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The Boundary Schwarz Lemma for Harmonic and Pluriharmonic Mappings and Some Generalizations

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Abstract

We use the improvement of the classical Schwarz lemmas for planar harmonic mappings into the sharp form, in order to provide some applications to sharp boundary Schwarz type lemmas for holomorphic and in particular pluriharmonic mappings between the unit balls in Hilbert and Banach spaces. In the second part of this article, using Burget's estimate we establish the sharp boundary Schwarz type lemmas for harmonic mappings between finite dimensional balls. Since here we do not suppose in general that maps fix the origin this is a generalization of the result, previously established by David Kalaj, for harmonic functions. At the end of this section, we derived some interesting conclusion considering hyperbolic-harmonic functions in the unit ball, which shows that Hopf's lemma is not applicable for those functions.

Keywords The boundary Schwarz lemma · Banach space · Harmonic functions in higher dimensions · Pluriharmonic mappings

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1 Introduction

The classical Schwarz lemma is a result in complex analysis about holomorphic functions from the open unit disk to itself. The lemma is less celebrated than deeper theorems, such as the Riemann mapping theorem, which it helps to prove. Although it is one of the simplest results showing the rigidity of holomorphic functions Schwarz lemma has been generalized in various directions and it has become a crucial theme in many branches of research in Mathematics for more than a hundred years to the present day. There is vast literature related to the subject, but here we cite mainly recent papers; for a more complete list of references see [2–5] and the references therein for more fundamental results.

We only briefly discuss recent results that have affected our work . We draw the reader's attention the result in subsection 1.2 was obtained before the result in the next subsection.

1.1 Schwarz Lemma and Hilbert Spaces

In [6], the first author of this paper in a joint paper with Li considered pluriharmonic and harmonic mappings f defined on the unit ball $\mathbb{B}^n$, $n \geq 2$, differentiable at a point a on the boundary of $\mathbb{B}^n$, and $f(\mathbb{B}^n)$ satisfies some convexity hypothesis at $f(a)$. For those mappings f, they obtained versions of boundary Schwarz lemma and the sharp estimate of the eigenvalue related to its Jacobian at a.

After writing the final version of the manuscript [6] Hamada turned attention [1] to the arxiv paper [3]. In [3], the authors generalize the classical Schwarz lemmas of planar harmonic mappings into the sharp forms for Banach spaces, and present some applications to sharp boundary Schwarz type lemmas for pluriharmonic mappings in Banach spaces. Recently, Hamada and Kohr published paper [7], where authors discussed rigidity theorems on the boundary for holomorphic mappings. They explained difference of the constants obtained for the unit ball and the unit polydisc and also presented a generalization for other bounded symmetric regions in $\mathbb{C}^n$ and balanced domains in complex Banach spaces.

In this paper we get further results using approaches from those papers.

The following result was obtained by I. Graham, H. Hamada and G. Kohr in [[8], Proposition 1.8] stated here as:

Theorem 1.1 ([8]) *Let $\mathbb{B}_j$ be the unit ball of a complex Hilbert space H_j for $j = 1, 2$, respectively. Let $f : \mathbb{B}_1 \to \mathbb{B}_2$ be a pluriharmonic mapping. Assume that f is of class C^1 at some point $z_0 \in \partial\mathbb{B}_1$ and $f(z_0) = w_0 \in \partial\mathbb{B}_2$. Then there exists a constant $\lambda \in \mathbb{R}$ such that $Df(z_0)^* w_0 = \lambda z_0$. Moreover,*

$$\lambda \geq \frac{1 - \mathrm{Re}\,(\langle f(0), w_0\rangle)}{2} > 0.$$

[1] in communication with M. Mateljevć

In Sect. 2 we will improve this estimate. Next S. Chen, H. Hamada, S. Ponnusamy, R. Vijayakumar in [3] observed that using [[8], Proposition 1.8] and the arguments similar to those in the proof of their Theorem 3.3 [3] one can obtain a better estimate:

Proposition 1.2

$$\lambda \geq \max\left\{\frac{2}{\pi} - |f(0)|, \frac{1 - \mathrm{Re}\,(\langle f(0), w_0\rangle)}{2}\right\}.$$

Note that under condition $f(0) = 0$, in Theorem 1.1 (ii) and (iii) in [6], it is shown that $\lambda \geq 2/\pi$. (But it also follows from the above Proposition 1.2.)

Next in [3] a version of the boundary Schwarz lemma for the complex Banach spaces was proved:

Theorem 1.3 (Theorem 3.3 [3]) *Suppose that B_X and B_Y are the unit balls of the complex Banach spaces X and Y, respectively, and $f : B_X \to B_Y$ is a pluriharmonic mapping. In addition, let f be differentiable at $b \in \partial B_X$ with $|f(b)|_Y = 1$. Then we have*

$$|Df(b)b|_Y \geq \max\left\{\frac{2}{\pi} - |f(0)|, \frac{1 - |f(0)|}{2}\right\}. \tag{1}$$

In Sect. 3 we proved Theorem 3.1 which yields better estimate. We leave to the interesting reader to check this claim. In Theorem 2.12, we establish Schwarz lemma on the boundary for harmonic functions, mapping the unit ball in $\mathbb{R}^n$ into unit ball in some Hilbert space, which maps origin to origin. This is a generalization of the work in paper [1].

1.2 Schwarz Lemma for Harmonic Functions in Several Variables

For a short discussion about Schwarz lemma for harmonic functions in the planar case see Sect. 1. As far as we know the study related to Schwarz lemma for real valued harmonic functions, defined on the unit ball in $\mathbb{R}^n$ with codomain $(-1, 1)$ was initiated by Khavinson, Burget, Axler at al., for more details see for example [5]. Generalizations of Schwarz lemma for functions of several variables were developed in the work of Burgeth [9] (see also the papers by H.A. Schwarz and E.J.P.G. Schmidt cited there), which were based on the integration of Poisson kernels over the so-called polar caps, using the spherical coordinates[2] and we used some formulas from that paper, which are described in the first part of Sect. 4. Khavinson [10], using also spherical caps, indicates an elementary argument that allows one to obtain sharp estimates of derivatives of bounded harmonic functions in the unit ball in $\mathbb{R}^n$ (explicitly stated for $n = 3$); this three-dimensional result has a physical interpretation. It is worth mentioning that a similar idea occurs in the book [11] for maps which fix the origin in which case the spherical cap is reduced to a hemisphere. Note that researches have often overlooked Burget's work (for more details see Sect. 1).

D. Kalaj [1] considered Heinz–Schwarz inequalities for harmonic mappings in the unit ball, which is a version of Schwarz lemma on the boundary.

[2] we refer to this method as Burget's spherical cap method

Recently, these ideas were discussed at the Belgrade Analysis Seminar, and several recent results in this subject were obtained by the first author and some of his associates: M. Svetlik, A. Khalfallah, M. Mhamdi, B. Purtić, H.P. Li and the second author of this paper, see ([12–14]). For more details see the introduction of paper [5] by the first author of this paper.

In particular we will use here [Proposition 4.4 [6]] which is a corollary of the estimate obtained in [13] (cf. also [14]), stated here as Proposition 2.4.

In Sect. 4 using Burget's estimate we establish Theorem 4.4 for harmonic mappings between finite dimensional unit balls. Since here we do not suppose that maps fix the origin this is a generalization of Theorem 2.5 in the mentioned Kalaj's paper.

At the end of this section, we derived some interesting conclusion considering hyperbolic-harmonic functions in the unit ball, which shows that Hopf's lemma is not applicable for those functions.

Chinese mathematicians have made a great contribution to this field but here we will mention only results that are related to the results presented here.[3]

2 Boundary Schwarz Lemma for Pluriharmonic Mappings in Hilbert Spaces

Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Then H can be regarded as a real Hilbert space with inner product Re $\langle \cdot, \cdot \rangle$. Let $|\cdot|$ be the induced norm in H. Let $\mathbb{B}$ be the unit ball of H. For each $z_0 \in \partial\mathbb{B}$, the tangent space $T_{z_0}(\partial\mathbb{B})$ is defined by

$$T_{z_0}(\partial\mathbb{B}) = \{\beta \in H : \text{Re}\,\langle z_0, \beta \rangle = 0\}.$$

Let H_1 and H_2 be complex Hilbert spaces and let Ω be a domain in H_1.

Definition 2.1 A mapping $f : \Omega \to H_2$ is said to be differentiable at $z \in \Omega$ if there exists a bounded linear map $Df(z) \in \mathcal{L}_{\mathbb{R}}(H_1, H_2)$ such that

$$f(z+h) = f(z) + Df(z)h + o(|h|), \text{ as } h \to 0.$$

If f is differentiable at each point of Ω, then f is said to be differentiable on Ω. In this case, the mapping

$$\mathcal{D}f : \Omega \to \mathcal{L}_{\mathbb{R}}(H_1, H_2), \; z \mapsto Df(z)$$

is called the derivative (or differential) of f on Ω. If $\mathcal{D}f$ is continuous in a neighborhood of z, the mapping f is said to be of class C^1 at z. If $Df(z)$ is bounded complex linear for each $z \in \Omega$, then f is said to be holomorphic on Ω.

Definition 2.2 A C^2−mapping $f : \mathbb{B}_1 \to H_2$ is said to be pluriharmonic if the restriction of the complex valued function $f_w(z) = \langle f(z), w \rangle$ to every complex line is harmonic for each $w \in H_2$.

[3] Z. Chen, Y. Liu and Y. Pan; S. Dai, H. Chen and Y. Pan;X. Tang, T. Liu and W. Zhang;J.F. Zhu, etc

Let $\mathbb{B}_j$ be the unit ball of a complex Hilbert space H_j for $j = 1, 2$, respectively. Note that if f is differentiable at $z_0 \in \partial\mathbb{B}_1$ with values in H_2, then the adjoint operator $Df(z_0)^*$ is defined by

$$\operatorname{Re}\left(\langle Df(z_0)^* w, z\rangle_{H_1}\right) = \operatorname{Re}\left(\langle w, Df(z_0)z\rangle_{H_2}\right) \quad \text{for} \quad z \in H_1, \; w \in H_2.$$

where $\langle\cdot,\cdot\rangle_{H_j}$ is the inner product of H_j, $j = 1, 2$. Here, by $D_r f(x)$ we denote directional derivative with respect to vector $\frac{x}{|x|}$, i. e. $D_r f(x) = \frac{\partial}{\partial r} f(x)$, where $r = |x|$.

For $a \in H_1$ and $v \in T_a(H_1)$ (the tangent space at the point a), we define the half space $H(a, v) = \{y \in H_1 : \operatorname{Re}\langle y - a, v\rangle < 0\}$. In order to shorten the notation, we will simply write H_a instead of $H(a, v)$. Also, we will use notation n_a to stress out the fact that this vector defines half-space $H_a = H(a, n_a)$. In this setting, we will assume that n_a is a unit vector. We also notice that in our approach the following simple result is useful:

Claim 2.3 Assume that f is differentiable at a point $a \in H_1$ and let $b = f(a) \in H_2$. Then by the definition of adjoint operator, we have

$$\operatorname{Re}\langle Df(a)Z, n_b\rangle = \operatorname{Re}\langle Z, Df(a)^* n_b\rangle,$$

for any $Z \in T_a(H_1)$. The following statements hold:

(i) If $Df(a)$ maps H_a into H_b, then $Df(a)^* n_b = \lambda n_a$, where $\lambda > 0$.
(ii) If further, f maps H_a into H_b, then $Df(a)^* n_b = \lambda n_a$, where $\lambda \geq 0$. In particular if $Df(a)^* n_b \neq 0$, then $\lambda > 0$.
(iii) In both cases (i) and (ii), we have $\lambda = |Df(a)^* n_b| = \operatorname{Re}\langle Df(a)n_a, n_b\rangle$, and $\lambda \leq |Df(a)n_a|$.
(iv) Let $|a| = 1$. Define $u(x) = \operatorname{Re}\langle f(x), n_b\rangle$. Then $\lambda = D_r u(a)$.

Proof Here it is convenient to identify H_a and H_b with subsets of $T_a(H_1)$ and $T_b(H_2)$, respectively. By hypothesis $Df(a)$ maps H_a into H_b, and therefore we have

$$0 = \operatorname{Re}\langle Df(a)X, n_b\rangle = \operatorname{Re}\langle X, Df(a)^* n_b\rangle$$

for all $X \in T_a(H_a)$. This shows that $X_0 = Df(a)^* n_b$ is orthogonal to $T_a(H_a)$. In our setting, it means that it equals to λn_b. Then by definition of the adjoint operator, one has

$$\operatorname{Re}\langle Df(a)n_a, n_b\rangle = \operatorname{Re}\langle n_a, Df(a)^* n_b\rangle = \operatorname{Re}\langle n_a, \lambda n_a\rangle = \lambda.$$

Since $n_a \in H_a$, $Df(a)n_a \in H_b$, by the definition of H_b, we first conclude that $\langle Df(a)n_a, n_b\rangle > 0$, and hence, $\lambda > 0$. This completes the proof of (i). For the proof of (ii), which is similar to (i), we leave it to the interested reader by considering two cases: $X_0 = 0$ and $X_0 \neq 0$.

(iii) is an immediate corollary of (i) and (ii). (iv) is consequence of the fact that $D_r u(a) = Re\langle Df(a)a, n_b\rangle = \lambda$. □

The proof of Proposition 2.5 and Theorem 3.1 is based on the following result.

Proposition 2.4 *(Proposition 4.4 [6]) Let $u : \mathbb{U} \to \mathbb{U}$ be a harmonic function such that $u(0) = b$. Assume that u has a continuously extension to the boundary point $z_0 \in \mathbb{T}$, $u(z_0) = c \in \mathbb{T}$ and $a = \tan \frac{|\operatorname{Re}(\bar{c}b)|\pi}{4}$. If u is differentiable at z_0, then $|D_r u(z_0)| \geq \frac{2}{\pi}\frac{1-|a|}{1+|a|}$.*

Let $s^-(x) = \frac{2}{\pi}\cot\left(\frac{\pi}{4}(1+x)\right), x \in (-1,1)$.

Proposition 2.5 *Let $\mathbb{B}_j$ be the unit ball of a complex Hilbert space H_j for $j = 1, 2$, respectively. Let $f : \mathbb{B}_1 \to \mathbb{B}_2$ be a pluriharmonic mapping. Assume that f is differentiable at some point $z_0 \in \partial\mathbb{B}_1$ and $f(z_0) = w_0 \in \partial\mathbb{B}_2$. Then there exists a constant $\lambda \in \mathbb{R}$ such that $Df(z_0)^* w_0 = \lambda z_0$. Moreover,*

$$\lambda \geq s^-(b) > 0, \ \textit{where } b = \operatorname{Re}(\langle f(0), w_0\rangle).$$

We note that $s^-(x) \geq \frac{1-x}{2}, \ x \in (-1,1)$.

Proof Let us consider function $u : \mathbb{U} \to (-1,1)$, defines with $u(z) = \operatorname{Re}\langle f(zz_0), w_0\rangle$. Function u will be a harmonic function and we have $u(0) = b$. Function u has continuous extension an point $z_0 \in \mathbb{T}$ and we can check that $u(1) = 1$. Applying Proposition 2.4, we get $|D_r u(1)| \geq s^-(b)$. Also, we have that $D_r u(1) = Re\langle Df(z_0)z_0, w_0\rangle = \lambda$. □

Suppose that Ω is a domain in H_1 and $f : \Omega \to H_2$ is a holomorphic map in Ω and $z_0 \in \Omega$ be any point. We define hermitian adjoint operator $Df(z_0)^\dagger$ in the next manner

$$\langle Df(z_0)^\dagger w, z\rangle_{H_1} = \langle w, Df(z_0)z\rangle_{H_2} \quad \text{for} \quad z \in H_1, \ w \in H_2,$$

where $\langle\cdot,\cdot\rangle_{H_j}$ is the inner product of $H_j, \ j = 1, 2$. Let $\mathbb{B}_1$ be the unit ball of a complex Hilbert space H_1.

Lemma 2.6 *([15])For $\xi \in \mathbb{B}_1$, let $\varphi_\xi(z) = A\frac{\xi - z}{1-\langle z,\xi\rangle}$ be the holomorphic automorphism of $\mathbb{B}_1$ where $A : H_1 \to H_1$ in the sense that $A(v) = s_\xi v + \frac{\xi\langle v,\xi\rangle}{1+s_\xi}$, $s_\xi = \sqrt{1-|\xi|^2}$ and $v \in H_1$. Then φ_ξ is biholomorphic in a neighborhood of $\overline{\mathbb{B}}_1$, and*

$$A^2 = s_\xi^2 Id + \xi\langle\cdot,\xi\rangle, \quad A\xi = \xi, \quad \varphi_\xi^{-1} = \varphi_\xi,$$
$$D\varphi_\xi(z) = A\left[-\frac{Id}{1-\langle z,\xi\rangle} + \frac{(\xi - z)\langle\cdot,\xi\rangle}{(1-\langle z,\xi\rangle)^2}\right].$$

If we denote $P(v) = \xi\langle v,\xi\rangle$ it can be checked that $P^\dagger = P$. From this $A^\dagger = A$ follows. Also, if $Q(v) = z\langle v,\xi\rangle$ and $R(v) = \xi\langle v, z\rangle$ then $Q^\dagger = R$. Now, we have

$$D\varphi_\xi(z)^\dagger = \left[-\frac{Id}{1-\overline{\langle z,\xi\rangle}} + \frac{\xi\langle\cdot,\xi - z\rangle}{(1-\overline{\langle z,\xi\rangle})^2}\right]A.$$

Let us denote $L_z = \left[-\frac{Id}{1-\overline{\langle z,\xi\rangle}} + \frac{\xi\langle\cdot,\xi-z\rangle}{(1-\overline{\langle z,\xi\rangle})^2}\right]$.

Lemma 2.7 *([15]) For every $z_0 \in \mathbb{B}_1$ we have $D\varphi_\xi(z_0)^\dagger\varphi_\xi(z_0) = \frac{1-|\xi|^2}{|1-\langle z_0,\xi\rangle|^2}z_0$.*

Proof By direct computation $D\varphi_\xi(z_0)^\dagger\varphi_\xi(z_0) = L_{z_0}A^2\frac{\xi-z_0}{1-\langle z_0,\xi\rangle}$. We can easily check that $A^2\frac{\xi-z_0}{1-\langle z_0,\xi\rangle} = \xi - \frac{s^2 z_0}{1-\langle z_0,\xi\rangle}$. According to this, we have

$$\begin{aligned}
D\varphi_\xi(z_0)^\dagger\varphi_\xi(z_0) &= L_{z_0}\left(\xi - \frac{s^2 z_0}{1-\langle z_0,\xi\rangle}\right)\\
&= -\frac{\xi}{1-\overline{\langle z_0,\xi\rangle}} + \frac{\xi\langle\xi,\xi-z_0\rangle}{(1-\overline{\langle z_0,\xi\rangle})^2} + \frac{s^2 z_0}{|1-\langle z_0,\xi\rangle|^2} - \frac{s^2\xi\langle z_0,\xi-z_0\rangle}{|1-\langle z_0,\xi\rangle|^2(1-\overline{\langle z_0,\xi\rangle})}\\
&= -\frac{\xi(1-\langle\xi,z_0\rangle)}{(1-\overline{\langle z_0,\xi\rangle})^2} + \frac{\xi\langle\xi,\xi-z_0\rangle}{(1-\overline{\langle z_0,\xi\rangle})^2} + \frac{s^2 z_0}{|1-\langle z_0,\xi\rangle|^2} - \frac{s^2\xi(\langle z_0,\xi\rangle-1)}{|1-\langle z_0,\xi\rangle|^2(1-\overline{\langle z_0,\xi\rangle})}\\
&= \frac{-\xi(1-|\xi|^2)}{(1-\overline{\langle z_0,\xi\rangle})^2} + \frac{s^2 z_0}{|1-\langle z_0,\xi\rangle|^2} - \frac{s^2\xi(\langle z_0,\xi\rangle-1)}{|1-\langle z_0,\xi\rangle|^2(1-\overline{\langle z_0,\xi\rangle})}\\
&= \frac{-\xi s^2(1-\langle z_0,\xi\rangle))}{(1-\overline{\langle z_0,\xi\rangle})^2(1-\langle z,\xi\rangle)} + \frac{s^2 z_0}{|1-\langle z_0,\xi\rangle|^2} - \frac{s^2\xi(\langle z_0,\xi\rangle-1)}{|1-\langle z_0,\xi\rangle|^2(1-\overline{\langle z_0,\xi\rangle})}\\
&= \frac{1-|\xi|^2}{|1-\langle z_0,\xi\rangle|^2}z_0.
\end{aligned}$$

□

Let V_1 and V_2 be two complex vector spaces. We define the sets or real linear, complex linear and complex antilinear operators between V_1 and V_2 in the following sense.

If $L : V_1 \to V_2$ is an additive linear operator, then

$$\begin{aligned}
L \in \mathcal{L}_{\mathbb{R}}(V_1, V_2) &\iff \forall\lambda \in \mathbb{R}, \zeta \in V_1 : L(\lambda\zeta) = \lambda L(\zeta),\\
L \in \mathcal{L}_{\mathbb{C}}(V_1, V_2) &\iff \forall z \in \mathbb{C}, \zeta \in V_1 : L(z\zeta) = zL(\zeta),\\
L \in \overline{\mathcal{L}}_{\mathbb{C}}(V_1, V_2) &\iff \forall z \in \mathbb{C}, \zeta \in V_1 : L(z\zeta) = \bar{z}L(\zeta).
\end{aligned}$$

It can be shown that the next statement holds: $\mathcal{L}_{\mathbb{R}}(V_1, V_2) = \mathcal{L}_{\mathbb{C}}(V_1, V_2) \oplus \overline{\mathcal{L}}_{\mathbb{C}}(V_1, V_2)$.

First, we check that $\mathcal{L}_{\mathbb{C}}(V_1, V_2) \cap \overline{\mathcal{L}}_{\mathbb{C}}(V_1, V_2) = \{0\}$. If we argue by contradiction, we assume that there exists complex both linear and antilinear operator L between V_1 and V_2 and $\zeta \in V_1$, such that $L(\zeta) \neq 0$. Then $L(i\zeta) = iL(\zeta) = -iL(\zeta)$ so $L(\zeta) = 0$, which is a contradiction.

Now, let us perceive arbitrary real linear operator L from V_1 into V_2. We can define operators $L_1, L_2 : V_1 \to V_2$ such that $L_1(\zeta) = \frac{1}{2}(L(\zeta) - iL(i\zeta))$ and $L_2(\zeta) = \frac{1}{2}(L(\zeta) + iL(i\zeta))$. We argue that $L = L_1 + L_2$ where $L_1 \in \mathcal{L}_{\mathbb{C}}(V_1, V_2)$, $L_2 \in \overline{\mathcal{L}}_{\mathbb{C}}(V_1, V_2)$. If we regard $z = x + iy$ as any complex number, and $\zeta \in V_1$ arbitrary, then

$$L_1(z\zeta)=\frac{1}{2}(L((x+iy)\zeta) - iL(i(x+iy)\zeta))=\frac{1}{2}(L(x\zeta + yi\zeta) - iL(-y\zeta + xi\zeta))$$

$$=\frac{1}{2}(xL(\zeta)+yL(i\zeta)+iyL(\zeta)-ixL(i\zeta))=\frac{1}{2}((x+iy)L(\zeta)$$
$$-(x+iy)iL(i\zeta))=(x+iy)L_1(\zeta)=zL_1(\zeta).$$

Analogously to this, we can get

$$L_2(z\zeta)=\frac{1}{2}(L((x+iy)\zeta)+iL(i(x+iy)\zeta))=\frac{1}{2}(L(x\zeta+yi\zeta)+iL(-y\zeta+xi\zeta))$$
$$=\frac{1}{2}(xL(\zeta)+yL(i\zeta)-iyL(\zeta)+ixL(i\zeta))=\frac{1}{2}((x-iy)L(\zeta)$$
$$+(x-iy)iL(i\zeta))=(x-iy)L_1(\zeta)=\bar{z}L_2(\zeta).$$

Claim 2.8 Let H_1 and H_2 be two complex Hilbert spaces and $L : H_1 \to H_2$ be a bounded complex linear operator. Then $L^* = L^\dagger$.

Proof Now, assume that L is bounded real linear operator from H_1 to H_2. Then, there are unique bounded operators L_1 and L_2, complex linear and complex antilinear, respectively, which satisfies $L = L_1 + L_2$. For these operators, we can find bounded complex linear operator $L_1^\dagger$ such that $\langle L_1^\dagger(w), z\rangle = \langle w, L_1(z)\rangle$, and bounded, complex antilinear operator $L_2^\ddagger$ defined with expression $\langle L_2^\ddagger(w), z\rangle = \langle w, L_2(z)\rangle$, for all $z \in H_1$, $w \in H_2$. We argue that $L_1^* = L_1^\dagger$ and $L_2^* = L_2^\ddagger$. First, since both complex linear and complex antilinear operators are real linear, we can define real adjonit for these operators. Also, if $\langle L_1^\dagger(w), z\rangle = \langle w, L_1(z)\rangle$ we get $\mathrm{Re}\,\langle L_1^\dagger(w), z\rangle = \mathrm{Re}\,\langle w, L_1(z)\rangle$, for all $z \in H_1$, $w \in H_2$. The same argument stands for the operator $L_2^\ddagger$. □

Proposition 2.9 *Let $\mathbb{B}_j$ be the unit ball of a complex Hilbert space H_j for $j = 1, 2$, respectively. Let $f : \mathbb{B}_1 \to \mathbb{B}_2$ be a pluriharmonic mapping such that $f(\xi) = 0$, for some $\xi \in \mathbb{B}_1$. Assume that f is differentiable at some point $z_0 \in \partial\mathbb{B}_1$ and $f(z_0) = w_0 \in \partial\mathbb{B}_2$. Then there exists a constant $\lambda \in \mathbb{R}$ such that*

$$Df(z_0)^* w_0 = \lambda \frac{1-|\xi|^2}{|1-\langle z_0, \xi\rangle|^2} z_0,$$

where $\lambda \geq \frac{2}{\pi}$.

Proof Let $\varphi_\xi(z) = A\frac{\xi - z}{1-\langle z,\xi\rangle}$ be the holomorphic automorphism of $\mathbb{B}_1$ where $A = s_\xi Id + \frac{\xi\langle\cdot,\xi\rangle}{1+s_\xi}$, $s_\xi = \sqrt{1-|\xi|^2}$.

Assume that $\varphi_\xi(z_0) = p \in \partial\mathbb{B}_2$. Let $g(z) = f \circ \varphi_\xi(z)$. Then g is a pluriharmonic mapping of $\mathbb{B}_1$ into $\mathbb{B}_2$ satisfying

$$g(0) = f \circ \varphi_\xi(0) = f(\xi) = 0,$$

and

$$g(p) = f \circ \varphi_\xi(p) = f(z_0) = w_0 \in \partial\mathbb{B}_2.$$

According to Proposition 2.5 we know that there exists a number $\lambda \geq \frac{2}{\pi}$ such that

$$D_g^*(p)w_0 = \lambda p.$$

From $\varphi_\xi^2 = Id$ it follows that $D\varphi_\xi(p)D\varphi_\xi(z_0) = Id$ and therefore (1) $D\varphi_\xi(z_0)^* D\varphi_\xi(p)^* = Id$. Since $Dg(p) = Df(z_0)D\varphi_\xi(p)$, we have $Dg(p)^* = (Df(z_0)D\varphi_\xi(p))^* = D\varphi_\xi(p)^* Df(z_0)^*$ and therefore (2) $D\varphi_\xi(p)^* D_f(z_0)^* w_0 = \lambda p$. By (1) and (2) we find $Df(z_0)^* w_0 = \lambda D\varphi_\xi(z_0)^* p$.

From Lemma 2.7 we conclude that $\langle z_0, D\varphi_\xi(z_0)^\dagger p\rangle = \mu$, where $\mu = \frac{1-|\xi|^2}{|1-\langle z_0,\xi\rangle|^2}$. Now, we can conclude that $\langle D\varphi_\xi(z_0)z_0, p\rangle = \mu$, from which $\mathrm{Re}\,\langle D\varphi_\xi(z_0)z_0, p\rangle = \mu$. From Theorem 1.1 we conclude that $D\varphi_\xi(z_0)^* p = \mu_1 z_0$, for some $\mu_1 > 0$. From the proof of Theorem 1.1 we have $\mu_1 = \mathrm{Re}\,\langle D\varphi_\xi(z_0)z_0, p\rangle = \langle D\varphi_\xi(z_0)z_0, p\rangle = \mu$. □

Suppose that $\Omega \subset \mathbb{R}^n$ is a domain and H is a Hilbert space. Let $f : \Omega \to H$ be a function such that $f \in C^2(\Omega)$. We define partial derivatives with respect to coordinates x_i, $i = 1, \dots n$ of the base $\{e_1, ..., e_n\}$ in $\mathbb{R}^n$ at the point $a \in \Omega$ with formula:

$$\frac{\partial f}{\partial x_i}(a) = Df(a)e_i.$$

Definition 2.10 Function f is harmonic in a domain Ω if $\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(a) = 0$ for every $a \in \Omega$.

Let us denote with $\mathbf{B}^n$ and $\mathbf{S}^{n-1}$ the unit ball and the unit sphere of $\mathbb{R}^n$.

It is well-known that a harmonic function $u \in L^\infty(\mathbf{B}^n)$ has the following integral representation

$$u(x) = \mathcal{P}[f](x) = \int_{\mathbf{S}^{n-1}} P(x,\zeta) f(\zeta) d\sigma(\zeta),$$

where f is the boundary function of $\mathbf{S}^{n-1}$, and

$$P[x,\zeta] = \frac{1-|x|^2}{|x-\zeta|^n} \quad , \quad \zeta \in \mathbf{S}^{n-1}$$

is the Poisson kernel and σ is the unique normalized rotation invariant Borel measure on $\mathbf{S}^{n-1}$. According to [1], we know that if u is a harmonic self-mapping of $\mathbf{B}^n$ such that $u(0) = 0$, then

$$|u(x)| \leq U(rN), \tag{2}$$

where $r = |x|$, $N = \{0, \cdots, 0, 1\}$ and U is a harmonic function of $\mathbf{B}^n$ into $[-1, 1]$ defined by

$$U(x) = P[\chi_{S^+} - \chi_{S^-}](x) \tag{3}$$

where χ is the indicator function and $S^+ = \{x \in \mathbf{S}^{n-1} : x_n \geq 0\}$, $S^- = \{x \in \mathbf{S}^{n-1} : x_n \leq 0\}$. We refer to [11, Chapter 6] for more details.

Recall that the *hypergeometric function* ${}_pF_q$ is defined for $|x| < 1$ by the power series ([16, (2.1.2)])

$$ {}_pF_q[a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; x] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!}. $$

Here $(a)_k$ is the *Pochhammer symbol* and given as follows $(a)_k = \frac{\Gamma(k+a)}{\Gamma(a)}$.

The following result is the so-called *Heinz-Schwarz inequality*.

Lemma 2.11 *[1, Lemma 2.3] The function* $V(r) = \frac{\partial U(rN)}{\partial r}$, $0 \le r \le 1$ *is decreasing on the interval* $[0, 1]$, *and we have*

$$ V(r) \ge V(1) = C_n =: \frac{n!\left(1 + n - (n-2){}_2F_1\left[\frac{1}{2}, 1; \frac{3+n}{2}; -1\right]\right)}{2^{3n/2}\Gamma\left[\frac{1+n}{2}\right]\Gamma\left[\frac{3+n}{2}\right]}. \tag{4} $$

We refer the readers to [1, Remark 2.7] for more details on the constant C_n and related functions, when $n = 2, 3, 4$.

A version of Theorem 1.2 [6] holds for harmonic functions, where codomain is the a ball $\mathbb{B}$ in a Hilbert space.

Theorem 2.12 *Suppose that* $f : \mathbf{B}^n \to \mathbb{B}$ *is a harmonic function, such that* $f(0) = 0$, *and* f *has a continuous extension to the point* $a \in \partial\mathbf{B}^n$ *such that* $f(a) = b \in \partial\mathbb{B}$. *Then* (1) $\limsup_{r\to 1_-} |D_r f(ra)| \ge C_n$.

Suppose in addition that f *has a differentiable extension to* a.

(i) Then there exists a positive number $\lambda \in \mathbb{R}$ *such that* $Df(a)^*b = \lambda a$ *and*
(ii) $\lambda \ge C_n$, *where* C_n *is given by (4).*
(iii) In particularly if $n = 2$, *we have* $\lambda \ge \frac{2}{\pi}$. *This is sharp.*

Proof (i) follows from Claim 2.3. Set $u = \mathrm{Re}\,\langle f, b\rangle$. Since u is harmonic and it maps $\mathbf{B}^n$ into $(-1, 1)$, $u(0) = 0$ and $u(a) = 1$,

using Theorem 6.24 [11] we have $u(x) \le U(rN)$ and therefore

$$ 1 - u(x) \ge 1 - U(rN), \quad \text{for} \quad r = |x| < 1. $$

Hence

$$ \frac{1 - u(x)}{1 - |x|} \ge \frac{1 - U(rN)}{1 - r}. $$

Next if we define $u_0(t) = u(ta)$ and $U_0(t) = U(tN)$, $0 < t < 1$, for every $0 < t < 1$ there are $c_t, d_t \in (t, 1)$ such that $1 - u_0(t) = u_0'(c_t)(1-t)$, $1 - U_0(t) = U_0'(d_t)(1-t)$ and $u_0'(c_t) \ge U_0'(d_t)$. Hence by Lemma 2.11, $u_0'(c_t) \ge C_n$ and therefore we get (1). If in addition f has differentiable extension to a, then

$$ D_r u(a) = \lim_{|x|\to 1^-} \frac{1 - u(x)}{1 - |x|} \ge \lim_{r\to 1^-} \frac{1 - U(rN)}{1 - r} = \left.\frac{\partial U(rN)}{\partial r}\right|_{r=1} = C_n. $$

Since by Claim 2.3 (iv), $\lambda = \text{Re}\,\langle D_r f(a), b\rangle = D_r u(a)$, (ii) follows. An application of Proposition 2.4 yields (iii). For additional details, see [13]. □

3 Boundary Schwarz lemma and Banach Spaces

We will use notation from [3]. Let X and Y be real or complex Banach spaces with norm $|\cdot|_X$ and $|\cdot|_Y$ respectively. We denote with $\mathcal{L}(X, Y)$ the space of all continuous linear operators from X into Y with the standard operator norm

$$|A| = \sup_{x \in X\setminus\{0\}} \frac{|Ax|}{|x|},$$

where $A \in \mathcal{L}(X, Y)$. Then $\mathcal{L}(X, Y)$ is a Banach space with respect to this norm. Denote by X^* the dual space of the real or complex Banach space X. For $x \in X \setminus \{0\}$, let

$$T(x) = \{l_x \in X^* : l_x(x) = |x| \text{ and } |l_x| = 1\}.$$

Then the well-known Hahn–Banach theorem implies that $T(x) \neq \emptyset$.

Let f be a mapping of a domain $\Omega \subset X$ into a real or complex Banach space Y, where X is a complex Banach space. We say that f is differentiable at $z \in \Omega$ if there exists a bounded real linear operator $Df(z) : X \to Y$ such that

$$\lim_{|h|\to 0^+} \frac{|f(z+h) - f(z) - Df(z)h|}{|h|} = 0.$$

Here $Df(z)$ is called the Fréchet derivative of f at z. If Y is a complex Banach space and $Df(z)$ is bounded complex linear for each $z \in \Omega$, then f is said to be holomorphic on Ω. Let Ω be a domain in a complex Banach space X. A mapping f of Ω into a real or complex Banach space Y is said to be pluriharmonic if the restriction of $l \circ f$ to every holomorphic curve is harmonic for any $l \in Y^*$.

Theorem 3.1 *Suppose that B_1 and B_2 are the unit balls of the complex Banach spaces X and Y, respectively, and $f : B_1 \to B_2$ is a pluriharmonic mapping. Assume that the function f is differentiable at $b \in \partial B_X$ with $|f(b)| = 1$. Then we have*

$$|Df(b)b| \geq s^-(|f(0)|).$$

Proof We consider the function $p(z) = \text{Re}\,(l_{f(b)}(f(zb)))$, for $z \in \mathbb{U}$. where $l_{f(b)} \in T(f(b))$. Since f is pluriharmonic we have that the function p is harmonic function on $\mathbb{U}$. Also, from $|l_{f(b)}| = 1$ we get $|\text{Re}\,(l_{f(b)}(f(zb)))| \leq |l_{f(b)}(f(zb))| \leq |f(zb)| < 1$, so we get that the function p maps the unit disc into interval $(-1, 1)$. From the definition of $l_{f(b)}$ we can conclude that $p(1) = 1$. Also, we have that $|p(0)| = |\text{Re}\, l_{f(b)}(f(0))| \leq |l_{f(b)}(f(0))| \leq |f(0)|$. Now we can conclude that

$$|D_r p(1)| \geq s^-(|p(0)|) \geq s^-(|f(0)|),$$

since the function s^- is decreasing on $(-1, 1)$. On the other hand, we have that $|D_r p(1)| \le |Df(b)b|$. Indeed, we have that

$$|D_r p(1)| = \lim_{r \to 1^-} \frac{|p(1) - p(r)|}{1 - r} = \lim_{r \to 1^-} \left| \operatorname{Re} l_{f(b)} \frac{f(b) - f(rb)}{1 - r} \right|$$
$$= |\operatorname{Re} l_{f(b)} Df(b)b| \le |Df(b)b|,$$

which concludes our proof. □

4 Hyperbolic Harmonic Functions in Higher Dimensions

We use notation from [9]. Let $\mathbf{B}^n$ be the unit ball in $\mathbb{R}^n$ and $\mathbf{S}^{n-1}$ be the boundary of the unit ball and Δ is Laplacian partial differential operator. Consider next Laplace-Beltrami operator

$$\Delta_0 = \frac{1 - |x|^2}{4} \left(\Delta + \frac{2(n-2)}{1 - |x|^2} \langle x, \nabla \rangle \right).$$

Any twice continuously differentiable function h which is defined on $\mathbf{B}^n$ and fulfills $\Delta_0 h = 0$ is said to be *hyperbolic harmonic* on $\mathbf{B}^n$.

In the sequel we will use some specific properties of both harmonic and hyperbolic-harmonic kernel, which are listed below:

a) There exists a Poisson formula for hyperbolic harmonic as well as for harmonic functions on $\mathbf{B}^n$. Let σ denote the usual surface measure on $\mathbf{S}^{n-1}$ and f be a σ-integrable function on $\mathbf{S}^{n-1}$. Set $x \in \mathbf{B}^n$ and $\eta \in \mathbf{S}^{n-1}$. Depending on whether we define $P(x, \eta)$ as

$$\frac{1}{\sigma(\mathbf{S}^{n-1})} \frac{1 - |x|^2}{|x - \eta|^n} \text{ or as } \frac{1}{\sigma(\mathbf{S}^{n-1})} \frac{(1 - |x|^2)^{n-1}}{|x - \eta|^{2(n-1)}},$$

we get a harmonic or a hyperbolic harmonic function on $\mathbf{B}^n$ by

$$h(x) = P[f](x) = \int_{\mathbf{S}^{n-1}} P(x, \eta) f(\eta) \mathrm{d}\sigma(\eta).$$

In the sequel we use the same notation P for both Poisson kernels.

b)
$$1 = P[\mathbf{1}](x) = \int_{\mathbf{S}^{n-1}} P(x, \eta) \mathrm{d}\sigma(\eta), \tag{5}$$

where $\mathbf{1}(\eta) = 1$ for all $\eta \in \mathbf{S}^{n-1}$ is a constant function.
This is an immediate consequence of the fact that constant functions belongs to classes of hyperbolic and hyperbolic functions, both respectively.

c) Harmonic (resp. hyperbolic-harmonic) functions possess the mean value property with respect to (hyperbolic) spheres.

d) The theorem of Fatou, concerning the $\sigma-$a.e, existence of non-tangential limits, is valid in both cases.

For convenience we set

$$M_c^n(|x|) = 2P[\chi_{S(c,\tilde{x})}](x) - 1 \tag{6}$$

$$m_c^n(|x|) = 2P[\chi_{S(c,-\tilde{x})}](x) - 1, \tag{7}$$

where $x \in \mathbf{B}^n$, $\tilde{x} = \frac{x}{|x|}$ for $x \neq 0$; $\tilde{x} = e_1$ for $x = 0$ and $S(c, \tilde{x})$ denotes the polar cap with center $\tilde{x}$ and $\sigma-$measure c. Also, χ_A is an indicator function of the set A. It is easy to verify that the expressions on the right-hand side of (6) inherit the rotational invariance of the measure σ.

For derivation on the explicit formula (8), we refer to the paper [17], specifically, to Proposition 5.10. In this proposition we use the following notation: σ_{n-1} is surface area of the sphere $\mathbf{S}^{n-1}$ and φ is an angle between radius vector of point $\eta \in \mathbf{S}^{n-1}$ and radius vector of the point $\tilde{x}$.

Proposition 4.1 *([17], Proposition 5.10) If f is a function on $\mathbf{S^{n-1}}$ depending only on φ, then*

$$\int_{\mathbf{S^{n-1}}} f(\eta) \mathrm{d}\sigma(\eta) = \sigma_{n-2} \int_0^\pi f(\varphi) \sin^{n-2} \varphi \mathrm{d}\varphi.$$

Since $|x-\eta|^2 = 1-2r\cos\varphi + r^2$ we have that both our kernels depend only on φ. Let us define $\sigma_*(n) = \frac{\sigma_{n-2}}{\sigma_{n-1}}$. Using formula $\sigma_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ we get $\sigma_*(n) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n/2)}{\Gamma((n-1)/2)}$.

Using Proposition 4.1 we can rewrite (6) as

$$M_c^n(|x|) = 2\sigma_*(n)(1-|x|^2)^\nu \int_0^{\alpha(c)} \frac{\sin^{n-2} t}{(1-2|x|\cos t + |x|^2)^\mu} \mathrm{d}t - 1, \tag{8}$$

$$m_c^n(|x|) = 2\sigma_*(n)(1-|x|^2)^\nu \int_{\pi-\alpha(c)}^{\pi} \frac{\sin^{n-2} t}{(1-2|x|\cos t + |x|^2)^\mu} \mathrm{d}t - 1, \tag{9}$$

where $(\nu, \mu) = (1, n/2)$ in harmonic case and $(\nu, \mu) = (n-1, n-1)$ in the hyperbolic-harmonic case and $\alpha(c)$ is the spherical angle of $S(c, \tilde{x})$.

Theorem 4.2 *[9] Let h be a harmonic or hyperbolic-harmonic function taking values in $(-1, 1)$ and $h(0) = a$, $-1 < a < 1$. Then for $c = \frac{a+1}{2}$ and all $x \in \mathbf{B}^n$*

$$m_c^n(|x|) \leq h(x) \leq M_c^n(|x|).$$

Equality on the right (resp., left) hand side for some $z \in \mathbf{B}^n \setminus \{0\}$ implies

$$h(x) = 2P[\chi_{S(c,\tilde{z})}](x) - 1 \quad (\textit{respectively, } h(x) = 2P[\chi_{S(c,-\tilde{z})}](x) - 1),$$

for all $x \in \mathbf{B}^n$

Lemma 4.3 *Let* $(\nu, \mu) = (1, n/2)$ *(harmonic case). Then*

$$\frac{\mathrm{d}M_c^n}{\mathrm{d}r}(r)\Big|_{r=1} = \frac{2^{2-n}}{\sqrt{\pi}} \frac{\Gamma(n/2)}{\Gamma((n-1)/2)} \int_{\alpha(c)}^{\pi} \frac{\sin^{n-2} t}{\sin^n(t/2)} \mathrm{d}t.$$

Proof We will use the following notation $T(r) = \frac{1-M_c^n(r)}{1-r}$. Then:

$$\frac{\mathrm{d}M_c^n}{\mathrm{d}r}(r)\Big|_{r=1} = \lim_{r\to 1^-} T(r).$$

By using formula (6) we have

$$T(r) = \frac{1-(2P[\chi_{S(c,\tilde{z})}](x)-1)}{1-r} = \frac{2(1-P[\chi_{S(c,\tilde{z})}](x))}{1-r}.$$

If we use formula (5) we get

$$T(r) = \frac{2P[\mathbf{1}-\chi_{S(c,\tilde{z})}](x)}{1-r} = \frac{2P[\chi_{\mathbf{S}^{n-1}\setminus S(c,\tilde{z})}](x)}{1-r}.$$

Now, by using version of Proposition 4.1 we get the following important result:

$$T(r) = 2\sigma_*(n)(1+r)\int_{\alpha(c)}^{\pi} \frac{\sin^{n-2} t}{(1-2r\cos t + r^2)^{n/2}} \mathrm{d}t.$$

We can reformulate this equation, in the following manner

$$T(r) = 2\sigma_*(n)(1+r)\int_{\alpha(c)}^{\pi} Q(r,t)\mathrm{d}t,$$

where $Q(r,t) = \frac{\sin^{n-2} t}{(1-2r\cos t + r^2)^{n/2}}$. Since we have limit of proper integral in the last expression, we can derive next formula

$$\frac{\mathrm{d}M_c^n}{\mathrm{d}r}(r)\Big|_{r=1} = 4\sigma_*(n)\int_{\alpha(c)}^{\pi} \frac{\sin^{n-2} t}{2^n \sin^n(t/2)} \mathrm{d}t.$$

□

Let us define

$$D_n(c) = \frac{2^{2-n}}{\sqrt{\pi}} \frac{\Gamma(n/2)}{\Gamma((n-1)/2)} \int_{\alpha(c)}^{\pi} \frac{\sin^{n-2} t}{\sin^n(t/2)} \mathrm{d}t.$$

Then the next theorem holds:

Theorem 4.4 *Suppose that* $f : \mathbf{B}^n \to \mathbf{B}^m$ *is a harmonic function, such that* $f(0) = a_0$, *and* f *has a continuous extension to the point* $x_0 \in \partial\mathbf{B}^n$ *such that* $f(x_0) = y_0 \in \partial\mathbf{B}^m$.

Then $\limsup_{r\to 1_-} |D_r f(rx_0)| \geq D_n(c)$, *where* $c = \frac{1+a}{2}$ *and* $a = \langle a_0, y_0\rangle$.

If, in addition, f has a differentiable continuation at point x_0, then there exists a positive number $\lambda \in \mathbb{R}$ such that $Df(x_0)^* y_0 = \lambda x_0$ and

$$\lambda \geq D_n(c).$$

This is sharp.

Proof Let us define the function $h(x) = \langle f(x), y_0 \rangle$. This function is harmonic in $\mathbf{B}^n$, with $h(0) = a$, and $h(x_0) = 1$. Since, by the Theorem of Fotou $M_c^n(1) = 1$, using Theorem 4.2 we have an implication

$$\frac{h(x_0) - h(rx_0)}{1 - r} \geq \frac{1 - M_c^n(r)}{1 - r}.$$

If $u(r) = h(rx_0), r \in [0, 1)$ then $u'(r) = Dh(rx_0)x_0 = D_r h(rx_0)$. From Lagrange's theorem we have that for every $r \in [0, 1)$ there exists $r_0 \in (r, 1)$ such that

$$\frac{1 - u(r)}{1 - r} = u'(r_0) = D_r h(r_0 x_0) \geq \frac{1 - M_c^n(r)}{1 - r}.$$

This means that $\limsup\limits_{r \to 1^-} D_r h(rx_0) \geq \liminf\limits_{r \to 1_-} \frac{1-u(r)}{1-r} \geq D_n(c)$. Cauchy–Schwarz inequality provides us that $|D_r f(x)| \geq D_r h(x)$, which gives us

$$\limsup_{r \to 1^-} |D_r f(rx_0)| \geq D_n(c).$$

□

At the end of this section we will investigate whether or not we can formulate the similar version of Schwarz lemma on the boundary, for hyperbolic harmonic functions.

Lemma 4.5 *Let $(\nu, \mu) = (n-1, n-1)$, where $n > 2$ (hyperbolic-harmonic case). Then*

$$\frac{\mathrm{d}M_c^n}{\mathrm{d}r}(r)\Big|_{r=1} = 0.$$

Proof Like in previous lemma, we have

$$\frac{\mathrm{d}M_c^n}{\mathrm{d}r}(r)\Big|_{r=1} = \lim_{r \to 1^-} T(r),$$

Let us define $Q_{hyp}(r, t) = \frac{\sin^{n-2} t}{(1-2r\cos t + r^2)^{n-1}}$. Then

$$T(r) = 2\sigma_*(n)(1-r)^{n-2}(1+r)^{n-1} \int_{\alpha(c)}^{\pi} Q_{hyp}(r, t)\mathrm{d}t.$$

Also, define $J_{hyp}(r) = \int_{\alpha(c)}^{\pi} Q_{hyp}(r,t)\mathrm{d}t$. We can pass with the limit, under proper integral sign, to get

$$\lim_{r\to 1^-} J_{hyp}(r) = J_{hyp} = \int_{\alpha(c)}^{\pi} q_{hyp}(t)dt,$$

where $q_{hyp}(t) = 4^{-n+1}\sin^{n-2} t \sin^{-2(n-1)} t/2$.
From this we can draw a conclusion $T(r) \sim d_n(1-r)^{n-2}$, $r \to 1^-$. This immediately gives our assertion. □

By this Lemma we conclude that we have different situation concerning hyperbolic-harmonic function, mappings of unit ball in $\mathbb{R}^n$ into unit ball in $\mathbb{R}^m$, in comparison with the harmonic function in same settings. Namely, we found explicit hyperbolic-harmonic function, that maps unit ball into interval $(-1, 1)$ such that $u(x_0) = 1$, for some x_0 on the boundary of the unit ball, but radial derivative in the point x_0 is vanishing.

At the first glance, this may look as surprise, having in mind famous Hopf lemma. Function u is satisfying $L(u) = 0$, where L is uniformly elliptical partial differential operator of second order, it has global maximum at point x_0 on the boundary of unit ball, so we expected that normal derivative in the point x_0 must be greater than zero.

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, $x \in \Omega$ be a point and u belongs to $C^2(\Omega)$. We define

$$Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u, a^{ij} = a^{ji}.$$

The summation convention that repeated indices indicate summation from 1 to n is followed here. We adopt the following definitions: operator L is *elliptic* in point $x \in \Omega$ if the coefficient matrix $A(x) = [a^{ij}(x)]$ is positive definite. If $\Lambda(x)$, $\lambda(x)$ are the greatest and the smallest eigenvalue of matrix $A(x)$ and Λ/λ is bounded in Ω we say that L is *uniformly elliptic* in Ω. Also we will need next condition. Let $k > 0$ is a constant and $x \in \Omega$ be an arbitrary

$$\frac{|b^i(x)|}{\lambda(x)} \leq k, i = 1, \dots, n. \tag{10}$$

Now, we can formulate Hopf Lemma.

Lemma 4.6 *(Hopf lemma, [18], Lemma 3.4) Suppose that L is uniformly elliptic operator, that satisfies condition (10), $c = 0$ and $Lu \geq 0$ in Ω. Let $x_0 \in \partial\Omega$ be such that*

(i) u is continuous at x_0;
(ii) $u(x_0) > u(x)$ for all $x \in \Omega$;
(iii) $\partial\Omega$ satisfies an interior sphere condition at x_0.

Then the outer normal derivative of u at x_0, if it exists, satisfies the strict inequality

$$\frac{\partial}{\partial \nu} u(x_0) > 0.$$

What turns out to be is that Hopf lemma demands some conditions on the coefficients standing by the first-order derivatives of the elliptic partial differential operator L, that hyperbolic-harmonic functions does not satisfies. We have that hyperbolic-harmonic functions satisfies $Lu = \Delta_0 u = 0$, where $A(x) = Id$ and $b_i(x) = \frac{2(n-2)}{1-|x|^2}, i = 1, \dots, n$. Since $\Lambda(x) = \lambda(x) = 1, x \in \mathbf{B}^n$, we conclude that operator Δ_0 does not satisfies condition (10) in $\mathbf{B}^n$, so we can not apply Hopf lemma in this situation.

A part of our result can be interpreted as a confirmation that condition (10) cannot be excluded from the statement of Hopf lemma.

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5. Appendix

Motivated by the role of the Schwarz lemma in Complex Analysis and numerous fundamental results, see for instance [4, 19] and references therein, in 2016, the first author [2] has posted on ResearchGate the project "Schwarz lemma, the Carathéodory and Kobayashi Metrics and Applications in Complex Analysis".[4]

In this project and in [4], cf. also [21], we developed the method related to holomorphic mappings with strip codomain (we refer to this method as the approach via the Schwarz–Pick lemma for holomorphic maps from the unit disc into a strip; shortly "planar strip method"). It is worth mentioning that the Schwarz lemma has been generalized in various directions; see [2, 3] and the references therein.

Even in planar case researches had some difficulties in handling Schwarz lemma for harmonic maps of the unit disc into self which does not fix the origin. It seems that the researchers have overlooked Burgeth and H. W. Hethcote results and they have had some difficulties to handle the case $f(0) \neq 0$ in this context; see for more details [5, 13, 14].

In joint paper of the first author with M. Svetlik [13] using "planar strip method" which is a completely different approach than B. Burgeth [9], we get a simple proof of an optimal version of the Schwarz lemma for real valued harmonic functions (without the assumption that 0 is mapped to 0 by the corresponding map), which improves H. W. Hethcote result[5].

In joint paper of the first author with A. Khalfallah and M. Mhamdi [12], some properties of mappings admitting a Poisson-type integral representations and the boundary Schwarz lemma were considered.

Presently on this project the first author works with some of his associates: A. Khalfallah, M. Arsenović, M. Svetlik, M. Mhamdi, B. Purtić, H.P. Li, J. Gajić and the second author of this paper.

Chinese mathematicians have made a great contribution to this field but here we will mention only some whose results are related to our results. For some interesting complex n-dimensional generalisations of classical Schwarz lemma type results see Jian-Feng Zhu's articles [22] and [23]. In paper [24] the authors proved Schwarz lemma

[4] Various discussions regarding the subject can also be found in the Q&A section on ResearchGate under the question "What are the most recent versions of the Schwarz lemma?" [20]; see also [30].

[5] Note here that Burget's spherical cap method yield optimal estimate in both planar and spatial case

on the boundary for holomorphic mappings between unit balls in $\mathbb{C}^n$, and some of theirs rigidity properties. Generalization of this theorem, for separable complex Hilbert space was given by Z. Chen, Y. Liu and Y. Pan in [15]. While proving Proposition 2.9, we independently proved Lemma 2.7, but later found that result proven in [15], as it can be seen in corresponding reference. In [25] the authors proved a higher order Schwarz-Pick lemma for holomorphic mappings between unit balls in complex Hilbert spaces.

For generalizations of Schwarz lemmas for planar harmonic mappings into the sharp forms of Banach spaces we refer the interested reader to Chen, Hamada et al. [3, 26] and literature cited there for the background. Recall the main purpose of the paper [3] is to develop some methods to investigate the Schwarz type lemmas for holomorphic mappings and pluriharmonic mappings in Banach spaces. Initially, they extend the classical Schwarz lemmas for holomorphic mappings to Banach spaces. Furthermore, they improve and generalize the classical Schwarz lemmas for planar harmonic mappings and obtain sharp versions for Banach spaces, and present some applications to sharp boundary Schwarz type lemmas for pluriharmonic mappings in Banach spaces. The obtained results provide improvements and generalizations of the corresponding results in [26] (cf. also [6]).

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