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Boolean Algebras Derived from a Quotient of a Distributive Lattice

H. Barzegar[1]

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Abstract
In this article, we introduce a lattice congruence θ_I^d with respect to a nonempty ideal I of a distributive lattice L and a derivation d on L. We investigate some necessary and sufficient conditions for the quotient algebra L/θ_I^d to be a Boolean algebra.

Keywords Distributive lattice · Boolean algebra · Congruence · Ideal · Filter

Mathematics Subject Classification Primary 03G10 · 06D05 · Secondary 03G05

1 Introduction and Preliminaries

In calculus a derivation is a linear map d with the additional property $d(f \cdot g) = d(f) \cdot g + f \cdot d(g)$. Based on this property, several authors have adapted the notion of derivation in different contexts. First, the notion of derivation has been studied in rings and near-rings [5, 12]. After that, some authors have considered the notion of derivation in other structures: Jun and Xin [10] in BCI-algebras, [2, 7, 15–17] in lattices and [3, 9, 13] in Leibnez Algebras. In [17], Xin et al, gave some equivalent conditions under which a derivation is isotone for lattices with a greatest element, modular lattices, and distributive lattices. They characterized modular lattices and distributive lattices in terms of isotone derivations. Also, Xin answered to some other questions about the relations among derivations, ideals, and sets of fixed points in [16].

Lattices and Boolean algebras play a significant role in computer science and logic as well. Recall that a Boolean algebra is a bounded complemented distributive lattice. So Boolean algebras can be seen as a special class of lattices. One of the common

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✉ H. Barzegar
barzegar@tafreshu.ac.ir

1 Department of Mathematics, Tafresh University, Tafresh 39518-79611, Iran

subjects in all kinds of algebras is congruences. The study of congruence relations on lattices and an the connections between ideals and congruences in a lattice have been investigated by many authors, see for example [1, 8, 11].

In [14], two types of congruences are introduced in distributive lattices, both of them defined in terms of derivations. After that, we are interested to generalize the work on a distributive lattice with a nonempty ideal. To this aim, in this section we briefly first recall some ingredients needed in the sequel. For more information see, for example, [6, 7, 17].

Throughout the paper L stands for a distributive lattice. The bottom element of a distributive lattice, if it exists, is denoted by $\bot_L$(or$\bot$) and the top element is denoted by $\top_L$(or$\top$). If both $\bot$ and $\top$ exist, L is called a bounded lattice. By a lattice map (or homomorphism) we mean a map $f : A \to B$ between two lattices which preserves binary operations $\vee$ and $\wedge$. Recall that a nonempty subset I of L is called an *ideal (filter)* of L if $a \vee b \in I$ $(a \wedge b \in I)$ and $a \wedge x \in I$ $(a \vee x \in I)$ whenever $a, b \in I$ and $x \in L$. An ideal I of L is called *prime ideal* if, for each $x, y \in L$, $x \wedge y \in I$ implies $x \in I$ or $y \in I$. An equivalence relation θ defined on L is said to be a lattice congruence on L if for all $a, b, c \in L$, $a\theta b$ implies that $(a \vee c)\theta(b \vee c)$ and $(a \wedge c)\theta(b \wedge c)$.

Definition 1.1 [7] For a lattice L, a function $d : L \to L$ is called a derivation on L, if for all $x, y \in L$: (i) $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$. (ii) $d(x \vee y) = d(x) \vee d(y)$.

In [17, Th. 3.21], it was shown that the condition (i) can be simplified in the following way which we use throughout the paper from now on.

Lemma 1.2 *[17] If L is a distributive lattice, then $d : L \to L$ is a derivation if and only if the following conditions hold:* (i) $d(x \wedge y) = d(x) \wedge y = x \wedge d(y)$. (ii) $d(x \vee y) = d(x) \vee d(y)$.

One can find the proof of the following lemma in [7] and [16].

Lemma 1.3 *Let $d : L \to L$ be a derivation and $x, y \in L$.*

(i) If L has a bottom element $\bot$, then $d(\bot) = \bot$.
(ii) $d(x) \leq x$.
(iii) $d(d(x)) = d(x)$.
(iv) If $x \leq y$, then $d(x) \leq d(y)$.
(v) If I is an ideal of L, then $d(I) \subseteq I$.
(vi) If L has a top element $\top$, then $d(x) = x \wedge d(\top)$.

As a consequence of Lemma 1.3.(iii), we have the following corollary.

Corollary 1.4 *Every derivation $d : L \to L$ is a lattice homomorphism.*

In [14], there are some notions concerning a distributive lattice with 0 (bottom element) such as ideals and congruences defined based on 0. In Section 2, replacing 0 by a nonempty ideal I of a distributive lattice, we general these notions and study their properties

Section 3 is devoted to the case where a distributive lattice L is an atomic lattice or, more generally, an I-atomic lattice. Our main results, can be found in Sect. 4. Here we show that for an ideal I, the identity map is a derivation such that L/θ_I^d become a Boolean algebra with maximal cardinality. Finally we demonstrate some necessary and sufficient conditions under which L/θ_I^d is a Boolean algebra.

2 Congruences and Ideals with Respect to a Derivation in a Distributive Lattice

In what follows, which is a generalization of the article [14], we introduce some special ideals and congruences with respect to a nonempty ideal and a derivation on distributive lattices. After that, we study some essential properties of this congruences, which will be used in Sects. 3 and 4. Note that most of the definitions of this section have been selected from [14].

Suppose L is a distributive lattice, I a nonempty ideal of L, $a \in L$ and d a derivation on L. By definition, we consider $ker_I d = d^{-1}(I) = \{x \in L \mid d(x) \in I\}$ and $(a)_I^d = \{x \in L \mid a \wedge x \in ker_I d\} = \{x \in L \mid d(a \wedge x) \in I\}$. Observe that we also have $(a)_I^d = (d \circ \lambda_a)^{-1}(I)$, where $\lambda_a : L \to L$ is the derivation defined by $\lambda_a(x) = a \wedge x$.

All parts of the following lemma will be used in what follows.

Lemma 2.1 *Let $a, b \in L$ and I be an ideal of L. Then,*

(i) $ker_I d$ *and* $(a)_I^d$ *are ideals of* L.
(ii) *if* $a \leq b$, *then* $(b)_I^d \subseteq (a)_I^d$.
(iii) $(a \vee b)_I^d = (a)_I^d \cap (b)_I^d$.
(iv) $I \subseteq ker_I d \subseteq (a)_I^d$.
(v) $a \in ker_I d$ iff $a \in (a)_I^d$ iff $(a)_I^d = L$.
(vi) $\bigcap_{a \in L} (a)_I^d = ker_I d$.
(vii) $a \in (b)_I^d$ *if and only if* $b \in (a)_I^d$.
(viii) *if* $(a)_I^d \neq L$, *then* $\bigcap_{b \in (a)_I^d} (b)_I^d \neq ker_I d$.
(ix) *if I and J are ideals of L in which $I \subseteq J$, then* $ker_I d \subseteq ker_J d$ *and* $(a)_I^d \subseteq (a)_J^d$, *for each* $a \in L$.

Now we introduce a binary relation on a distributive lattice with respect to an ideal and a derivation. The following proposition, which has an easy proof, shows that this binary relation is a lattice congruence.

Proposition 2.2 *For an ideal I of L, the binary relation θ_I^d defined as*

$$x \theta_I^d y \text{ iff } (x)_I^d = (y)_I^d$$

is a lattice congruence.

An element $a \in L$ is called a *kernel element* with respect to an ideal I, if $(a)_I^d = ker_I d$. Let us denote the set of all kernel elements with respect to the ideal I of L by $\mathcal{K}_I^d$.

For the ideal L, it is not difficult to check that $L = ker_L d = (a)_L^d = \mathcal{K}_L^d$ and hence $\theta_L^d = \nabla = \{(a, b) \mid a, b \in L\}$, which implies that L/θ_L^d is a singleton. So, from now on, all the ideals I will be assumed to be nontrivial ($I \neq L$).

Lemma 2.3 *(i)* $\mathcal{K}_I^d \neq \emptyset$, *then* $\mathcal{K}_I^d$ *is a filter of* L.
(ii) $ker_I d = L$ *if and only if* $\mathcal{K}_I^d = L$.
(iii) *If* $(a)_I^d$ *and* $\mathcal{K}_I^d$ *are nontrivial, then* $\mathcal{K}_I^d \cap (a)_I^d = \emptyset$.

(iv) $(x)_I^d = (d(x))_I^d$ *and* $x\theta_I^d d(x)$, *for all* $x \in L$.
(v) *If* $x\theta_I^d y$, *then* $d(x)\theta_I^d d(y)$.

Proof (i) Let $a, b \in \mathcal{K}_I^d$ and $c \in L$. By Lemma 2.1(iv), $ker_I d \subseteq (a \wedge b)_I^d$. For the converse, let $x \in (a \wedge b)_I^d$. Then $d((a \wedge b) \wedge x) \in I$ and hence $b \wedge x \in (a)_I^d = ker_I d$. So $d(b \wedge x) \in I$, which implies $x \in (b)_I^d = ker_I d$. Thus $(a \wedge b)_I^d = ker_I d$, hence $a \wedge b \in \mathcal{K}_I^d$. Also $a \vee c \in \mathcal{K}_I^d$, by Lemma 2.1(iii) and (iv).

To prove (ii), apply Lemma 2.1(v) and for (iii), let $b \in \mathcal{K}_I^d \cap (a)_I^d$. Appling Lemma 2.1(vii), $a \in (b)_I^d = ker_I d$. So by Lemma 2.1(v), $(a)_I^d = L$, which is impossible.

(iv) By Lemma 1.3(ii), $d(x) \leq x$ and hence $(x)_I^d \subseteq (d(x))_I^d$. Let $y \in (d(x))_I^d$. Hence $d(y \wedge x) = d(y \wedge d(x)) \in I$, which implies $y \in (x)_I^d$. Thus $(x)_I^d = (d(x))_I^d$. □

The following proposition shows that the quotient lattice L/θ_I^d is a bounded lattice.

Proposition 2.4 *For a nontrivial ideal I of L, the distributive lattice* L/θ_I^d *is a bounded lattice with*

(i) $\perp_{L/\theta_I^d} = ker_I d$,
(ii) $\top_{L/\theta_I^d} = \mathcal{K}_I^d$ *whenever* $\mathcal{K}_I^d \neq \emptyset$.

Proof (i) Let $a \in ker_I d$. By Lemma 2.1, for each $b \in ker_I d$, $(a)_I^d = L = (b)_I^d$ and hence $a\theta_I^d b$. Thus $ker_I d \subseteq [a]_{\theta_I^d}$. For the converse, let $c \in [a]_{\theta_I^d}$. Again, by Lemma 2.1, $(c)_I^d = (a)_I^d = L$ and hence $c \in (c)_I^d$. So $d(c) = d(c \wedge c) \in I$, which implies $c \in ker_I d$. Thus $ker_I d = [a]_{\theta_I^d}$. Since $ker_I d$ is an ideal of L, for each $[y]_{\theta_I^d} \in L/\theta_I^d$, we get that $a \wedge y \in ker_I d$ and hence $ker_I d = [a]_{\theta_I^d} = [a \wedge y]_{\theta_I^d} \leq [y]_{\theta_I^d}$. Therefore $\perp_{L/\theta_I^d} = ker_I d$.
(ii) The proof is similar to (i). □

The following theorem is another version of [14, Th. 3.4].

Theorem 2.5 *Let I be an ideal of L and d a derivation on L. Then the following are equivalent:*

(i) $\theta_I^d = \nabla$.
(ii) $ker_I d = L$
(iii) *For each* $x \in L$, $I \cap [x]_{ker(d)}$ *is a singleton set.*

Proof (i) ⇒ (ii) Let $x \in L$ and $a \in ker_I d$. Since $\theta_I^d = \nabla$, $x\theta_I^d a$ and, by Lemma 2.1(v), $x \in ker_I d$. So $ker_I d = L$.

(ii) ⇒ (iii) From the part one of the proof of [14, Th. 3.4].

(iii) ⇒ (i) Let $x, y \in L$. Consider $I \cap [x]_{ker(d)} = \{x_0\}$ and $I \cap [y]_{ker(d)} = \{y_0\}$. By Lemma 1.3(ii), $d(x) = d(x_0) \leq x_0$ and, since I is an ideal, $d(x) \in I$. By Lemma 1.3(ii), $d(x) \in I \cap [x]_{ker(d)}$, which implies $d(x) = x_0$. Similarly $d(y) = y_0$. Using Lemma 2.3(iv) and Proposition 2.4(i), $x\theta_I^d x_0 \theta_I^d y_0 \theta_I^d y$. Thus $\theta_I^d = \nabla$. □

As seen in Lemma 2.3(i), $\mathcal{K}_I^d$ is a filter, whenever $\mathcal{K}_I^d \neq \emptyset$. So in the following lemma we investigate some conditions over which $\mathcal{K}_I^d \neq \emptyset$.

Lemma 2.6 *(i) If $\top \in L$, then $\top, d(\top) \in \mathcal{K}_I^d$.*

(ii) If I or $ker_I d$ is a prime nontrivial ideal of L, then $\mathcal{K}_I^d \neq \emptyset$ and if $ker_I d \neq L$, then L is the disjoint union of $ker_I d$ and $\mathcal{K}_I^d$. Also $\theta_I^d = \{(a, b) \mid \{a, b\} \subseteq ker_I d$ or $\{a, b\} \subseteq \mathcal{K}_I^d\}$.

(iii) If L is a chain and I a nontrivial ideal of L, then $\mathcal{K}_I^d \neq \emptyset$.

Proof We just prove (ii). If $ker_I d = L$, then $ker_I d = \mathcal{K}_I^d = L$. Let $ker_I d \neq L$ and $b \notin ker_I d$ and $x \in (b)_I^d$. Then $x \wedge d(b) \in I$ and $d(b) \notin I$. If I is prime, then $x \in I \subseteq ker_I d$. Now let $ker_I d$ is prime. Since $b \notin ker_I d$ and $x \wedge d(b) \in I \subseteq ker_I d$, $x \in ker_I d$. Thus $b \in \mathcal{K}_I^d$. So $L = ker_I d \cup \mathcal{K}_I^d$ and the first part of the proof will be complete by using Lemma 2.3.(iii). Now by Proposition 2.4, $\theta_I^d = \{(a, b) \mid \{a, b\} \subseteq ker_I d$ or $\{a, b\} \subseteq \mathcal{K}_I^d\}$. □

As a consequence of Lemma 2.6(ii), we conclude that, if $I \subseteq J$, then there is no information on the relation (with respect to containment) between $\mathcal{K}_I^d$ and $\mathcal{K}_J^d$ at all. For example, let $I \subseteq J$ be two prime ideals of L and d be the identity derivation. By Lemmas 2.6(ii) and 2.1(x), $\mathcal{K}_J^d \subseteq \mathcal{K}_I^d$. For another example, assume that L has a bottom element $\bot$. Consider $\bot \neq a \in L$, $I = \{\bot\}$, $J = \downarrow a$ and a derivation d defined by $d(x) = a \wedge x$. Clearly $ker_J d = L$ and, since $d(a) = a \wedge a = a \neq \bot$, $a \notin ker_I d$. So, by Lemma 2.3(ii), $\mathcal{K}_I^d \subseteq \mathcal{K}_J^d$.

In the lattice $Con(L)$, all of congruences of L, $\theta_1 \leq \theta_2$ if $\theta_1 \subseteq \theta_2$.

Proposition 2.7 *For a nontrivial ideal I of L, the congruence θ_I^d is the greatest congruence relation having $ker_I d$ as a whole class.*

Proof Let θ be a lattice congruence on L such that $ker_I d$ is a congruence class and $x\theta y$. At first suppose that $\mathcal{K}_I^d \neq \emptyset$. By Proposition 2.4, $\mathcal{K}_I^d$ and $ker_I d$ are two congruence classes, i.e., elements of L/θ_I^d. The following cases may occur:

Case 1. $x, y \in \mathcal{K}_I^d$. Hence $(x)_I^d = ker_I d = (y)_I^d$ and $x\theta_I^d y$.

Case 2. $x, y \notin \mathcal{K}_I^d$. For each $a \in (x)_I^d$, $(x \wedge a)\theta(y \wedge a)$ and $x \wedge a \in ker_I d$. Then $[y \wedge a]_\theta = [x \wedge a]_\theta = ker_I d$. So $y \wedge a \in ker_I d$ and $a \in (y)_I^d$. Thus $(x)_I^d \subseteq (y)_I^d$ and, analogously, $(y)_I^d \subseteq (x)_I^d$, which implies that $x\theta_I^d y$.

Case 3. $x \in \mathcal{K}_I^d$ and $y \notin \mathcal{K}_I^d$ (or similarly $y \in \mathcal{K}_I^d$ and $x \notin \mathcal{K}_I^d$). This case cannot occur. For, consider $b \in (y)_I^d \setminus (x)_I^d$. Then $b \wedge y \in ker_I d$ and $b \wedge x \notin ker_I d$. Also $(b \wedge x)\theta(b \wedge y)$. So $b \wedge x \in ker_I d$, which is impossible. Therefore $\theta \subseteq \theta_I^d$. Now let $\mathcal{K}_I^d = \emptyset$. So only the case 2 may be occured, which implies $x\theta_I^d y$. □

From now on, up to Lemma 2.11, we investigate some conditions over ideals and derivations to get a smallest congruence θ_I^d. The smallest one infer that the quotient lattice L/θ_I^d has the maximal cardinality.

Proposition 2.8 *For an ideal I and a derivation d on L, $\theta_I^{id} \subseteq \theta_I^d$.*

Proof Let $a\theta_I^{id} b$ and $x \in (a)_I^d$. Then $d(x) \in (a)_I^{id} = (b)_I^{id}$. So $d(b \wedge x) = b \wedge id(d(x)) \in I$. Thus $x \in (b)_I^d$ which implies $(a)_I^d \subseteq (b)_I^d$ and, by similar way, $(b)_I^d \subseteq (a)_I^d$. So $a\theta_I^d b$. □

The following example shows that the maps that assign each ideal I to θ_J^d and $\mathcal{K}_J^d$ need not be increasing or decreasing.

Example 2.9 (i) Let $L = \{a, b, c, d\}$ in which $a \prec b \prec c \prec d$ where $\prec$ denotes the covering relation, $I = \{a\}$ and $J = \{a, b, c\}$. So $I \subset J$. It is not difficult to check that $(a, b) \in \theta_J^{id} \setminus \theta_I^{id}$ and $(c, d) \in \theta_I^{id} \setminus \theta_J^{id}$. Thus $\theta_I^{id} \nsubseteq \theta_J^{id}$ and $\theta_J^{id} \nsubseteq \theta_I^{id}$. Also, we can conclude that $\mathcal{K}_J^{id} = \{d\}$ and $d \in \mathcal{K}_I^{id}$. So $\mathcal{K}_J^{id} \subseteq \mathcal{K}_I^{id}$.

(ii) Consider the four element Boolean algebra $L = \{a, b, c, d\}$ in which a and d are bottom and top elements, respectively. Consider $I = \{a\}$ and $J = \{a, b\}$ and id the identity map. So $I \subset J$ and clearly $\mathcal{K}_I^{id} = \{d\} \subset \{c, d\} = \mathcal{K}_J^{id}$. Also it is not difficult to check that $(a)_I^{id} = L$, $(b)_I^{id} = (c)_I^{id} = (d)_I^{id} = \{a\}$, $(a)_J^{id} = (b)_J^{id} = L$ and $(c)_J^{id} = (d)_J^{id} = J$. So $(b, c) \in \theta_I^{id} \setminus \theta_J^{id}$ and $(a, b) \in \theta_J^{id} \setminus \theta_I^{id}$.

Lemma 2.10 *For ideals $I \subseteq J$ and a derivation d on L, if there exists a derivation d_1 on L such that $ker_I d_1 = J$, then $\theta_I^d \subseteq \theta_J^d$ and the equality holds if $d_1 = d$.*

Proof Let $a\theta_I^d b$ and $x \in (a)_J^d$. Then $d(x \wedge a) \in J = ker_I d_1$, which implies $d_1(x) \wedge d(a) = d_1(d(x \wedge a)) \in I$. So $d_1(x) \in (a)_I^d = (b)_I^d$ and hence $d(x \wedge b) \in ker_I d_1 = J$. Thus $x \in (b)_J^d$. This gives that $\theta_I^d \subseteq \theta_J^d$.

Now let $d_1 = d$. Consider $a\theta_{ker_I d}^d b$ and $x \in (a)_I^d$. Since $ker_I d$ is an ideal and $d(x) \leq x$, $d(x) \wedge a = d(x \wedge a) \in ker_I d$. So $x \in (a)_{ker_I d}^d = (b)_{ker_I d}^d$ and $d(x \wedge b) \in ker_I d$. Now it is not difficult to show that $x \in (b)_I^d$. Thus $\theta_{ker_I d}^d \subseteq \theta_I^d$. □

Lemma 2.11 *Let I be an ideal of L and $a \in L$. If $J = (a)_I^d$ and K is an ideal of L such that $I \subseteq K \subseteq J$, then*

(i) $(a)_I^d = (a)_K^d$.
(ii) $a \in \mathcal{K}_J^d$.
(iii) $\theta_K^d \subseteq \theta_J^d$.
(iv) $\theta_K^d = \theta_J^d$ *whenever* $a \in \mathcal{K}_I^d$.

Proof (i) By Lemma 2.1(x), $(a)_I^d \subseteq (a)_K^d$. Now let $x \in (a)_K^d$. Then $d(x \wedge a) \in K \subseteq (a)_I^d$, which implies $a \wedge d(x) = d(d(a \wedge x) \wedge a) \in I$. So $x \in (a)_I^d$.
(ii) By part (i), it is enough to consider $K = J$. So $(a)_J^d = (a)_I^d = J$, which means $a \in \mathcal{K}_J^d$.
(iii) Let $x\theta_K^d y$ and $z \in (x)_J^d$. Then $d(x \wedge z) \in J$, which implies $d(x) \wedge (d(z) \wedge a) = d(d(x \wedge z) \wedge a) \in I \subseteq K$. So $d(z) \wedge a \in (x)_K^d$ and since $(x)_K^d = (y)_K^d$, $d(d(y \wedge z) \wedge a) = d(y) \wedge (d(z) \wedge a) \in K$, hence $d(y \wedge z) \in (a)_K^d = (a)_I^d = J$. Thus $z \in (y)_J^d$. Similarly, we can prove $(y)_J^d \subseteq (x)_J^d$. So $(x)_J^d = (y)_J^d$, which implies that $\theta_K^d \subseteq \theta_J^d$.
(iv) By (i), $(a)_K^d = (a)_I^d = ker_I d \subseteq ker_K d$. So by Lemma 2.1.(iv), $a \in \mathcal{K}_K^d$. Now the proof is straightforward, using Lemma 2.10. □

Note that the following example shows that in Lemma 2.11(iii), θ_K^d can be a strict subset of θ_J^d. Consider I a nontrivial prime ideal of L and $a \in I$. Then $J = (a)_I^d = L$

and hence for each $x \in L$, $(x)^d_J = L$. So $\theta^d_J = \nabla$ and, by Lemma 2.6, $\theta^d_I = \{(a, b) \mid \{a, b\} \subseteq ker_I d \text{ or } \{a, b\} \subseteq \mathcal{K}^d_I\}$. Thus $\theta^d_I \neq \theta^d_J$.

In the rest of this section we investigate some relationships between prime ideals and ideals of the form $(x)^d_I$. First note that, if I is a prime ideal, then so is $ker_I d$.

Lemma 2.12 (i) *If I is a prime ideal of L, then $ker_I d = L$ or for each $x \notin ker_I d$, $I = ker_I d = (x)^d_I$.*

(ii) *If $(x)^d_I$ is not a subset of the prime ideal $(y)^d_I$, then $x \wedge y \in ker_I d$.*

(iii) *If $(x)^d_I \neq (y)^d_I$ are prime ideals, then $x \wedge y \in ker_I d$.*

Proposition 2.13 *There exist prime ideals P_1, P_2 in L in which $P_1 \cup P_2 = L$ and $P_1 \cap P_2 = ker_I d$ if and only if there exist two classes $[a]_{\theta^d_I}$ and $[b]_{\theta^d_I}$ such that for each $x \in [a]_{\theta^d_I}$ and $y \in [b]_{\theta^d_I}$, $x \wedge y \in ker_I d$ and L/θ^d_I is of the form $\{ker_I d, [a]_{\theta^d_I}, [b]_{\theta^d_I}\}$.*

Proof Let $L/\theta^d_I = \{ker_I d, [a]_{\theta^d_I}, [b]_{\theta^d_I}\}$. First note that, by Lemma 2.1(v), for each $x \in [a]_{\theta^d_I}$, $x \wedge a \notin ker_I d$. The subsets $P_1 = [a]_{\theta^d_I} \cup ker_I d$ and $P_2 = [b]_{\theta^d_I} \cup ker_I d$ of L are prime ideals. For, let $x, y \in P_1$. In the case where $x \in ker_I d$ or $y \in ker_I d$, by Lemma 2.1(i), $x \vee y \in P_1$, else, $(x \vee y)^d_I = (x)^d_I \cap (y)^d_I = (a)^d_I$. Thus $x \vee y \in P_1$. Consider $x \in P_1$, $z \in L$ and $z \leq x$. Then $z \wedge a \leq x \wedge a \in ker_I d$. Thus $z \in (a)^d_I$ and hence $z \in P_1$. Now let $x \wedge y \in P_1$ and $y \in [b]_{\theta^d_I}$. So $y \wedge b \notin ker_I d$. If $y \wedge b \in [a]_{\theta^d_I}$, then $y \wedge b = (y \wedge b) \wedge b \in ker_I d$, which is impossible. So $y \wedge b \in [b]_{\theta^d_I}$. Consider $x \in [b]_{\theta^d_I}$, so $(x)^d_I = (b)^d_I = (y \wedge b)^d_I$. If $x \wedge y \in P_1$, then $(x \wedge y) \wedge b \in ker_I d$ and hence $x \in (b \wedge y)^d_I = (x)^d_I$. So $x \in ker_I d$ which is a contradiction. Thus $x \in P_1$.

For the converse, consider $V_1 = P_1 \setminus ker_I d$ and $V_2 = P_2 \setminus ker_I d$. The subset V_1 is a class, for, let $a \in V_1$. We show $V_1 = [a]_{\theta^d_I}$. Let $x \in V_1$. For each $y \in (a)^d_I$, $a \wedge y \in ker^d_I \subseteq P_2$ and, since $a \notin P_2$, $y \in P_2$. If $y \in ker_I d$, then $y \in (x)^d_I$, else, $y \in V_2 \subseteq P_2$, which implies $x \wedge y \in P_1 \cap P_2 = ker_I d \subseteq (x)^d_I$. So $y \in (x)^d_I$ and hence $(a)^d_I \subseteq (x)^d_I$. The proof of $(x)^d_I \subseteq (a)^d_I$ is similar. Thus $(a)^d_I = (x)^d_I$, which implies $V_1 \subseteq [a]^d_I$. Now let $x \in [a]^d_I$. Then $(x)^d_I = (a)^d_I$ and, since $a \notin ker_I d$, then $x \notin ker_I d$, too. If $x \notin P_1$, then $x \in P_2$ and hence $a \wedge x \in P_1 \cap P_2 = ker_I d$. Thus $a \in (x)^d_I = (a)^d_I$. By Lemma 2.1(iv), $(a)^d_I = L$, which is a contradiction. Thus $x \in P_1$ and hence $x \in V_1$. So $V_1 = [a]_{\theta^d_I}$. Similarly, $V_2 = [b]_{\theta^d_I}$. □

Definition 2.14 For a nontrivial ideal I of L, an ideal P is called I-minimal, if it is minimal in the set of ideals containing I and it is called an I-minimal prime ideal, if P is a least prime ideal containing I.

From now on, we consider the set $\Sigma = \{(x)^d_I \mid x \in L \setminus ker_I d\}$. The set Σ is a poset under the inclusion relations.

Theorem 2.15 *Let I be an ideal of L and $a \in I$. The following assertions are equivalent:*

(i) $(a)^d_I$ *is a maximal element in* Σ.

(ii) $(a)^d_I$ *is a prime ideal.*

(iii) $(a)^d_I$ *is a* $ker_I d$*-minimal prime ideal.*

Proof (i)⇒ (ii) Let $x \wedge y \in (a)^d_I$ and $x \notin (a)^d_I$. Since $a \wedge x \leq a$, using Lemma 2.1(ii), $(a)^d_I \subseteq (a \wedge x)^d_I$. By the hypothesis, $(a)^d_I = (a \wedge x)^d_I$ or $(a \wedge x)^d_I = L$. If $(a \wedge x)^d_I = L$, then $a \wedge x \in ker_I d$, which is a contradiction. Thus $(a)^d_I = (a \wedge x)^d_I$, which gives that $y \in (a)^d_I$. Now, the proof is complete using Lemma 2.1(i).

(ii)⇒ (iii) Since $(a)^d_I$ is a prime ideal, it is a proper ideal of L and, by Lemma 2.1(v), $a \notin ker_I d$. If $ker_I d$ is a prime ideal, we are done, by Lemma 2.12(i). Let Q be a prime ideal of L containing $ker_I d$ such that $Q \subseteq (a)^d_I$ and $x \in (a)^d_I \setminus Q$. Then $x \wedge a \in ker_I d \subseteq Q$. Since $x \notin Q$ and Q is a prime ideal, $a \in Q \subseteq (a)^d_I$. Now, by Lemma 2.1(v), $(a)^d_I = L$, which is a contradiction.

(iii)⇒ (i) Let $(a)^d_I \subseteq (x)^d_I \neq L$. Consider $y \in (x)^d_I \setminus (a)^d_I$. Then $y \wedge x \in ker_I d \subseteq (a)^d_I$, which implies that $x \in (a)^d_I \subseteq (x)^d_I$. Again, by Lemma 2.1(v), $(x)^d_I = L$, which is a contradiction. □

Lemma 2.16 *In the following assertions we have,* (i)⇒ (ii)⇒ (iii).

(i) The set Σ *satisfies the descending chain condition with respect to inclusion.*
(ii) L *does not have an infinite* $M \subseteq L \setminus ker_I d$ *such that for each* $x, y \in M$, $x \wedge y \in ker_I d$.
(iii) The set Σ *satisfies the ascending chain condition with respect to inclusion.*

Proof (i)⇒(ii) Let L have an infinite $M \subseteq L \setminus ker_I d$ such that for each $x, y \in M$, $x \wedge y \in ker_I d$ and consider $x_1, x_2 \in M$. By Lemma 2.1(ii), $(x_1 \vee x_2)^d_I \subseteq (x_1)^d_I$ and clearly $x_2 \in (x_1)^d_I \setminus (x_1 \vee x_2)^d_I$. Thus the following proper descending chain is induced, which is a contradiction:

$$(x_1)^d_I \supset (x_1 \vee x_2)^d_I \supset (x_1 \vee x_2 \vee x_3)^d_I \supset \cdots$$

(ii)⇒(iii) Let $(a_1)^d_I \subset (a_2)^d_I \subset \cdots$ be a proper chain and $x_j \in (a_j)^d_I \setminus (a_{j-1})^d_I$ for $j = 2, 3, \cdots$. Consider $y_j = x_j \wedge a_{j-1} \notin ker_I d$. For each $i < j$, since $x_i \in (a_i)^d_I \subseteq (a_{j-1})^d_I$, it is not difficult to show that $y_i \wedge y_j \in ker_I d$. Also, if $y_i = y_j$, then $y_i = y_i \wedge y_j \in ker_I d$, a contradiction. Thus the set $M = \{y_i \mid i = 2, 3, \cdots\}$ is an infinite set such that for each $x, y \in M$, $x \wedge y \in ker_I d$, which is a contradiction. □

We say that the lattice L satisfies the condition $(*)$, if L does not have an infinite $M \subseteq L \setminus ker_I d$ such that for each $x, y \in M$, $x \wedge y \in ker_I d$.

Lemma 2.17 *Suppose that L satisfies the condition* $(*)$*, then L has only a finite number of distinct* $ker_I d$*-minimal prime ideals of the form* $(a_i)^d_I (1 \leq i \leq n)$*. Also* $\bigcap_{i=1}^n (a_i)^d_I = ker^d_I$.

Proof By Lemma 2.16, Σ has maximal elements. Let $(a)^d_I \neq (b)^d_I$ be two maximal element in the set Σ. By Lemma 2.1(ii), $(a)^d_I \subseteq (a \wedge b)^d_I$ and $(b)^d_I \subseteq (a \wedge b)^d_I$. Since $(a)^d_I$ and $(b)^d_I$ are maximal elements in Σ, $(a \wedge b)^d_I = L$. Using Lemma 2.1(v), $a \wedge b \in ker_I d$. So if Σ has an infinite number of maximal element, then L has an infinite $M \subseteq L \setminus ker_I d$ such that for each $x, y \in M$, $x \wedge y \in ker_I d$, which is a contradiction. So Σ has a finite number of maximal elements. Now Theorem 2.15 completes the first part of the proof.

Now we show $\bigcap_{i=1}^{n}(a_i)_I^d = ker_I^d$. Using Lemma 2.16 and Zorn's lemma, for each $a\ ker_I d$ the set $\{(b)_I^d \mid (a)_I^d \subseteq (b)_I^d \ \ and\ b \notin ker_I d\}$ has a maximal element. Thus every proper ideal $(a)_I^d$ is contained in a maximal ideal $(a_i)_I^d$ in Σ, $1 \leq i \leq n$. Consider $x \in \bigcap_{i=1}^{n}(a_i)_I^d$. If $(x)_I^d \neq L$, there exists $1 \leq i \leq n$ such that $a_i \in (x)_I^d \subseteq (a_i)_I^d$. So $(a_i)_I^d = L$, which is not true. Thus $(x)_I^d = L$ and hence $x \in ker_I d$. □

Corollary 2.18 *If L satisfies the condition* $(*)$*, then every* $ker_I d$*-minimal prime ideal of L is of the form* $(a)_I^d$*, for some* $a \in L$*.*

Proof Let P be a $ker_I d$-minimal prime ideal of L. By Lemma 2.17, $\bigcap_{i=1}^{n}(a_i)_I^d = ker_I d$. Thus $\bigcap_{i=1}^{n}(a_i)_I^d \subseteq P$ and, since P is a prime ideal, there exists $j \in J$ such that $(a_j)_I^d \subseteq P$, which implies $(a_j)_I^d = P$. □

We close this section by the following important result, which is an immediate consequence of Corollary 2.18.

Theorem 2.19 *If L is a distributive lattice with a bottom element* $\bot$ *and satisfies the condition* $(*)$ *for* $ker_\bot(id)$*, then every minimal prime ideal of L is of the form* $(a)_\bot^{id}$*, for some* $a \in L$*.*

A special case of the previous theorem is the case where L is an atomic distributive lattice with a finite number of atoms.

3 Atomic Distributive Lattices

In this section the lattice L will be assumed to be a $ker_I d$-atomic distributive lattice which will be defined in the following definition.

Definition 3.1 For an ideal I of L, an element $a \in L \setminus I$ is called I-atom, if $\downarrow a \setminus \{a\} = \{x \in L \mid x < a\} \subseteq I$ and the lattice L is called I-atomic if for each $a \in L$ there exists an I-atom a_0 less than or equal to a.

From now on we denote with $A_I^d(L)$ the set of all $ker_I d$-atoms of L and we set, $A_I^d(a) = A_I^d(L) \cap \downarrow a$ and $A_I^d(a)^c = A_I^d(L) \setminus A_I^d(a)$.

Lemma 3.2 *(i) Let L have a top element* $\top$*. If* $\bigvee_{j \in J} a_j = \top$*, for some* $ker_I d$*-atoms* a_j*, then* $L \setminus \{\top\} \subseteq \bigcup_{j \in J}(a_j)_I^d$*.*

(ii) $\bigcap_{a_i \in A_I^d(L)}(a_i)_I^d = ker_I d$*.*

Proof (i) Let $x \in L$, $x \neq \top$. There exists a $ker_I d$-atom a_j such that $a_j \not\leq x$. Then $a_j \wedge x \in \downarrow a_j \setminus \{a_j\} \subseteq ker_I d$, which means $x \in (a_j)_I^d$.

(ii) By Lemma 2.1, $ker_I d \subseteq \bigcap_{a_i \in A_I^d(L)}(a_i)_I^d$. For the converse let $x \in \bigcap_{a_i \in A_I^d(L)}(a_i)_I^d \setminus ker_I d$. Then there exists a $ker_I d$-atom $a \leq x$ in such a way that $x \wedge a = a \notin ker_I d$. So $x \notin (a)_I^d$, which is impossible. □

In part (i) of the following lemma, for a $ker_I d$-atomic distributive lattice, we get a characterization of the congruence θ_I^d.

Lemma 3.3 *If $a, b \in L$, then;*

(i) $a\theta_I^d b$ if and only if $A_I^d(a) = A_I^d(b)$.
(ii) $a \wedge b \in ker_I d$ if and only if $A_I^d(a) \cap A_I^d(b) = \emptyset$.
(iii) For an element $a \in L$, if $A_I^d(L) = A_I^d(a)$, then $a \in \mathcal{K}_I^d$.
(iv) If $x = \bigvee_{a_i \in A_{Id}(L)} a_i$, then $x \in \mathcal{K}_I^d$.

Proof (i) Let $a\theta_I^d b$ and $x \in A_I^d(a)$. So $(a)_I^d = (b)_I^d$, $x \leq a$, $x \notin ker_I d$ and $\downarrow x \setminus \{x\} \subseteq ker_I d$. If $x \nleq b$, then $b \wedge x < x$ and $b \wedge x \in ker_I d$ which implies $x \in (b)_I^d = (a)_I^d$. So $x = x \wedge a \in ker_I d$ which is a contradiction. Thus $x \leq b$ which implies $x \in A_I^d(b)$ and $A_I^d(a) \subseteq A_I^d(b)$. By a similar way $A_I^d(b) \subseteq A_I^d(a)$.

For the converse, let $A_I^d(a) = A_I^d(b)$ and $x \in (a)_I^d$. Then $a \wedge x \in ker_I d$. Consider $b \wedge x \notin ker_I d$. Since L is $ker_I d$-atomic distributive lattice, there exists $x_0 \in A_I^d(b \wedge x) \subseteq A_I^d(b) = A_I^d(a)$. So $x_0 = a \wedge x_0 \leq a \wedge (b \wedge x) \in ker_I d$, which contradicts $x_0 \notin ker_I d$. Thus $b \wedge x \in ker_I d$ and hence $x \in (b)_I^d$. By a similar way $(b)_I^d \subseteq (a)_I^d$.

The proof of (ii) is clear.

(iii) Since $A_I^d(L) = A_I^d(a)$, for each $a_i \in A_I^d(L)$, $a_i \leq a$. By Lemmas 2.1(ii) and 3.2(ii), $(a)_I^d \subseteq \bigcap_{a_i \in A_I^d(L)} (a_i)_I^d = ker_I d \subseteq (a)_I^d$. Then $a \in \mathcal{K}_I^d$.

(iv) Straightforward, by (iii). □

Lemma 3.4 *If $a \in A_I^d(L)$, then $(a)_I^d$ is a maximal element in the set Σ.*

Proof Let $(a)_I^d \subseteq (b)_I^d \subsetneqq L (b \notin ker_I d)$. If $a \wedge b \in ker_I d$, then $b \in (a)_I^d \subseteq (b)_I^d$ and, by Lemma 2.1(v), $(b)_I^d = L$. So $a \wedge b \notin ker_I d$. Since a is $ker_I d$-atom, $a = a \wedge b$ and hence $a \leq b$. By Lemma 2.1(ii), $(b)_I^d \subseteq (a)_I^d$, which implies $(b)_I^d = (a)_I^d$. Thus $(a)_I^d$ is a maximal element in the set Σ. □

Theorem 3.5 *For an element $a \in L$, $(a)_I^d$ is a maximal element in the set Σ if and only if there exists a $ker_I d$-atom a_0 such that $A_I^d(a) = \{a_0\}$.*

Proof Let $(a)_I^d$ is a maximal element in the set Σ. By Lemma 2.1(v), $a \notin ker_I d$. Since L is a $ker_I d$-atomic distributive lattice, there exists $a_0 \in A_I^d(a)$ which Lemma 2.1(ii) implies that $(a)_I^d \subseteq (a_0)_I^d$. So $(a)_I^d = (a_0)_I^d$ and hence $A_I^d(a) = A_I^d(a_0) = \{a_0\}$.

For the converse, let $(a)_I^d \subseteq (b)_I^d \subsetneqq L$. By Lemma 2.1, $a \notin ker_I d$ and $b \leq a$. So $A_I^d(b) \neq \emptyset$ and since $A_I^d(b) \subseteq A_I^d(a) = \{a_0\}$, $A_I^d(b) = A_I^d(a)$. Now by Lemma 3.3.(i), $(a)_I^d = (b)_I^d$ which implies $(a)_I^d$ is a maximal element in Σ. □

Lemma 3.6 *Let L satisfy the condition $(*)$. Then*

(i) *Every $ker_I d$-minimal prime ideal of L is of the form $(a)_I^d$, for some $a \in A_I^d(L)$.*

(ii) *If L is atomic, then every minimal prime ideal of L is of the form $(a)_I^d$, for some atom a.*

Proof (i) Let P be a $ker_I d$-minimal prime ideal of L. Using Corollary 2.18, $P = (a_j)_I^d$ for some $a_j \in L$. Now, by Theorems 2.15 and 3.5, there exists $a \in A_I^d(L)$ such that $A_I^d(a_j) = \{a\}$. Thus, $A_I^d(a_j) = A_I^d(a)$ and by Lemma 3.3(i), $P = (a_j)_I^d = (a)_I^d$.

Item (ii) is a particular case of item (i) in the case where $I = \{0\}$. □

Theorem 3.7 *Let L satisfy the condition $(*)$, then L has only a finite number of distinct $ker_I d$-minimal prime ideals $P_i (1 \leq i \leq n)$. Furthermore, $\bigcap_{i=1}^n P_i = ker_I d$, $\bigcap_{i \neq j} P_i \neq ker_I d$ for all $1 \leq j \leq n$ and $L \setminus \bigcup_{i=1}^n P_i = \mathcal{K}_I^d$.*

Proof By Lemmas 2.17 and 3.6, L has only a finite number of distinct $ker_I d$-minimal prime ideals $P_i (1 \leq i \leq n)$, in which $\bigcap_{i=1}^n P_i = ker_I d$. Let $\bigcap_{i \neq j} P_i = ker_I d$, for some index j. By Lemma 3.6, each P_i is of the form $(a_i)_I^d$, for some $a_i \in A_I^d(L)$. Consider for all $i \neq j$, $x_i \in (a_i)_I^d \setminus (a_j)_I^d$. Then $\bigwedge_{i \neq j} x_i \in \bigcap_{i \neq j} P_i = ker_I d \subseteq (a_j)_I^d$. Since $(a_j)_I^d$ is a prime ideal, there is an $i \neq j$ such that $x_i \in (a_j)_I^d$, which is a contradiction. Therefore $\bigcap_{i \neq j} P_i \neq ker_I d$ for all $1 \leq j \leq n$. Now we show $L \setminus \bigcup P_i = \mathcal{K}_I^d$. Let $x \in L \setminus \bigcup P_i$ and $y \notin ker_I d$. If $x \wedge y \in ker_I d$, then $x \in (y)_I^d$ and by Lemma 2.16(iii), there exists a maximal element $(a_i)_I^d \in \Sigma$ such that $(y)_I^d \subseteq (a_i)_I^d$. By Theorem 2.15, $(a_i)_I^d$ is a $ker_I d$-minimal prime ideal. So $x \in (a_i)_I^d$ which contradicts the property $x \in L \setminus \bigcup P_i$. Thus $y \notin (x)_I^d$ and hence $(x)_I^d = ker_I d$. So $x \in \mathcal{K}_I^d$. Now consider $x \in \mathcal{K}_I^d$. If there exists $1 \leq i \leq n$ such that $x \in (a_i)_I^d$, then $a_i \in (x)_I^d = ker_I d \subseteq (a_i)_I^d$, which is a contradiction. Thus $x \in L \setminus \bigcup P_i$ and hence $L \setminus \bigcup P_i = \mathcal{K}_I^d$. □

Corollary 3.8 *If L has a bottom element $\perp$ and does not have an infinite $M \subseteq L \setminus \{\perp\}$ such that for each $x, y \in M$, $x \wedge y = \perp$, then L has only a finite number of minimal prime ideals.*

Theorem 3.9 *The following assertions are equivalent:*

(i) *L satisfies the condition $(*)$.*

(ii) *There exists a finite number of minimal $ker_I d$-prime ideals $P_i (1 \leq i \leq n)$ such that $\bigcap_{i=1}^n P_i = ker_I d$.*

Proof (i)⇒(ii) This is Theorem 3.7.

(ii)⇒(i) Let $M \subseteq L \setminus ker_I d$ such that for each $x, y \in M$, $x \wedge y \in ker_I d$ and $\mid M \mid \geq n$. By Pigeonhole principle, there exist $x, y \in M$ and P_i such that $x, y \in P_i^c$, which is a contradiction, because, P_i is prime and $x \wedge y \in ker_I d \subseteq P_i$. □

In the following, the subset $\bigcup (a_i)_I^d \setminus ker_I d$ of L in which $a_i \notin ker_I d$ is denoted by $\Gamma_I^d(L)$. As an immediate consequence of Lemma 2.1(ii), If L is a $ker_I d$-atomic lattice, then $\Gamma_I^d(L) = \bigcup (a_i)_I^d \setminus ker_I d$ for all $a_i \in A_I^d(L)$. For a subset A of L, the upset generated by A is denoted by $\Uparrow A$, which is the set $\{x \in L \mid \exists a \in A \; s.t \; a \leq x\}$. In the following theorem we use the notation, $A_I^d(a)^c = A_I^d(L) \setminus A_I^d(a)$.

Theorem 3.10 *Let L be a $ker_I d$-atomic distributive lattice. Then for each $a \in \Gamma_I^d(L)$, $(a)_I^d = \Uparrow A_I^d(a)^c \setminus \Uparrow A_I^d(a)$, where $A_I^d(a)^c = A_I^d(L) \setminus A_I^d(a)$.*

Proof Let $x \in (a)_I^d$. If $x \in \Uparrow A_I^d(a)$, then there exists $c \in A_I^d(a)$ such that $c \leq x$. Hence $c \leq x \wedge a \in ker_I d$, which is impossible. So $x \in \Uparrow A_I^d(a)^c \setminus \Uparrow A_I^d(a)$. For the converse, assume that $x \in \Uparrow A_I^d(a)^c \setminus \Uparrow A_I^d(a)$. If $x \notin (a)_I^d$, then $a \wedge x \notin ker_I d$ and so $x \in \Uparrow A_I^d(a \wedge x) \subseteq \Uparrow A_I^d(a)$, which is a contradiction. □

Consider $C_I^d(L) = \{B \subseteq L \setminus ker_I d \mid \forall x, y \in B, x \wedge y \in ker_I d\}$. It is easy to check that $A_I^d(L) \in C_I^d(L)$.

Theorem 3.11 *If L is a $ker_I d$-atomic lattice, then for each $B \in C_I^d(L)$, $\mid B \mid \leq \mid A_I^d(L) \mid$.*

Proof Let $B \in C_I^d(L)$ and $x, y \in B$. By Lemma 3.3(ii), $A_I^d(x) \cap A_I^d(y) = \emptyset$ such that $A_I^d(x)$ and $A_I^d(y)$ are nonempty set. By the axiom of choice, for each $b \in B$, choose and fix $a_b \in A_I^d(b) \neq \emptyset$. So the map $f : B \to A_I^d(L)$, defined by $f(b) = a_b$, is a one-to-one map. Hence $\mid B \mid \leq \mid A_I^d(L) \mid$. □

4 When a Quotient of a Distributive Lattice is a Boolean Algebra

In this section some necessary and sufficient conditions are derived for the quotient algebra L/θ_I^d to be a Boolean algebra. For a distributive lattice L and a lattice congruence θ on L, It can be easily observed that L/θ is a distributive lattice in which $[x]_\theta \wedge [y]_\theta = [x \wedge y]_\theta$ and $[x]_\theta \vee [y]_\theta = [x \vee y]_\theta$.

Theorem 4.1 *Let I be a nontrivial ideal of L. Then L/θ_I^d is a Boolean algebra if and only if for each $x \in L$, there exists $y \in (x)_I^d$ such that $x \vee y \in \mathcal{K}_I^d$.*

Proof Let L be a distributive lattice and θ be a lattice congruence on L. It is not difficult to check that, the distributive lattice L/θ is a Boolean algebra if and only if the following conditions hold:

(i) There exists $a_0, b_0 \in L$ such that for each $x \in L$, $[a_0]_\theta \leq [x]_\theta \leq [b_0]_\theta$, which means that $\bot_{L/\theta} = [a_0]_\theta$ and $\top_{L/\theta} = [b_0]_\theta$.

(ii) For each $x \in L$ there exists $y \in L$ such that $(x \wedge y)\theta a_0$ and $(x \vee y)\theta b_0$.

The proof is now complete using Propositions 2.4 and 2.7. □

For a particular case of the previous theorem see [14, Th.2.8], where $I = \{\bot\}$. The complement of an element x in a Boolean algebra L is denoted by x^{-1}.

Corollary 4.2 *Let L/θ_I^d be a Boolean algebra. Then $[x]_{\theta_I^d}^{-1} = [y]_{\theta_I^d}$ if and only if $x \wedge y \in ker_I d$ and $x \vee y \in \mathcal{K}_I^d$.*

In the following proposition we give some conditions under which L/θ_I^d is a Boolean algebra.

Proposition 4.3 (i) *If I or $ker_I d$ is a prime ideal of L, then L/θ_I^d is a Boolean algebra.*

(ii) *If each $(x)_I^d$ has a maximum element, then L/θ_I^d is a Boolean algebra.*

Proof (i) If $ker_I d = L$, then $ker_I d = \mathcal{K}_I^d = L$. Thus $\theta_I^d = \nabla$ and L/θ_I^d is a singleton set. Let $ker_I d \neq L$ and $x \in L$. By Lemma 2.6(ii), $\mathcal{K}_I^d \neq \emptyset$ and L is a disjoint union of $ker_I d$ and $\mathcal{K}_I^d$. It follows that $L/\theta_I^d \simeq \{0, 1\}$, with $0 < 1$, which is trivially a Boolean algebra.

(ii) If $ker_I d = L$, then $ker_I d = \mathcal{K}_I^d = L$. Thus $\theta_I^d = \nabla$ and L/θ_I^d is a singleton set. If $\mathcal{K}_I^d = L$, then for each $a, b \in L$, $(a)_I^d = ker_I d = (b)_I^d$. Thus $\theta_I^d = \nabla$ and L/θ_I^d is a singleton set. Let $ker_I d$ and $\mathcal{K}_I^d$ be nontrivial and $x \in L$. Consider $a_0 \in ker_I d$ and $b_0 \in \mathcal{K}_I^d$. If $x \in ker_I d$, then $x \wedge b_0 \in ker_I d$ and $x \vee b_0 \in \mathcal{K}_I^d$. If $x \in \mathcal{K}_I^d$, then $x \wedge a_0 \in ker_I d$ and $x \vee a_0 \in \mathcal{K}_I^d$. Now, let $x \notin ker_I d \cup \mathcal{K}_I^d$ and y be the maximum element of $(x)_I^d$. Then $x \wedge y \in ker_I d$. We show that $x \vee y \in \mathcal{K}_I^d$. Let $z \in (x \vee y)_I^d = (x)_I^d \cap (y)_I^d$. Since y is a maximum element of $(x)_I^d$, $z = (x \wedge z) \vee z = (x \wedge z) \vee (y \wedge z) = (x \vee y) \wedge z \in ker_I d$. Thus $(x \vee y)_I^d \subseteq ker_I d$ and, by Lemma 2.1(iv), $x \vee y \in \mathcal{K}_I^d$. So, Theorem 4.1 completes the proof. □

One of the important special case of Proposition 4.3(i) is when L is a chain, indeed, each ideal in a chain is prime ideal.

Lemma 4.4 *If L is a Boolean algebra with a bottom element $\perp$, then $\theta_\perp^{id} = \Delta = \{(a, a) \mid a \in L\}$.*

Proof It is clear that $ker_\perp(id) = \{\perp\}$ and $(a)_\perp^{id} =\downarrow a^{-1}$. If $a\theta_\perp^{id}b$, then $\downarrow a^{-1} =\downarrow b^{-1}$ and hence $a^{-1} = b^{-1}$. Thus $a = b$, which implies $\theta_\perp^{id} = \Delta$. □

By Corollary 1.4, every derivation is a lattice homomorphism. So for a derivation d, $ker(d) = \{(a, b) \mid d(a) = d(b)\}$ is a lattice congruence on L.

It is not difficult to show that for a nontrivial ideal I and a derivation d, $ker(d) \subseteq \theta_I^d$, but the converse is not generally true. For example, consider $I \neq \perp$ and $d = id$. Then $ker(d) = \Delta$ and for each $x, y \in I (x \neq y)$, $(x)_I^d = (y)_I^d = L$. So $x\theta_I^d y$. In the case where $I = \{\perp\}$, using Lemma 2.1(v), $\theta_\perp^d = \nabla$ implies that $ker(d) = \nabla$. The following lemma show that, when L is a Boolean algebra with a bottom element $\perp$, then $\theta_\perp^d = ker(d)$ in general.

Lemma 4.5 *Let L be a Boolean algebra with a bottom element $\perp$ and d a derivation on L. Then $ker(d) = \theta_\perp^d$.*

Proof Let $x\theta_\perp^d y$. Since L is a Boolean algebra, y has a complement element y^{-1} and $y^{-1} \in (y)_\perp^d = (x)_\perp^d$. Thus $d(x) \wedge d(y^{-1}) = \perp$. Also $x \vee y \leq \top$, implies $d(x) \vee d(y) = d(x \vee y) \leq d(\top) = d(y) \vee d(y^{-1})$. Hence $d(y) \vee d(x) = (d(y) \vee d(x)) \wedge (d(y) \vee d(y^{-1})) = d(y) \vee (d(x) \wedge d(y^{-1})) = d(y) \vee \perp = d(y)$. So $d(x) \leq d(y)$ and, similarly, $d(y) \leq d(x)$. Therefore $(x, y) \in ker(d)$.

Proposition 4.6 *The Boolean algebra L/θ_I^d is isomorphic to* **2** *if and only if $ker_I d$ is a prime ideal of L.*

Proof Let L/θ_I^d is isomorphic to **2**, $x \wedge y \in ker_I d$ and $x, y \in L \setminus ker_I d$. So by Proposition 2.4(i), $x\theta_I^d y$. So $x \in (y)_I^d = (x)_I^d = ker_I d$. This implies $x \in (x)_I^d$, which contradicts Lemma 2.1(v).

The converse one gets using Lemma 2.6. □

Here we provide an example in which $L/\theta_I^d = \mathbf{2}$, but I is not prime. Consider $L = \{\bot, a, b, \top\}$, in which $\bot$ and $\top$ are bottom and top element, respectively, and a, b are not comparable. The map $d : L \to L$ defined by $d(x) = \begin{cases} \bot, & \text{if } x = \bot, b \\ a, & \text{if } x = a, \top \end{cases}$ is a derivation. It is clear that $ker_I d = \{\bot, b\}$ and $\mathcal{K}_I^d = \{a, \top\}$. So $L/\theta_I^d = \mathbf{2}$, but $I = \{\bot\}$ is not a prime ideal.

The set $\Sigma = \{(x)_I^d \mid x \in L\}$ by an order given by, for each $x, y \in L$, $(x)_I^d \leq (y)_I^d$ if and only if $(y)_I^d \subseteq (x)_I^d$, is a poset. Also by the usual operations, $(x)_I^d \vee (y)_I^d = (x \vee y)_I^d$ and $(x)_I^d \wedge (y)_I^d = (x \wedge y)_I^d$, Σ is a bounded distributive lattice. The bottom and the top elements are of the form, $\bot_\Sigma = (x)_I^d = L$ for each $x \in ker_I d$ and $\top_\Sigma = (x)_I^d = ker_I d$ for each $x \in \mathcal{K}_I^d$. The map $f : L \to \Sigma$ defined by $f(x) = (x)_I^d$ is a lattice epimorphism, in which $ker f = \theta_I^d$. Thus, by the Isomorphism Theorem, $L/\theta_I^d \cong \Sigma$.

Lemma 4.7 *If the quotient lattice L/θ_I^d is a Boolean algebra then for each $x \in L$, the set $\{(z)_I^d \mid z \in (x)_I^d\}$ has a maximum element.*

Proof Let L/θ_I^d be a Boolean algebra and $x \in L$. By Theorem 4.1, there exists $y \in L$ such that $x \wedge y \in ker_I d$ and $x \vee y \in \mathcal{K}_I^d$. Consider $z \in (x)_I^d$. Since $x \wedge z \in ker_I d$, applying Proposition 2.7, $(x \wedge y)\theta_I^d(x \wedge z)$. Thus $y\theta_I^d[y \vee (x \wedge z)] = [(x \vee y) \wedge (y \vee z)]\theta_I^d(y \vee z)$. So $(y)_I^d = (y \vee z)_I^d = (y)_I^d \cap (z)_I^d$ and hence $(y)_I^d \subseteq (z)_I^d$, which implies that $(z)_I^d \leq (y)_I^d$. □

Theorem 4.8 *Let L be a $ker_I d$-atomic distributive lattice. The lattice L/θ_I^d is a Boolean algebra if and only if for each $x \in L$, there exists $y \in L$ such that $A_I^d(x)$ and $A_I^d(y)$ are a partition of $A_I^d(L)$ and $[y]_{\theta_I^d}$ is a complement of $[x]_{\theta_I^d}$ in L/θ_I^d.*

Proof ($\Leftarrow$) It is clear that $x \wedge y \in ker_I d$ and, by Lemma 3.3, $x \vee y \in \mathcal{K}_I^d$. Hence, Theorem 4.1 completes the proof.

($\Rightarrow$) Consider $x \in L$. Since L/θ_I^d is a Boolean algebra, by Theorem 4.1, there exists $y \in L$ such that $x \wedge y \in ker_I d$ and $x \vee y \in \mathcal{K}_I^d$. Clearly $A_I^d(x) \cap A_I^d(y) = \emptyset$. Let $a \in A_I^d(L) \setminus (A_I^d(x) \cup A_I^d(y))$. Using Lemma 2.3(i), $(x \vee a) \vee y \in \mathcal{K}_I^d$. Also $(x \vee a) \wedge y \in ker_I d$. So, by Corollary 4.2, $[y]_{\theta_I^d}$ has two different complements $[x]_{\theta_I^d}$ and $[x \vee a]_{\theta_I^d}$, which is a contradiction, because $a \in (x)_I^d$ and $a \notin (x \vee a)_I^d$. □

Theorem 4.9 *If L/θ_I^d is a Boolean algebra, then the congruence θ_I^d is the only congruence relation having $ker_I d$ as a whole class.*

Proof Let θ be a lattice congruence on L such that $ker_I d$ is a whole class. By Proposition 2.7, $\theta \subseteq \theta_I^d$. For the converse, let $x\theta_I^d y$. Then there exists $z \in L$ such that $[x]_{\theta_I^d}^{-1} = [y]_{\theta_I^d}^{-1} = [z]_{\theta_I^d}$. By Proposition 2.4, $[x \wedge z]_{\theta_I^d} = [x]_{\theta_I^d} \wedge [z]_{\theta_I^d} = \bot_{L/\theta_I^d} = ker_I d$. Thus $x \wedge z \in ker_I d$ and also $y \wedge z \in ker_I d$, which implies $(x \wedge z)\theta(y \wedge z)$. By a similar way, $(x \vee z)\theta(y \vee z)$. Now we have $x = x \vee (x \wedge z)\theta[x \vee (y \wedge z)]\theta[(x \vee y) \wedge (x \vee z)]\theta[(x \vee y) \wedge (y \vee z)] = [y \vee (x \wedge z)]\theta[y \vee (y \wedge z)] = y$. Thus $\theta_I^d \subseteq \theta$ and hence $\theta_I^d = \theta$. □

Corollary 4.10 *For a congruence θ, if L/θ_I^d and L/θ are Boolean algebras such that the congruence θ having $ker_I d$ as a whole class, then $\theta_I^d = \theta$.*

Corollary 4.11 *If L is a distributive lattice with a least element $\perp$, $ker_I d = \{\perp\}$ and L/θ_I^d is a Boolean algebra, then $\theta_I^d = \Delta$.*

Conclusion

In this final section, for an ideal I, we conclude that the lattice congruence θ_I^{id} is the smallest congruence in the set of all congruences of the form θ_I^d and so the best congruence in the sense that the Boolean algebra L/θ_I^{id} has the maximum cardinality in the set of all Boolean algebras L/θ_I^d.

(i) Consider an ideal I and a derivation d on L. By Proposition 2.8, $\theta_I^{id} \subseteq \theta_I^d$. Thus the map $\pi : L/\theta_I^{id} \to L/\theta_I^d$ defined by $\pi([a]_{\theta_I^{id}}) = [a]_{\theta_I^d}$ is a lattice homomorphism. Using the first isomorphism theorem, if L/θ_I^{id} is a Boolean algebra, then so is L/θ_I^d. Thus the lattice congruence θ_I^{id} is the best congruence in the set $\{\theta_I^d \mid \text{d is a derivation}\}$.

(ii) Combining Theorem 4.9 and Proposition 2.8, it is concluded that θ_I^{id} is the smallest congruence in the set of all congruences having $ker_I d$ as a whole class.

(iii) Using Lemma 2.10, θ_I^{id} is the smallest congruence in the set $\{\theta_J^d\}$ in which there exists a derivation d on L such that $ker_I d = J$.

(iv) Using Lemma 2.11, θ_I^{id} is the smallest congruence in the set $\{\theta_J^d\}$ in which $J = (a)_I^d$, for all $a \in L$.

(v) Using Theorem 4.9, θ_I^{id} is the smallest congruence in the set of all congruences having I as a whole class.

(vi) In the case where L is a $ker_I d$-atomic distributive lattice such that for each $x \in L$, there exists $y \in L$ such that $A_I^d(x)$ and $A_I^d(y)$ are a partition of $A_I^d(L)$, then θ_I^{id} is the smallest congruence in which L/θ_I^d is a Boolean algebra.

There is still an open question concerning θ_I^d:

Is there a necessary and sufficient condition on an ideal I such that θ_I^d is the smallest congruence in which L/θ_I^d is a Boolean algebra?

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