Bull. Malays. Math. Sci. Soc. (2022) 45:2175–2193
https://doi.org/10.1007/s40840-022-01374-1

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Toeplitz Kernels and Finite-Rank Commutators of Truncated Toeplitz Operators

Xiaoyuan Yang[1] · **Yufeng Lu**[2]

Received: 13 January 2022 / Revised: 10 August 2022 / Accepted: 12 August 2022 /
Published online: 29 August 2022

Abstract

In this paper, using some properties about Toeplitz kernels, we present some results about finite-rank properties of the commutator $[A_f,\ A_g]$. Firstly, we show that $[A_{B_n},\ A_v^*]$ must have a finite rank on the model space K_u^2, where B_n is a finite Blaschke product and v is an inner function. Next, we present that when $\ker T_{\overline{u}B_n}$ is an invariant subspace of T_ϕ^*, then $[A_{B_n},\ A_\phi^*]$ has a finite rank on K_u^2 for $\phi \in H^\infty$. Finally, we prove that $[A_{B_n},\ A_\phi^*]$ must have a finite rank on K_u^2 when $u = B_n u_1$ for an inner function u_1.

Keywords Model spaces · Truncated Toeplitz operators · Commutators · Finite Blaschke products · Finite-rank

Mathematics Subject Classification 47B35 · 47B47

1 Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $\mathbb{T}$ denote the unit circle. Denote by $L^2 = L^2(\mathbb{T},\ \mathrm{d}m)$ the Hilbert space of square integrable functions with respect to the Lebesgue measure $\mathrm{d}m$ on $\mathbb{T}$, normalized so that the measure of the

Communicated by Mohammad Sal Moslehian.

✉ Xiaoyuan Yang
Yangxyxy@163.com

Yufeng Lu
lyfdlut@dlut.edu.cn

1 School of Science, Jiangsu Ocean University, Lianyungang 222005, Jiangsu, People's Republic of China

2 School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, People's Republic of China

entire circle is 1. Let L^∞ be the space of essentially bounded functions on the unit circle. The Hardy space H^2 denotes the Hilbert space of all holomorphic functions in $\mathbb{D}$ having square-summable Taylor coefficients at the origin, and it will be identified with the space of boundary functions, the subspace of L^2 consisting of functions whose Fourier coefficients with negative indices vanish. Let H^∞ denote the space of all bounded holomorphic functions in $\mathbb{D}$ and $C(\mathbb{T})$ denote the space of all continuous functions on $\mathbb{T}$.

Every function in H^2, other than the constant function 0, can be factorized into the product of an inner function and an outer function. An inner function is a function $u \in H^\infty$ such that $|u(e^{i\theta})| = 1$ almost everywhere with respect to the Lebesgue measure. Every inner function can be factorized into the product of a Blaschke product and a singular inner function. A Blaschke product is an analytic function $B \in H^\infty$ of the form

$$B(z) = z^m \prod_{k=1}^{\infty} \frac{\overline{z_k}}{z_k} \frac{z_k - z}{1 - \overline{z_k} z},$$

where $\{z_k\}$ are zeros of B counting multiplicity which satisfy that $\sum_k (1 - |z_k|) < \infty$. A nonconstant inner function that has no zeros in $\mathbb{D}$ is called a singular inner function S_μ, which has the following form

$$S_\mu(z) = c \exp\left(-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta)\right),$$

where μ is a finite positive regular Borel measure on $[0, 2\pi]$, singular with respect to the Lebesgue measure and c is a constant of modulus 1. The function $F \in H^2$ is an outer function if F is a cyclic vector of the unilateral shift S. That is,

$$\bigvee_0^\infty \{S^k F\} = H^2.$$

For more properties about Hardy spaces, we can refer to [14].

By Beurling's theorem [4], the invariant subspace of the unilateral shift operator $Sf = zf$ on H^2 has the form uH^2, where u is an inner function. It is easy to check that $K_u^2 = H^2 \ominus uH^2$ is the invariant subspace of the backward shift operator S^* on H^2, which is called the model space. Let P denote the orthogonal projection from L^2 onto H^2 and P_u denote the orthogonal projection from L^2 onto K_u^2. For $f \in L^\infty$, the Toeplitz operator T_f induced by the symbol f is defined on H^2 by

$$T_f g = P(fg), \;\; g \in H^2.$$

Obviously, $T_f^* = T_{\overline{f}}$. Toeplitz operators acting on H^2 have very simple and natural matrix representations via infinite Toeplitz matrices that have constant entries on diagonals parallel to the main one. The Hankel operator H_f induced by the symbol f is defined on H^2 by

$$H_f g = (I - P)(fg),\ g \in H^2.$$

Then $H_f^* h = P(\overline{f} h)$ for $h \in (H^2)^\perp$. Compressions of Toeplitz operators on K_u^2 are called truncated Toeplitz operators, for $\psi \in L^\infty$, which are defined by

$$A_\psi f = P_u(\psi f),\ f \in K_u^2.$$

The function ψ is called the symbol of A_ψ. Clearly, $A_\psi^* = A_{\overline{\psi}}$.

Truncated Toeplitz operators represent a far reaching generalization of classical Toeplitz matrices. Although a particular case had appeared before in the literature, the general theory has been initiated in the seminal paper [21]. Since then, truncated Toeplitz operators have constituted an active area of research. We mention only a few relevant papers [5, 11, 17, 20] and so on. On the operator theory level, Nagy and others showed that A_z is a model for a certain class of contraction operators [24]. Every contraction operator T on the Hilbert space H having defect indices (1, 1) and such that $\lim_{n\to\infty} T^{*n} = 0$ (SOT) is unitarily equivalent to A_z for an inner function u, where SOT denotes the strong operator topology. In terms of function theory, Douglas et al. [9] in 1970 showed that functions in the model space and inner functions have the same analytic continuation neighborhood, that is, assuming that u could be analytically extended on $E \subseteq \mathbb{T}$, then all functions in the model space K_u^2 could be analytically extended on E. Therefore, the rational functions approximate the functions in the model space, the boundary value of functions in the model space, the angular derivative of inner functions and the relationship between them are deeply studied, refer to [2] and [3]. Thus, the research on truncated Toeplitz operators is of representative significance.

D. Sarason once proposed in [22]: what $f,\ g \in H^2$ can make $T_f T_{\overline{g}}$ to be a bounded operator. From this, many scholars begin to study the boundedness of the product of Toeplitz operators and Hankel operators. By $T_{fg} - T_f T_g = H_{\overline{f}}^* H_g$, properties of commutators of Toeplitz operators also gradually begin to enter the research line of sight. The map $\tau :\ f \to T_f$ is a contractive $*$-linear mapping from L^∞ to $L(H^2)$, where $L(H^2)$ is the algebra of all bounded linear operators on H^2. But this mapping is not multiplicative. When f is a continuous function, by Hartman's theorem (see Corollary 4.3.3 in [18]) we get that $T_{fg} - T_f T_g = H_{\overline{f}}^* H_g$ is compact for any $g \in L^\infty$. Thus, $\tau(fg) = \tau(f)\tau(g) + K$ for $f \in C(\mathbb{T})$, where K is a compact operator. This makes the multiplicability problem very interesting, and many scholars begin to study the compactness of commutators of Toeplitz operators. In addition, the compactness of commutators of Toeplitz operators also originated from the research on Fredholm theory of Toeplitz operators in 1970s by Douglas, Sarason and others.

The results about compact or finite-rank (semi-)commutators of T_f or H_f are quite complete, corresponding results are summarized in [26]. But there are very few results for compact or finite-rank (semi-)commutators of truncated Toeplitz operators. In [11], Garcia concluded that $A_{fg} - A_f A_g$ is compact for $f,\ g \in C(\mathbb{T})$. In [7], authors described the kernels and ranks of commutators of truncated Toeplitz operators with symbols induced by finite Blaschke products. By $L^2 = K_u^2 \bigoplus (K_u^2)^\perp$, for $f \in L^\infty$, the multiplication operator M_f is expressed as an operator matrix

$$M_f = \begin{pmatrix} A_f & B^*_{\overline{f}} \\ B_f & D_f \end{pmatrix},$$

where D_f denotes the dual truncated Toeplitz operator on $L^2 \ominus K^2_u$ defined by

$$D_f h = (I - P_u)(fh), \ h \in L^2 \ominus K^2_u.$$

It is easy to see that $D^*_f = D_{\overline{f}}$. The truncated Hankel operator B_f is defined by

$$B_f \varphi = (I - P_u)(f\varphi), \ \varphi \in K^2_u.$$

Moreover, $B^*_f h = P_u(\overline{f}h)$ for $h \in L^2 \ominus K^2_u$. By $M_f M_g = M_{fg}$, we get that

$$A_{fg} = A_f A_g + B^*_{\overline{f}} B_g. \tag{1}$$

In [19], authors introduced sufficient and necessary conditions for B_f to be of finite-rank or compact. By (1), we know some sufficient conditions for $A_f A_g - A_{fg}$ to be of finite-rank or compact. From this, we can study the compact commutator $[A_f, \ A_g]$ by truncated Hankel operators B_f. In [26], authors of this article and others gave some results that commutators of truncated Toeplitz operators are compact or of finite-rank operators on model spaces. In our paper, using the structure of Toeplitz kernels, we present some results about finite-rank properties of commutators $[A_f, \ A_g]$.

The paper is organized as follows. In Sect. 2, we recall some necessary definitions and properties about model spaces and truncated Toeplitz operators. In Sect. 3, we obtain that $[A_{B_n}, \ A^*_v]$ must have a finite rank on K^2_u, where B_n is a finite Blaschke product and v is an inner function. In Sect. 4, using some properties of Toeplitz kernels, when $\ker T_{\overline{u}B_n}$ is an invariant subspace of T^*_ϕ, we show that $[A_{B_n}, \ A^*_\phi]$ has a finite rank on K^2_u for $\phi \in H^\infty$. In particular, we present that $[A_{B_n}, \ A^*_\phi]$ has a finite rank on K^2_u when $u = B_n u_1$ for an inner function u_1.

2 Preliminaries

In this section we introduce some basic properties of truncated Toeplitz operators. The reproducing kernel of H^2 at $\lambda \in \mathbb{D}$ is the function $k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$, and it is easy to check that the reproducing kernel of K^2_u at $\lambda \in \mathbb{D}$ is the function

$$k^u_\lambda(z) = (P_u k_\lambda)(z) = \frac{1 - \overline{u(\lambda)}u(z)}{1 - \overline{\lambda}z}.$$

It is well known that K_u^2 carries a natural conjugation C, antiunitary, involution operator, defined by $Cf = \overline{zf}u$ for $f \in K_u^2$. We have that

$$\widetilde{k_\lambda^u}(z) = (Ck_\lambda^u)(z) = \frac{u(z) - u(\lambda)}{z - \lambda},$$

which is the conjugation reproducing kernel of K_u^2 at $\lambda \in \mathbb{D}$. That is,

$$\widetilde{f}(\lambda) = (Cf)(\lambda) = \langle \widetilde{k_\lambda^u}, f \rangle,$$

for $f \in K_u^2$. A bounded linear operator A on K_u^2 is called C-symmetric if

$$CAC = A^*.$$

S. R. Garcia and M. Putinar introduced some properties of C-symmetric operators in [12], and they showed that all truncated Toeplitz operators are C-symmetric. More complex symmetric operators can be found in [13].

The systematic study of truncated Toeplitz operators was initiated by Sarason [21], and an intensive study revealed many interesting and different properties about these operators. For example, unbounded symbols may give bounded truncated Toeplitz operators, see Sarason's example in [21]. For $f \in L^2$, Sarason in [21] proved that $A_f = 0$ if and only if $f \in uH^2 + \overline{uH^2}$. Therefore, the symbols of truncated Toeplitz operators are not unique.

It is well known that the model space K_u^2 is the kernel space of $T_{\overline{u}}$. There are many important connections between model spaces and Toeplitz kernels. In [6], authors presented some classical results about the relationship between Toeplitz kernels and model spaces. Using the relationship between Toeplitz kernels and model spaces, authors showed maximal vectors for model spaces by maximal vectors for Toeplitz kernels. Moreover, they also discussed the multiplier between Toeplitz kernels by the multiplier between model spaces.

For Toeplitz kernels, one classical result is the Coburn theorem (see Proposition 7.24 in [8]). It is said that either $\ker T_g = \{0\}$ or $\ker T_g^* = \{0\}$ for $g \in L^\infty$. In 1986, Hayashi [15] showed that the kernel of T_g can be written as ϕK_η^2, where ϕ is an outer function and η is an inner function with $\eta(0) = 0$, and the function ϕ multiplies K_η^2 isometrically onto $\ker T_g$. It is easy to check that the Toeplitz kernel is nearly S^*-invariant. In 1988, Hitt [16] showed that any nearly S^*-invariant of subspace M is of form hK_θ^2, where $h \in M$ meeting some conditions and θ is an inner function. Sarason [23] gave a new proof of Hitt's theorem and presented better description of h and θ. More about the research process of Toeplitz kernels can be found in [6].

The following lemma is well known and we provide a proof for the sake of completeness.

Lemma 1 *If u and v are inner functions, then*

$$\ker T_{\overline{u}v} = \{\varphi \in K_u^2 : \ v\varphi \in K_u^2\}. \tag{2}$$

Moreover, $\ker T_{\overline{u}v} \neq \{0\}$ *if and only if* $vH^2 \cap K_u^2 \neq \{0\}$.

Proof Denote $E = \{\varphi \in K_u^2 : \; v\varphi \in K_u^2\}$. For $\phi \in E \subset K_u^2$, since the model space K_u^2 has the conjugation, there exists $\eta \in K_u^2$ such that $v\phi = u\overline{z\eta}$. Then

$$T_{\overline{u}v}\phi = P(\overline{u}v\phi) = P(\overline{u}u\overline{z\eta}) = P(\overline{z\eta}) = 0.$$

It implies that

$$\phi \in \ker T_{\overline{u}v} \text{ and } E \subseteq \ker T_{\overline{u}v}.$$

For $\varphi \in \ker T_{\overline{u}v}$, we have that $T_{\overline{u}v}\varphi = P(\overline{u}v\varphi) = 0$, and there exists $x \in H^2$ such that

$$\overline{u}v\varphi = \overline{zx}. \tag{3}$$

Then $v\varphi = u\overline{zx} \in H^2$. Since

$$K_u^2 = uz\overline{H^2} \cap H^2, \tag{4}$$

we get that $v\varphi \in K_u^2$. By (3), we have that $\varphi = u\overline{zxv} \in H^2$. By (4), we conclude that $\varphi \in K_u^2$ and $\varphi \in E$. Thus $\ker T_{\overline{u}v} \subseteq E$ and we have proved that $\ker T_{\overline{u}v} = E$.

Suppose that $vH^2 \cap K_u^2 \neq \{0\}$. There exists $0 \neq h \in H^2$ such that $vh \in K_u^2$. Since K_u^2 has a conjugation, there is $0 \neq g \in K_u^2$ such that $vh = u\overline{zg}$. Then

$$T_{\overline{u}v}h = P(\overline{u}vh) = P(\overline{u}u\overline{zg}) = 0.$$

It implies that $h \in \ker T_{\overline{u}v}$, and $\ker T_{\overline{u}v} \neq \{0\}$. By (2), it is easy to get that $vH^2 \cap K_u^2 \neq \{0\}$ when $\ker T_{\overline{u}v} \neq \{0\}$. The proof is completed. □

3 The Finite-Rank Property of $[A_{B_n}, \; A_v^*]$ for an Inner Function v

By [11] we know that $A_f A_g - A_{fg}$ is compact for $f, \; g \in C(\mathbb{T})$. Thence $[A_{B_n}, \; A_v^*]$ must be compact for any finite Blaschke product B_n and any inner function v. This makes us want to discuss when $[A_{B_n}, \; A_v^*]$ has a finite rank on K_u^2. In the following we show that $[A_{B_n}, \; A_v^*]$ must have a finite rank for any inner function v.

In the following we will frequently use the following relationship:

$$T_\psi^* K_u^2 \subseteq K_u^2 \text{ and } A_\psi^* = T_\psi^*|_{K_u^2},$$

for any $\psi \in H^\infty$ and an inner function u.

We use $\text{Hol}(\mathbb{D})$ to denote the set of all holomorphic functions in $\mathbb{D}$. For a pair of inner functions v and η, we explore multipliers

$$\mathcal{M}(v, \; \eta) = \{\phi \in \text{Hol}(\mathbb{D}) : \; \phi K_v^2 \subseteq K_\eta^2\}$$

between model spaces K_v^2 and K_η^2.

Lemma 2 *(Corollary 3.3 in [10]) If $v,\ \eta$ are inner functions, then*

$$\mathcal{M}(v,\ \eta) \cap H^\infty = \ker T_{\overline{z\eta}v} \cap H^\infty \subseteq \mathcal{M}(v,\ \eta) \subseteq \ker T_{\overline{z\eta}v}.$$

A finite Blaschke product is a function of the form $B_n(z) = c \prod_{i=1}^{n} \frac{z-z_i}{1-\overline{z_i}z}$ for $z_i \in \mathbb{D}$. The degree of a finite Blaschke product B_n is its number of zeros.

Lemma 3 *(Theorem 4.3 in [10]) If B is a finite Blaschke product and v is any inner function with the infinite degree, then $\mathcal{M}(B,\ v) \cap H^\infty \neq \{0\}$.*

Lemma 4 *(Lemma 2.1 in [7]) Let $a_1,\ a_2,\ \cdots,\ a_n$ be points in $\mathbb{D}$ and put*

$$B_n(z) = \prod_{i=1}^{n} \frac{z-a_i}{1-\overline{a_i}z},$$

then

$$K_{B_n}^2 = \frac{\mathcal{P}_{n-1}}{\prod_{i=1}^{n}(1-\overline{a_i}z)},$$

where $\mathcal{P}_k$ denotes the set of all analytic polynomials with the degree less than or equal to k. In particular, $\dim K_{B_n}^2 = n$.

Theorem 1 *Let u be a nonconstant inner function and K_u^2 be the infinite dimensional model space. If B_n is a finite Blaschke product with degree n, then $[A_{B_n},\ A_v^*]$ has a finite rank on K_u^2 for any inner function v and* $\operatorname{rank}[A_{B_n},\ A_v^*] \leq 2n$. *Moreover, if* $\ker T_{\overline{u}B_n v} \neq \{0\}$, *then*

$$v\ker T_{\overline{u}B_n v} \subseteq \ker\, [A_{B_n},\ A_v^*] \text{ and } \operatorname{ran}\, [A_{B_n},\ A_v^*] \subseteq K_u^2 \ominus B_n \ker T_{\overline{u}B_n v}.$$

Proof Since zB_n is a finite Blaschke product, by Lemma 3, we get that

$$\mathcal{M}(zB_n,\ u) \cap H^\infty \neq \{0\}.$$

Then by Lemma 2,

$$\ker T_{\overline{zu}zB_n} = \ker T_{\overline{u}B_n} \neq \{0\}.$$

By Lemma 1, we know that

$$\ker T_{\overline{u}B_n} = \{f \in K_u^2 :\ B_n f \in K_u^2\}.$$

Then $B_n g \in K_u^2$ for $g \in \ker T_{\overline{u}B_n}$. By $H^2 = vH^2 \oplus K_v^2$, there exist $g_1 \in H^2$ and $g_2 \in K_v^2$ such that $g = vg_1 + g_2$. Then

$$\begin{aligned}[A_{B_n},\ A_v^*]g &= (A_{B_n}A_v^* - A_v^*A_{B_n})g \\ &= A_{B_n}T_v^*g - A_v^*A_{B_n}g \\ &= P_u(B_nP(\overline{v}g)) - P_u(\overline{v}P_u(B_ng)) \\ &= P_u(B_nP(\overline{v}g)) - P_u(\overline{v}B_ng) \\ &= P_u(B_nP(\overline{v}vg_1)) - P_u(\overline{v}B_nvg_1) + P_u(B_nP(\overline{v}g_2)) - P_u(\overline{v}B_ng_2) \\ &= P_u(B_ng_1) - P_u(B_ng_1) + P_u(B_nP(\overline{v}g_2)) - P_u(\overline{v}B_ng_2) \\ &= P_u(B_nP(\overline{v}g_2)) - P_u(\overline{v}B_ng_2).\end{aligned}$$

Since $g_2 \in K_v^2$, we get that $\overline{v}g_2 \in \overline{zH^2}$ and $P(\overline{v}g_2) = 0$. Then

$$[A_{B_n},\ A_v^*]g = -P_u(\overline{v}B_ng_2) = -P_uP(\overline{v}B_ng_2).$$

We claim that $P(\overline{v}B_nK_v^2) \subseteq K_{B_n}^2$. In fact, for any $f \in K_v^2$ and $h \in H^2$, we have that

$$\langle P(\overline{v}B_nf),\ B_nh\rangle = \langle f,\ vh\rangle = 0.$$

Thence

$$[A_{B_n},\ A_v^*]\ker T_{\overline{u}B_n} \subseteq \operatorname{ran} P_uP_{B_n}.$$

By Lemma 4, we obtain that $\dim K_{B_n}^2 = n$. Then $\dim(\operatorname{ran} P_uP_{B_n}) \le n$, and

$$\dim\left([A_{B_n},\ A_v^*]\ker T_{\overline{u}B_n}\right) \le n < \infty. \tag{5}$$

Since

$$K_u^2 = \ker T_{\overline{u}B_n} \oplus (K_u^2 \ominus \ker T_{\overline{u}B_n}), \tag{6}$$

in the following we consider the dimension of $[A_{B_n},\ A_v^*](K_u^2 \ominus \ker T_{\overline{u}B_n})$. By

$$H^2 = B_nH^2 \oplus K_{B_n}^2,$$

there exist $h_1 \in H^2$ and $h_2 \in K_{B_n}^2$ such that $h = B_nh_1 + h_2$ for any $h \in H^2$. Then

$$T_{\overline{B_n}u}h = P(\overline{B_n}uB_nh_1) + P(\overline{B_n}uh_2) = uh_1 + P(\overline{B_n}uh_2).$$

By Lemma 4, we know that $\dim K_{B_n}^2 = n$. It follows that

$$\dim P(\overline{B_n}uK_{B_n}^2) \le n < \infty. \tag{7}$$

It is easy to check that $P(\overline{B_n}u\phi) \in K_u^2$ for any $\phi \in K_{B_n}^2$. Thus

$$T_{\overline{B_n}u}H^2 \subseteq uH^2 \oplus P(\overline{B_n}uK_{B_n}^2).$$

Consequently,

$$cl\ (\text{ran}\ T_{\overline{B_n}u}) \subseteq uH^2 \oplus P(\overline{B_n}uK_{B_n}^2),$$

where the abbreviation " cl " denotes the closure of a set. Since

$$(\ker T_{\overline{u}B_n})^{\perp} = cl\ (\text{ran}\ T_{\overline{B_n}u}),$$

we get that

$$K_u^2 \ominus \ker T_{\overline{u}B_n} = cl\ (\text{ran}\ T_{\overline{B_n}u}) \cap K_u^2 \subseteq P(\overline{B_n}uK_{B_n}^2) \cap K_u^2.$$

By $P(\overline{B_n}u\phi) \in K_u^2$ for any $\phi \in K_{B_n}^2$, we get that $P(\overline{B_n}uK_{B_n}^2) \subset K_u^2$. It follows that

$$P(\overline{B_n}uK_{B_n}^2) \cap K_u^2 = P(\overline{B_n}uK_{B_n}^2).$$

Then

$$K_u^2 \ominus \ker T_{\overline{u}B_n} \subseteq P(\overline{B_n}uK_{B_n}^2). \tag{8}$$

By (7) and (8), we have that

$$\dim\ (K_u^2 \ominus \ker T_{\overline{u}B_n}) \leq n < \infty.$$

Then

$$\dim\ ([A_{B_n},\ A_v^*](K_u^2 \ominus \ker T_{\overline{u}B_n})) \leq n < \infty. \tag{9}$$

By (5), (6) and (9), we get that

$$\begin{aligned}\dim\ \left([A_{B_n},\ A_v^*]K_u^2\right) &= \dim\ \left([A_{B_n},\ A_v^*](\ker T_{\overline{u}B_n} \oplus (K_u^2 \ominus \ker T_{\overline{u}B_n}))\right)\\ &= \dim\ \left([A_{B_n},\ A_v^*]\ker T_{\overline{u}B_n} + [A_{B_n},\ A_v^*](K_u^2 \ominus \ker T_{\overline{u}B_n})\right)\\ &\leq \dim\ [A_{B_n},\ A_v^*]\ker T_{\overline{u}B_n} + \dim\ [A_{B_n},\ A_v^*](K_u^2 \ominus \ker T_{\overline{u}B_n})\\ &\leq 2n < \infty.\end{aligned}$$

Thus $[A_{B_n},\ A_v^*]$ has a finite rank on K_u^2 and $\text{rank}[A_{B_n},\ A_v^*] \leq 2n$.

Suppose that $\ker T_{\overline{u}B_nv} \neq \{0\}$. By Lemma 1, we get that $vB_n\phi \in K_u^2$ for any $\phi \in \ker T_{\overline{u}B_nv}$. Then there exists $\psi \in K_u^2$ such that

$$vB_n\phi = u\overline{z\psi}.$$

That is, $v\phi = uz\overline{\psi B_n}$. By $K_u^2 = H^2 \cap uz\overline{H^2}$, we have that $v\phi \in K_u^2$. Then

$$\begin{aligned}
[A_{B_n},\ A_v^*]v\phi &= (A_{B_n}A_v^* - A_v^*A_{B_n})v\phi \\
&= A_{B_n}T_v^*v\phi - A_v^*A_{B_n}v\phi \\
&= P_u(B_nP(\overline{v}v\phi)) - P_u(\overline{v}P_u(B_nv\phi)) \\
&= P_u(B_n\phi) - P_u(\overline{v}B_nv\phi) \\
&= P_u(B_n\phi) - P_u(B_n\phi) \\
&= 0.
\end{aligned}$$

Thus $v\ker T_{\overline{u}B_nv} \subseteq \ker\,[A_{B_n},\ A_v^*]$. By the same way, we get that

$$B_n\ker T_{\overline{u}B_nv} \subseteq \ker\,[A_v,\ A_{B_n}^*]. \tag{10}$$

Since

$$K_u^2 \ominus \ker\,[A_v,\ A_{B_n}^*] = cl\ (\text{ran}\,[A_v,\ A_{B_n}^*]^*) = cl\ (\text{ran}\,[A_{B_n},\ A_v^*]),$$

and $[A_{B_n},\ A_v^*]$ has a finite rank, we obtain that

$$K_u^2 \ominus \ker\,[A_v,\ A_{B_n}^*] = \text{ran}\,[A_{B_n},\ A_v^*].$$

Then by (10),

$$\text{ran}\,[A_{B_n},\ A_v^*] \subseteq K_u^2 \ominus B_n\ker T_{\overline{u}B_nv}.$$

□

In the following we give an example illustrating Theorem 1 and use span$\{h\}$ to denote the space generated by the function h.

Example 1 Let $B_1(z) = \frac{z-a}{1-\overline{a}z}$ for $a \in \mathbb{D}$ and $v(z) = \exp\frac{z+1}{z-1}$ be a singular inner function. If $u = B_1v$ and K_u^2 is the corresponding model space, then

$$[A_{B_1},\ A_v^*]K_u^2 \subseteq \text{span}\,\{k_a\} \oplus \text{span}\,\{P_u\ (B_1k_a)\},$$

and rank$[A_{B_1},\ A_v^*] \le 2$, where $k_a(z) = \frac{1}{1-\overline{a}z}$.

Proof By Lemma 4, we know that

$$K_{B_1}^2 = \text{span}\,\{k_a\}\ \text{and}\ \dim K_{B_1}^2 = 1. \tag{11}$$

Since $u = B_1v$, we have that $K_{B_1}^2 \subsetneq K_u^2$. Using the proof of Theorem 1 and (11), we get that

$$[A_{B_1},\ A_v^*]\ker T_{\overline{u}B_1} \subseteq \text{ran}\ P_uP_{B_1} = \text{span}\,\{P_u\ (k_a)\} = \text{span}\,\{k_a\} = K_{B_1}^2. \tag{12}$$

By (8) and (11), we have that

$$K_u^2 \ominus \ker T_{\overline{u}B_1} \subseteq P(\overline{B_1} u K_{B_1}^2) = \operatorname{span}\left\{k_a \cdot \exp \frac{z+1}{z-1}\right\}.$$

It implies that

$$[A_{B_1},\ A_v^*](K_u^2 \ominus \ker T_{\overline{u}B_1}) \subseteq [A_{B_1},\ A_v^*]\operatorname{span}\left\{k_a \cdot \exp \frac{z+1}{z-1}\right\}.$$

In fact,

$$\begin{aligned}
[A_{B_1},\ A_v^*](k_a \cdot \exp \frac{z+1}{z-1}) &= [A_{B_1},\ A_v^*](vk_a) \\
&= (A_{B_1} A_{\overline{v}} - A_{\overline{v}} A_{B_1})(vk_a) \\
&= A_{B_1} P(\overline{v}vk_a) - A_{\overline{v}} P_u(B_1 v k_a) \\
&= A_{B_1} k_a - A_{\overline{v}} P_u(uk_a) \\
&= P_u(B_1 k_a).
\end{aligned}$$

Hence

$$[A_{B_1},\ A_v^*](K_u^2 \ominus \ker T_{\overline{u}B_1}) \subseteq \operatorname{span}\{P_u\,(B_1 k_a)\}. \tag{13}$$

By (12) and (13), we conclude that

$$\begin{aligned}
[A_{B_1},\ A_v^*]K_u^2 &= [A_{B_1},\ A_v^*](\ker T_{\overline{u}B_1} \oplus (K_u^2 \ominus \ker T_{\overline{u}B_1})) \\
&\subseteq K_{B_1}^2 \oplus \operatorname{span}\{P_u\,(B_1 \cdot k_a)\}.
\end{aligned}$$

Then by (11),

$$\dim\,([A_{B_1},\ A_v^*]K_u^2) \leq 2.$$

□

Remark 1 From the proof of Theorem 1, we know that $P_u P_{B_n}$ has a finite rank because B_n is the finite Blaschke product. In fact, we have the following claim.

Claim: The projection $P_u P_\theta$ has a finite rank if and only if u or θ is a finite Blaschke product.

Using the following lemma, we give a further proof of the claim.

Lemma 5 *(Section 6 in* [1]*) Hankel operators H_f have a finite rank if and only if $f \in \overline{b}H^\infty$, where b is the finite Blaschke product.*

Proof If u or θ is a finite Blaschke product, it is obvious that $P_u P_\theta$ has a finite rank. Suppose that $P_u P_\theta$ has a finite rank. It is easy to get that

$$P_u P_\theta|_{H^2} = H^*_{\overline{u}} H_{\overline{u}} H^*_{\overline{\theta}} H_{\overline{\theta}}. \tag{14}$$

By (14), we have that

$$H_{\overline{\theta}} H^*_{\overline{u}} H_{\overline{u}} H^*_{\overline{\theta}} H_{\overline{\theta}} H^*_{\overline{u}} H_{\overline{u}} H^*_{\overline{\theta}} = (H_{\overline{\theta}} H^*_{\overline{u}})(H_{\overline{\theta}} H^*_{\overline{u}})^*(H_{\overline{\theta}} H^*_{\overline{u}})(H_{\overline{\theta}} H^*_{\overline{u}})^*$$

has a finite rank. Then $H_{\overline{\theta}} H^*_{\overline{u}}$ has a finite rank. Lemma 1 in [1] says that the operator $H_{\overline{\theta}} H^*_{\overline{u}}$ is compact if and only if $H^*_{\overline{u}} H_{\overline{\theta}}$ is compact. By the same way as in Lemma 1 in [1], we can get that $H_{\overline{\theta}} H^*_{\overline{u}}$ has a finite rank if and only if $H^*_{\overline{u}} H_{\overline{\theta}}$ has a finite rank. Using the fact that $H_{\overline{\theta}} H^*_{\overline{u}}$ has a finite rank if and only if $H_{\overline{\theta}}$ or $H_{\overline{u}}$ has a finite rank (see Theorem 4 in [1]), by Lemma 5, we get that $\overline{\theta} = \overline{b}h$ or $\overline{u} = \overline{b_1}h_1$, where b and b_1 are finite Blaschke products and $h, \ h_1 \in H^\infty$. It implies that θ or u is a finite Blaschke product. □

For u and v inner functions, the compactness of $P_u P_v$ reflects the asymptotically orthogonal relationship of K^2_u and K^2_v. We say that model spaces K^2_u and K^2_v are asymptotically orthogonal if $P_u P_v$ is a compact operator. Moreover, the following statements are equivalent.

(a) Model spaces K^2_u and K^2_v are asymptotically orthogonal;
(b) $T_u T^*_v - T_{u\overline{v}}$ is compact;
(c) $H^\infty[\overline{u}] \cap H^\infty[\overline{v}] = H^\infty + C$, where $H^\infty[\overline{u}]$ denotes the Douglas algebra generated by $\overline{u}$ and H^∞;
(d) For each support set S, either $u|_s$ or $v|_s$ is a constant;
(e) $\lim_{|z|\to 1} \max(|u(z)|, \ |v(z)|) = 1$;
(f) $H_{\overline{v}}|_{K^2_u}$ is compact;
(g) $T_{\overline{v}}|_{K^2_u}$ is a compact perturbation of an isometry.

In fact, by $T_u T^*_v - T_{u\overline{v}} = -H^*_{\overline{u}} H_{\overline{v}}$, using the same meaning as in the proof that $P_u P_v$ has a finite rank, we get that $(a) \Leftrightarrow (b)$. The equivalence of (b) and (c) comes from Theorem 1 in [1] and [25]. By Lemma 3 in [1], we get that $(c) \Leftrightarrow (d) \Leftrightarrow (e)$. The proof of $(b) \Leftrightarrow (f) \Leftrightarrow (g)$ can be found in Theorem 2 in [1].

4 The Finite-Rank Property of $[A_{B_n}, \ A^*_\phi]$ for $\phi \in H^\infty$

It is well known that $[A_{B_n}, \ A^*_\phi]$ is compact when B_n is a finite Blaschke product and $\phi \in H^\infty$. In this section assuming that $\ker T_{\overline{u}B_n}$ is an invariant subspace of T^*_ϕ, we present that $[A_{B_n}, \ A^*_\phi]$ has a finite rank on K^2_u.

Theorem 2 *Let u be a nonconstant inner function and K^2_u be the infinite dimensional model space. If $T^*_\phi(\ker T_{\overline{u}B_n}) \subseteq \ker T_{\overline{u}B_n}$, then $[A_{B_n}, \ A^*_\phi]$ has a finite rank on K^2_u and* $\operatorname{rank}[A_{B_n}, \ A^*_\phi] \le 2n$, *where $\phi \in H^\infty$ and B_n is a finite Blaschke product with degree n.*

Proof By Lemmas 2 and 3, we get that $\ker T_{\overline{u}B_n} \neq \{0\}$. For $f \in \ker T_{\overline{u}B_n}$, by Lemma 1, we have that $B_n f \in K_u^2$. Then

$$\begin{aligned}[A_{B_n},\ A_\phi^*]f &= (A_{B_n}A_\phi^* - A_\phi^* A_{B_n})f \\ &= A_{B_n}T_\phi^* f - T_\phi^* A_{B_n} f \\ &= P_u(B_n T_\phi^* f) - P(\overline{\phi} P_u(B_n f)) \\ &= P_u(B_n T_\phi^* f) - P(\overline{\phi} B_n f) \\ &= P_u(B_n T_\phi^* f) - T_{B_n\overline{\phi}} f.\end{aligned}$$

Since $T_\phi^*(\ker T_{\overline{u}B_n}) \subseteq \ker T_{\overline{u}B_n}$, by Lemma 1, we get that $B_n T_\phi^* f \in K_u^2$. Then

$$[A_{B_n},\ A_\phi^*]f = B_n T_\phi^* f - T_{B_n\overline{\phi}} f = (T_{B_n}T_\phi^* - T_{B_n\overline{\phi}})f = -H_{\overline{B_n}}^* H_{\overline{\phi}} f.$$

It is easy to check that $\ker H_{\overline{B_n}} = B_n H^2$. Then

$$cl(\text{ran } H_{\overline{B_n}}^*) = (\ker H_{\overline{B_n}})^\perp = H^2 \ominus B_n H^2 = K_{B_n}^2.$$

By Lemma 4, we obtain that $\dim\,(K_{B_n}^2) = n$. It implies that

$$\dim\,[A_{B_n},\ A_\phi^*](\ker T_{\overline{u}B_n}) \leq n < \infty. \tag{15}$$

Since

$$K_u^2 = \ker T_{\overline{u}B_n} \oplus (K_u^2 \ominus \ker T_{\overline{u}B_n}), \tag{16}$$

in the following we consider the dimension of $[A_{B_n},\ A_\phi^*](K_u^2 \ominus \ker T_{\overline{u}B_n})$. By

$$H^2 = B_n H^2 \oplus K_{B_n}^2,$$

there exist $h_1 \in H^2$ and $h_2 \in K_{B_n}^2$ such that $h = B_n h_1 + h_2$ for any $h \in H^2$. Then

$$T_{\overline{B_n}u} h = P(\overline{B_n} u B_n h_1) + P(\overline{B_n} u h_2) = u h_1 + P(\overline{B_n} u h_2).$$

We claim that

$$P(\overline{B_n} u \varphi) \in K_u^2 \ \text{ for any } \varphi \in K_{B_n}^2. \tag{17}$$

In fact, for any $\psi \in H^2$, we have that

$$\langle P(\overline{B_n} u \varphi), u\psi\rangle = \langle \varphi, B_n \psi\rangle = 0.$$

Thus

$$T_{\overline{B_n}u} H^2 \subseteq uH^2 \oplus P(\overline{B_n} u K^2_{B_n}).$$

This implies that

$$cl \ (\text{ran } T_{\overline{B_n}u}) \subseteq uH^2 \oplus P(\overline{B_n} u K^2_{B_n}).$$

Since $(\ker T_{\overline{u}B_n})^{\perp} = cl \ (\text{ran } T_{\overline{B_n}u})$, we get that

$$K^2_u \ominus \ker T_{\overline{u}B_n} = cl \ (\text{ran } T_{\overline{B_n}u}) \cap K^2_u \subseteq (uH^2 \oplus P(\overline{B_n} u K^2_{B_n})) \cap K^2_u. \tag{18}$$

We claim that

$$(uH^2 \oplus P(\overline{B_n} u K^2_{B_n})) \cap K^2_u \subseteq P(\overline{B_n} u K^2_{B_n}). \tag{19}$$

For $g \in (uH^2 \oplus P(\overline{B_n} u K^2_{B_n})) \cap K^2_u$, there are $g_1 \in H^2$ and $g_2 \in P(\overline{B_n} u K^2_{B_n})$ such that

$$g = ug_1 + g_2 \in K^2_u,$$

we have that

$$0 = \langle ug_1, ug_1 + g_2 \rangle = \|g_1\|^2 + \langle ug_1, g_2 \rangle.$$

By (17), we obtain that $\langle ug_1, g_2 \rangle = 0$. Then $\|g_1\|^2 = 0$ and $g_1 = 0$. This implies that $g = g_2$ and (19) holds. By (17), (18) and (19), we obtain that

$$K^2_u \ominus \ker T_{\overline{u}B_n} \subseteq P(\overline{B_n} u K^2_{B_n}).$$

By Lemma 4, we get that $\dim K^2_{B_n} = n$ and $\dim (K^2_u \ominus \ker T_{\overline{u}B_n}) \le n < \infty$. Then

$$\dim ([A_{B_n}, \ A^*_{B_\phi}](K^2_u \ominus \ker T_{\overline{u}B_n})) \le n < \infty. \tag{20}$$

By (15), (16) and (20), we get that

$$\begin{aligned}
\dim ([A_{B_n}, \ A^*_{\phi}]K^2_u) &= \dim \Big([A_{B_n}, \ A^*_{\phi}](\ker T_{\overline{u}B_n} \oplus (K^2_u \ominus \ker T_{\overline{u}B_n}))\Big) \\
&= \dim \Big([A_{B_n}, \ A^*_{\phi}]\ker T_{\overline{u}B_n} + [A_{B_n}, \ A^*_{\phi}](K^2_u \ominus \ker T_{\overline{u}B_n})\Big) \\
&\le \dim [A_{B_n}, \ A^*_{\phi}]\ker T_{\overline{u}B_n} + \dim [A_{B_n}, \ A^*_{\phi}](K^2_u \ominus \ker T_{\overline{u}B_n}) \\
&\le 2n < \infty.
\end{aligned}$$

Thus $[A_{B_n}, \ A^*_{\phi}]$ has a finite rank on K^2_u and $\text{rank}[A_{B_n}, \ A^*_{\phi}] \le 2n$.

□

Remark 2 For the condition of Theorem 2, we give the following explanation. For any $f \in \ker T_{\overline{u}B_n}$, we have that $T_\phi^*(\ker T_{\overline{u}B_n}) \subseteq \ker T_{\overline{u}B_n}$ if and only if $B_n T_\phi^* f \in K_u^2$ if and only if $P_u(\phi \widetilde{f}) \in B_n \ker T_{\overline{u}B_n}$, where $\widetilde{f} = C(f)$.

Proof By $\ker T_{\overline{u}B_n} = \{f \in K_u^2 : B_n f \in K_u^2\}$, it is easy to get that $\ker T_{\overline{u}B_n}$ is invariant under T_ϕ^* if and only if $B_n T_\phi^* f \in K_u^2$.

Suppose that $B_n T_\phi^* f \in K_u^2$ for $f \in \ker T_{\overline{u}B_n}$. There exists $g \in K_u^2$ such that

$$B_n T_\phi^* f = u\overline{zg}.$$

By $K_u^2 = H^2 \cap u\overline{zH^2}$, we get that

$$T_\phi^* f = P(\overline{\phi} f) = u\overline{zgB_n} \in K_u^2. \tag{21}$$

There is $g_1 \in H^2$ such that

$$\overline{\phi} f = u\overline{zgB_n} + \overline{zg_1}.$$

Since $f \in K_u^2$, there exists $f_1 \in K_u^2$ such that $f = u\overline{zf_1}$. Then $\overline{\phi}u\overline{zf_1} = u\overline{zgB_n} + \overline{zg_1}$. This implies that

$$\phi f_1 = gB_n + ug_1 \text{ and } P_u(\phi f_1) = P_u(gB_n).$$

By (21), we have that

$$B_n g \in K_u^2 \text{ and } g \in \ker T_{\overline{u}B_n}.$$

Then $P_u(\phi f_1) = gB_n$. By $f = u\overline{zf_1}$, we get that $f_1 = u\overline{zf} = C(f) = \widetilde{f}$. Thus, the function $P_u(\phi \widetilde{f})$ belongs to $B_n \ker T_{\overline{u}B_n}$.

Suppose that $P_u(\phi \widetilde{f}) \in B_n \ker T_{\overline{u}B_n}$ for $f \in \ker T_{\overline{u}B_n}$. There exist functions $g_1 \in \ker T_{\overline{u}B_n}$ and $g_2 \in H^2$ such that

$$\phi \widetilde{f} = \phi C(f) = B_n g_1 + ug_2.$$

Then

$$\overline{\phi} u\overline{zC(f)} = u\overline{zB_n g_1} + \overline{zg_2}.$$

By $C(f) = u\overline{zf}$, we have that

$$\overline{\phi} u\overline{zu\overline{zf}} = u\overline{zB_n g_1} + \overline{zg_2}.$$

That is,

$$\overline{\phi} f = u\overline{zB_n g_1} + \overline{zg_2}.$$

This implies that

$$P(\overline{\phi} f) = P(uz\overline{B_n g_1}).$$

Since $g_1 \in \ker T_{\overline{u}B_n}$, we get that $B_n g_1 \in K_u^2$ and $uz\overline{B_n g_1} \in K_u^2$. Then

$$P(\overline{\phi} f) = uz\overline{B_n g_1} \text{ and } B_n T_\phi^* f = uz\overline{g_1}.$$

By $K_u^2 = H^2 \cap uz\overline{H^2}$, we have that $B_n T_\phi^* f = uz\overline{g_1} \in K_u^2$. The proof is completed. □

Corollary 1 *Let u be a nonconstant inner function and K_u^2 be the infinite dimensional model space. If $u = B_n u_1$ for a finite Blaschke B_n and an inner function u_1, then $[A_{B_n}, A_\phi^*]$ has a finite rank on K_u^2 for $\phi \in H^\infty$.*

Proof Since $u = B_n u_1$, we get that

$$T_\phi^*(\ker T_{\overline{u}B_n}) = T_\phi^*(\ker T_{\overline{u_1}}) = T_\phi^* K_{u_1}^2 \subseteq K_{u_1}^2 = \ker T_{\overline{u_1}} = \ker T_{\overline{u}B_n}.$$

By Theorem 2, we obtain that $[A_{B_n}, A_\phi^*]$ has a finite rank on K_u^2. □

Example 2 Let $B_1(z) = \frac{z-a}{1-\overline{a}z}$ for $a \in \mathbb{D}$ and $v(z) = \exp \frac{z+1}{z-1}$ be a singular inner function. If $u = B_1 v$ and K_u^2 is the corresponding model space, then the following statements hold.

(a) If $\phi = 1 + B_1$ is an outer function, then

$$[A_{B_1}, A_\phi^*]K_u^2 \subseteq \text{span}\{k_a\} \oplus \text{span}\left\{P_u\left(B_1 P(\overline{B_1} v k_a)\right)\right\},$$

and $\text{rank}[A_{B_1}, A_\phi^*] \leq 2$, where $k_a(z) = \frac{1}{1-\overline{a}z}$.

(b) If $\phi = 1 + v$ is an outer function, then

$$[A_{B_1}, A_\phi^*]K_u^2 \subseteq \text{span}\{k_a\} \oplus \text{span}\{P_u(B_1 k_a)\},$$

and $\text{rank}[A_{B_1}, A_\phi^*] \leq 2$, where $k_a(z) = \frac{1}{1-\overline{a}z}$.

Proof By Lemma 4, we know that

$$K_{B_1}^2 = \text{span}\{k_a\} \text{ and } \dim K_{B_1}^2 = 1, \tag{22}$$

Since $u = B_1 v$, we have that $K_{B_1}^2 \subsetneq K_u^2$. Using the proof of Theorem 2, we get that

$$[A_{B_1}, A_\phi^*]\ker T_{\overline{u}B_1} \subseteq cl(\text{ran } H_{\overline{B_1}}^*) = K_{B_1}^2, \tag{23}$$

and

$$[A_{B_1}, A_\phi^*](K_u^2 \ominus \ker T_{\overline{u}B_1}) \subseteq [A_{B_1}, A_\phi^*]P(\overline{B_1} u K_{B_1}^2) = [A_{B_1}, A_\phi^*]\text{span}\{v k_a\}. \tag{24}$$

(a) If $\phi = 1 + B_1$, then

$$\begin{aligned}[A_{B_1},\ A_\phi^*](vk_a) &= (A_{B_1} A_{\overline{1+B_1}} - A_{\overline{1+B_1}} A_{B_1})(vk_a) \\ &= A_{B_1} P(\overline{1+B_1} vk_a) - A_{\overline{1+B_1}} P_u(B_1 vk_a) \\ &= A_{B_1}(vk_a) + A_{B_1} P(\overline{B_1} vk_a) \\ &= P_u\left(B_1 P(\overline{B_1} vk_a)\right).\end{aligned}$$

By (23) and (24), we conclude that

$$\begin{aligned}[A_{B_1},\ A_\phi^*]K_u^2 &= [A_{B_1},\ A_\phi^*](\ker T_{\overline{u}B_1} \oplus (K_u^2 \ominus \ker T_{\overline{u}B_1})) \\ &\subseteq K_{B_1}^2 \oplus \operatorname{span}\left\{P_u\left(B_1 P(\overline{B_1} vk_a)\right)\right\}.\end{aligned}$$

Then by (22),

$$\dim\left([A_{B_1},\ A_\phi^*]K_u^2\right) \leq 2.$$

(b) If $\phi = 1 + v$, then

$$\begin{aligned}[A_{B_1},\ A_\phi^*](vk_a) &= (A_{B_1} A_{\overline{1+v}} - A_{\overline{1+v}} A_{B_1})(vk_a) \\ &= A_{B_1} P(\overline{1+v} vk_a) - A_{\overline{1+v}} P_u(B_1 vk_a) \\ &= A_{B_1}(vk_a) + A_{B_1} P(\overline{v} vk_a) \\ &= P_u(B_1 k_a).\end{aligned}$$

By (23) and (24), we conclude that

$$\begin{aligned}[A_{B_1},\ A_\phi^*]K_u^2 &= [A_{B_1},\ A_\phi^*](\ker T_{\overline{u}B_1} \oplus (K_u^2 \ominus \ker T_{\overline{u}B_1})) \\ &\subseteq K_{B_1}^2 \oplus \operatorname{span}\{P_u(B_1 k_a)\}.\end{aligned}$$

Then by (22),

$$\dim\left([A_{B_1},\ A_\phi^*]K_u^2\right) \leq 2.$$

□

Acknowledgements This research is supported by National Natural Science Foundation of China No. 12031002, 11671065.

Declarations

Conflict of interest Not applicable.

References

1. Axler, S., Chang, S.-Y.A., Sarason, D.: Integral equations operator theory. Prod. Toeplitz Op. **1**(3), 285–309 (1978)
2. Ahern, P.R., Clark, D.N.: Radial limits and invariant subspaces. Am. J. Math. **92**, 332–342 (1970)
3. Ahern, P.R., Clark, D.N.: On functions orthogonal to invariant subspaces. Acta Math. **124**, 191–204 (1970)
4. Beurling, A.: On two problems concerning linear transformations in Hilbert space. Acta Math. **81**, 239–255 (1948)
5. Bessonov, R.: Truncated Toeplitz operators of finite rank. Proc. Am. Math. Soc. **142**(4), 1301–1313 (2014)
6. Cristina, C.M., Partington, J.R.: Toeplitz kernels and model spaces. The diversity and beauty of applied operator theory, 139-153, Operator Theory: Advances and Applications, 268, (2018)
7. Chen, Y., Lee, Y.J., Zhao, Y.: Ranks of commutators for a class of truncated Toeplitz operators. Banach J. Math. Anal. **15**(1), 16 (2021)
8. Douglas, R.G.: Banach algebra techniques in the theory of Toeplitz operators. Regional Conference Series in Mathematics, no. 15, American Mathematical Society, (1973)
9. Douglas, R.G., Shapiro, H.S., Lowell, Allen: Cyclic vectors and invariant subspaces for the backward shift operator. Université de Grenoble. Ann. de ıInst. Fourier **20**(1), 37–76 (1970)
10. Fricain, E., Hartmann, A., Ross, W.T.: Multipliers between model spaces. Stud. Math. **240**(2), 177–191 (2018)
11. Garcia, S.R., Ross, W.T., Wogen, W.R.: *C^*-algebras generated by truncated Toeplitz operators.* Concrete operators, spectral theory, operators in harmonic analysis and approximation, 181-192. Operator Theory Advances and Applications, 236, (2014)
12. Garcia, S.R., Putinar, M.: Complex symmetric operators and applications. Trans. Am. Math. Soc. **358**(3), 1285–1315 (2006)
13. Garcia, S.R., Putinar, M.: Complex symmetric operators and applications II. Trans. Am. Math. Soc. **359**(8), 3913–3931 (2007)
14. Hoffman, K.: Banach spaces of analytic functions. Englewood Cliffs, Prentice-Hall, Prentice-Hall Series in Modern Analysis, (1962)
15. Hayashi, E.: The kernel of a Toeplitz operator. Integr. Equs. Oper. Theory **9**(4), 588–591 (1986)
16. Hitt, D.: Invariant subspaces of H^2 of an annulus. Pac. J. Math. **134**(1), 101–120 (1988)
17. Jurasik, J., Łanucha, B.: Asymmetric truncated Toeplitz operators equal to the zero operator Annales. Universitatis Mariae Curie-Sk↕odowska, **7**(2), 51–62 (2016)
18. Martinez-Avendaño, R.A., Rosenthal, P.M.: An introduction to operators on the Hardy-Hilbert space. Graduate Texts in Mathematics, 237. Springer, New York (2007)
19. Ma, P., Yan, F., Zheng, D.: Zero, finite rank, and compact big truncated Hankel operators on model spaces. Proc. Am. Math. Soc. **146**(12), 5235–5242 (2018)
20. Ma Pan; Zheng Dechao: Compact truncated Toeplitz operators. J. Funct. Anal. **270**(11), 4256–4279 (2016)
21. Sarason, D.: Algebraic properties of truncated Toeplitz operators. Oper. Matrices **1**(4), 491–526 (2007)
22. Sarason, D.: Products of Toeplitz operators. In 'Linear and Complex Analysis Problem Book 3', Part I, Vol. 1573, 318–319, Springer-Verlag, (1994)
23. Sarason, D.: Nearly invariant subspaces of the backward shift. Contributions to operator theory and its applications, 481–493, Operator Theory: Advances and Applications, 35, (1988)
24. Szőkefalvi-Nagy, B., Foias, C., Bercovici, H., Kérchy, L.: Harmonic analysis of operators on Hilbert space, 2nd edn. Universitext, Springer, New York, Revised and enlarged edition (2010)
25. Volberg, Alexander: Two remarks concerning the theorem of S. Axler, S.-Y. A. Chang and D. Sarason. J. Oper. Theory **7**(2), 209–218 (1982)
26. Yang, X., Lu, Y., Yang, Y.: Compact commutators of truncated Toeplitz operators on the model space. Ann. Funct. Anal. **13**(3), 49 (2022)