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A Note on Hyperspaces by Closed Sets with Vietoris Topology

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Abstract

For a topological space X, let $CL(X)$ be the set of all non-empty closed subset of X, and denote the set $CL(X)$ with the Vietoris topology by $(CL(X), \mathbb{V})$. In this paper, we mainly discuss the hyperspace $(CL(X), \mathbb{V})$ when X is an infinite countable discrete space. As an application, we first prove that the hyperspace with the Vietoris topology on an infinite countable discrete space contains a closed copy of nth power of Sorgenfrey line for each $n \in \mathbb{N}$. Then we investigate the tightness of the hyperspace $(CL(X), \mathbb{V})$ and prove that the tightness of $(CL(X), \mathbb{V})$ is equal to the set-tightness of X. Moreover, we extend some results about the generalized metric properties on the hyperspace $(CL(X), \mathbb{V})$. Finally, we give a characterization of X such that $(CL(X), \mathbb{V})$ is a γ-space.

Keywords Hyperspace · Countable set-tightness · Compact metrizable · γ-space · Weakly first-countable · D_1-space · D_0-space

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1 Introduction

It is well known that the topics of the hyperspace has been the focus of much research, see [7–11, 15–25, 27]. There are many results on the hyperspace $CL(X)$ of closed subsets of a topological space equipped with various topologies. In this paper, we endow $CL(X)$ with the Vietoris topology $\mathbb{V}$, or the so-called finite topology, the base of which consists of all subsets of the following form:

$$\langle U_1, ..., U_k\rangle = \{K \in CL(X) : K \subset \bigcup_{i=1}^{k} U_i \text{ and } K \cap U_j \neq \emptyset, 1 \leq j \leq k\},$$

where each U_i is open in X and $k \in \mathbb{N}$. We denote the hyperspace $CL(X)$ with Vietoris topology by $(CL(X), \mathbb{V})$. In 1997, Holá and Levi in [16, Corollary 1.8] gave a characterization of the first countability of $(CL(X), \mathbb{V})$; in 2003, Holá, Pelant and Zsilinszky in [15, Theorem 3.1] proved that $(CL(X), \mathbb{V})$ is developable iff $(CL(X), \mathbb{V})$ is Moore iff $(CL(X), \mathbb{V})$ is metrizable iff $(CL(X), \mathbb{V})$ has a σ-discrete network iff X is compact and metrizable. So it is natural for us to consider the following two problems:

Problem 1.1 *Let $\mathcal{C}$ be a proper subclass of the class of first-countable spaces, and let $\mathcal{P}$ be a topological property. If $(CL(X), \mathbb{V}) \in \mathcal{C}$, does X have the property $\mathcal{P}$?*

Problem 1.2 *Let $\mathcal{C}$ be a class of generalized metrizable spaces. If $(CL(X), \mathbb{V}) \in \mathcal{C}$, is X compact and metrizable?*

The paper is organized as follows. In Sect. 2, we introduce the necessary notation and terminology which are used in the paper. In Sect. 3, we mainly discuss the hyperspace $(CL(D(\omega)), \mathbb{V})$ and prove that $(CL(D(\omega)), \mathbb{V})$ contains a closed copy of $\mathbb{S}^n$ for each $n \in \mathbb{N}$, where $\mathbb{S}$ is the Sorgenfrey line. In Sect. 4, we prove that the tightness of $(CL(X), \mathbb{V})$ is equal to the set-tightness of X; moreover, we give a characterization of $(CL(X), \mathbb{V})$ which is Fréchet-Urysohn. In Sect. 5, we give some answers to Problems 1.1 and 1.2, respectively. In particular, we prove that $(CL(X), \mathbb{V})$ is quasi-developable iff $(CL(X), \mathbb{V})$ is a semi-stratifiable space iff $(CL(X), \mathbb{V})$ is symmetrizable iff $(CL(X), \mathbb{V})$ is a D_1-space iff X is compact and metrizable; moreover, we prove that $(CL(X), \mathbb{V})$ is a γ-space iff X is a separable metrizable space and $S(X)$ is compact, where $S(X)$ is the set of all non-isolated points of X.

2 Preliminaries

In this paper, the base space X is always supposed to be regular. Let $\mathbb{N}$ and ω denote the sets of all positive integers and all nonnegative integers, respectively. Let $\mathbb{S}$ be the real line endowed with half open interval topology, that is, Sorgenfrey line. For a space X, $S(X)$ is the set of all non-isolated points of X. For undefined notations and terminologies, the reader may refer to [6], [12] and [22].

Let X be a topological space and $A \subseteq X$ be a subset of X. The *closure* of A in X is denoted by $\overline{A}$. A subset P of X is called a *sequential neighborhood* of $x \in X$, if each sequence converging to x is eventually in P. A subset U of X is called *sequentially open* if U is a sequential neighborhood of each of its points. A subset F of X is called *sequentially closed* if $X \setminus F$ is sequentially open. The space X is called a *sequential space* if each sequentially open subset of X is open. The space X is said to be *Fréchet-Urysohn* if, for each $x \in \overline{A} \subset X$, there exists a sequence $\{x_n\}$ in A such that $\{x_n\}$ converges to x.

Definition 2.1 Let $\mathscr{P}$ be a cover of a space X such that (i) $\mathscr{P} = \bigcup_{x\in X} \mathscr{P}_x$; (ii) for each $x \in X$, if $U, V \in \mathscr{P}_x$, then $W \subseteq U \cap V$ for some $W \in \mathscr{P}_x$; (iii) $x \in \bigcap \mathscr{P}_x$ for each $x \in X$; and (iv) for each point $x \in X$ and each open neighborhood U of x there is some $P \in \mathscr{P}_x$ such that $x \in P \subseteq U$.

• The family $\mathscr{P}$ is called a *weak base* for X if, for every $G \subset X$, the set G must be open in X whenever for each $x \in G$ there exists $P \in \mathscr{P}_x$ such that $P \subset G$, and X is *weakly first-countable* if X has a weak base $\mathscr{P}$ and $\mathscr{P}_x$ is countable for each $x \in X$.

Definition 2.2 Let $\mathscr{P}$ be a family of subsets of a space X. The family $\mathscr{P}$ is called a *k-network* if for every compact subset K of X and an arbitrary open set U containing K in X there is a finite subfamily $\mathscr{P}' \subseteq \mathscr{P}$ such that $K \subseteq \bigcup \mathscr{P}' \subseteq U$.

A space X is said to be *Lašnev* if it is the continuous closed image of some metric space. The following Lašnev space in Definition 2.3 plays an important role in the study of the generalized metric theory.

Definition 2.3 Let κ be an infinite cardinal. For each $\alpha \in \kappa$, let T_α be a sequence converging to $x_\alpha \notin T_\alpha$. Let $T = \bigoplus_{\alpha\in\kappa}(T_\alpha \cup \{x_\alpha\})$ be the topological sum of $\{T_\alpha \cup \{x_\alpha\} : \alpha \in \kappa\}$. Then $S_\kappa = \{x\} \cup \bigcup_{\alpha\in\kappa} T_\alpha$ is the quotient space obtained from T by identifying all the points $x_\alpha \in T$ to the point x. The space S_κ is called a *sequential fan*.

The following space is not a Lašnev space.

Definition 2.4 A space X is called an S_2-*space* (*Arens' space*) if

$$X = \{\infty\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_{n,m} : m, n \in \omega\}$$

and the topology is defined as follows: Each $x_{n,m}$ is isolated; a basic neighborhood of x_n is $\{x_n\} \cup \{x_{n,m} : m > k\}$, where $k \in \omega$; a basic neighborhood of ∞ is

$$\{\infty\} \cup (\bigcup \{V_n : n > k\}) \text{ for some } k \in \omega,$$

where V_n is a neighborhood of x_n for each $n \in \omega$.

Given a topological space X, we define its *hyperspace* as the following set:

$$CL(X) = \{H : H \text{ is non-empty, closed in } X\}.$$

We endow $CL(X)$ with *Vietoris topology* defined as the topology generated by the following family

$$\{\langle U_1, \ldots, U_k\rangle : U_1, \ldots, U_k \text{ are open subsets of } X, k \in \mathbb{N}\},$$

where $\langle U_1, ..., U_k\rangle = \{H \in CL(X) : H \subset \bigcup_{i=1}^{k} U_i \text{ and } H \cap U_j \neq \emptyset, 1 \leq j \leq k\}$. We denote this hyperspace with Vietoris topology by $(CL(X), \mathbb{V})$.

If U is a subset of X, then

$$U^- = \{H \in CL(X) : H \cap U \neq \emptyset\}$$

and

$$U^+ = \{H \in CL(X) : H \subset U\}.$$

Sometimes, we denote U^- by U^{-X} in order to prevent the confusion.

Let X be a space. The *closed set character* (resp. *compact set character*) of X is the minimal cardinal $\tau \geq \omega$ such that for each closed (resp. compact) set A of X the cardinal of the character of A in X is at most τ. The closed set character (resp. compact set character) of X is denoted by $cl\chi(X)$ (resp. $co\chi(X)$). If $cl\chi(X) = \omega$, then X is called a *D_1-space* [3] if $\sup\{\chi(H) : H \in CL(X)\} \leq \omega$; if $co\chi(X) = \omega$, then X is called a *D_0-space* [26] if $\sup\{\chi(H) : H \text{ is compact in } X\} \leq \omega$. Clearly, each D_1-space is a D_0-space.

3 The Topological Properties of Hyperspace on an Infinite Countable Discrete Space

In this section, we mainly discuss the topological properties of hyperspace on an infinite countable discrete space. First, we recall a concept.

A proper subset C of the rational number $\mathbb{Q}$ is called a *cut* if C has no largest element and $(-\infty, p] \cap \mathbb{Q} \subset C$ for each $p \in C$. If C, D are cuts and C is a proper subset of D, then denoted by $C < D$.

In this paper, we always denote any countable infinite discrete space by $D(\omega)$. The following lemma is a simple modification of [14, Theorem 4.11].

Lemma 3.1 *The hyperspace $(CL(D(\omega)), \mathbb{V})$ contains a closed copy of Sorgenfrey line $\mathbb{S}$.*

Proof Let $\mathbb{Q}$ be the set of rational number with the discrete topology; then $D(\omega)$ is homeomorphic to $\mathbb{Q}$. Therefore, we may assume that $D(\omega)$ is $\mathbb{Q}$. Let

$$\mathbb{X} = \{C \in CL(D(\omega)) : C \text{ is a cut}\}.$$

It was proved that the subspace $\mathbb{X}$ of $(CL(D(\omega)), \mathbb{V})$ is homeomorphic to the Sorgenfrey line by [14, Theorem 4.11]. Now we only prove that $\mathbb{X}$ is closed in $(CL(D(\omega)), \mathbb{V})$.

Take any $C \in CL(D(\omega)) \setminus \mathbb{X}$; then C is not a cut. Hence, C has a largest element or there exist $p \in C$ such that $((-\infty, p] \cap \mathbb{Q}) \setminus C \neq \emptyset$. In order to find an open neighborhood $\widehat{U}$ of C in $CL(D(\omega))$ such that $\widehat{U} \cap \mathbb{X} = \emptyset$, we divide the proof into the following two cases.

Case 1: C has a largest element p.

Then $p \in C$ such that $r \leq p$ for any $r \in C$. Clearly, $\langle C, \{p\}\rangle$ is an open neighborhood of C in $(CL(D(\omega)), \mathbb{V})$; hence, it easily follows that $\langle C, \{p\}\rangle \cap \mathbb{X} = \emptyset$. Now put $\widehat{U} = \langle C, \{p\}\rangle$, as desired.

Case 2: There exist $p \in C$ such that $((-\infty, p] \cap \mathbb{Q}) \setminus C \neq \emptyset$.

Pick any $q \in ((-\infty, p] \cap \mathbb{Q}) \setminus C$; then $q < p$. Clearly, $\langle C, \{p\}\rangle$ is an open neighborhood of C. We claim that $\langle C, \{p\}\rangle \cap \mathbb{X} = \emptyset$. Indeed, if not, there exists a cut $D \in \mathbb{X}$ such that $p \in D$ and $D \subset C$, then $q \in D$ since $q < p$ and D is a cut. This is a contradiction since $q \notin C$. Now put $\widehat{U} = \langle C, \{p\}\rangle$, as desired.

Therefore, it follows from Cases 1 and 2 that $\mathbb{X}$ is closed in $(CL(D(\omega)), \mathbb{V})$. □

Proposition 3.2 *Let X be a space and $X = \bigoplus_{i\in\mathbb{N}} X_i$, where $X_i \cap X_j = \emptyset$ for any distinct $i \in \mathbb{N}$ and $j \in \mathbb{N}$. Then the box product $\prod_{i\in\mathbb{N}}(CL(X_i), \mathbb{V})$ is homeomorphic to a closed subspace of $(CL(X), \mathbb{V})$.*

Proof Let

$$\mathbb{X}' = \{H \in CL(X) : H \cap X_i \neq \emptyset, i \in \mathbb{N}\}.$$

We claim that $\mathbb{X}'$ is a closed subspace of $(CL(X), \mathbb{V})$. Indeed, take any $K \in CL(X) \setminus \mathbb{X}'$; then $K \cap X_i = \emptyset$ for some $i \in \mathbb{N}$. Put $Y = \bigcup_{j\in\mathbb{N}\setminus\{i\}} X_j$. Then Y^+ is a neighborhood of K and $Y^+ \cap \mathbb{X}' = \emptyset$. Now we prove that the box product $\prod_{i\in\mathbb{N}}(CL(X_i), \mathbb{V})$ is homeomorphic to $\mathbb{X}'$.

Indeed, define the mapping $f : \prod_{i\in\mathbb{N}}(CL(X_i), \mathbb{V}) \to \mathbb{X}'$ by $f(\prod_{i\in\mathbb{N}} C_i) = \bigcup_{i\in\mathbb{N}} C_i$ for any $\prod_{i\in\mathbb{N}} C_i \in \prod_{i\in\mathbb{N}}(CL(X_i), \mathbb{V})$. Clearly, f is a bijection. Next it suffices to prove that f is an open continuous mapping.

(1) The mapping f is continuous.
Take any nonempty open subset V of X. Then there exists a subset $A \subset \mathbb{N}$ such that $V \cap X_n \neq \emptyset$ for each $n \in A$ and $V \cap X_m = \emptyset$ for each $m \in \mathbb{N} \setminus A$. Then

$$f^{-1}(V^{-X} \cap \mathbb{X}') = \bigcup_{i\in A}\left((V \cap X_i)^{-X_i} \times \prod_{j\in\mathbb{N}\setminus\{i\}} X_j^+\right),$$

and then

$$f^{-1}(V^+) = \prod_{i\in\mathbb{N}}(X_i \cap V)^+$$

if $A = \mathbb{N}$. Hence f is continuous.

(2) The mapping f is open.

Let $V_i \subset X_i$ be a nonempty open subset of X_i for each $i \in \mathbb{N}$. For any subset $B \subset \mathbb{N}$, we have

$$f\left(\prod_{i\in B} V_i^{-X_i} \times \prod_{j\in\mathbb{N}\setminus B} V_j^{+}\right) = \bigcap_{i\in B} V_i^{-X} \cap \left(\bigcup_{i\in B} X_i \cup \bigcup_{j\in\mathbb{N}\setminus B} V_j\right)^{+} \cap \mathbb{X}'.$$

Therefore, f is a homeomorphism. □

By Proposition 3.2, we have the following theorem.

Theorem 3.3 *The hyperspace $(CL(D(\omega)), \mathbb{V})$ contains a closed copy of the box product $\prod_{n\in\mathbb{N}} \mathbb{S}_n$, where each $\mathbb{S}_n$ is homeomorphic the Sorgenfrey line $\mathbb{S}$.*

Proof We can write $D(\omega) = \bigcup_{i\in\mathbb{N}} E_i$ such that each E_i is infinite and $E_i \cap E_j = \emptyset$ for distinct i and j. From Proposition 3.2, it follows that the box product $\prod_{i\in\mathbb{N}}(CL(E_i), \mathbb{V})$ is homeomorphic to a closed subspace of $(CL(D(\omega)), \mathbb{V})$. By Lemma 3.1, each $(CL(E_i), \mathbb{V})$ contains a closed copy of Sorgenfrey line, hence $(CL(D(\omega)), \mathbb{V})$ contains a closed copy of the box product $\prod_{n\in\mathbb{N}} S_n$. □

From Theorem 3.3, we easily see the following corollary.

Corollary 3.4 *The hyperspace $(CL(D(\omega)), \mathbb{V})$ contains a closed copy of $\mathbb{S}^n$ for each $n \in \mathbb{N}$.*

Remark 3.5 It is well known that Sorgenfrey line $\mathbb{S}$ is a non-metrizable space which is hereditarily Lindelöf, hereditarily separable, first-countable, perfect[1] and non-developable; moreover, it has the Baire property and a regular G_δ-diagonal. However, the square of Sorgenfrey line is not normal. Therefore, $(CL(D(\omega)), \mathbb{V})$ is not normal. Further, we have the following proposition.

Proposition 3.6 *The Sorgenfrey line $\mathbb{S}$ does not belong to any one of the following classes of spaces.*

(1) β-spaces;[2]
(2) spaces with a point-countable k-network;
(3) spaces with a BCO;[3]
(4) p-spaces;[4]

[1] A space X is called *perfect* if every closed subset of X is a G_δ-set.

[2] A space (X, τ) is called a *β-space* if there exists a function $g : \mathbb{N} \times X \to \tau$ such that (i) for any $x \in X$, we have $g(n+1, x) \subset g(n, x)$ for any $n \in \mathbb{N}$, (ii) for any $x \in X$ and sequence $\{x_n\}$ in X, if $x \in g(n, x_n)$ for each $n \in \mathbb{N}$, then $\{x_n\}$ has an accumulation point in X

[3] A space X is said to have a *base of countable order* if there is a sequence $\{\mathcal{B}_n\}$ of bases for X such that: Whenever $x \in b_n \in \mathcal{B}_n$ and $\{b_n\}$ is decreasing, then $\{b_n : n \in \omega\}$ is a base at x. We use 'BCO' to abbreviate 'base of countable order'.

[4] A regular space X is called a *p-space* if there is a sequence $\{\mathscr{U}_n\}$ of families of open sets in βX such that (1) each $\mathscr{U}_n$ covers X; (2) for each $x \in X$, $\bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{U}_n) \subset X$. If we also have (3) for each $x \in X$, $\bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{U}_n) = \bigcap_{n\in\mathbb{N}} \overline{\mathrm{st}(x, \mathscr{U}_n)}$, then X is called a *strict p-space*.

(5) symmetrizable[5]
(6) quasi-developable spaces;[6]
(7) D_1-spaces.

Therefore, $(CL(D(\omega)), \mathbb{V})$ *does not belong to any one of the classes of spaces* (1)–(7).

Proof (1) If the Sorgenfrey line $\mathbb{S}$ is a β-space, then it is a Moore space[7] hence, $\mathbb{S}$ is metrizable since a paratopological group which is a β-space is developable. This is a contradiction.

(2) If the Sorgenfrey line $\mathbb{S}$ has a point-countable k-network, then it has a point-countable base [13, Corollary 3.6] since $\mathbb{S}$ is a first-countable space. Since a separable space with a point-countable base has a countable base by [12, Theorem 7.2], it follows that $\mathbb{S}$ has a countable base, thus it is metrizable, this is a contradiction.

(3) If the Sorgenfrey line $\mathbb{S}$ has a BCO, then it follows that it is developable since each submetacompact space with a BCO is developable [12, Theorem 6.6], hence it is metrizable. This is a contradiction.

(4) If the Sorgenfrey line $\mathbb{S}$ is a p-space, then it is a Lindelöf p-space with a G_δ-diagonal[8], hence $\mathbb{S}$ is metrizable by [12, Corollaries 3.4 and 3.20]. This is a contradiction.

(5) If the Sorgenfrey line $\mathbb{S}$ is symmetrizable, then it is a semi-stratifiable[9] space by [12, Theorem 9.6] and [12, Theorem 9.8], hence a β-space [12, Page 475], this is a contradiction to (1).

(6) If the Sorgenfrey line $\mathbb{S}$ is quasi-developable, then it is developable by [12, Theorem 8.6] since $\mathbb{S}$ is perfect, this is a contradiction.

(7) If the Sorgenfrey line $\mathbb{S}$ is a D_1-space, then $\mathbb{S}$ is metrizable [5, Theorem 7(4)] since $\mathbb{S}$ has a G_δ-diagonal, this is a contradiction.

□

Theorem 3.7 *The hyperspace* $(CL(D(\omega)), \mathbb{V})$ *is non-Archimedean quasi-metrizable; thus it is quasi-metrizable.*

Proof Let $D(\omega) = \{r_n : n \in \mathbb{N}\}$ endowed with a discrete topology τ. Now we define a g-function from $\mathbb{N} \times CL(D(\omega)) \to \mathbb{V}$ as follows (1) and (2):

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[5] A function $d : X \times X \to \mathbb{R}^+$ is called *symmetric* on a set X if for each $x, y \in X$, we have (1) $d(x, y) = 0$ if and only if $x = y$ and (2) $d(x, y) = d(y, x)$. A space (X, τ) is called *symmetrizable* if there exists a symmetric d on X such that the topology τ given on X is generated by the symmetric d, that is, a subset $U \in \tau$ if and only if for every $x \in U$, there is $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$.

[6] A space (X, τ) is called *quasi-developable* if there exists a sequence $\{\mathscr{U}_n\}$ of families consisting of open sets in X such that for each $x \in U \in \tau$ there exists $n \in \mathbb{N}$ such that $x \in \mathrm{st}(x, \mathscr{U}_n) \subset U$.

[7] A space (X, τ) is called *developable* if there exists a sequence $\{\mathscr{U}_n\}$ of families of open covers of X such that, for each $x \in X$, $\{\mathrm{st}(x, \mathscr{U}_n)\}$ is an open neighborhood base of x in X. A regular developable space is called a *Moore space*;

[8] A space X is said to have a *G_δ-diagonal* if there is a sequence $\{\mathscr{U}_n\}$ of open covers of X, such that, for each $x \in X$, $\{x\} = \bigcap_{n \in \mathbb{N}} \mathrm{st}(x, \mathscr{U}_n)$.

[9] A space (X, τ) is called a *semi-stratifiable* if, there exists a function $F : \mathbb{N} \times \tau \to \tau^c$ satisfying the following conditions: (1) $U \in \tau \Rightarrow U = \bigcup_{n \in \mathbb{N}} F(n, U)$; (2) $V \subset U \Rightarrow F(n, V) \subset F(n, U)$, where $\tau^c = \{F : F \subset X, X \setminus F \in \tau\}$.

(1) If $A \in CL(D(\omega))$ is a finite subset of $D(\omega)$, then there exist $k_A \in \mathbb{N}$ and a finite subset $\{n(1, A), \ldots, n(k_A, A)\}$ of $\mathbb{N}$ with $n(1, A) < \ldots < n(k_A, A)$ such that $A = \{r_{n(1,A)}, \cdots, r_{n(k_A,A)}\}$; then put

$$G(m, A) = \langle \{r_{n(1,A)}\}, \ldots, \{r_{n(k_A,A)}\} \rangle$$

for each $m \in \mathbb{N}$.

(2) If $A \in CL(D(\omega))$ is an infinite subset of $D(\omega)$, then there exists a strictly increasing sequence $\{n(i, A)\}_{i \in \mathbb{N}}$ of $\mathbb{N}$ such that $A = \{r_{n(i,A)} : i \in \mathbb{N}\}$; then put

$$G(m, A) = \langle \{r_{n(1,A)}\}, \ldots, \{r_{n(m,A)}\}, A \rangle$$

for each $m \in \mathbb{N}$.

Now it easily check the following two conditions hold.

(i) For each $A \in CL(D(\omega))$ the family $\{G(m, A)\}_{m \in \mathbb{N}}$ is a base at A in $(CL(D(\omega)), \mathbb{V})$.
(ii) For each $A \in CL(D(\omega))$, if $B \in G(m, A)$, then $G(m, B) \subset G(m, A)$. Therefore, it follows from [12, Theorem 10.2] that $(CL(D(\omega)), \mathbb{V})$ is non-Archimedean quasi-metrizable.

□

Let $C_\omega = \{\infty\} \cup \{x_{mn} : n, m \in \mathbb{N}\}$ be a countable infinite set. Endow C_ω with a topology υ as follows:

(1) Each single point set $\{x_{mn}\}$ is open in C_ω;
(2) For each $k \in \mathbb{N}$, put $U_k = \{x_{mn} : m \in \mathbb{N}, n \geq k + 1\} \cup \{\infty\}$; the family $\{U_k\}$ is a base at the point ∞.

From Theorem 5.17, it follows that $(CL(C_\omega), \mathbb{V})$ is a γ-space[10]. However, the following question is still unknown for us.

Question 3.8 *Is the hyperspace* $(CL(C_\omega), \mathbb{V})$ *quasi-metrizable?*

4 The Characterizations of Tightness in Hyperspaces

In this section, we mainly give a characterization of tightness in hyperspace; in particular, we give a characterization of hyperspace which is Fréchet–Urysohn. First, we recall and introduce some concepts.

The *tightness* of a space X is the minimal cardinal $\tau \geq \omega$ such that if any x is a cluster point of any subset A of X, then there is a subset B of A such that $|B| \leq \tau$ and x is a cluster point of B. The tightness of X is denoted by $t(X)$.

Definition 4.1 Let X be a space, $\mathcal{F} \subset CL(X)$ and $A \in CL(X)$.

[10] A space (X, τ) is a γ-*space* if there exists a function $g : \omega \times X \to \tau$ such that (i) $\{g(n, x) : n \in \omega\}$ is a base at x; (ii) for each $n \in \omega$ and $x \in X$, there exists $m \in \omega$ such that $y \in g(m, x)$ implies $g(m, y) \subset g(n, x)$. By [12, Theorem 10.6(iii)], each γ-space is a D_0-space.

(1) The set A is called a *cluster set* of $\mathcal{F}$ in X if for any finite open subsets $\{V_i : i \leq k\}$ with $V_i \cap A \neq \emptyset$ $(i \leq k)$ and any open neighborhood U of A, there is a $F \in \mathcal{F}$ such that $F \subset U$ and $F \cap V_i \neq \emptyset$ for any $i \leq k$.
(2) The *set-tightness* of X is the minimal cardinal $\tau \geq \omega$ such that if A is a cluster set of any $\mathcal{F} \subset CL(X)$, then there is a subfamily $\mathcal{F}' \subset \mathcal{F}$ such that $|\mathcal{F}'| \leq \tau$ and A is a cluster set of $\mathcal{F}'$. The set-tightness of X is denoted by $st(X)$.
(3) The sequence $\{A_j : j \in \mathbb{N}\}$ of $CL(X)$ is called *strongly converging* to A in X if for any finite open subsets $\{V_i : i \leq k\}$ with $A \cap V_i \neq \emptyset$ $(j \leq k)$ and any open neighborhood U of A, there exists $N \in \mathbb{N}$ such that $A_j \subset U$ and $A_j \cap V_i \neq \emptyset$ $(i \leq k)$ whenever $j \geq N$.
(4) The space X has *set-FU property* if whenever A is a cluster set of $\mathcal{F} \subset CL(X)$, there is a countable subfamily $\{A_j : j \in \mathbb{N}\}$ of $\mathcal{F}$ such that $\{A_j : j \in \mathbb{N}\}$ strongly converges to A in X.

From the definition of the set-FU property, it follows that if elements of $\mathcal{F}$ and A are all singleton, then X is Fréchet–Urysohn. Therefore, it easily see that there exists a countable set-tightness space X such that X is not set-FU property, such as Arens space S_2. Now we can use the concepts of set-FU property and set-tightness to characterize the Fréchet–Urysohn and tightness of $(CL(X), \mathbb{V})$, respectively. First, the following proposition gives a characterization of X such that $t((CL(X), \mathbb{V})) \leq \tau$.

Proposition 4.2 *Let X be a space. Then $t((CL(X), \mathbb{V})) \leq \tau$ if and only if $st(X) \leq \tau$.*

Proof Sufficiency. Assume $t((CL(X), \mathbb{V})) \leq \tau$. Let A be a cluster set of $\mathcal{F} \subset CL(X)$. For any finite open subsets $\{V_i : i \leq k\}$ with $A \cap V_i \neq \emptyset$ $(i \leq k)$ and any open neighborhood U of A, the set $\langle V_1 \cap U, ..., V_k \cap U, U\rangle$ is a neighborhood of A in $(CL(X), \mathbb{V})$. Since $t((CL(X), \mathbb{V})) \leq \tau$, there exists a subfamily $\mathcal{F}' \subset \mathcal{F}$ such that $|\mathcal{F}'| \leq \tau$ and $A \in \overline{\mathcal{F}'}$ in $(CL(X), \mathbb{V})$. Since $A \in \langle V_1 \cap U, ... V_k \cap U, U\rangle$, there exists $F \in \mathcal{F}'$ such that $F \in \langle V_1 \cap U, ..., V_k \cap U, U\rangle$; then $F \subset U$, $F \cap V_i \neq \emptyset$ for any $i \leq k$. Therefore, A be a cluster set of $\mathcal{F}'$. Thus $st(X) \leq \tau$.

Necessity. Assume $st(X) \leq \tau$, and suppose that A belongs to the closure of $\mathcal{F}$ in $(CL(X), \mathbb{V})$, where $\mathcal{F} \subset CL(X)$. We claim that A is a cluster set of $\mathcal{F}$. Indeed, for any finite open subsets $\{V_i : i \leq k\}$ and any open neighborhood U of A, the set $\langle V_1 \cap U, ..., V_k \cap U, U\rangle$ is a neighborhood of A in $(CL(X), \mathbb{V})$. Then there exists $F \in \mathcal{F}$ such that $F \in \langle V_1 \cap U, ..., V_k \cap U, U\rangle$, which implies that $F \cap V_i \neq \emptyset$ for $i \leq k$ and $F \subset U$. Hence, A is a cluster set of $\mathcal{F}$. Since $st(X) \leq \tau$, there is a subfamily $\mathcal{F}_1 \subset \mathcal{F}$ such that $|\mathcal{F}_1| \leq \tau$ and A is a cluster set of $\mathcal{F}_1$ in X. Finally it suffices to prove the following claim.

Claim: $A \in \overline{\mathcal{F}_1}$ in $(CL(X), \mathbb{V})$. Let $\langle W_1, ..., W_m\rangle$ be a neighborhood of A in $(CL(X), \mathbb{V})$, and let $W = \cup\{W_i : i \leq m\}$. Then $A \cap W_i \neq \emptyset$ for any $i \leq m$ and $A \subset W$. Since A is a cluster set of $\mathcal{F}_1$, there exists $F \in \mathcal{F}_1$ such that $F \cap W_i \neq \emptyset$ for any $i \leq m$ and $F \subset W$. Hence, $F \in \langle W_1, ..., W_m\rangle$. Therefore, $A \in \overline{\mathcal{F}_1}$ in $(CL(X), \mathbb{V})$. □

Corollary 4.3 *Let X be a space. Then $(CL(X), \mathbb{V})$ is of countable tightness if and only if X is of countable set-tightness.*

Proposition 4.4 *Let X be a (regular) space. Then we have the following statements:*

(1) $co\chi(X) \leq st(X)$*;*
(2) If X *is a normal space, then* $cl\chi(X) \leq st(X)$*.*

Proof We only prove (2), and the proof of (1) is similar. Let $st(X) = \tau$, let A be an arbitrary closed subset of X, and let $\mathcal{B}_A = \{U_\alpha : \alpha \in I\}$ be an open neighborhood base at A in X. Since X is normal, it follows that $\mathcal{B}_A = \{\overline{U_\alpha} : \alpha \in I\}$ be a neighborhood base at A in X, hence it easily check that A is a cluster set of $\mathcal{B}_A$. Because $st(X) \leq \tau$, there exists a subfamily $\mathcal{B}'_A = \{\overline{U_\alpha} : \alpha \in I_1\}$ of $\mathcal{B}_A$ such that $|I_1| \leq \tau$ and A is a cluster set of $\mathcal{B}'_A$. Therefore, for any open neighborhood U of A in X, there exists $\alpha \in I_1$ such that $A \subset \overline{U_\alpha} \subset U$. Hence $\mathcal{B}'_A$ is a neighborhood of A in X. Hence $cl\chi(X) \leq st(X)$. □

Corollary 4.5 *If* X *is a (regular) space with countable set-tightness, then* X *is a* D_0*-space; in particular,* X *is a* D_1*-space if* X *is normal.*

By Proposition 4.2 and Corollary 4.5, we have the following corollary.

Corollary 4.6 *If* X *is a space and* $(CL(X), \mathbb{V})$ *has countable tightness, then* X *is a* D_0*-space; in particular,* X *is a first-countable space.*

By [1, Proposition 3], it is natural to pose the following question.

Question 4.7 *Let* X *be a space. If* $(CL(X), \mathbb{V})$ *has countable tightness, does then* $(CL(X), \mathbb{V})$ *contain a copy of* S_ω*?*

Question 4.8 *Under what conditions of a space* X*, we have* $t(X) = st(X)$*.*

The following proposition gives a partial answer to Question 4.8.

Proposition 4.9 *Let* X *be a normal space. Then* X *has countable set-tightness if and only if* X *has the following properties:*

(1) X *is perfectly normal;*
(2) the set $X \setminus S(X)$ *is countable;*
(3) $S(X)$ *is countably compact, hereditarily separable and* $\chi(S(X), X) \leq \aleph_0$*.*

Proof Suppose that X has the properties (1)-(3), then it follows from [16, Corollary 1.8] that $(CL(X), \mathbb{V})$ is first-countable, hence X has countable set-tightness. Now it suffices to prove the necessity. Let X have countable set-tightness. By Proposition 4.2, $(CL(X), \mathbb{V})$ has countable tightness. Since X is normal, it follows from [14, Proposition 2.6] that $(CL(X), \mathbb{V})$ is first-countable. Then the necessity holds by [16, Corollary 1.8]. □

Remark 4.10 By Proposition 4.9, there exists a metrizable space X such that X is not countable set-tightness. Indeed, let X be an arbitrary non-compact metrizable space such that any point of X is not isolated. By Proposition 4.9, X is not countable set-tightness.

The gap between D_1-spaces and D_0-spaces is large, see [5, Theorem 4]. The following proposition gives some relations between D_0-spaces and other generalized metric spaces.

Proposition 4.11 *Let X be a developable space or a space with a point-countable base. Then X is a D_0-space.*

Proof Fix an arbitrary compact subset $K \subset X$.

(1) Assume that X is a space with a point-countable base. Let $\mathcal{B}$ be a point-countable base of X. Then K is metrizable by [12, Theorem 7.6]. Let D be a countable dense subset of K, and put $\mathcal{B}' = \{B \in \mathcal{B} : B \cap K \neq \emptyset\}$; then $|\mathcal{B}'| \leq \omega$. Let

$$\mathcal{B}'' = \{\cup\mathcal{F} : \mathcal{F} \subset \mathcal{B}' \text{ is a finite cover of } K\}.$$

We prove that $\mathcal{B}''$ is a countable base of K. Indeed, if $K \subset U$ with U open, then, for any $x \in K$, pick $B_x \in \mathcal{B}'$ such that $x \in B_x \subset U$. Since $\{B_x : x \in K\}$ is an open cover of K, there exists $n \in \mathbb{N}$ such that $\{B_{x_i} : i \leq n\}$ is a finite open cover of K, then $K \subset \bigcup_{i \leq n} B_{x_i} \subset U$ and $\bigcup_{i \leq n} B_{x_i} \in \mathcal{B}''$.

(2) Let X be a developable space, and let Y be the quotient space by identifying K to a point z with the canonical map f. It is easy to see that f is a perfect map. Since developable spaces are preserved by perfect maps, then Y is developable. Let $\{U_n : n \in \mathbb{N}\}$ be a countable local base at z, and put $V_n = f^{-1}(U_n)$ for each $n \in \mathbb{N}$. Then $\{V_n : n \in \mathbb{N}\}$ is a countable base of K. Hence X is a D_0-space.

□

From Theorems 5.1 and 5.17, there exists a space X such that $(CL(X), \mathbb{V})$ is a D_0-space, but $(CL(X), \mathbb{V})$ is not a D_1-space. Indeed, let X be the space of topological sum of a compact metrizable space C and a countable infinite discrete space D, that is, $X = C \bigoplus D$. Then it follows that $(CL(X), \mathbb{V})$ is a D_0-space and not a D_1-space.

The following proposition gives a characterization of X such that $(CL(X), \mathbb{V})$ is Fréchet–Urysohn, which could be proved by a similar proof of Proposition 4.2.

Proposition 4.12 *Let X be a space. Then $(CL(X), \mathbb{V})$ is Fréchet–Urysohn if and only if X has set-FU property.*

It is well known that a strongly Fréchet–Urysohn space is Fréchet–Urysohn, but not vice versa. It is natural to pose the following two questions. Clearly, if Question 4.14 is positive, then Question 4.13 is also positive.

Question 4.13 *Let X be a space. If $(CL(X), \mathbb{V})$ is Fréchet–Urysohn, is then $(CL(X), \mathbb{V})$ strongly Fréchet–Urysohn?*

Question 4.14 *Let X be a space. If $(CL(X), \mathbb{V})$ contains a (closed) copy of S_ω, does then $(CL(X), \mathbb{V})$ contain a (closed) copy of S_2?*

5 Some Generalized Metric Properties on Hyperspaces

In this section, we mainly give the characterizations of some generalized metric properties on hyperspaces, such as semi-stratifiable spaces, quasi-developable spaces, D_1-spaces, symmetrizable spaces, and γ-spaces.

First, we prove the first main theorem in this section as follows, which gives a partial answer to Problem 1.2.

Theorem 5.1 *Let X be a space. Then the following statements are equivalent.*

(1) $(CL(X), \mathbb{V})$ *is a semi-stratifiable space;*
(2) $(CL(X), \mathbb{V})$ *is quasi-developable;*
(3) $(CL(X), \mathbb{V})$ *is a* D_1*-space;*
(4) $(CL(X), \mathbb{V})$ *is symmetrizable;*
(5) X *is a compact metrizable space.*

In order to give the proof, we give some technique lemmas and theorems.

Lemma 5.2 *Let $\mathcal{P}$ be a topological property that is closed hereditary, and let there exist $n \in \mathbb{N}$ such that $\mathbb{S}^n$ does not have the property $\mathcal{P}$. If $(CL(X), \mathbb{V})$ has the property $\mathcal{P}$, then X is countably compact.*

Proof Suppose X is not countably compact, then there exists a closed, countable infinite discrete subset $D(\omega) \subset X$. Then $(CL(D(\omega)), \mathbb{V})$ is a closed subspace of $(CL(X), \mathbb{V})$. By Corollary 3.4, $(CL(X), \mathbb{V})$ contains a closed copy of $\mathbb{S}^n$ for each $n \in \mathbb{N}$, then $\mathbb{S}^n$ has the property $\mathcal{P}$, this is a contradiction. Hence X is countably compact. □

Since all properties in Proposition 3.6 are closed hereditary, it follows from Lemma 5.2 that we have the following theorem.

Theorem 5.3 *If $(CL(X), \mathbb{V})$ belongs to any one of spaces in Proposition 3.6, then X is countably compact.*

Since each strict p-space is a β-space [12, page475], it follows from Theorem 5.3 that we have the following corollary.

Corollary 5.4 *A space X is compact if and only if $(CL(X), \mathbb{V})$ is a strict p-space.*

Proof If $(CL(X), \mathbb{V})$ is a strict p-space, then it follows from Theorem 5.3 that X is countably compact. Since X is a strict p-space, X is submetacompact, hence X is compact. If X is compact, then it follows from [4, Corollary 13] that $(CL(X), \mathbb{V})$ is compact, thus it is a strict p-space by [12, Theorem 3.19]. □

Remark 5.5 It is well known that $(CL(X), \mathbb{V})$ is locally compact if and only if X is compact if and only if $(CL(X), \mathbb{V})$ is compact, see [4, Corollary 13]. Both locally compact spaces and strict p-spaces are p-spaces, it is natural to ask the following question.

Question 5.6 *If $(CL(X), \mathbb{V})$ is a p-space, is then X compact?*

Lemma 5.7 *Let $\mathcal{P}$ be a property that is closed hereditary, and let there exists some $n \in \mathbb{N}$ such that $\mathbb{S}^n$ does not have the property $\mathcal{P}$. Then a space X is compact metrizable if and only if $(CL(X), \mathbb{V})$ is perfect and has property $\mathcal{P}$.*

Proof It suffices to prove the sufficiency. By Lemma 5.2, X is countably compact. Next we prove that X has a G_δ-diagonal. Since X is a closed subset of $(CL(X), \mathbb{V})$ (indeed, X is the set $\{\{x\} : x \in X\}$), there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of $(CL(X), \mathbb{V})$ such that $X = \bigcap_{n\in\mathbb{N}} U_n$. Put $F_2(X) = \{B \subset X : |B| \leq 2\}$; then $\{F_2(X) \cap U_n : n \in \mathbb{N}\}$ is a countable family of open subsets of $F_2(X)$. Define $f_2 : X \times X \to F_2(X)$ by $f_2(x, y) = \{x, y\}$; then it is well known that f_2 is an open and closed continuous mapping from X^2 to $F_2(X)$. Note that $f_2^{-1}(X) = \{(x, x) : x \in X\} = \Delta$ is the diagonal of X, and that $\{f_2^{-1}(F_2(X) \cap U_n) : n \in \mathbb{N}\}$ is a countable family of open neighborhoods of Δ and $\bigcap_{n\in\mathbb{N}} f_2^{-1}(U_n \cap F_2(X)) = \Delta$, hence X has a G_δ-diagonal; therefore, X is compact metrizable by [12, Theorem 2.14]. □

Now we can prove our first theorem.

Proof of Theorem 5.1 Clearly, it suffices to prove that (1), (2), (3), (4) ⇒ (5). By Lemma 5.7, we have (1) ⇒ (5) and (3) ⇒ (5).

(2) ⇒ (5). Assume that $(CL(X), \mathbb{V})$ is quasi-developable. Then, by Theorem 5.3, X is countably compact. Since each countably compact space is a M-space, it follow from [12, Theorem 8.5] and [12, Corollary 8.3(ii)] that X is metrizable.

(4) ⇒ (5). Let $(CL(X), \mathbb{V})$ be symmetrizable; then $(CL(X), \mathbb{V})$ has countable tightness, hence from Theorem 5.3 and Corollary 4.6, it follows that X is first-countable and countably compact. Since a first-countable, symmetrizable space is semi-stratifiable, it concludes that X has a G_δ-diagonal. Hence X is compact metrizable by [12, Theorem 2.14]. The proof is completed. □

It was proved that if $(CL(X), \mathbb{V})$ is a σ-space (i.e., a regular space with a σ-discrete network[11]), then X is compact metrizable by [14, Theorem 4.14]. So it is natural to ask the following question.

Question 5.8 *If $(CL(X), \mathbb{V})$ has a σ-locally countable network, is then X compact metrizable?*

We give a partial answer to Question 5.8. First, we give a lemma.

Lemma 5.9 *If X is a (regular) space having a σ-locally countable network, then each singleton is a G_δ-set.*

Proof Let $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ be a σ-locally countable network of X, where each $\mathcal{P}_n$ is locally countable. Since X is regular, we may assume that each element of $\mathcal{P}$ is closed. Fix any $x \in X$. For $n \in \mathbb{N}$, let U_n be an open neighborhood of x such that U_n intersects at most countably many elements of $\mathcal{P}_n$, and let $\mathcal{P}'_n = \{P \in \mathcal{P}_n : P \cap U_n \neq \emptyset\}$. For each $n \in \mathbb{N}$, enumerate $\{P \in \mathcal{P}'_n, x \notin P\}$ as $\{P_{n,i} : i \in \mathbb{N}\}$, and let $V_{n,i} = X \setminus P_{n,i}$ for each $i \in \mathbb{N}$; then $V_{n,i}$ is open and $x \in V_{n,i}$ for each $i \in \mathbb{N}$. Now it suffices to prove the following claim.

Claim: $\{x\} = (\bigcap_{n\in\mathbb{N}} U_n) \cap (\bigcap_{n,i\in\mathbb{N}} V_{n,i})$.

Suppose not, then there exists $y \neq x$ such that $y \in (\bigcap_{n\in\mathbb{N}} U_n) \cap (\bigcap_{n,i\in\mathbb{N}} V_{n,i})$. Let V be an open neighborhood of y with $x \notin V$. Pick $P \in \mathcal{P}$ with $y \in P \subset V$. Then $P = P_{k,j}$ for some $k, j \in \mathbb{N}$, and $y \notin V_{k,j} = X \setminus P_{k,j}$. This is a contradiction. □

Theorem 5.10 *(MA(ω_1) + TOP) A (regular) space X is compact metrizable if and only if $(CL(X), \mathbb{V})$ has a σ-locally countable network.*

[11] A family $\mathscr{P}$ in a space X is called a *network* for X if, for each $x \in U$ with U open in X, there exists $P \in \mathscr{P}$ such that $x \in P \subset U$.

Proof It suffices to prove the sufficiency. From Theorem 5.3, it follows that X is countably compact. By Lemma 5.9, each singleton of $(CL(X), \mathbb{V})$ is a G_δ-set, then X is hereditarily separable by [15, Proposition 4.3]. Under MA(ω_1)+TOP, X is Lindelöf. Since a Lindelöf space with a σ-locally countable network has a countable network, X is a countably compact space with a countable network, hence it is compact metrizable by [12, Corollary 4.7(ii)]. □

If X is a k-space,[12] we have the following result.

Theorem 5.11 *Let X be a (regular) k-space. Then X is compact metrizable if and only if $(CL(X), \mathbb{V})$ has a point-countable k-network.*

Proof By Theorem 5.3, X is countably compact. Since X is k-space with a point-countable k-network, it follows that X is compact metrizable space [13, Theorem 4.1]. □

We do not know whether we can delete the condition 'regular k-space' in Theorem 5.11, hence we have the following question.

Question 5.12 *Suppose $(CL(X), \mathbb{V})$ has a point-countable k-network, is X metrizable?*

The following theorem gives a characterization of X such that $(CL(X), \mathbb{V})$ has a BCO under the assumption of MA$(\omega_1) + TOP$.

Theorem 5.13 *(MA$(\omega_1) + TOP$) A (regular) space X is compact metrizable if and only if $(CL(X), \mathbb{V})$ has a BCO.*

Proof By Theorem 5.3, X is countably compact. Moreover, it is obvious that each singleton of $(CL(X), \mathbb{V})$ is a G_δ-set, then X is hereditarily separable by [15, Proposition 4.3], hence it is Lindelöf under MA(ω_1)+TOP. A Lindelöf space having a BCO is metrizable by [12, Theorem 6.6], therefore, X is compact and metrizable. □

Next we prove the second main theorems in this section, see Theorem 5.15. First, we give some concepts.

A family $\mathcal{B}$ of open subsets of a space X is called an *external π-base of a subset A* if whenever $A \cap U \neq \emptyset$ with U open in X, there is $B \in \mathcal{B}$ such that $A \cap B \neq \emptyset$ and $A \cap B \subset U$. We denote

$$e\pi w(A) = \inf\{|\mathcal{B}| : \mathcal{B} \text{ is an external } \pi\text{-base of } A\}.$$

If $\mathcal{B}$ is an external π-base of A, then it easily see that $\{B \cap A : B \in \mathcal{B}\}$ is a π-base of A.

It was proved that

$$\chi(CL(X), \mathbb{V}) = hd(X) \cdot \sup\{\chi(H, X) : H \in CL(X)\}$$

[14, Theorem 2.2(5)]. We describe this result in terms of external π-base.

[12] A space X is called a *k-space* if, for each $A \subset X$, A is closed in X provided $K \cap A$ is closed for each compact subset K of X.

Proposition 5.14 *For a space X, we have*

$$\chi(CL(X), \mathbb{V}) = \sup\{\chi(H, X) : H \in CL(X)\} \cdot \sup\{e\pi w(H) : H \in CL(X)\}.$$

Proof Suppose $\chi(CL(X), \mathbb{V}) \leq \kappa$. Fix any $H \in CL(X)$, and let $\{\widehat{U}_\alpha : \alpha < \kappa\}$ be a local base at H in $(CL(X), \mathbb{V})$. We write $\widehat{U}_\alpha = \langle U_1(\alpha), ..., U_{k_\alpha}(\alpha)\rangle$ for any $\alpha < \kappa$, where each $k_\alpha \in \mathbb{N}$. Let $W_\alpha = \bigcup_{j \leq k_\alpha} U_j(\alpha)$ for each α; then, it is easy to check that $\{W_\alpha : \alpha < \kappa\}$ is a local base at H in X. Therefore, $\sup\{\chi(H, X) : H \in CL(X)\} < \kappa$. Next we prove that the family $\mathcal{B} = \{U_j(\alpha) : \alpha < \kappa, j \leq k_\alpha\}$ is an external π-base of H. Indeed, let V be an open subset of X with $V \cap H \neq \emptyset$; then $\langle V, X\rangle$ is a neighborhood of H, hence there exists $\alpha < \kappa$ such that $\widehat{U}_\alpha \subset \langle V, X\rangle$, then it follows from [22, Lemma 2.3.1] that V contains $U_j(\alpha)$ for some $j \leq k_\alpha$. Therefore, $\mathcal{B}$ is an external π-base of H, that is, $\sup\{e\pi w(H) : H \in CL(X)\} \leq \kappa$.

Suppose $\sup\{\chi(H, X) : H \in CL(X)\} \leq \kappa$ and $\sup\{e\pi w(H) : H \in CL(X)\} \leq \kappa$. Fix any $H \in CL(X)$, let $\mathcal{W}$ be an external π-base of H in X with $|\mathcal{W}| < \kappa$, and let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be a local base at H in X. We claim that

$$\{\langle W_1 \cap U, ..., W_r \cap U, U\rangle : \{W_1, ..., W_r\} \in \mathcal{W}^{<\omega}, U \in \mathcal{U}\}$$

is a local base at H in $(CL(X), \mathbb{V})$.

Indeed, let $\langle V_1, ..., V_p\rangle$ be an open neighborhood of H in $(CL(X), \mathbb{V})$; then $H \subset \bigcup_{j \leq p} V_j$ and $H \cap V_j \neq \emptyset$ for each $j \leq p$. Pick $U' \in \mathcal{U}$ such that $U' \subset \bigcup_{j \leq p} V_j$, and $W_j \in \mathcal{W}$ such that $W_j \subset V_j$ for each $j \leq p$. Then $H \in \langle W_1 \cap U', ..., W_p \cap U', U'\rangle \subset \langle V_1, ..., V_p\rangle$. □

By Proposition 5.14, it is easily seen that the second main theorem holds, which gives a partial answer to Problem 1.1.

Theorem 5.15 *Let X be a space. Then $(CL(X), \mathbb{V})$ is first-countable if and only if X is a D_1-space and each closed subset of X has a countable external π-base.*

It is well known that each first-countable space is weakly first-countable. The next theorem shows that weak first-countability is equivalent to first-countability in $(CL(X), \mathbb{V})$.

Theorem 5.16 *Let X be a (regular) space. Then $(CL(X), \mathbb{V})$ is first-countable if and only if $(CL(X), \mathbb{V})$ is weakly first-countable.*

Proof Clearly, it suffices to prove the sufficiency. Assume that $(CL(X), \mathbb{V})$ is weakly first-countable, so it has countable tightness. Then X is first-countable by Corollary 4.6. Moreover, we claim that $d(A) \leq \omega$ for each $A \in CL(X)$. Indeed, take any $A \in CL(X)$, and let $\mathcal{F} = \{C : C \subset A, |C| < \omega\}$. Then A belongs to the closure of $\mathcal{F}$ in $(CL(X), \mathbb{V})$. In fact, for any open neighborhood $\langle U_1, \ldots, U_n\rangle$ of A in $(CL(X), \mathbb{V})$, we have $U_i \cap A \neq \emptyset$ for any $i \leq n$; hence pick an arbitrary $x_i \in U_i \cap A$ for any $i \leq n$. Then $\{x_1, \ldots, x_n\} \in \langle U_1, \ldots, U_n\rangle \cap \mathcal{F} \neq \emptyset$. Therefore, A belongs to the closure of $\mathcal{F}$ in $(CL(X), \mathbb{V})$. Since $(CL(X), \mathbb{V})$ has a countable tightness, there exists a countable subset $\mathcal{F}_1 = \{C_n : n \in \mathbb{N}\}$ of $\mathcal{F}$ such that A belongs to the closure of $\mathcal{F}_1$

in $(CL(X), \mathbb{V})$. Put $D = \bigcup \mathcal{F}_1$. Then the closure of D in X is just A. In fact, let U be an arbitrary open subset of X such that $U \cap A \neq \emptyset$; then U^- is an open neighborhood of A in $(CL(X), \mathbb{V})$, hence U^- contains some element $F \in \mathcal{F}_1$, then $F \cap U \neq \emptyset$. Therefore, $D \cap U \neq \emptyset$. Thus $d(A) \leq \omega$.

Take any $A \in CL(X)$; then, by Proposition 5.14, it suffices to prove that $e\pi w(A) = \omega$ and $\chi(A, X) = \omega$.

(1) $e\pi w(A) = \omega$. Let D be a countable dense subset of A; for each $d \in D$, let $\mathcal{B}_d$ be a countable base at d in X. Then it is easy to check that $\bigcup\{\mathcal{B}_d : d \in D\}$ is an external π-base of A, hence $e\pi w(A) = \omega$.

(2) $\chi(A, X) = \omega$. Let $\{\mathcal{U}_i : i \in \mathbb{N}\}$ be a countable weak base at A, and let $U_i = \bigcup \mathcal{U}_i$ for each $i \in \mathbb{N}$. obviously, $A \subset U_i$ for each $i \in \mathbb{N}$. We prove that $\{\text{int}(U_i) : i \in \mathbb{N}\}$ is a countable base at A in X.

First, we prove that each U_i is a sequential neighborhood of A in X. Indeed, let $x_n \to x \in A$ as $n \to \infty$, and let $A_n = A \cup \{x_n\}$ for each $n \in \mathbb{N}$; then $A_n \to A$ as $n \to \infty$ in $(CL(X), \mathbb{V})$. Since $\mathcal{U}_i$ is a weak neighborhood of A in $(CL(X), \mathbb{V})$, there exists $k \in \mathbb{N}$ such that $A_n \in \mathcal{U}_i$ whenever $n > k$, it implies $A_n \subset U_i$ for $n > k$, hence $\{x_n : n > N\} \subset U_i$.

Second, for any $A \subset U$ with U open in X, the set $\langle U \rangle$ is an open neighborhood of A in $(CL(X), \mathbb{V})$, hence there exists $\mathcal{U}_i$ such that $A \in \mathcal{U}_i \subset \langle U \rangle$, then $A \subset U_i \subset U$.

Finally, we prove $A \subset \text{int}(U_i)$ for each $i \in \mathbb{N}$. Suppose not, pick any $x \in A \setminus \text{int}(U_i)$. Since X is first-countable, there is a sequence $\{x_n : n \in \mathbb{N}\} \subset X \setminus U_i$ such that $x_n \to x$ as $n \to \infty$, which is a contradiction because U_i is a sequential neighborhood of A.

Therefore, $\{\text{int}(U_i) : i \in \mathbb{N}\}$ is a countable base at A, i.e., $\chi(A, X) = \omega$. □

Finally we prove the third main theorem in this section (see Theorem 5.17), which also gives a partial answer to Problem 1.1. Recall that the set of non-isolated points of a space X is denoted by $S(X)$.

Theorem 5.17 *Let X be a space. Then $(CL(X), \mathbb{V})$ is a γ-space if and only if X is a separable metrizable space and $S(X)$ is compact.*

Proof Necessity. Clearly, $(CL(X), \mathbb{V})$ is first-countable, then it follows from Theorem 5.15 that X is a D_1-space; moreover, X is a γ-space since the property of γ-space is hereditary. Therefore, X is metrizable by [5, Theorem 7(8)], and $S(X)$ is also countably compact by [5, Theorem 1], thus $S(X)$ is compact. Since X has a countable external π-base by Theorem 5.15, it follows that X is separable.

Sufficiency. Assume that X is a separable metrizable space and $S(X)$ is compact, and assume that d is the metric on X. Let $X = I(X) \cup S(X)$, where $I(X)$ is the set of all isolated points of X. Clearly, $I(X)$ is countable, and we write $I(X) = \{r_1, r_2, ...\}$. Let $\mathcal{C}'$ be a countable base of X, and let $\mathcal{C} = \{C \in \mathcal{C}' : C \cap S(X) \neq \emptyset\}$; then $\mathcal{C}$ is an external base[13] of $S(X)$, and we write $\mathcal{C} = \{C_1, C_2, ..., C_n, ...\}$. For any subset $A \subset I(X)$, if A is finite then there exist $k_A \in \mathbb{N}$ and a finite subset $\{n(1, A), \ldots, n(k_A, A)\}$ of $\mathbb{N}$ with $n(1, A) < \ldots < n(k_A, A)$ such that $A = \{r_{n(1,A)}, ..., r_{(k_A,A)}\}$; if A is

[13] A family $\mathcal{B}$ of open subsets of a space X is called an external base [2, Page 467] of a set $Y \subset X$ if for every point $y \in Y$ and every neighborhood U of y in X there exists $V \in \mathcal{B}$ such that $y \in V \subset U$.

infinite, then there exists a strictly increasing sequence $\{n(i,A)\}_{i\in\mathbb{N}}$ of $\mathbb{N}$ such that $A = \{r_{n(i,A)} : i \in \mathbb{N}\}$.

For each $A \in CL(X)$ and $n \in \mathbb{N}$, we define a function $G : \mathbb{N} \times CL(X) \to \tau$ as follows, where τ is the topology of $(CL(X), \mathbb{V})$.

Case 1: $A \subset I(X)$. If A is finite, then put

$$G(n, A) = \langle \{r_{n(1,A)}\}, ..., \{r_{n(k_A,A)}\} \rangle = \{A\}$$

for each $n \in \mathbb{N}$. If A is infinite, then put

$$G(m, A) = \langle \{r_{n(1,A)}\}, \ldots, \{r_{n(m,A)}\}, A \rangle$$

for each $m \in \mathbb{N}$. We verify that the family $\{G(n, A) : n \in \mathbb{N}\}$ satisfies the conditions (i) and (ii) of the definition of γ-space.

(i) Let $\mathbb{U} = \langle U_1, .., U_m \rangle$ be an arbitrary open neighborhood of A. Pick $r_{n(j_i,A)} \in U_i$ for $i \le m$, and let $k = \max\{j_i : i \le m\}$; then

$$A \in \langle \{r_{n(1,A)}\}, ..., \{r_{n(k,A)}\}, A \rangle = G(k, A) \subset \mathbb{U}.$$

Hence $\{G(n, A) : n \in \mathbb{N}\}$ is a local base at A.

(ii) For any $m \in \mathbb{N}$, let $B \in G(m+1, A)$; then $\{r_{n(1,A)}, ..., r_{n(m,A)}, r_{n(m+1,A)}\} \subset B \subset A$. If B is finite, it is obvious that $G(k_B + 1, B) = \{B\} \subset G(m + 1, A)$; if B is infinite, then $r_{n(i,B)} = r_{n(i,A)}$ for any $i \le m + 1$, hence

$$G(m + 1, B) = \langle \{r_{n(1,B)}\}, ..., \{r_{n(m+1,B)}\}, B \rangle \subset \langle \{r_{n(1,A)}\}, ..., \{r_{n(m+1,A)}\}, A \rangle,$$

that is, $G(m + 1, B) \subset G(m + 1, A) \subset G(m, A)$.

Case 2: $A \setminus I(X) \neq \emptyset$. Then $A = A_1 \cup A_2$, where $A_1 = A \cap I(X)$, $A_2 = A \cap S(X)$. Clearly, A_2 is compact. For each $n \in \mathbb{N}$, let $B_{1/n}(A_2) = \{x \in X, d(A_2, x) < 1/n\}$, and put $\mathcal{D} = \{D \in \mathcal{C} : D \cap A_2 \neq \emptyset\}$. Then we write $\mathcal{D} = \{D_1, D_2, ..., D_k, ...\}$ such that $D_i = C_{q_i}$ for $i \in \mathbb{N}$ and $\{q_i : i \in \mathbb{N}\}$ is increasing. For each $n \in \mathbb{N}$, let $V_n = B_{1/n}(A_2) \cup A_1$. If A_1 is finite, then $A_1 = \{r_{n(1,A_1)}, ..., r_{(k_{A_1},A_1)}\}$, then put

$$G(m, A) = \langle D_1 \cap V_m, ..., D_m \cap V_m, \{r_{n(1,A_1)}\}, ..., \{r_{n(k_{A_1},A_1)}\}, V_m \rangle;$$

if A_1 is infinite, then $A_1 = \{r_{n(i,A_1)} : i \in \mathbb{N}\}$, then for each $m \in \mathbb{N}$ put

$$G(m, A) = \langle D_1 \cap V_m, ..., D_m \cap V_m, \{r_{n(1,A_1)}\}, ..., \{r_{n(m,A_1)}\}, V_m \rangle.$$

Now it suffices to prove $G(n, A)$ satisfies (i) and (ii) in the definition of γ-space as A_1 is infinite; for the case that A_1 is finite, we may use a similar way to prove it.

(i′) Let $\mathbb{U} = \langle U_1, .., U_m \rangle$ be an arbitrary open neighborhood of A. Since $A_2 \subset \bigcup\{U_i : i \le m\}$, there is $n' \in \mathbb{N}$ such that $A_2 \subset B_{1/n'}(A_2) \subset \bigcup\{U_i : i \le m\}$. For each $i \leq m$, if $U_i \cap A_2 \neq \emptyset$, then we can find $D_{j_i} \in \mathcal{D}$ such that $D_{j_i} \cap A_2 \neq \emptyset$ and $D_{j_i} \subset U_i$. Let $n'' = \max\{j_i : i \leq m\}$, and let $m' = \max\{n', n''\}$. Then

$$\begin{aligned} &A \in G(m', A) \\ &\quad = \langle D_1 \cap V_{m'}, ..., D_{m'} \cap V_{m'}, \{r_{n(1,A_1)}\}, ..., \{r_{n(m',A_1)}\}, V_{m'}\rangle \subset \langle U_1, ..., U_m\rangle \end{aligned}$$

by [22, Lemma 2.3.1]. Hence $\{G(m, A) : m \in \mathbb{N}\}$ is a countable local base at A.

(ii$'$) For any $m \in \mathbb{N}$, let $s = q_m$, then $s > m$. We claim that, for any $B \in G(s, A)$, we have $G(s, B) \subset G(m, A)$. Indeed, it is obvious that $B \cap (D_i \cap V_s) \neq \emptyset$ for each $i \leq s$ and $r_{n(i,B)} = r_{n(i,A)}$ for each $i \leq s$ and $B \subset V_s \subset V_m$. Let $\mathcal{E} = \{C \in \mathcal{C} : C \cap B \neq \emptyset\}$; then we write $\mathcal{E} = \{E_i : i \in \mathbb{N}\}$ such that $E_i = C_{l_i}$ for $i \in \mathbb{N}$ and $\{l_i\}$ is increasing. Note that $B \cap D_i \neq \emptyset$ and $B \cap E_j \neq \emptyset$ for any $i \leq m$, $j \leq s$, we can see that $\{D_1, ..., D_m\} \subset \{E_1, ..., E_s\}$. Therefore, it follows from [22, Lemma 2.3.1] that

$$\begin{aligned} G(s, B) &= \langle E_1 \cap V_s, ..., E_s \cap V_s, \{r_{n(1,A)}\}, ..., \{r_{n(s,A)}\}, V_s\rangle \\ &\subset \langle D_1 \cap V_m, ..., D_2 \cap V_m, \{r_{n(1,A)}\}, ..., \{r_{n(m,A)}\}, V_m\rangle \\ &= G(m, A). \end{aligned}$$

Therefore, $(CL(X), \mathbb{V})$ is a γ-space. □

Corollary 5.18 *The following statements are equivalent for a space X.*

(1) $(CL(X), \mathbb{V})$ is a γ-space;
(2) $(CL(X), \mathbb{V})$ is a weakly first-countable and submetrizable space;
(3) $(CL(X), \mathbb{V})$ is weakly first-countable and has a G_δ-diagonal;
(4) X is a separable metrizable space and $S(X)$ is compact.

Proof (1) $\Longleftrightarrow$ (4) by Theorem 5.17. (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4). By Theorems 5.15 and 5.16, X is a separable D_1-space with a G_δ-diagonal, then $S(X)$ is countably compact by [5, Theorem 1], hence $S(X)$ is compact metrizable. Thus X is metrizable by [5, Theorem 7(8)].

(4) $\Rightarrow$ (2). By [15, Proposition 8(2)], $(CL(X), \mathbb{V})$ is submetrizable. Moreover, X is also first-countable by [14, Theorem 2.3]. □

By Theorem 5.16 and Corollary 5.18, we have the following corollary.

Corollary 5.19 *Let X be a (regular) space. Then $(CL(X), \mathbb{V})$ is a γ-space if and only if $(CL(X), \mathbb{V})$ is weakly first-countable and has a G_δ-diagonal.*

The following theorem shows that the classes of D_0-spaces and γ-spaces are equivalent in $(CL(X), \mathbb{V})$ under the assumption of $MA + \neg CH$.

Theorem 5.20 *($MA + \neg CH$) Let X be a space. Then $(CL(X), \mathbb{V})$ is a D_0-space if and only if $(CL(X), \mathbb{V})$ is a γ-space.*

Proof By [12, Theorem 10.6 (iii)], every γ-space is a D_0-space, so the necessity is done.

Sufficiency. Assume $(CL(X), \mathbb{V})$ is a D_0-space, then, by Theorem 5.17, it suffices to prove that X is a separable metrizable space and $S(X)$ is compact.

By Theorem 5.15, X is a D_1-space and every closed subset of X has countable external π-base, hence $S(X)$ is countably compact by [5, Theorem 1] and X is hereditarily separable. Under $MA + \neg CH$, X is strongly paracompact by [5, Theorem 5 (2)], which implies that $S(X)$ is compact. Moreover, $S(X)$ is a closed subset of X, then $(CL(S(X)), \mathbb{V})$ is a closed subspace of $(CL(X), \mathbb{V})$. Since $S(X)$ is compact, it follows that $(CL(S(X)), \mathbb{V})$ is a compact D_0-space. Then $(CL(S(X)), \mathbb{V})$ is a D_1-space since every closed subset of $(CL(S(X)), \mathbb{V})$ is compact. By Theorem 5.1, $S(X)$ is compact metrizable, hence X is metrizable by [5, Theorem 7 (2)]. Therefore, $(CL(X), \mathbb{V})$ is a γ-space by Theorem 5.17. □

Since each quasi-metrizable space is a γ-space, we have the following conjecture.

Conjecture 1 Let X be a space. Then $(CL(X), \mathbb{V})$ is quasi-metrizable if and only if $X = C \oplus D$, where C is a compact metrizable space and D is a countable discrete space.

Remark 5.21 If Question 3.8 is affirmative, then it is obvious that this Conjecture 1 does not hold. If Question 3.8 is negative, then this Conjecture 1 holds. Indeed, assume that $(CL(X), \mathbb{V})$ is quasi-metrizable, then it follows from Theorem 5.17 that X is a separable metrizable space and $S(X)$ is compact. Since $(CL(C_\omega), \mathbb{V})$ is not quasi-metrizable, it follows that X is locally compact, which implies that $S(X)$ is open in X. Therefore, X is the topological sum of a compact metrizable space and a countable discrete space. Moreover, from Proposition 3.2 and Theorem 3.7, it follows that $(CL(X), \mathbb{V})$ is quasi-metrizable if X is the topological sum of a compact metrizable space and a countable discrete space.

From Theorem 3.7, we also have the following conjecture.

Conjecture 2 Let X be a space. Then $(CL(X), \mathbb{V})$ is quasi-metrizable if and only if $(CL(X), \mathbb{V})$ is non-Archimedean quasi-metrizable.

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