilon^{\tau'+5/2}\|w_n\|. \tag{2.7}$$

If $h_\varepsilon(x, t) \geq \phi(x)$, then by the definition of $g_\varepsilon(x, t)$ and $G_\varepsilon(x, t)$, we have

$$G_\varepsilon(x, t) = \frac{1}{2}\phi(x)t^2 - \frac{1}{2}\phi(x)t_0^2 + H_\varepsilon(x, t_0), \quad h_\varepsilon(x, t_0) = \phi(x),$$

where $H_\varepsilon(x, t_0) = \int_0^{t_0} h_\varepsilon(x, s)s\mathrm{d}s = \frac{1}{p}K(\varepsilon x)t_0^p + \frac{1}{3}Q(\varepsilon x)t_0^p(m_\varepsilon(t_0^2))^{\frac{3-p}{2}}$. So there holds

$$\left|\frac{1}{2}g_\varepsilon(x, t_0)t_0^2 - G_\varepsilon(x, t_0)\right| \leq \left|\frac{1}{2}\phi(x)t_0^2 - H_\varepsilon(x, t_0)\right| \tag{2.8}$$

Since $h_\varepsilon(x, t_0) = \phi(x)$, i.e.,

$$K(\varepsilon x)t_0^{p-2} + \frac{p}{3}Q(\varepsilon x)t_0^{p-2}\left(m_\varepsilon(t_0^2)\right)^{\frac{3-p}{2}}$$
$$+ \frac{3-p}{3}Q(\varepsilon x)t_0^p\left(m_\varepsilon(t_0^2)\right)^{\frac{3-p}{2}-1} b_\varepsilon(t_0^2) = \phi(x).$$

If $t_0 \gg 1$, then we have

$$K(\varepsilon x)t_0^{p-2} \le K(\varepsilon x)t_0^{p-2} + \frac{p}{3}Q(\varepsilon x)t_0^{p-2}\left(m_\varepsilon(t_0^2)\right)^{\frac{3-p}{2}} = \phi(x).$$

From above, we know that t_0 has an upper bound, i.e., there exists a constant $M > 0$, such that $t_0 \le M$. Similarly, if $t_0 \ll 1$, then there holds $K(\varepsilon x)t_0^{p-2} + Q(\varepsilon x)t_0^p = \phi(x)$. It follows that

$$\left|\frac{1}{2}g_\varepsilon(x, t_0)t_0^2 - G_\varepsilon(x, t_0)\right| \le \frac{1}{2}\,|K(\varepsilon x)|^{-\frac{2}{p-2}}\,|\phi(x)|^{\frac{p}{p-2}}.$$

Then

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}(1-\chi(\varepsilon x))\left(\frac{1}{2}g_\varepsilon(x,|w_n|)|w_n|^2 - G_\varepsilon(x,|w_n|)\mathrm{d}x\right)\right|\\ &\quad\le C\int_{\mathbb{R}^3\setminus(\Lambda^\delta)_\varepsilon}|K(\varepsilon x)|^{-\frac{2}{p-2}}|\phi(x)|^{\frac{p}{p-2}} \le C\varepsilon^{\frac{p(\tau'+4)}{p-2}-3}.\end{aligned} \tag{2.9}$$

Injecting (2.7), (2.8) and (2.9) into (2.6), we have

$$\begin{aligned}C(1+\|w_n\|) &\ge \int_{\mathbb{R}^3}\left(\frac{1}{2}f_\varepsilon(x,|w_n|)|w_n|^2 - F_\varepsilon(x,|w_n|)\right)\mathrm{d}x + \frac{1}{2}P'_\varepsilon(w_n)w_n - P_\varepsilon(w_n)\mathrm{d}x\\ &\ge \int_{\mathbb{R}^3}\left(\frac{1}{2}\chi(\varepsilon x)h_\varepsilon(x,|w_n|)|w_n|^2 - \chi(\varepsilon x)H_\varepsilon(x,|w_n|)\right)\mathrm{d}x - C\varepsilon^{\tau'+5/2}\|w_n\|\\ &= \int_{\mathbb{R}^3}\chi(\varepsilon x)\left(\frac{1}{2}h_\varepsilon(x,|w_n|)|w_n|^2 - H_\varepsilon(x,|w_n|)\right)\mathrm{d}x - C\varepsilon^{\tau'+5/2}\|w_n\|,\end{aligned} \tag{2.10}$$

where $H_\varepsilon(x,|w_n|) = \int_0^{|w_n|}h_\varepsilon(x,s)s\mathrm{d}s$. By Hölder inequality and (2.10), we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\chi(\varepsilon x)h_\varepsilon(x,|w_n|)w_n\cdot(w_n^+ - w_n^-)\mathrm{d}x\right|\\ &\quad\le\left|\int_{\mathbb{R}^3}K(\varepsilon x)\chi(\varepsilon x)|w_n|^{p-2}w_n\cdot(w_n^+ - w_n^-)\mathrm{d}x\right|\\ &\qquad+\left|\int_{\mathbb{R}^3}\chi(\varepsilon x)\widetilde{h}_\varepsilon(x,|w_n|)w_n\cdot(w_n^+ - w_n^-)\mathrm{d}x\right|\\ &\quad\le\|K^{1/p}|w_n|\|^{p-1}_{L^p((\Lambda^\delta)_\varepsilon)}\|K^{1/p}|w_n^+ - w_n^-|\|_{L^p((\Lambda^\delta)_\varepsilon)}\\ &\qquad+\left|\int_{\mathbb{R}^3}\chi(\varepsilon x)\widetilde{h}_\varepsilon(x,|w_n|)w_n\cdot(w_n^+ - w_n^-)\mathrm{d}x\right|\\ &\quad\le C\left(1+\|w_n\|\right)^{\frac{p-1}{p}}\varepsilon^{\frac{p-3}{p}}\|K\|^{\frac{1}{p}}_{L^{\frac{3-p}{3}}(\Lambda^\delta)}\|w_n^+ - w_n^-\|\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_{\mathbb{R}^3}\chi(\varepsilon x)\widetilde{h}_\varepsilon(x,|w_n|)w_n\cdot(w_n^+-w_n^-)\mathrm{d}x\right| \\
&\le C\varepsilon^{\frac{p-3}{p}}\left(\|w_n\|+\|w_n\|^{\frac{2p-1}{p}}\right)+\left(\int_{\mathbb{R}^3}\chi(\varepsilon x)(\widetilde{h}_\varepsilon(x,|w_n|)|w_n|)^{\frac{3}{2}}\mathrm{d}x\right)^{\frac{2}{3}} \\
&\times\left(\int_{\mathbb{R}^3}|(w_n^+-w_n^-)|^3\mathrm{d}x\right)^{\frac{1}{3}},
\end{aligned}
$$

where $\widetilde{h}_\varepsilon(x,t)=\frac{p}{3}Q(\varepsilon x)t^{p-2}\left(m_\varepsilon(t^2)\right)^{\frac{3-p}{2}}+\frac{3-p}{3}Q(\varepsilon x)t^p\left(m_\varepsilon(t^2)\right)^{\frac{3-p}{2}-1}b_\varepsilon(t^2)$. Then (2.10) can be rewrite

$$
\begin{aligned}
C(1+\|w_n\|) &\ge -C\varepsilon^{\tau'+5/2}\|w_n\|+\int_{\mathbb{R}^3}\chi(\varepsilon x)\left(\frac{1}{2}\widetilde{h}_\varepsilon(x,|w_n|)|w_n|^2-\widetilde{H}_\varepsilon(x,|w_n|)\right)\mathrm{d}x \\
&\ge -C\varepsilon^{\tau'+5/2}\|w_n\|+c\int_{\mathbb{R}^3}\chi(\varepsilon x)\left(\widetilde{h}_\varepsilon(x,|w_n|)|w_n|\right)^{\frac{3}{2}}\mathrm{d}x. \qquad (2.11)
\end{aligned}
$$

By the definition of $g_\varepsilon(x,t)$, we have

$$
\left|\int_{\mathbb{R}^3}(1-\chi(\varepsilon x))\,g_\varepsilon(x,|w_n|)w_n\cdot(w_n^+-w_n^-)\mathrm{d}x\right|\le C\varepsilon^{\tau'+3}\|w_n\|^2. \qquad (2.12)
$$

Combining (2.4), (2.5), (2.11) and (2.12), we have

$$
\begin{aligned}
\min\left\{1-\theta,\frac{1}{8}\right\}\|w_n\|^2-C\varepsilon^{\tau'+3}\|w_n\|^2 &\le C\varepsilon^{\frac{p-3}{p}}\left(\|w_n\|+\|w_n\|^{\frac{2p-1}{p}}\right) \\
&+C(1+\|w_n\|)^{\frac{2}{3}}\|w_n\|.
\end{aligned}
$$

This implies the bounded-ness of $\{w_n\}$ in E for small ε_0 and $\varepsilon\in(0,\varepsilon_0)$.

Next we prove $w_n\to w$ in E as $n\to\infty$, denoting $z_n=w_n-w$, we have

$$
\Phi'_\varepsilon(w_n)(z_n^+-z_n^-)=o_n(1),\quad \Phi'_\varepsilon(w)(z_n^+-z_n^-)=o_n(1).
$$

It follows that

$$
\begin{aligned}
o_n(1)=&\mathrm{Re}(w_n^+,z_n^+)+\mathrm{Re}(w_n^-,z_n^-)+\mathrm{Re}\int_{\mathbb{R}^3}V(\varepsilon x)w_n\cdot(z_n^+-z_n^-)\mathrm{d}x \\
&-\mathrm{Re}\int_{\mathbb{R}^3}\frac{a}{8}\widetilde{\chi}(\varepsilon x)\widetilde{\xi}(x,|w_n|)w_n\cdot(z_n^+-z_n^-)+f_\varepsilon(x,|w_n|)w_n\cdot(z_n^+-z_n^-)\mathrm{d}x;
\end{aligned}
$$

and

$$
\begin{aligned}
0=&\mathrm{Re}(w^+,z_n^+)+\mathrm{Re}(w^-,z_n^-)+\mathrm{Re}\int_{\mathbb{R}^3}V(\varepsilon x)w\cdot(z_n^+-z_n^-)\mathrm{d}x \\
&-\mathrm{Re}\int_{\mathbb{R}^3}\frac{a}{8}\widetilde{\chi}(\varepsilon x)\widetilde{\xi}(x,|w|)w\cdot(z_n^+-z_n^-)+f_\varepsilon(x,|w|)w\cdot(z_n^+-z_n^-)\mathrm{d}x.
\end{aligned}
$$

Then there holds

$$\begin{aligned} o_n(1) = \Phi'_\varepsilon(w_n)(z_n^+ - z_n^-) - \Phi'_\varepsilon(w)(z_n^+ - z_n^-) = \|z_n\|^2 \\ + \operatorname{Re}\int_{\mathbb{R}^3} V(\varepsilon x) z_n \cdot (z_n^+ - z_n^-)\mathrm{d}x \\ - \operatorname{Re}\int_{\mathbb{R}^3} \frac{a}{8}\widetilde{\chi}(\varepsilon x)\widetilde{\xi}(x,|w_n|)w_n \cdot (z_n^+ - z_n^-)\mathrm{d}x \\ + \operatorname{Re}\int_{\mathbb{R}^3} \frac{a}{8}\widetilde{\chi}(\varepsilon x)\widetilde{\xi}(x,|w|)w \cdot (z_n^+ - z_n^-)\mathrm{d}x \\ - \operatorname{Re}\int_{\mathbb{R}^3} \chi(\varepsilon x)(h_\varepsilon(x,|w_n|)w_n - h_\varepsilon(x,|w|)w) \cdot (z_n^+ - z_n^-)\mathrm{d}x \\ - \operatorname{Re}\int_{\mathbb{R}^3} (1-\chi(\varepsilon x))(g_\varepsilon(x,|w_n|)w_n - g_\varepsilon(x,|w|)w) \cdot (z_n^+ - z_n^-)\mathrm{d}x. \end{aligned} \tag{2.13}$$

By the definition of $\widetilde{\chi}$ and $\widetilde{\xi}(x,t)$, it follows that

$$\begin{aligned} \lim_{n\to\infty} \operatorname{Re}\int_{\mathbb{R}^3} \widetilde{\chi}(\varepsilon x)\widetilde{\xi}(x,|w_n|)w \cdot (z_n^+ - z_n^-)\mathrm{d}x \\ = \lim_{n\to\infty} \operatorname{Re}\int_{\mathbb{R}^3} \widetilde{\chi}(\varepsilon x)\widetilde{\xi}(x,|w|)w \cdot (z_n^+ - z_n^-)\mathrm{d}x = 0. \end{aligned}$$

Moreover, we have

$$(g_\varepsilon(x,|w_n|) - g_\varepsilon(x,|w|)) \cdot (z_n^+ - z_n^-) \rightharpoonup 0 \ \text{ in } L^2(\mathbb{R}^3,\mathbb{C}^4),$$

which leads to

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}^3} \big(1-\chi(\varepsilon x)\,(g_\varepsilon(x,|w_n|) - g_\varepsilon(x,|w|))\,w \cdot (z_n^+ - z_n^-)\big)\,\mathrm{d}x\right| = 0.$$

Hence, (2.13) can be rewritten as follows

$$\begin{aligned} o_n(1) =& \|z_n\|^2 + \operatorname{Re}\int_{\mathbb{R}^3} \left[V(\varepsilon x) - \frac{a}{8}\widetilde{\chi}(\varepsilon x)\widetilde{\xi}(x,|w_n|)\right] z_n \cdot (z_n^+ - z_n^-)\mathrm{d}x \\ & - \operatorname{Re}\int_{\mathbb{R}^3} \chi(\varepsilon x)\,(h_\varepsilon(x,|w_n|)w_n - h_\varepsilon(x,|w|)w) \cdot (z_n^+ - z_n^-)\mathrm{d}x. \end{aligned} \tag{2.14}$$

By [34], we know that

$$\begin{aligned} \int_{\mathbb{R}^3} V(\varepsilon x)|z_n^+|^2 - \frac{a}{8}\widetilde{\chi}(\varepsilon x)\widetilde{\xi}(x,|w_n|)|z_n^+|^2\mathrm{d}x \\ \ge \int_{\mathbb{R}^3} -\theta a(1-\chi(\varepsilon x))|z_n^+|^2\mathrm{d}x - \frac{7a}{8}\int_{|w_n|\ge 3\phi(x)} \chi_-(\varepsilon x)|z_n^+|^2\mathrm{d}x \end{aligned}$$

$$+\frac{a}{2}\int_{\mathbb{R}^3}\chi_+(\varepsilon x)|z_n^+|^2\mathrm{d}x - a\int_{\mathbb{R}^3\setminus(|w_n|\geq 3\phi(x))}\chi_-(\varepsilon x)|z_n^+|^2\mathrm{d}x + o_n(1)$$

and

$$\begin{aligned}&\int_{\mathbb{R}^3} -V(\varepsilon x)|z_n^-|^2 + \frac{a}{8}\widetilde{\chi}(\varepsilon x)\widetilde{\xi}(x,|w_n|)|z_n^-|^2\mathrm{d}x\\ &\geq \int_{\mathbb{R}^3} -\theta a(1-\chi(\varepsilon x))|z_n^-|^2\mathrm{d}x - \frac{7a}{8}\int_{|w_n|\geq 3\phi(x)}\chi_+(\varepsilon x)|z_n^-|^2\mathrm{d}x\\ &\quad+\frac{a}{2}\int_{\mathbb{R}^3}\chi_-(\varepsilon x)|z_n^-|^2\mathrm{d}x.\end{aligned}$$

Combining the above two inequalities and (2.14), we obtain

$$\min\left\{1-\theta,\frac{1}{8}\right\}\|z_n\|^2 - \mathrm{Re}\int_{\mathbb{R}^3}\chi(\varepsilon x)(h_\varepsilon(x,|w_n|)w_n - h_\varepsilon(x,|w|)w)\cdot(z_n^+ - z_n^-)\mathrm{d}x \leq 0. \tag{2.15}$$

By mean value theorem, there exists a function θ_n such that

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\chi(\varepsilon x)\left(K(\varepsilon x)|w_n|^{p-2}w_n - K(\varepsilon x)|w|^{p-2}w\right)\cdot(z_n^+ - z_n^-)\mathrm{d}x\right|\\ &\leq (p-1)\left|\int_{\mathbb{R}^3}\chi(\varepsilon x)(K(\varepsilon x)|\theta_n|^{p-2}z_n\cdot(z_n^+ - z_n^-)\mathrm{d}x\right|\\ &\leq (p-1)\int_{\mathbb{R}^3}\chi(\varepsilon x)(K(\varepsilon x)|\theta_n|^{p-2}|z_n|\cdot|z_n^+ - z_n^-|\mathrm{d}x\\ &\leq (p-1)\left(\int_{\mathbb{R}^3}\chi(\varepsilon x)(K(\varepsilon x)|\theta_n|^{p-2})^{\frac{p}{p-2}}\mathrm{d}x\right)^{\frac{p-2}{p}}\\ &\quad\times\left(\int_{\mathbb{R}^3}|z_n|^p\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^3}|z_n^+ - z_n^-|^p\mathrm{d}x\right)^{\frac{1}{p}}\\ &\leq (p-1)\left\{\left(\int_{\mathbb{R}^3}(\chi(\varepsilon x)|K(\varepsilon x)|^{\frac{p}{p-2}})^{\frac{3}{3-p}}\mathrm{d}x\right)^{\frac{3-p}{3}}\cdot|\theta_n|_3^p\right\}^{\frac{p-2}{p}}\cdot|z_n|_p\cdot|z_n^+ - z_n^-|_p\\ &= o_n(1).\end{aligned} \tag{2.16}$$

Similarly, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\chi(\varepsilon x)Q(\varepsilon x)\left(|w_n|^{p-2}\left(m_\varepsilon(|w_n|^2)\right)^{\frac{3-p}{2}}w_n - |w|^{p-2}\left(m_\varepsilon(|w|^2)\right)^{\frac{3-p}{2}}w\right)\right.\\ &\quad\left.\cdot\left(z_n^+ - z_n^-\right)\mathrm{d}x\right| = o_n(1),\end{aligned} \tag{2.17}$$

and

$$\Big| \int_{\mathbb{R}^3} \chi(\varepsilon x) Q(\varepsilon x) |w_n|^{p-2} \left(m_\varepsilon(|w_n|^2) \right)^{\frac{3-p}{2}-1} b_\varepsilon(|w_n|^2) w_n \cdot (z_n^+ - z_n^-) \mathrm{d}x \\ - \int_{\mathbb{R}^3} \chi(\varepsilon x) Q(\varepsilon x) |w|^{p-2} \left(m_\varepsilon(|w|^2) \right)^{\frac{3-p}{2}-1} b_\varepsilon(|w|^2) w \cdot (z_n^+ - z_n^-) \mathrm{d}x \Big| = o_n(1). \tag{2.18}$$

Taking (2.16), (2.17) and (2.18) into (2.15), and we can obtain

$$\min\{1-\theta, \frac{1}{8}\} \|z_n\|^2 \le o_n(1).$$

Therefore, $\{w_n\}$ has a convergent subsequence in E, and the proof is completed. □

3 The Solutions of Modified Equation

In this section, we will use an abstract linking theorem [12] to obtain nontrivial critical points for the modified variational functional. Let's write the modified equation as follows

$$-i\alpha \cdot \nabla w + a\beta w + V(\varepsilon x) w - \frac{a}{8} \widetilde{\chi}(\varepsilon x) \widetilde{\xi}(x, |w|) w = f_\varepsilon(x, |w|) w. \tag{3.1}$$

For the convenience, we give the following notations.

$$B_r = \{w \in E : \|w\| \le r\}, \quad S_r = \{w \in E : \|w\| = r\};$$
$$E(e) = \{w \in E : w = se + v, \ s \ge 0 \text{ and } v \in E^-\}.$$

In order to obtain the linking structure of the modified functional, we first give the following lemma.

Lemma 3.1 *([34, Lemma 3.1.]) Assume that (V) holds. Then there exists a constant $C > 0$ which independent of ε, such that for any $w \in E$,*

$$\left| \int_{\mathbb{R}^3} V(\varepsilon x)|w|^2 - \frac{a}{4} \widetilde{\chi}(\varepsilon x) \widehat{\xi}(x, |w|) dx \right| \le \max\left\{ \theta, \frac{7}{8} \right\} \|w\|^2 + C\varepsilon^{2\tau'+5}.$$

Lemma 3.2 *Assume that (V), (K) and (Q) hold, then there exist constants $r_0 > 0$ and $\rho > 0$, such that*

$$\inf_{w \in E^+, \|w\| = r_0} \Phi_\varepsilon(w) \ge \rho, \quad for\ any \ \ \varepsilon \in (0, \varepsilon_0).$$

Proof Taking $w \in E^+$, by Lemma 3.1, there holds

$$
\begin{aligned}
\Phi_\varepsilon(w) &= \frac{1}{2}\|w\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)|w|^2 \mathrm{d}x - P_\varepsilon(w) - \int_{\mathbb{R}^3} F_\varepsilon(x,|w|)\mathrm{d}x \\
&= \frac{1}{2}\|w\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)|w|^2 - \frac{a}{4}\widetilde{\chi}(\varepsilon x)\widehat{\xi}(x,|w|)\mathrm{d}x - \int_{\mathbb{R}^3} F_\varepsilon(x,|w|)\mathrm{d}x \\
&\geq \frac{1}{2}\min\left\{1-\theta, \frac{1}{8}\right\}\|w\|^2 - C\varepsilon^{2\tau'+5} - \int_{\mathbb{R}^3} F_\varepsilon(x,|w|)\mathrm{d}x.
\end{aligned}
$$

By the definition of $F_\varepsilon(x,t)$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3} F_\varepsilon(x,|w|)\mathrm{d}x &\leq \int_{(\Lambda^\delta)_\varepsilon} \frac{1}{p}|K(\varepsilon x)||w|^p + \frac{1}{3}Q(\varepsilon x)|w|^p\left(m_\varepsilon(|w|^2)\right)^{\frac{3-p}{2}}\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3\setminus(\Lambda^\delta)_\varepsilon} G_\varepsilon(x,|w|)\mathrm{d}x \\
&\leq \frac{1}{p}\left(\int_{(\Lambda^\delta)_\varepsilon}|K(\varepsilon x)|^{\frac{3}{3-p}}\mathrm{d}x\right)^{\frac{3-p}{3}} \cdot \left(\int_{(\Lambda^\delta)_\varepsilon}|w|^3\mathrm{d}x\right)^{\frac{p}{3}} \\
&\quad + \int_{(\Lambda^\delta)_\varepsilon}|w|^3\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3\setminus(\Lambda^\delta)_\varepsilon}\phi|w|^2\mathrm{d}x \\
&\leq C\varepsilon^{p-3}\|w\|^p + \|w\|^3 + C\varepsilon^{\tau'+4}\|w\|^2.
\end{aligned}
$$

Therefore, by the above two estimates, we have

$$
\begin{aligned}
\Phi_\varepsilon(w) &\geq \frac{1}{2}\min\left\{1-\theta,\frac{1}{8}\right\}\|w\|^2 - C\varepsilon^{2\tau'+5} - C\varepsilon^{p-3}\|w\|^p - \|w\|^3 - C\varepsilon^{\tau'+4}\|w\|^2 \\
&\geq \frac{1}{4}\min\left\{1-\theta,\frac{1}{8}\right\}\|w\|^2 - C\varepsilon^{2\tau'+5} - C\varepsilon^{p-3}\|w\|^p - \|w\|^3.
\end{aligned}
$$

Let $\|w\| = \varepsilon < 1$, in the light of $p \in (5/2, 3)$, then

$$
\begin{aligned}
\Phi_\varepsilon(w) &\geq \frac{1}{4}\min\left\{1-\theta,\frac{1}{8}\right\}\varepsilon^2 - C\varepsilon^{2\tau'+5} - C\varepsilon^{2p-3} - \varepsilon^3 \\
&\geq \frac{1}{4}\min\left\{1-\theta,\frac{1}{8}\right\}\varepsilon^2 - C'\varepsilon^{2p-3}.
\end{aligned}
$$

We complete the proof of this lemma. □

Lemma 3.3 *Assume that (V), (K) and (Q) hold. Fix $e_0 \in E^+$, then there exist $\varepsilon_0 > 0$ and $R_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, there holds*

$$
\sup_{w\in E(e_0),\|w\|\geq R_0} \Phi_\varepsilon(w) \leq 0.
$$

Moreover, $\sup_{w\in E(e_0)} \Phi_\varepsilon(w) \leq 2R_0^2$.

Proof Taking $w\in E(e_0)$, denote $w=se_0+v$ with $s\geq 0,\ v\in E^-$, we deduce

$$\Phi_\varepsilon(w)=\frac{s^2}{2}\|e_0\|^2-\frac{1}{2}\|v\|^2+\frac{1}{2}\int_{\mathbb{R}^3}V(\varepsilon x)|se_0+v|^2\mathrm{d}x\\ -\int_{\mathbb{R}^3}\frac{a}{8}\widetilde{\chi}(\varepsilon x)\widehat{\xi}(x,|se_0+v|)\mathrm{d}x-\int_{\mathbb{R}^3}F_\varepsilon(x,|se_0+v|)\mathrm{d}x. \tag{3.2}$$

Now we will discuss three cases:

Case 1: If $s=0$ and $v\neq 0$, then by (3.2) and Lemma 3.1, we have

$$\begin{aligned}\Phi_\varepsilon(w)&=-\frac{1}{2}\|v\|^2+\frac{1}{2}\int_{\mathbb{R}^3}\left[V(\varepsilon x)|v|^2\mathrm{d}x-\frac{a}{4}\widetilde{\chi}(\varepsilon x)\widehat{\xi}(x,|v|)\right]\mathrm{d}x-\int_{\mathbb{R}^3}F_\varepsilon(x,|v|)\mathrm{d}x\\ &\leq-\frac{1}{2}\|v\|^2+\frac{1}{2}\left(\max\left\{\theta,\frac{7}{8}\right\}\|v\|^2+C\varepsilon^{2\tau'+5}\right)-\int_{\mathbb{R}^3}F_\varepsilon(x,|v|)\mathrm{d}x\\ &\leq-\frac{1}{2}\min\left\{1-\theta,\frac{1}{8}\right\}\|v\|^2+C\varepsilon^{2\tau'+5}.\end{aligned}$$

It follows that $\Phi_\varepsilon(w)\to-\infty$ as $\|w\|=\|v\|\to\infty$.

Case 2: If $w=se_0\neq 0$, then by (3.2) and Lemma 3.1, there holds

$$\begin{aligned}\Phi_\varepsilon(w)&=\frac{s^2}{2}\|e_0\|^2+\frac{s^2}{2}\int_{\mathbb{R}^3}\left[V(\varepsilon x)|e_0|^2\mathrm{d}x-\frac{a}{4}\widetilde{\chi}(\varepsilon x)\widehat{\xi}(x,|se_0|)\right]\mathrm{d}x-\int_{\mathbb{R}^3}F_\varepsilon(x,|se_0|)\mathrm{d}x\\ &\leq\frac{s^2}{2}\|e_0\|^2+\frac{s^2}{2}\left(\max\left\{\theta,\frac{7}{8}\right\}\|e_0\|^2+C\varepsilon^{2\tau'+5}\right)-\int_{\mathbb{R}^3}F_\varepsilon(x,|se_0|)\mathrm{d}x\\ &\leq\frac{1}{2}\max\left\{1+\theta,\frac{15}{8}\right\}\|se_0\|^2+C\varepsilon^{2\tau'+5}-\frac{1}{p}\int_{\mathbb{R}^3}\chi(\varepsilon x)K(\varepsilon x)|se_0|^p\mathrm{d}x\\ &\leq C_1\|e_0\|^2s^2-C_2\|e_0\|^ps^p+C\varepsilon^{2\tau'+5}.\end{aligned}$$

Therefore, $\Phi_\varepsilon(w)\to-\infty$ as $s\to\infty$. Define $\varrho_1:=\frac{1}{2}\max\{1+\theta,\frac{15}{8}\}$, $\varrho_2:=\frac{1}{2}\min\{1-\theta,\frac{1}{8}\}$.

Case 3: If $se_0\neq 0$ and $v\neq 0$, then (3.2) and Lemma 3.1 leads to

$$\begin{aligned}\Phi_\varepsilon(w)=\Phi_\varepsilon(se_0+v)&\leq\frac{1}{2}\|se_0\|^2-\frac{1}{2}\|v\|+\frac{1}{2}\max\left\{\theta,\frac{7}{8}\right\}\|se_0+v\|^2\\ &\quad-\int_{\mathbb{R}^3}F_\varepsilon(x,|se_0+v|)\mathrm{d}x+C\varepsilon^{2\tau'+5}\\ &\leq\frac{1}{2}\max\left\{1+\theta,\frac{15}{8}\right\}\|se_0\|^2-\frac{1}{2}\min\left\{1-\theta,\frac{1}{8}\right\}\|v\|^2\\ &\quad-\int_{\mathbb{R}^3}F_\varepsilon(x,|se_0+v|)\mathrm{d}x+C\varepsilon^{2\tau'+5}\\ &\leq\varrho_1\|se_0\|^2-\varrho_2\|v\|^2\\ &\quad-\int_{\mathbb{R}^3}\chi(\varepsilon x)\left(\frac{1}{p}K(\varepsilon x)|se_0+v|^p+\frac{1}{3}Q(\varepsilon x)\left(m_\varepsilon(|se_0+v|^2)\right)^{\frac{3-p}{2}}\right)\mathrm{d}x\end{aligned}$$

$$-\int_{\mathbb{R}^3}(1-\chi(\varepsilon x))G_\varepsilon(x,|se_0+v|)\mathrm{d}x+C\varepsilon^{2\tau'+5}$$
$$\le \|w\|^2\left(\varrho_1\|\frac{se_0}{\|w\|}\|^2-\varrho_2\|\frac{v}{\|w\|}\|^2-\frac{\|w\|^{p-2}}{p}\int_{\mathbb{R}^3}\chi(\varepsilon x)K(\varepsilon x)\frac{|w|^p}{\|w\|^p}\mathrm{d}x\right)$$
$$+C\varepsilon^{2\tau'+5}. \tag{3.3}$$

If $\Phi_\varepsilon(w)\to-\infty$ as $\|w\|\to\infty$, we can get the conclusion. Otherwise there exist $M>0$ and a sequence $\{w_n\}\subset E(e_0)$, such that $\Phi_\varepsilon(w_n)>-M$ as $\|w_n\|\to\infty$. Hence, by (3.3) we can get

$$-\frac{M}{\|w_n\|^2}\le\varrho_1\cdot\|\frac{s_ne_0}{\|w_n\|}\|^2-\varrho_2\cdot\|\frac{v_n}{\|w_n\|}\|^2-\frac{\|w_n\|^{p-2}}{p}\int_{\mathbb{R}^3}\chi(\varepsilon x)K(\varepsilon x)$$
$$\times\frac{|w_n|^p}{\|w_n\|^p}\mathrm{d}x+o_n(1). \tag{3.4}$$

Denote $\frac{w_n}{\|w_n\|}=\frac{s_ne_0}{\|w_n\|}+\frac{v_n}{\|w_n\|}$, $\|\frac{w_n}{\|w_n\|}\|=1$, by (3.4) and (K), we know that $\frac{s_ne_0}{\|w_n\|}\to w_0\neq0$ since $p\in(5/2,3)$. Otherwise we can get $1=\|\frac{w_n}{\|w_n\|}\|\to0$. Therefore, by (3.3), we have

$$0\le\frac{\Phi_\varepsilon(w_n)}{\|w_n\|^2}\le\varrho_1\cdot\|\frac{s_ne_0}{\|w_n\|}\|^2+C_1-\frac{\|w_n\|^{p-2}}{p}\int_{\mathbb{R}^3}\chi(\varepsilon x)K(\varepsilon x)\frac{|w_n|^p}{\|w_n\|^p}\mathrm{d}x\to-\infty.$$

This is a contradiction, so we have $\Phi_\varepsilon(w)\to-\infty$ as $\|w\|\to\infty$. Combining the above three cases, we can get $\sup_{w\in E(e_0),\|w\|\ge R_0}\Phi_\varepsilon(w)\le0$. Furthermore, for any $w\in B_{R_0}$, there holds

$$\Phi_\varepsilon(w)\le\frac{1}{2}\|w^+\|^2-\frac{1}{2}\|w\|^2+\frac{3a}{4}\int_{\mathbb{R}^3}|w|^2\mathrm{d}x\le2R_0.$$

Now the proof is complete. □

Let X be a reflexive Banach space, and X can be decompose to $X=X^+\oplus X^-$. Take $\mathcal{S}\subset(X^-)^*$ be a dense subset and $\mathcal{P}$ be the family of semi-norms on X, it consisting of all semi-norm as follow

$$p_s:X=X^+\oplus X^-\to\mathbb{R},\quad p_s(x^++x^-):=|s(x^-)|+\|x^+\|,\ s\in\mathcal{S}.$$

Thus $\mathcal{P}$ induces the product topology on X, it is contained in the product topology $(X^-,\mathcal{T}_w)\times(X^+,\|\cdot\|)$ on X. The associated topology is denote $\mathcal{T}_{\mathcal{P}}$. We denote the weak* topology on X^* by $(X^*,\mathcal{T}_{w^*})$. For more detail about the $\mathcal{T}_{\mathcal{P}}$ topology, one can see [12, Chapter4]. From now on, we take $X=E$ and denote $\Phi_{\varepsilon,c}=\{w\in E:\Phi_\varepsilon\ge c\}$.

Lemma 3.4 *Assume that (V), (K) and (Q) hold , then the functional $\Phi_\varepsilon:E\to\mathbb{R}$ is sequence $\mathcal{P}$-upper semicontinuous and $\Phi'_\varepsilon:(\Phi_{\varepsilon,c},\mathcal{T}_{\mathcal{P}})\to(E^*,\mathcal{T}_{w^*})$ is continuous for every $c\in\mathbb{R}$.*

Proof The argument is similar to [32, Lemma 3.4], so we omit it. □

Combining above lemmas and Lemma 2.1, we have the following theorem.

Theorem 3.5 *([12, Theorem 4.4]) Suppose that assumptions* $(V),(K),(Q)$ *hold. Then for every* $(0,\varepsilon_0)$*, the modified Eq.* (3.1) *has a nontrivial solution* w_ε *which satisfy* $\Phi_\varepsilon(w_n)\in[\rho,\sup_{w\in E(e_0)}\Phi_\varepsilon]$*. Moreover, there holds* $\rho_0\le\|w_\varepsilon\|\le C_{R_0}$*, where* $\rho_0>0$ *and* $C_{R_0}>0$.

4 Profile Decomposition of Solutions and Regularity

By Theorem 3.5, we know that for any $\varepsilon\in(0,\varepsilon_0)$, the modified Eq. (3.1) has a nontrivial solution w_ε. In order to show these solutions are actually solutions of the original problem (1.1), we need following several lemmas. Firstly, since $\rho_0\le\|w_\varepsilon\|\le C_{R_0}$, then we have the following profile decomposition with respect to $\{w_\varepsilon\}$.

Lemma 4.1 *Assume* $\{\varepsilon_n\}\subset\mathbb{R}^+$ *is a sequence of real numbers, and* $\varepsilon_n\to0$ *as* $n\to\infty$*. Then there exist a sequence* $\{\sigma_{j,n}\}\subset\mathbb{R}^+$ *and sequence* $\{x_{i,n}\}\subset\mathbb{R}^3$*,* $\{x_{j,n}\}\subset\mathbb{R}^3$*, such that* $\lim_{n\to\infty}\sigma_{j,n}=\infty$ *and* $\{w_{\varepsilon_n}\}$ *has following properties.*

$$w_{\varepsilon_n}=\sum_{i\in\Lambda_1}W_i(\cdot-x_{i,n})+\sum_{j\in\Lambda_\infty}\sigma_{j,n}W_j(\sigma_{j,n}(\cdot-x_{j,n}))+r_n,$$

where Λ_1 *and* Λ_∞ *are finite index sets. In addition,*

$$\lim_{n\to\infty}|x_{i,n}-x_{i',n}|=\infty\quad for\ i,i'\in\Lambda_1\ and\ i\neq i'.$$

Moreover,

(i) For any $i\in\Lambda_1$*,* $w_{\varepsilon_n}(\cdot+x_{i,n})\rightharpoonup W_i\neq0$ *in* $H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$ *as* $n\to\infty$*, and for any* $j\in\Lambda_\infty$*,* $\sigma_{j,n}^{-1}w_{\varepsilon_n}(\sigma_{j,n}^{-1}\cdot+x_{j,n})\rightharpoonup W_j\neq0$ *in* $\dot H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$ *as* $n\to\infty$*, where* $\dot H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$ *is defined by*

$$\dot H^{1/2}(\mathbb{R}^3,\mathbb{C}^4):=\{w\in L^3(\mathbb{R}^3,\mathbb{C}^4):(-\Delta)^{1/4}w\in L^2(\mathbb{R}^3,\mathbb{C}^4)\}$$

with the inner product $(w,v)=((-\Delta)^{1/4}w,(-\Delta)^{1/4}v)_2$ *and the norm* $\|w\|^2_{\dot H^{1/2}}=(w,w)$ *for any* $w,v\in\dot H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$.

(ii) There holds

$$\sum_{i\in\Lambda_1}\int_{\mathbb{R}^3}|W_i|^3dx+\sum_{j\in\Lambda_\infty}\int_{\mathbb{R}^3}|W_j|^3dx\le\liminf_{n\to\infty}\int_{\mathbb{R}^3}|w_{\varepsilon_n}|^3dx.$$

(iii) $r_n\to0$ *in* $L^3(\mathbb{R}^3,\mathbb{C}^4)$ *as* $n\to\infty$.

(iv) W_j *satisfies the equation*

$$-i\alpha\cdot\nabla W_j=\chi(x_j)E_j(x,|W_j|)W_j,$$

where $E_j(x,t)$ *is defined below* (4.1). $x_j = \lim_{n\to\infty}\varepsilon_n x_{j,n}$, $x_j \in \Lambda^{\delta_0}$. *Moreover, there holds*

$$|W_j(x)| \leq \frac{C}{1+|x|^2} \quad for\ any\ x \in \mathbb{R}^3.$$

(iv) W_i *satisfies the equation*

$$-i\alpha\cdot\nabla W_i + a\beta W_i + V(x_i)W_i - \frac{a}{4}\widetilde{\chi}(x_i)W_i = \widetilde{E}(x_i,|W_i|)W_i,$$

where $\widetilde{E}(x,t)$ *is given by* (4.17), $x_i = \lim_{n\to\infty}\varepsilon_n x_{i,n}$, $x_i \in \Lambda^{\delta}$. *Moreover, there holds*

$$|W_i(x)| \leq C\exp(-c|x|) \quad for\ any\ x \in \mathbb{R}^3,$$

where C *and* c *are positive constants.*

Remark 4.2 For more information about the homogeneous Sobolev space $\dot{H}^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$ and the relationship between $\dot{H}^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$ and $L^p(\mathbb{R}^3,\mathbb{C}^4)$, one can refer to [30]. For details of operator $(-\Delta)^{1/4}$, we refer to [23].

Proof According to [6, Lemma 4.2], it is not difficult to know that (i), (ii) and (iii) are hold. Hence we only need to prove (iv) and (v). We first introduce the following piecewise function, which will be used to construct the equation satisfied by W_j. Denote $\rho_j = \lim_{n\to\infty}\varepsilon_n\sigma_{j,n}^2$. We define

$$E_j(x,t) := \begin{cases} 0, & \rho_j = +\infty; \\ Q(0)t, & \rho_j = 0; \\ \frac{p}{3}Q(0)\rho_j^{-\frac{3-p}{2}}t^{p-2}A^{\frac{3-p}{2}} + \frac{3-p}{3}Q(0)\rho_j^{\frac{p-1}{2}}t^pA^{\frac{3-p}{2}-1}\varphi(\rho_j t^2), & 0 < \rho_j < +\infty, \end{cases} \tag{4.1}$$

where $A = \Psi(\rho_j t^2)$ and $\Psi(t) = \int_0^t \varphi(s)\mathrm{d}s$. By (Q), we know that

$$\sup_{x\in\mathbb{R}^3}\sup_{t>0} t^{-1}E_j(x,t) < +\infty.$$

Let $u_{j,n} = \sigma_{j,n}^{-1}w_{\varepsilon_n}(\sigma_{j,n}^{-1}\cdot + x_{j,n})$. Since w_{ε_n} satisfies Eq. (3.1) with $\varepsilon = \varepsilon_n$, then $u_{j,n}$ satisfies the equation

$$\begin{aligned} &-i\alpha\cdot\nabla u_{j,n} + \sigma_{j,n}^{-1}a\beta u_{j,n} + \sigma_{j,n}^{-1}V\left(\varepsilon_n(\sigma_{j,n}^{-1}\cdot + x_{j,n})\right)u_{j,n} \\ &-\sigma_{j,n}^{-1}\frac{a}{8}\widetilde{\chi}\left(\varepsilon_n(\sigma_{j,n}^{-1}\cdot + x_{j,n})\right)\cdot\widetilde{\xi}\left(\sigma_{j,n}^{-1}\cdot + x_{j,n}, \sigma_{j,n}|u_{j,n}|\right)u_{j,n} \\ &= \sigma_{j,n}^{-1}f_{\varepsilon_n}\left(\sigma_{j,n}^{-1}\cdot + x_{j,n}, \sigma_{j,n}|u_{j,n}|\right)u_{j,n}. \end{aligned} \tag{4.2}$$

Since $\sigma_{j,n} \to \infty$ as $n \to \infty$, hence, for any $\varphi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$,

$$\left|\sigma_{j,n}^{-1} \int_{\mathbb{R}^3} a\beta u_{j,n} \cdot \varphi \mathrm{d}x\right| \le \sigma_{j,n}^{-1} \left(a \int_{\mathbb{R}^3} |u_{j,n}|^2 \mathrm{d}x\right)^{1/2} \cdot \left(\int_{\mathbb{R}^3} |\varphi|^2 \mathrm{d}x\right)^{1/2}$$
$$\le a^{1/2} \sigma_{j,n}^{-1} C_{R_0}^{1/2} \left(\int_{\mathbb{R}^3} |\varphi|^2 \mathrm{d}x\right)^{1/2} \to 0 \text{ as } n \to \infty. \quad (4.3)$$

By the definition of $\widetilde{\xi}$, we know that $\widetilde{\xi}(x,t) \in C^1(\mathbb{R}^3 \times \mathbb{R}, [0,2])$, then

$$\Bigg|\sigma_{j,n}^{-1} \int_{\mathbb{R}^3} \Big[V\left(\varepsilon_n(\sigma_{j,n}^{-1}x + x_{j,n})\right)$$
$$-\frac{a}{8}\widetilde{\chi}\left(\varepsilon_n(\sigma_{j,n}^{-1}x + x_{j,n})\right)\widetilde{\xi}\left(\sigma_{j,n}^{-1}x + x_{j,n}, \sigma_{j,n}|u_{j,n}|\right)\Big] u_{j,n} \cdot \varphi \mathrm{d}x\Bigg|$$
$$\le \sigma_{j,n}^{-1}\left(\frac{3}{2}a \int_{\mathbb{R}^3} |u_{j,n}|^2 \mathrm{d}x\right)^{1/2} \cdot \left(\int_{\mathbb{R}^3} |\varphi|^2 \mathrm{d}x\right)^{1/2}$$
$$\le \left(\frac{3a}{2}\right)^{1/2} \sigma_{j,n}^{-1} C_{R_0}^{1/2} \left(\int_{\mathbb{R}^3} |\varphi|^2 \mathrm{d}x\right)^{1/2} \to 0 \text{as } n \to \infty. \quad (4.4)$$

Similarly,

$$\left|\sigma_{j,n}^{-1} \int_{\mathbb{R}^3} \left(1 - \chi\left(\varepsilon_n\left(\sigma_{j,n}^{-1}x + x_{j,n}\right)\right)\right) g_{\varepsilon_n}\left(\sigma_{j,n}^{-1}x + x_{j,n}, \sigma_{j,n}|u_{j,n}|\right) u_{j,n} \cdot \varphi \mathrm{d}x\right|$$
$$\le \sigma_{j,n}^{-1}\left|\int_{\mathbb{R}^3} \left(1 - \chi\left(\varepsilon_n\left(\sigma_{j,n}^{-1}x + x_{j,n}\right)\right)\right) \frac{1}{1 + |\sigma_{j,n}^{-1}x + x_{j,n}|^{\tau'+4}} u_{j,n} \cdot \varphi \mathrm{d}x\right|$$
$$\le \sigma_{j,n}^{-1}\left|\int_{\mathbb{R}^3} u_{j,n} \cdot \varphi \mathrm{d}x\right| \le \sigma_{j,n}^{-1} C_{R_0}^{1/2}\left(\int_{\mathbb{R}^3} |\varphi|^2 \mathrm{d}x\right)^{1/2} \to 0 \text{as } n \to \infty. \quad (4.5)$$

Now we prove $x_j = \lim_{n\to\infty} \varepsilon_n x_{j,n} \in \Lambda^{\delta_0}$. We assume that $|\varepsilon_n x_{j,n}| \to \infty$ or $\varepsilon_n x_{j,n} \to x_0 \notin \Lambda^{\delta_0}$ as $n \to \infty$, then

$$\left|\sigma_{j,n}^{-1} \int_{\mathbb{R}^3} \chi\left(\varepsilon_n\left(\sigma_{j,n}^{-1}x + x_{j,n}\right)\right) h_{\varepsilon_n}\left(\sigma_{j,n}^{-1}x + x_{j,n}, \sigma_{j,n}|u_{j,n}|\right) u_{j,n} \cdot \varphi \mathrm{d}x\right|$$
$$\le \Bigg|\sigma_{j,n}^{-1} \int_{\mathbb{R}^3} \chi\left(\varepsilon_n\left(\sigma_{j,n}^{-1}x + x_{j,n}\right)\right) K\left(\varepsilon_n\left(\sigma_{j,n}^{-1}x + x_{j,n}\right)\right)$$
$$\times \left(\sigma_{j,n}|u_{j,n}|\right)^{p-2} u_{j,n} \cdot \varphi \mathrm{d}x\Bigg| + o_n(1)$$
$$+ \Bigg|\sigma_{j,n}^{-1} \int_{\mathbb{R}^3} \chi\left(\varepsilon_n\left(\sigma_{j,n}^{-1}x + x_{j,n}\right)\right) Q\left(\varepsilon_n\left(\sigma_{j,n}^{-1}x + x_{j,n}\right)\right)$$
$$\times \left(\sigma_{j,n}|u_{j,n}|\right)^{p-2}\left(m_\varepsilon\left(\left(\sigma_{j,n}|u_{j,n}|\right)^2\right)\right)^{\frac{3-p}{2}} u_{j,n} \cdot \varphi \mathrm{d}x\Bigg|$$

$$\leq \left|\sigma_{j,n}^{-1}\int_{\mathbb{R}^3}\chi\left(\varepsilon_n\left(\sigma_{j,n}^{-1}x+x_{j,n}\right)\right)\right.$$
$$\left.\times K\left(\varepsilon_n\left(\sigma_{j,n}^{-1}x+x_{j,n}\right)\right)\left(\sigma_{j,n}|u_{j,n}|\right)^{p-2}u_{j,n}\cdot\varphi\mathrm{d}x\right|$$
$$+\left|\sigma_{j,n}^{-1}\int_{\mathbb{R}^3}\chi\left(\varepsilon_n\left(\sigma_{j,n}^{-1}x+x_{j,n}\right)\right)\right.$$
$$\left.\times Q\left(\varepsilon_n\left(\sigma_{j,n}^{-1}x+x_{j,n}\right)\right)\left(\sigma_{j,n}|u_{j,n}|\right)u_{j,n}\cdot\varphi\mathrm{d}x\right|+o_n(1)$$
$$\leq \sigma_{j,n}^{p-3}\int_{\mathbb{R}^3}\chi\left(\varepsilon_n\left(\sigma_{j,n}^{-1}x+x_{j,n}\right)\right)K\left(\varepsilon_n\left(\sigma_{j,n}^{-1}x+x_{j,n}\right)\right)|u_{j,n}|^{p-1}\cdot|\varphi|\mathrm{d}x$$
$$+\int_{\mathbb{R}^3}\chi\left(\varepsilon_n\left(\sigma_{j,n}^{-1}x+x_{j,n}\right)\right)Q\left(\varepsilon_n\left(\sigma_{j,n}^{-1}x+x_{j,n}\right)\right)|u_{j,n}|^2\cdot|\varphi|\mathrm{d}x+o_n(1)$$
$$\leq \int_{\mathbb{R}^3}\chi(x_j)\left[K(x_j)\sigma_{j,n}^{p-3}|u_{j,n}|^{p-1}|\varphi|+Q(x_j)|u_{j,n}|^2|\varphi|\right]\mathrm{d}x$$
$$+o_n(1)\to 0 \text{ as } n\to\infty. \tag{4.6}$$

Thus, combining (4.2), (4.3), (4.4), (4.5) and (4.6), we can get

$$\int_{\mathbb{R}^3}-i\alpha\cdot\nabla u_{j,n}\cdot\varphi\mathrm{d}x\to 0 \ \text{ for any } \varphi\in C_0^\infty(\mathbb{R}^3,\mathbb{C}^4).$$

By (i), there holds $u_{j,n}\rightharpoonup W_j$, consequently,

$$-i\alpha\cdot W_j=0.$$

It follows that $W_j=0$, which contradicts (i). Therefore, $x_j=\lim_{n\to\infty}\varepsilon_n x_{j,n}\in\Lambda^{\delta_0}$. By the definition of $h_{\varepsilon_n}(x,t)$ and $E_j(x,t)$, we claim that

$$\sigma_{j,n}^{-1}h_{\varepsilon_n}(x,\sigma_{j,n}t)\to E_j(x,t) \ \text{ for any } x\in\mathbb{R}^3,\ t\in[0,\infty) \text{ as } n\to\infty. \tag{4.7}$$

If $\rho_j:=\lim_{n\to\infty}\varepsilon_n\sigma_{j,n}^2\in(0,\infty)$, then for any $x\in\mathbb{R}^3$ and $t\in[0,\infty)$, there holds

$$\sigma_{j,n}^{-1}h_{\varepsilon_n}(x,\sigma_{j,n}t)$$
$$=\sigma_{j,n}^{-1}\left\{K(\varepsilon_n x)(\sigma_{j,n}t)^{p-2}+\frac{p}{3}Q(\varepsilon_n x)(\sigma_{j,n}t)^{p-2}(m_{\varepsilon_n}(\sigma_{j,n}t)^2)^{\frac{3-p}{2}}\right\}$$
$$+\sigma_{j,n}^{-1}\frac{3-p}{3}Q(\varepsilon_n x)(\sigma_{j,n}t)^p(m_{\varepsilon_n}(\sigma_{j,n}t)^2)^{\frac{3-p}{2}-1}b_{\varepsilon_n}((\sigma_{j,n}t)^2). \tag{4.8}$$

Observed that

$$m_{\varepsilon_n}((\sigma_{j,n}t)^2)=\int_0^{(\sigma_{j,n}t)^2}b_{\varepsilon_n}(s)\mathrm{d}s=\frac{1}{\varepsilon_n}\int_0^{\varepsilon_n\sigma_{j,n}^2t^2}\varphi(s)\mathrm{d}s$$

Hence, we have

$$\begin{aligned}&\sigma_{j,n}^{-1}\left\{K(\varepsilon_n x)(\sigma_{j,n}t)^{p-2}+\frac{p}{3}Q(\varepsilon_n x)(\sigma_{j,n}t)^{p-2}\left(m_{\varepsilon_n}(\sigma_{j,n}t)^2\right)^{\frac{3-p}{2}}\right\}\\&=\sigma_{j,n}^{p-3}t^{p-2}\left\{K(\varepsilon_n x)+\frac{p}{3}Q(\varepsilon_n x)(\varepsilon_n)^{\frac{p-3}{2}}\left(\int_0^{\varepsilon_n\sigma_{j,n}^2t^2}\varphi(s)\mathrm{d}s\right)^{\frac{3-p}{2}}\right\}\\&\to\frac{p}{3}Q(0)\rho_j^{-\frac{3-p}{2}}\left(\Psi(\rho_j t^2)\right)^{\frac{3-p}{2}}\quad\text{as } n\to\infty.\end{aligned}\tag{4.9}$$

and

$$\begin{aligned}&\sigma_{j,n}^{-1}\frac{3-p}{3}Q(\varepsilon_n x)(\sigma_{j,n}t)^p\left(m_{\varepsilon_n}\left(\sigma_{j,n}t\right)^2\right)^{\frac{3-p}{2}-1}b_{\varepsilon_n}\left((\sigma_{j,n}t)^2\right)\\&=\frac{3-p}{3}Q(\varepsilon_n x)(\sigma_{j,n})^{p-1}t^p(\varepsilon_n)^{\frac{1-p}{2}}\left(\int_0^{\varepsilon_n\sigma_{j,n}^2t^2}\varphi(s)\mathrm{d}s\right)^{\frac{3-p}{2}-1}\varphi(\varepsilon_n\sigma_{j,n}^2t^2)\\&\to\frac{3-p}{3}Q(0)\rho_j^{\frac{p-1}{2}}t^p\left(\Psi(\rho_j t^2)\right)^{\frac{3-p}{2}-1}\varphi(\rho_j t^2)\ \text{ as } n\to\infty.\end{aligned}\tag{4.10}$$

Taking (4.9) and (4.10) into (4.8), we obtain that for any $x\in\mathbb{R}^3$, $t\in[0,\infty)$,

$$\lim_{n\to\infty}\sigma_{j,n}^{-1}h_{\varepsilon_n}(x,\sigma_{j,n}t)=E_j(x,t)\quad\text{for } 0<\rho_j<+\infty.$$

Similarly, we can derive that

$$\lim_{n\to\infty}\sigma_{j,n}^{-1}h_{\varepsilon_n}(x,\sigma_{j,n}t)=0\ \text{ with } \rho_j=+\infty.$$

and

$$\lim_{n\to\infty}\sigma_{j,n}^{-1}h_{\varepsilon_n}(x,\sigma_{j,n}t)=Q(0)t\ \text{ with } \rho_j=0.$$

Then the claim is true. From (i), we know that

$$u_{j,n}=\sigma_{j,n}^{-1}w_{\varepsilon_n}(\sigma_{j,n}^{-1}\cdot+x_{j,n})\to W_j(x)\ \text{ a.e. } x\in\mathbb{R}^3 \text{ as } n\to\infty.\tag{4.11}$$

Combining the (4.7) and (4.11), there holds

$$\lim_{n\to\infty}\sigma_{j,n}^{-1}h_{\varepsilon_n}(x,\sigma_{j,n}|u_{j,n}|)u_{j,n}=E_j(x,|W_j|)W_j\ \text{ a.e. } x\in\mathbb{R}^3 \text{ as } n\to\infty.$$

Then from Lebesgue dominated convergence theorem, it follows that

$$\sigma_{j,n}^{-1}\int_{\mathbb{R}^3}\chi(\varepsilon_n(\sigma_{j,n}^{-1}x+x_{j,n})h_{\varepsilon_n}(\sigma_{j,n}^{-1}x+x_{j,n},\sigma_{j,n}|u_{j,n}|)u_{j,n}\cdot\varphi\mathrm{d}x$$
$$\to\int_{\mathbb{R}^3}\chi(x_j)E_j(x,|W_j|)W_j\cdot\varphi\mathrm{d}x \text{ as } n\to\infty. \tag{4.12}$$

By (4.2), (4.3), (4.4), (4.5) and (4.12), we have

$$-i\alpha\cdot\nabla W_j=\chi(x_j)E_j(x,|W_j|)W_j.$$

Thus, from [3, Theorem 1.1], we can obtain

$$|W_j(x)|\le\frac{C}{1+|x|^2}\ \text{ for any } x\in\mathbb{R}^3.$$

We finish the proof of (iv).

To prove (v). Since w_{ε_n} satisfies Eq. (3.1) with $\varepsilon=\varepsilon_n$, i.e.,

$$-i\alpha\cdot\nabla w_{\varepsilon_n}+a\beta w_{\varepsilon_n}+V(\varepsilon_nx)w_{\varepsilon_n}-\frac{a}{8}\widetilde{\chi}(\varepsilon_nx)\widetilde{\xi}(x,|w_{\varepsilon_n}|)w_{\varepsilon_n}=f_{\varepsilon_n}(x,|w_{\varepsilon_n}|)w_{\varepsilon_n}.$$

From (i), we know that $w_{\varepsilon_n}(\cdot+x_{i,n})\rightharpoonup W_i\neq 0$ in $H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$ as $n\to\infty$. Denote $u_{i,n}:=w_{\varepsilon_n}(\cdot+x_{i,n})$, then

$$-i\alpha\cdot\nabla u_{i,n}+a\beta u_{i,n}+V(\varepsilon_n(x+x_{i,n}))u_{i,n}-\frac{a}{8}\widetilde{\chi}(\varepsilon_n(x+x_{i,n}))\widetilde{\xi}(x+x_{i,n},|u_{i,n}|)u_{i,n}$$
$$=f_{\varepsilon_n}((x+x_{i,n}),|u_{i,n}|)u_{i,n}. \tag{4.13}$$

If $x_i:=\lim_{n\to\infty}\varepsilon_nx_{i,n}\in\Lambda^{\delta_0}$, then for any $\varphi\in C_0^\infty(\mathbb{R}^3,\mathbb{C}^4)$, we have

$$\int_{\mathbb{R}^3}-i\alpha\cdot\nabla u_{i,n}\cdot\varphi\mathrm{d}x\to\int_{\mathbb{R}^3}-i\alpha\cdot\nabla W_i\cdot\varphi\mathrm{d}x,$$
$$\int_{\mathbb{R}^3}a\beta u_{i,n}\cdot\varphi\mathrm{d}x\to\int_{\mathbb{R}^3}a\beta W_i\cdot\varphi\mathrm{d}x,$$
$$\int_{\mathbb{R}^3}V(\varepsilon_n(x+x_{i,n}))u_{i,n}\cdot\varphi\mathrm{d}x\to\int_{\mathbb{R}^3}V(x_i)W_i\cdot\varphi\mathrm{d}x,$$
$$\int_{\mathbb{R}^3}\frac{a}{8}\widetilde{\chi}(\varepsilon_n(x+x_{i,n}))\widetilde{\xi}(x+x_{i,n},|u_{i,n}|)u_{i,n}\cdot\varphi\mathrm{d}x$$
$$\to\int_{\mathbb{R}^3}\frac{a}{8}\widetilde{\chi}(x_i)2\cdot W_i\cdot\varphi\mathrm{d}x=\frac{a}{4}\int_{\mathbb{R}^3}\widetilde{\chi}(x_i)W_i\cdot\varphi\mathrm{d}x. \tag{4.14}$$

In additional, there holds

$$\int_{\mathbb{R}^3}f_{\varepsilon_n}\left((x+x_{i,n}),|u_{i,n}|\right)u_{i,n}\cdot\varphi\mathrm{d}x$$
$$=\int_{\mathbb{R}^3}\chi\left(\varepsilon_n(x+x_{i,n})h_{\varepsilon_n}((x+x_{i,n}),|u_{i,n}|\right)u_{i,n}\cdot\varphi\mathrm{d}x$$

$$+\int_{\mathbb{R}^3}\left(1-\chi(\varepsilon_n(x+x_{i,n})\right)g_{\varepsilon_n}((x+x_{i,n}),|u_{i,n}|)u_{i,n}\cdot\varphi\mathrm{d}x. \tag{4.15}$$

Since

$$\begin{aligned}&h_{\varepsilon_n}((x+x_{i,n}),|u_{i,n}|)\\&=K(\varepsilon_n(x+x_{i,n}))|u_{i,n}|^{p-2}+\frac{p}{3}Q(\varepsilon_n(x+x_{i,n}))|u_{i,n}|^{p-2}\left(m_{\varepsilon_n}(|u_{i,n}|^2)\right)^{\frac{3-p}{2}}\\&+\frac{3-p}{3}Q(\varepsilon_n(x+x_{i,n}))|u_{i,n}|^{p}\left(m_{\varepsilon_n}(|u_{i,n}|^2)\right)^{\frac{3-p}{2}-1}b_{\varepsilon_n}(|u_{i,n}|^2)\\&\to K(x_i)|W_i|^{p-2}+Q(x_i)|W_i|^2\ \text{ a.e. } x\in\mathbb{R}^3\ \text{ as } n\to\infty,\end{aligned}$$

and

$$\begin{aligned}&g_{\varepsilon_n}((x+x_{i,n}),|u_{i,n}|)=\min\left\{h_{\varepsilon_n}((x+x_{i,n}),|u_{i,n}|),\phi(x+x_{i,n})\right\}\\&\to\min\left\{K(x_i)|W_i|^{p-2}+Q(x_i)|W_i|^2,0\right\}\ \text{ a.e. } x\in\mathbb{R}^3\ \text{ as } n\to\infty,\end{aligned}$$

Consequently, by (4.15), there holds

$$\begin{aligned}&\int_{\mathbb{R}^3}f_{\varepsilon_n}\left((x+x_{i,n}),|u_{i,n}|\right)u_{i,n}\cdot\varphi\mathrm{d}x\\&\to\int_{\mathbb{R}^3}\chi(x_i)\left(K(x_i)|W_i|^{p-2}+Q(x_i)|W_i|^2\right)W_i\cdot\varphi\mathrm{d}x\\&+\int_{\mathbb{R}^3}(1-\chi(x_i))\min\left\{K(x_i)|W_i|^{p-2}+Q(x_i)|W_i|^2,0\right\}W_i\cdot\varphi\mathrm{d}x \text{ as } n\to\infty.\end{aligned}$$

By (K) and (Q), it follows that

$$\begin{aligned}&\int_{\mathbb{R}^3}f_{\varepsilon_n}\left((x+x_{i,n}),|u_{i,n}|\right)u_{i,n}\cdot\varphi\mathrm{d}x\\&\to\int_{\mathbb{R}^3}\chi(x_i)\left(K(x_i)|W_i|^{p-2}+Q(x_i)|W_i|^2\right)W_i\cdot\varphi\mathrm{d}x \text{ as } n\to\infty.\end{aligned} \tag{4.16}$$

We define

$$\widetilde{E}(x,t)=\chi(x)K(x)|t|^{p-2}+\chi(x)Q(x)|t|^2. \tag{4.17}$$

Combining (4.13), (4.14), (4.15), (4.16) and (4.17), there holds

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(-i\alpha\cdot\nabla u_{i,n}+a\beta u_{i,n}+V(\varepsilon_n(x+x_{i,n}))u_{i,n}\right)\cdot\varphi\mathrm{d}x\\&-\int_{\mathbb{R}^3}\left(\frac{a}{8}\widetilde{\chi}\left(\varepsilon_n(x+x_{i,n})\right)\widetilde{\xi}(x+x_{i,n},|u_{i,n}|)+f_{\varepsilon_n}\left((x+x_{i,n}),|u_{i,n}|\right)u_{i,n}\right)\cdot\varphi\mathrm{d}x\end{aligned}$$

$$\to \int_{\mathbb{R}^3} \left[-i\alpha \cdot \nabla W_i + a\beta W_i + V(x_i)W_i - \frac{a}{4}\widetilde{\chi}(x_i)W_i - \widetilde{E}(x_i, |W_i|)W_i \right] \cdot \varphi \mathrm{d}x.$$

Then, we have

$$-i\alpha \cdot \nabla W_i + a\beta W_i + V(x_i)W_i - \frac{a}{4}\widetilde{\chi}(x_i)W_i = \widetilde{E}(x_i, |W_i|)W_i. \tag{4.18}$$

Now we will show that $x_i := \lim_{n\to\infty} \varepsilon_n x_{i,n} \in \Lambda^{\delta_0}$. We assume that $x_i \notin \Lambda^{\delta_0}$, by the definition of f_{ε_n} and $\widetilde{\xi}$, then W_i satisfies the equation

$$-i\alpha \cdot \nabla W_i + a\beta W_i + V(x_i)W_i - \frac{a}{4}\widetilde{\chi}(x_i)W_i = 0.$$

Take the scalar product with $\left(W_i^+ - W_i^-\right)$ and integrate in $\mathbb{R}^3$, we have

$$\begin{aligned} 0 &= \|W_i\|^2 + \mathrm{Re}\int_{\mathbb{R}^3} V(x_i)W_i \cdot \left(W_i^+ - W_i^-\right) \mathrm{d}x \\ &\quad - \mathrm{Re}\frac{a}{4}\int_{\mathbb{R}^3} \widetilde{\chi}(x_i)W_i \cdot \left(W_i^+ - W_i^-\right) \mathrm{d}x \\ &\geq a\|W_i\|_2^2 - \frac{3a}{4}\|W_i\|_2^2 = \frac{a}{4}\|W_i\|_2^2. \end{aligned}$$

Therefore, we obtain $W_i = 0$, which contradicts (i). Consequently, $x_i \in \Lambda^{\delta_0}$. Since W_i satisfies (4.18), then according to [31, Lemma 4.6], there holds

$$|W_i(x)| \leq C\exp(-c|x|) \text{ for any } x \in \mathbb{R}^3.$$

The proof is now completed. □

Lemma 4.3 *Assume that* (P), (Q) *and* (K) *hold,* $5/2 < p < 3$, *then the index set* $\Lambda_\infty = \emptyset$.

Proof The proof of this lemma is similar to the one of [6, Lemma 4.21] with the help of Lemma 4.1 in this paper, therefore, we omit its proof. □

Now we give the L^∞ estimate for the solutions which solves modified Eq. (3.1).

Lemma 4.4 *Assume that* (P), (Q) *and* (K) *hold,* $5/2 < p < 3$, *let* $\{w_\epsilon\}$ *be a family of critical points of* (3.1) *which obtained in Theorem 3.5. Then there exist* $M > 0$ *and* $\varepsilon_0 > 0$, *such that for any* $0 < \varepsilon < \varepsilon_0$,

$$\sup_{x\in\mathbb{R}^3} |w_\varepsilon(x)| \leq M.$$

Before prove the Lemma 4.4, we need the following two lemmas.

Lemma 4.5 *[25, Lemma 4.2] For any $p \in (1, \infty)$, there exists a constant $C > 0$ such that*

$$\|\nabla\psi\|_{L^p(\mathbb{R}^3)} \le C\|i\alpha\cdot\nabla\psi\|_{L^p(\mathbb{R}^3)} \ \text{for any } \psi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4).$$

Lemma 4.6 *Let ζ be a cut-off function such that $\zeta(x) = 1$ for $x \in B_{R/2}(0)$, $\zeta(x) = 0$ for $x \notin B_R(0)$ and $|\nabla\zeta(x)| \le R/4$, w_{ε_n} is solution of (3.1), then there holds*

$$\|\zeta w_{\varepsilon_n}\|_{W^{1,p}(\mathbb{R}^3)} \le C_{p,R}\|w_{\varepsilon_n}\|_{L^p(B_R(0))} + C_p\|\zeta|w_{\varepsilon_n}|^2\|_{L^p(B_R(0))}.$$

Proof Since $\{w_{\varepsilon_n}\}$ solves Eq. (3.1), i.e.,

$$-i\alpha\cdot\nabla w_{\varepsilon_n} + a\beta w_{\varepsilon_n} + V(\varepsilon_n x)w_{\varepsilon_n} - \frac{a}{8}\widetilde{\chi}(\varepsilon_n x)\widetilde{\xi}(x, |w_{\varepsilon_n}|)w_{\varepsilon_n} = f_{\varepsilon_n}(x, |w_{\varepsilon_n}|)w_{\varepsilon_n}.$$

By multiplying w_{ε_n} with ζ and substituting the product into above formula, there holds

$$\begin{aligned} &a\beta(\zeta w_{\varepsilon_n}) + V(\varepsilon_n x)(\zeta w_{\varepsilon_n}) - \frac{a}{8}\widetilde{\chi}(\varepsilon_n x)\widetilde{\xi}(x, |\zeta w_{\varepsilon_n}|)(\zeta w_{\varepsilon_n}) - f_{\varepsilon_n}(x, |\zeta w_{\varepsilon_n}|)(\zeta w_{\varepsilon_n}) \\ &= i\alpha\cdot\nabla(\zeta w_{\varepsilon_n}) = \zeta(i\alpha\cdot\nabla w_{\varepsilon_n}) + i\sum_{k=1}^{3}(\partial_k\zeta)\alpha_k\cdot w_{\varepsilon_n}. \end{aligned} \tag{4.19}$$

It is clear that

$$\int_{\mathbb{R}^3}|a\beta(\zeta w_{\varepsilon_n})|^p\mathrm{d}x \le a^p\int_{B_R(0)}|w_{\varepsilon_n}|^p\mathrm{d}x.$$

By the definition of $\widetilde{\chi}$ and $\widetilde{\xi}$, we know that

$$\begin{aligned} &\int_{\mathbb{R}^3}\left|V(\varepsilon_n x)(\zeta w_{\varepsilon_n}) - \frac{a}{8}\widetilde{\chi}(\varepsilon_n x)\widetilde{\xi}(x, |\zeta w_{\varepsilon_n}|)(\zeta w_{\varepsilon_n})\right|^p \mathrm{d}x \\ &\le \int_{B_R(0)}\left|V(\varepsilon_n x)w_{\varepsilon_n}\right|^p \mathrm{d}x + \int_{B_R(0)}\left|\frac{a}{8}\widetilde{\chi}(\varepsilon_n x)\widetilde{\xi}(x, |w_{\varepsilon_n}|)w_{\varepsilon_n}\right|^p \mathrm{d}x \\ &\le a^p\int_{B_R(0)}|w_{\varepsilon_n}|^p\mathrm{d}x + \left(\frac{a}{4}\right)^p\int_{B_R(0)}|w_{\varepsilon_n}|^p\mathrm{d}x \\ &= \left(1 + \frac{1}{4^p}\right)a^p\int_{B_R(0)}|w_{\varepsilon_n}|^p\mathrm{d}x. \end{aligned}$$

Combining this and (4.19), there holds

$$\|i\alpha\cdot\nabla(\zeta w_{\varepsilon_n})\|_{L^p(\mathbb{R}^3)} \le C\|w_{\varepsilon_n}\|_{L^p(B_R(0))} + \|f_{\varepsilon_n}(x, |\zeta w_{\varepsilon_n}|)|\zeta w_{\varepsilon_n}|\|_{L^p(\mathbb{R}^3)}. \tag{4.20}$$

By the definition of f_{ε_n}, we have

$$\|f_{\varepsilon_n}(x, |\zeta w_{\varepsilon_n}|)|\zeta w_{\varepsilon_n}|\|^p_{L^p(\mathbb{R}^3)} = \int_{\mathbb{R}^3} f_{\varepsilon_n}(x, |\zeta w_{\varepsilon_n}|)|\zeta w_{\varepsilon_n}|^p\mathrm{d}x$$

$$
\begin{aligned}
&\leq \int_{B_R(0)} f_{\varepsilon_n}(x, |\zeta w_{\varepsilon_n}|)|w_{\varepsilon_n}|^p \mathrm{d}x \leq C \int_{B_R(0)} |(K(\varepsilon_n x) + Q(\varepsilon_n x)|\zeta w_{\varepsilon_n}|)|\zeta w_{\varepsilon_n}||^p \mathrm{d}x \\
&\leq C_1 \int_{B_R(0)} |K(\varepsilon_n x) w_{\varepsilon_n}|^p \mathrm{d}x + C_2 \int_{B_R(0)} |Q(\varepsilon_n x)\zeta |w_{\varepsilon_n}|^2|^p \mathrm{d}x \\
&\leq C_3 \int_{B_R(0)} |w_{\varepsilon_n}|^p + |\zeta |w_{\varepsilon_n}|^2|^p \mathrm{d}x. \qquad (4.21)
\end{aligned}
$$

Using Lemma 4.5, there holds

$$
\begin{aligned}
\|\zeta w_{\varepsilon_n}\|_{W^{1,p}(\mathbb{R}^3)} &= \|\zeta w_{\varepsilon_n}\|_{L^p(\mathbb{R}^3)} + \|\nabla(\zeta w_{\varepsilon_n})\|_{L^p(\mathbb{R}^3)} \\
&\leq \|w_{\varepsilon_n}\|_{L^p(B_R(0))} + C_p\|\alpha \cdot \nabla(\zeta w_{\varepsilon_n})\|_{L^p(\mathbb{R}^3)}. \qquad (4.22)
\end{aligned}
$$

Combining (4.20), (4.21) and (4.22), there holds

$$
\|\zeta w_{\varepsilon_n}\|_{W^{1,p}(\mathbb{R}^3)} \leq C_{p,R}\|w_{\varepsilon_n}\|_{L^p(B_R(0))} + C_p\|\zeta |w_{\varepsilon_n}|^2\|_{L^p(B_R(0))}.
$$

The proof of Lemma 4.6 is now complete. □

Proof of Lemma 4.4 We assume that there exist a sequence of $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and a sequence of critical points $\{w_{\varepsilon_n}\} \subset E$ of (3.1) such that

$$
\sup_{x \in \mathbb{R}^3} |w_{\varepsilon_n}(x)| \to \infty \quad \text{as } n \to \infty.
$$

By Lemma 4.3 and (i) of Lemma 4.1, we have

$$
w_{\varepsilon_n} = \sum_{i \in \Lambda_1} W_i(\cdot - x_{i,n}) + r_n.
$$

Moreover, by (iii) of Lemma 4.1, there holds

$$
r_n \to 0 \;\text{ in } L^3(\mathbb{R}^3, \mathbb{R}) \text{ as } n \to \infty.
$$

Since $\{w_{\varepsilon_n}\}$ solves Eq. (3.1), i.e.,

$$
-i\alpha \cdot \nabla w_{\varepsilon_n} + a\beta w_{\varepsilon_n} + V(\varepsilon_n x) w_{\varepsilon_n} - \frac{a}{8}\widetilde{\chi}(\varepsilon_n x)\widetilde{\xi}(x, |w_{\varepsilon_n}|) w_{\varepsilon_n} = f_{\varepsilon_n}(x, |w_{\varepsilon_n}|) w_{\varepsilon_n}. \qquad (4.23)
$$

By (v) of Lemma 4.1, we know that $|W_i| \in L^\infty(\mathbb{R}^3, \mathbb{R})$ for any $i \in \Lambda_1$. Using this and (4.20), we can deduce there exist $N_\gamma > 0$ and $\varrho > 0$, such that

$$
\sup_{y \in \mathbb{R}^3} \int_{B_\varrho(y)} |w_{\varepsilon_n}|^3 \mathrm{d}x \leq \gamma \;\text{ for any } n > N_\gamma. \qquad (4.24)
$$

Define $\eta \in C_0^\infty(\mathbb{R}^3, [0,1])$ such that $\eta(x) = 1$ for $x \in B_{\varrho/2}(y)$, $\eta(x) = 0$ for $x \notin B_\varrho(y)$ and $|\nabla\eta(x)| \le 4/\varrho$ for $x \in \mathbb{R}^3$. By multiplying w_{ε_n} with η and substituting the product into (4.23), there holds

$$a\beta(\eta w_{\varepsilon_n}) + V(\varepsilon_n x)(\eta w_{\varepsilon_n}) - \frac{a}{8}\widetilde{\chi}(\varepsilon_n x)\widetilde{\xi}(x, |\eta w_{\varepsilon_n}|)(\eta w_{\varepsilon_n}) - f_{\varepsilon_n}(x, |\eta w_{\varepsilon_n}|)(\eta w_{\varepsilon_n})$$
$$= i\alpha \cdot \nabla w_{\varepsilon_n} = \eta(i\alpha \cdot \nabla w_{\varepsilon_n}) + i\sum_{k=1}^{3}(\partial_k \eta)\alpha_k \cdot w_{\varepsilon_n}.$$

By Lemma 4.6, we have

$$\|\eta w_{\varepsilon_n}\|_{W^{1,p}(\mathbb{R}^3)} \le C_{p,\varrho}\|w_{\varepsilon_n}\|_{L^p(B_\varrho(y))} + C_p\|\eta|w_{\varepsilon_n}|^2\|_{L^p(B_\varrho(y))}.$$

Then Hölder inequality and (4.24) implies

$$\|\eta|w_{\varepsilon_n}|^2\|_{L^p(B_\varrho(y))} \le \|w_{\varepsilon_n}\|_{L^3(B_\varrho(y))} \cdot \|\eta|w_{\varepsilon_n}|\|_{L^{p^*}(B_\varrho(y))} \le \gamma^{1/3}\|\eta|w_{\varepsilon_n}|\|_{L^{p^*}(B_\varrho(y))},$$

where $p^* = \frac{3p}{3-p}$. Hence, when $\gamma > 0$ small enough,

$$\begin{aligned}\|w_{\varepsilon_n}\|_{L^{p^*}(B_{\varrho/2}(y))} &\le \|\eta|w_{\varepsilon_n}|\|_{L^{p^*}(B_\varrho(y))} \le \frac{1}{S_p}\|\eta w_{\varepsilon_n}\|_{W^{1,p}(\mathbb{R}^3)} \\ &\le \frac{1}{S_p}\left[C_{p,\varrho}\|w_{\varepsilon_n}\|_{L^p(B_\varrho(y))} + C_p\gamma^{1/3}\|\eta w_{\varepsilon_n}\|_{L^{p^*}(B_\varrho(y))}\right],\end{aligned}$$

where S_p is Sobolev constant, which deduce that

$$\|w_{\varepsilon_n}\|^3_{L^{p^*}(B_{\varrho/2}(y))} \le \frac{C}{S_p}\|w_{\varepsilon_n}\|_{L^p(B_\varrho(y))} \le C'\|w_{\varepsilon_n}\|^3_{L^3(B_\varrho(y))} \le C'\gamma. \tag{4.25}$$

Since $p \in \left(\frac{5}{2}, 3\right)$, it follows that $p^* = \frac{3p}{3-p} \in (15, +\infty)$. Therefore, by (4.25), there holds

$$\|w_{\varepsilon_n}\|^3_{L^{15}(B_{\varrho/2}(y))} \le C'\gamma. \tag{4.26}$$

Denote $\widetilde{\eta}(x) = \eta(2x)$, then using Lemma 4.6 again, we can get

$$\|\widetilde{\eta} w_{\varepsilon_n}\|_{W^{1,p}(\mathbb{R}^3)} \le C\|w_{\varepsilon_n}\|_{L^p(B_{\varrho/2}(y))} + C_p\|\widetilde{\eta}|w_{\varepsilon_n}|^2\|_{L^p(B_{\varrho/2}(y))}. \tag{4.27}$$

If we take $3 < p' < 15/2$, then by (4.26), (4.27) and Hölder inequality, there holds

$$\begin{aligned}\|\widetilde{\eta} w_{\varepsilon_n}\|_{W^{1,p'}(\mathbb{R}^3)} &\le C\|w_{\varepsilon_n}\|_{L^{15}(B_{\varrho/2}(y))} + C_{p'}\|w_{\varepsilon_n}\|^2_{L^{15}(B_{\varrho/2}(y))} \\ &\le C_{\varrho,\gamma}.\end{aligned}$$

By Sobolev embedding theorem, $W^{1,p'}(\mathbb{R}^3) \hookrightarrow C^0(\mathbb{R}^3)$ is continuous. Therefore, we have $\|\widetilde{\eta} w_{\varepsilon_n}\|_{L^\infty(\mathbb{R}^3)} \le C_{\varrho,\gamma}$, i.e., $\|w_{\varepsilon_n}\|_{L^\infty(B_{\varrho/4}(y))} \le C_{\varrho,\gamma}$. By the arbitrariness of y, there holds

$$\|w_{\varepsilon_n}\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)} \le C_{\varrho,\gamma}.$$

This contradicts $\sup_{x\in\mathbb{R}^3} |w_{\varepsilon_n}(x)| \to \infty$ as $n \to \infty$. Consequently, there exists a constant $M > 0$ such that $\sup_{x\in\mathbb{R}^3} |w_{\varepsilon_n}(x)| \le M$. The proof is completed. □

5 Proof of Theorem 1.1

Proof of Theorem 1.1 By Lemma 4.4, we know that there exists a $\varepsilon_0 > 0$, such that for any $0 < \varepsilon < \varepsilon_0$, $|w_\varepsilon(x)| \le M$. Recalling the definition of the $b_\varepsilon(t)$ and $m_\varepsilon(t)$, it is clear that

$$m_\varepsilon(|w_\varepsilon|^2) = \int_0^{|w_\varepsilon|^2} b_\varepsilon(s)\mathrm{d}s = |w_\varepsilon|^2, \quad b_\varepsilon(|w_\varepsilon|^2) = \varphi(\varepsilon|w_\varepsilon|^2) = 1,$$

then we deduce that

$$\begin{aligned} h_\varepsilon(x,|w_\varepsilon|) &= K(\varepsilon x)|w_\varepsilon|^{p-2} + \frac{p}{3}Q(\varepsilon x)|w_\varepsilon|^{p-2}\left(|w_\varepsilon|^2\right)^{\frac{3-p}{2}} \\ &\quad + \frac{3-p}{3}Q(\varepsilon x)|w_\varepsilon|^p(|w_\varepsilon|^2)^{\frac{3-p}{2}-1} \\ &= K(\varepsilon x)|w_\varepsilon|^{p-2} + Q(\varepsilon x)|w_\varepsilon|. \end{aligned}$$

Using this and the definition of f_ε, we obtain

$$f_\varepsilon(x,|w_\varepsilon|) = \chi(\varepsilon x)(K(\varepsilon x)|w_\varepsilon|^{p-2} + Q(\varepsilon x)|w_\varepsilon|) + (1-\chi(\varepsilon x))g_\varepsilon(x,|w_\varepsilon|).$$

By Lemma 4.3 and Lemma 4.1 (i), we know that for any sequence of solutions $\{w_{\varepsilon_n}\}$ will not concentrate at a single point, then we can treat the situation as the subcritical equations like [32]. By the similar argument as [32, Lemma 4.6, Proposition 5.2], we can get

$$|w_\varepsilon| \le C_1 \exp\left(-C_2\left(\frac{\mathrm{dist}(x,\mathcal{O}(\delta))}{\varepsilon}\right)^{\frac{2-\tau}{2}}\right), \tag{5.1}$$

where C_1, C_2 are positive constants. Then, by choose κ large enough, we have

$$\begin{aligned} g_\varepsilon(x,|w_\varepsilon|) &= \min\left\{K(\varepsilon x)|w_\varepsilon|^{p-2} + Q(\varepsilon x)|w_\varepsilon|, \frac{\kappa}{1+|x|^{\tau'+4}}\right\} \\ &= K(\varepsilon x)|w_\varepsilon|^{p-2} + Q(\varepsilon x)|w_\varepsilon|. \end{aligned}$$

Therefore, $f_\varepsilon(x, |w_\varepsilon|) = K(\varepsilon x)|w_\varepsilon|^{p-2} + Q(\varepsilon x)|w_\varepsilon|$. Then (3.1) can be rewritten as follows

$$- i\alpha \cdot \nabla w_\varepsilon + a\beta w_\varepsilon + V(\varepsilon x)w_\varepsilon - \frac{a}{8}\widetilde{\chi}(\varepsilon x)\widetilde{\xi}(x, |w_\varepsilon|)w_\varepsilon$$
$$= K(\varepsilon x)|w_\varepsilon|^{p-2}w_\varepsilon + Q(\varepsilon x)|w_\varepsilon|w_\varepsilon.$$

By the definition of $\widetilde{\chi}$, $\widetilde{\xi}$ and (5.1), it is not difficult to know that

$$\widetilde{\chi}(\varepsilon x)\widetilde{\xi}(x, |w_\varepsilon|) = 0.$$

Then

$$-i\alpha \cdot \nabla w_\varepsilon + a\beta w_\varepsilon + V(\varepsilon x)w_\varepsilon = K(\varepsilon x)|w_\varepsilon|^{p-2}w_\varepsilon + Q(\varepsilon x)|w_\varepsilon|w_\varepsilon.$$

This means that we can obtain the desire result and the proof of Theorem 1.1 is completed. □

Acknowledgements The authors sincerely express their gratitude to the anonymous referees and the editor for their very valuable suggestions and comments which greatly improved the manuscript. The third author is supported by National Natural Science Foundation of China (No. 11471147).

Declarations

Conflict of interest Authors declare that they have no conflict of interest.

References

1. Bartsch, T., Ding, Y.: Deformation theorems on non-metrizable vector spaces and applications to critical point theory. Math. Nachr. **279**, 1267–1288 (2006)
2. Bartsch, T., Ding, Y.: Solutions of nonlinear Dirac equations. J. Differ. Equ. **226**, 210–249 (2006)
3. Borrelli, W., Frank, R.L.: Sharp decay estimates for critical Dirac equations. Trans. Am. Math. Soc. **373**, 45–70 (2020)
4. Byeon, J., Wang, Z.: Standing waves with a critical frequency for nonlinear Schrödinger equations. Calc. Var. Partial Differ. Equ. **II**(18), 207–219 (2003)
5. Chen, Y., Ding, Y., Xu, T.: Potential well and multiplicity of solutions for nonlinear Dirac equations, Commun. Pure. Appl. Anal. **19**, 587–607 (2020)
6. Chen, S., Gou, T.: Infinitely many localized semiclassical states for critical nonlinear Dirac equations. Nonlinearity **34**, 6358–6397 (2021)
7. Chen, S., Gou, T.: Higher topological type semiclassical states for Sobolev critical Dirac equationas with degenerate potential. J. Geom. Anal. **32**, 231 (2022)
8. Chen, S., Liu, J., Wang, Z.: Localized nodal solutions for a critical nonlinear Schrödinger equation. J. Funct. Anal. **277**, 594–640 (2019)
9. Chen, S., Wang, Z.: Localied nodal solutions of higher topological type for semiclassical nonlinear Schrödinger equation. Calc. Var. Partial Differ. Equ. **56**, 26 (2017)
10. del Pino, M., Felmer, P.: Semi-classical states for nonlinear Schröinger equations. J. Funct. Anal. **149**, 245–265 (1997)
11. del Pino, M., Felmer, P.: Multi-peak bound states for nonlinear Schröinger equations. Ann. Inst. H. Poincaré Anal. Non Lináire **15**, 127–149 (1998)

12. Ding, Y.: Variational methods for strongly indefinite problems. World Scientific Publishing, Singapore (2007)
13. Ding, Y.: Semi-classical ground states concentrating on the nonlinear potential for a Dirac equation. J. Differ. Equ. **249**, 1015–1034 (2010)
14. Ding, Y., Guo, Q., Yu, Y.: Existence of semiclassical solutions for some critical Dirac equation. J. Math. Phys. **62**, 22 (2021)
15. Ding, Y., Liu, X.: Semiclassical limits of ground states of a nonlinear Dirac equation. J. Differ. Equ. **252**, 4962–4987 (2012)
16. Ding, Y., Ruf, B.: Solutions of a nonlinear Dirac equation with external fields. Arch. Ration. Mech. Anal. **190**, 57–82 (2008)
17. Ding, Y., Ruf, B.: Existence and concentration of semiclassical solutions for Dirac equations with critical nonlinearities. SIAM J. Math. Anal. **44**, 3755–3785 (2012)
18. Ding, Y., Xu, T.: Localized concentration of semiclassical states for nonlinear Dirac equation. Arch. Ration. Mech. Anal. **216**, 415–447 (2015)
19. Esteban, M.J., Séré, E.: Stationary states of the nonlinear Dirac equation: a variational approach. Comm. Math. Phys. **171**, 323–350 (1995)
20. Esteban, M.J., Lewin, M., Séré, E.: Variational methods in relativistic quantum mechanics. Bull. Amer. Math. Soc. **45**, 535–593 (2008)
21. Feynman, R.P.: Quantum electrodynamics. Benjamin, New York (1961)
22. Finkelstein, R., Fronsdal, C., Kaus, P.: Nonlinear spinor field. Phys. Rev. **103**, 1571–1579 (1956)
23. Halmos, P.: Introduction to Hilbert spaces and the theory of spectral multiplicity. Chelsea Pub. Co., New York (1951)
24. Heisenberg, W.: Quantum theory of fields and elementary particles. Rev. Mod. Phys. **29**, 269–278 (1957)
25. Ichinose, T., Saitō, Y.: Improved Sobolev embedding theorems for vector-valued functions. Funkc. Ekvacioj. **57**, 45–95 (2014)
26. Rabinowitz, P.H.: On a class of nonlinear Schrödinger equations. Z Angew. Math. Phys. **43**, 270–291 (1992)
27. Soler, M.: Classical, stable, nonlinear spinor filed with positive rest energy. Phys. Rev. D. **1**, 2766–2769 (1970)
28. Thaller, B.: The Dirac equation. Texts and monographs in physics. Springer, Berlin (1992)
29. Thirring, W.E.: A soluble relativistic field theory. Ann. Phys. **3**, 91–112 (1958)
30. Tintarev, C.: Concentration analysis and cocompactness concentration analysis and applications to PDE, pp. 117–141. Birkhäuser, Basel (2013)
31. Wang, Z., Zhang, X.: An infinite sequence of localized semiclassical bound states for nonlinear Dirac equations. Calc. Var. Partial Differ. Equ. **57**, 30 (2018)
32. Wang, Z., Zhang, X.: Semiclassical states for nonlinear Dirac equations with singular potential. Calc. Var. Partial Differ. Equ. **60**, 29 (2021)
33. Willem, M.: Minimax theorems, progress in nonlinear differential equations and applications. Birkhäuser, Boston (1996)
34. Zhang, X., Wang, Z.: Semiclassical states of nonlinear Dirac equations with degenerate potential. Ann. Mat. Pura Appl. **198**, 1955–1984 (2019)