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Rank One Perturbation of Unitary Operators with Full Measure of Hypercyclic Vectors

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Abstract

In this note, we find a class of unitary operators, denoted by $\mathcal{U}$, on a complex separable infinite-dimensional Hilbert space $\mathcal{H}$ such that for any $U \in \mathcal{U}$, there exists an operator R of rank 1 on $\mathcal{H}$ such that $U + R$ is hypercyclic and the hypercyclic vectors are of full measure. Then, these results are applied to the controllability of discrete-time linear control systems, where the rank one perturbation is used as a one-dimensional feedback control law.

Keywords Unitary operator · Hilbert space · Hypercyclic · Perturbation · Gaussian measure · Controllability

Mathematics Subject Classification 47A16 · 37A05 · 47A35

1 Introduction

Let $T : X \longrightarrow X$ be a bounded linear operator on an infinite dimensional separable Banach space X, write $T \in \mathcal{B}(X)$. T is said to be hypercyclic if there exists a point $x \in X$ such that the corresponding orbit $\{x, T(x), T^2(x), \ldots\}$ is dense in X. Such a point x is a hypercyclic point for T.

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Hypercyclicity is one of central notions in both operator theory and dynamical systems. Rolewicz [18] gave the first example of hypercyclic operator by showing that the operator λB on l^p $(1 \leq p < \infty)$ (or c_0) is hypercyclic for any complex number λ with $|\lambda| > 1$. Here, B is the backward shift, defined by $B(x_i)_{i\in\mathbb{N}} = (x_{i+1})_{i\in\mathbb{N}}$. Since then more and more examples of hypercyclic operators on different kinds of spaces have been found. See monographs [6, 15]. Among others, an important theoretical result was from Ansari [1] and Bernal-Gonzalez [7] who showed that any separable infinite-dimensional Banach space X supports a hypercyclic operator. Such an operator has the form $T = I + N$, where I is the identity map and N is a nuclear operator on X. This shows that the identity can be perturbed to be hypercyclic by nuclear operators. Motivated by this fact and the work of Salas [19] on supercyclicity of weighted shifts, Shkarin [21] proposed the following question:

Question 1 *Can a finite rank perturbation of a unitary operator on a complex separable infinite-dimensional Hilbert space be hypercyclic?*

Shkarin gave in [21] an affirmative answer on this question and proved the following proposition:

Proposition 1.1 [21, Theorem 1] *There exist a unitary operator V and a bounded linear operator R of rank at most* 2 *acting on a Hilbert space H such that $T = V + R$ is hypercyclic.*

The idea to construct such operator T in [21] is as follows. Let U be the multiplication operator by z on $L^2(\mathbb{T})$, that is, $(Uf)(z) = zf(z)$, and S be a rank 1 operator on $L^2(\mathbb{T})$ defined by $Sf = \langle f, g\rangle h$ for some $g, h \in L^2(\mathbb{T})$. Then, the key is to construct $g, h \in L^2(\mathbb{T})$ and a closed linear subspace $\mathcal{K}$ of $L^2(\mathbb{T})$ which is invariant for $U + S$ such that the restriction $T \in \mathcal{B}(\mathcal{K})$ of the operator $U + S$ to $\mathcal{K}$ is hypercyclic and T can be decomposed as $T = V + R$, where $V \in \mathcal{B}(\mathcal{K})$ and $R \in \mathcal{B}(\mathcal{K})$ has rank at most 2. As a by-product, this also provides an example of a contraction A and an operator R of rank 1 on H such that $A + R$ is hypercyclic. A further question remained is: can a unitary operator be perturbed to be hypercyclic by an operator of rank 1. Thus, Grivaux [14] proposed the following question:

Question 2 *Does there exist a rank 1 perturbation of a unitary operator on a Hilbert space which is hypercyclic?*

Grivaux answered in [14] this question in the affirmative and proved the following result.

Proposition 1.2 [14, Theorem 1.2] *There exist a unitary operator U and a rank* 1 *operator R on the complex Hilbert space l^2 such that the operator $T = U + R$ is hypercyclic on l^2.*

The approach to construct the operator T in [14] with the properties in Proposition 1.2 is much more elementary than that in [21]. In the construction scheme, the unitary operator U can be chosen to be a diagonal operator D on l^2 defined by $De_n = \lambda_n e_n, n \geq 1$, where $\lambda_n \in \mathbb{T}$ and $(e_n)_{n\geq1}$ is the canonical basis on the space

l^2. And the rank one operator has the form $Rx = \langle x, b\rangle a$ for any $x \in l^2$, where $a = \sum_{n\geq 1} a_n e_n$ and $b = \sum_{n\geq 1} b_n e_n$ are two elements of l^2. The key technical point is to construct the coefficients λ_n, a_n and b_n $(n \geq 1)$ by induction such that $T = U + R$ satisfies the hypercyclic criterion Lemma 2.1 in next section.

Though the proofs of Propositions 1.1 and 1.2 are constructive, the operators V and U in the propositions have not explicitly been given. This is not sufficient in application. For example, from the viewpoint of control theory, one may concern the following question: Given an operator U on some Hilbert space can we find a perturbation operator R of rank one such that $U + R$ is hypercyclic?

In this note, we find a class of unitary operators, denoted by $\mathcal{U}$, on a complex separable infinite-dimensional Hilbert space $\mathcal{H}$ such that for any $U \in \mathcal{U}$, there exists an operator R of rank one on $\mathcal{H}$ so that $U + R$ is hypercyclic. The main result in this paper is as follows.

Main Theorem *Let U be a bounded linear operator on an infinite-dimensional complex separable Hilbert space $\mathcal{H}$. Assume U satisfies the following conditions:*

(1) *U has distinct eigenvalues $\lambda_i \in \mathbb{T}$, $(i \geq 1)$ and the set $\{\lambda_i | i \geq 1\}$ contains at most finite number of isolated points;*
(2) *The eigenvectors $\{e_i\}_{i\geq 1}$ of U form an orthogonal basis of $\mathcal{H}$, where e_i is the eigenvector corresponding to the eigenvalue λ_i.*

Then, there exist $p, q \in \mathcal{H}$ and an operator $R(\cdot) = \langle p, \cdot\rangle q$ such that the operator $T = U + R$ is weak-mixing with respect to a nondegenerate Gaussian measure.

The demonstration strategy is similar to the one employed in [14], in which the coefficients of a unitary operator U on l^2 as well as the two vectors p and q in l^2 which define $R(\cdot) = \langle p, \cdot\rangle q$ are alternatively constructed by induction such that $T = U + R$ has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues. In our situation, since the unitary operator U is specified in advance, we have to choose two vectors p and q in l^2 such that $T = U + R$ has the same properties as in [14]. This is more challenging and so the construction of p and q is a rather sophisticated procedure.

We also note that Baranov et al. [2] gave another proof of Proposition 1.2 using function theory method. In [3], they further show that any countable union of perfect Carleson sets on the unit circle can be the spectrum of a hypercyclic operator which is a rank one perturbation of some unitary operator U, and got some information about the spectral measure of U. But they did not propose a characterization of unitary operators which have a hypercyclic rank one perturbation.

The proof of the main theorem is presented in Sect. 2. As an application, in Sect. 3 we discuss feedback controllability for a class of linear discrete-time control systems on Hilbert spaces by hypercyclicity.

2 Rank One Perturbation of Non-hypercyclic Operators

2.1 Criteria for Hypercyclicity

The earliest forms of criteria for hypercyclicity were established independently by Kitai [17] and by Gethner and Shapiro [12], which are called Kitai's criterion and Gethner and Shapiro's criterion, respectively. Since then, various necessary or sufficient conditions for hypercyclicity have been stated in terms of different characteristics of the operators. See, for example [6, 15]. Here, the criterion we will use to prove our main theorem is characterized in terms of eigenvectors associated to eigenvalues of modulus 1 of the operator.

Let X be a complex separable infinite-dimensional Banach space. Recall that a bounded linear operator $T \in \mathcal{B}(X)$ is called having *a perfectly spanning set of eigenvectors associated to unimodular eigenvalues* if there exists a continuous probability measure σ on the unit circle $\mathbb{T}$ such that for any σ-measurable subset $B \subset \mathbb{T}$ with $\sigma(B) = 1$, the set $\operatorname{span}\{\ker(T - \lambda) : \lambda \in B\}$ is dense in X [4].

The following lemma gives a sufficient condition for an operator having a perfectly spanning set of eigenvectors.

Lemma 2.1 [13] *Let X be a complex separable infinite-dimensional Banach space and T be a bounded linear operator on X. Suppose that there exists a sequence $(u_i)_{i\geq 1}$ of vectors of X with the following conditions:*

- (i) *for each $i \geq 1$, u_i is an eigenvector of T associated to an eigenvalue μ_i with $\mu_i \in \mathbb{T}$ and the μ_i's all distinct;*
- (ii) $\operatorname{span}\{u_i : i \geq 1\}$ *is dense in X;*
- (iii) *for any $i \geq 1$ and any $\varepsilon > 0$, there exists an $n \neq i$ such that $\|u_n - u_i\| < \varepsilon$.*

Then, T has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues. Therefore, T is hypercyclic.

In case of Hilbert spaces, Lemma 2.1 also implies that T is weak-mixing with respect to an invariant measure m on X. Recall the definition of weak-mixing:

- (i) m is an invariant measure for T, i.e. $m(T^{-1}A) = m(A)$ for any measurable subset $A \subset X$;
- (ii)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} |m(T^{-k}A \cap B) - m(A)m(B)| = 0,$$

  for any two measurable subsets $A, B \subset X$.

Lemma 2.2 [5, Theorem 3.22] *Let $\mathcal{H}$ be a complex separable infinite-dimensional Hilbert space and T a bounded linear operator on $\mathcal{H}$. If T has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues. There exists a nondegenerate Gaussian invariant measure m on $\mathcal{H}$ such that T is weak-mixing.*

Remark 2.3 A Gaussian measure is a Borel measure, and the nondegenerate Gaussian measure m has full support, i.e. any open set on $\mathcal{H}$ has positive measure. When T is weak-mixing with respect to m, the set of all hypercyclic points has full measure.

2.2 Rank One Perturbation of Non-hypercyclic Operators

Now we consider rank one perturbation of operators on Hilbert spaces. Let $\mathcal{H}$ be a complex separable infinite-dimensional Hilbert space, and U a bounded linear operator on $\mathcal{H}$. We always assume that the operator U has eigenvalues $(\lambda_i)_{i\geq 1}$ and eigenvectors $(e_i)_{i\geq 1}$ with the following properties:

- (A_1) $\lambda_i (i \geq 1)$ are distinct points on $\mathbb{T}$ and the set $\{\lambda_i | i \geq 1\}$ contains at most finite number of isolated points;
- (A_2) $Ue_i = \lambda_i e_i$ and $(e_i)_{i\geq 1}$ is an orthonormal basis of $\mathcal{H}$.

Denote by $\mathcal{U}$ the set of all bounded linear operators on $\mathcal{H}$ which satisfies the assumptions (A_1) and (A_2).

Remark 2.4 The following two facts are obvious for $U \in \mathcal{U}$:

(a) U is a unitary operator;
(b) Denote by K the closure of $\{\lambda_i | i \geq 1\}$. For any non-isolated point λ_j in $\{\lambda_i | i \geq 1\}$ and any $\varepsilon > 0$, there exists a $\mu \in O(\lambda_j, \varepsilon) \cap (K \backslash \{\lambda_i | i \geq 1\})$ such that $O(\mu, \delta) \cap \{\lambda_i | i \geq 1\} \neq \emptyset$ for any $\delta > 0$. Here, $O(\mu, \delta)$ denotes the δ neighbourhood of μ in $\mathbb{T}$.

The rank one operator R has the form,

$$R : \mathcal{H} \longrightarrow \mathcal{H}, \quad R(x) = \langle p, x \rangle q,$$

for some $q, p \in \mathcal{H}$.

For a given $U \in \mathcal{U}$ with eigenvalues $(\lambda_i)_{i\geq 1}$ and eigenvectors $(e_i)_{i\geq 1}$, we expand q and p by the orthonormal basis $(e_i)_{i\geq 1}$ as

$$q = \sum_{j\geq 1} q_j e_j, \quad p = \sum_{j\geq 1} p_j e_j. \tag{2.1}$$

In order to make the operator $T = U + R$ to be hypercyclic, it suffices to construct vectors q and p such that T satisfies all the conditions in Lemma 2.1. To do this end, we need the following proposition about the eigenvalues and eigenvectors of T.

Proposition 2.5 [14, Lemma 2.3] *Let $\mu \in \mathbb{T} \backslash \{\lambda_j | j \geq 1\}$. Then, μ is an eigenvalue of $T = U + R$ if and only if*

$$\sum_{j\geq 1} \left| \frac{q_j}{\mu - \lambda_j} \right|^2 < \infty \quad \textit{and} \quad \sum_{j\geq 1} \frac{q_j \bar{p}_j}{\mu - \lambda_j} = 1.$$

In this case, the corresponding eigenvector is given by

$$u = \sum_{j \geq 1} \frac{q_j}{\mu - \lambda_j} e_j.$$

Theorem 2.6 *Let U be a bounded linear operator on an infinite-dimensional separable complex Hilbert space $\mathcal{H}$. Assume that*

- (A_1) *λ_i $(i \geq 1)$ are distinct eigenvalues of U with $\lambda_i \in \mathbb{T}$, and $\{\lambda_i | i \geq 1\}$ contains at most finite number of isolated points;*
- (A_2) *$Ue_i = \lambda_i e_i$ $(i \geq 1)$, and $\{e_i | i \geq 1\}$ is an orthonormal basis of $\mathcal{H}$.*

Then, there exists $p, q \in \mathcal{H}$ and $R(\cdot) = \langle p, \cdot \rangle q$ such that $T = U + R$ satisfies the following conditions:

- (i) *μ_j $(j \geq 1)$ are distinct points on $\mathbb{T}$;*
- (ii) *$Tu_j = \mu_j u_j$, and $\overline{\mathrm{span}\{u_j : j \geq 1\}} = \mathcal{H}$;*
- (iii) *for each u_m and any $\varepsilon > 0$, there exists a u_n $(n > m)$ s.t. $\|u_n - u_m\| < \varepsilon$.*

Therefore, T is weak-mixing with respect to a nondegenerate Gaussian measure by Lemmas 2.1 and 2.2.

Remark 2.7 By Proposition 2.5, we need to solve equations to get p and q. At first, we select values of μ_j and $q_j (j \geq 1)$, then get p by solving a system of linear equations. The inverse of the coefficient matrix of these equations has good properties if each eigenvalue μ_j is in the closure of $\{\lambda_j | j \geq 1\}$. In order to make the set of eigenvectors self-dense, these eigenvalues $\{\mu_j | j \geq 1\}$ also need to be self-dense. During the construction process, odd items have much freedom and even items are difficult to select to satisfy many constraints. For odd items, we let $q_n \ll |\mu_n - \lambda_n|$ such that each eigenvector u_m can be approximated by some $u_n (n \neq m)$. But for even items, we let $q_n \gg |\mu_n - \lambda_n|$ in order to make $p_n \to 0$.

To prove Theorem 2.6, we need some preliminaries. Assume that all the isolated points in $\{\lambda_i | i \geq 1\}$ are $\lambda_1, \lambda_2, \ldots, \lambda_{n_0}$.

1. Rearrange the sequence $(\lambda_j)_{j > n_0}$ by a bijective map

   $$\sigma : \mathbb{N} \longrightarrow \{j \in \mathbb{N} | j > n_0\},$$

   such that we can choose a series of points $(\mu_j)_{j \geq 1}$ on $\mathbb{T}$ corresponding to the rearranged sequence $(\lambda_{\sigma(j)})_{j \geq 1}$ as follows, by the fact (b) in Remark 2.4. For $1 \leq n \leq n_0$, let $\sigma(n) = n_0 + n$, take a $\mu_n \in K \backslash (\{\lambda_j | j \geq 1\} \cup \{\mu_j | j < n\})$ which is close to $\lambda_{n_0 + n}$. If $n_0 = 0$, let $\sigma(1) = 1$ and $\sigma(2) = 2$, take a $\mu_1 \in K \backslash \{\lambda_j | j \geq 1\}$ which is close to λ_1, then a $\mu_2 \in K \backslash (\{\lambda_j | j \geq 1\} \cup \{\mu_1\})$ which is close to λ_2;
   Inductively, for $n = 2k - 1 \geq \max\{n_0 + 1, 3\}$, take a $\lambda_{\sigma(2k-1)}$ in a small neighbourhood of some $\mu_m (m < 2k - 1)$, and take $\mu_{2k-1} \in K \backslash (\{\lambda_j | j \geq 1\} \cup \{\mu_j | j < 2k - 1\})$ in a small neighbourhood of $\lambda_{\sigma(2k-1)}$. For $n = 2k \geq \max\{n_0 + 1, 3\}$, define $\sigma(2k)$ by the least number in $\{n_0 + 1, n_0 + 2, \ldots, 2k\} \backslash \sigma(\{1, 2, \ldots, 2k -$

1}) (This assures that the map $\sigma : \mathbb{N} \longrightarrow \{j \in \mathbb{N} | j > n_0\}$ is bijective), and take a $\mu_{2k} \in K \backslash (\{\lambda_j | j \geq 1\} \cup \{\mu_j | j < 2k\})$ in a small neighbourhood of $\lambda_{\sigma(2k)}$. These two sequences $\lambda_{\sigma(n)}$ and μ_n can be further modified so that the following conditions are satisfied.

2. Take μ_n sufficiently close to $\lambda_{\sigma(n)}$, such that

$$|\mu_n - \lambda_{\sigma(n)}| < \frac{1}{n^7} |\mu_j - \lambda_{\sigma(n)}|, \quad \forall j < n.$$

3. Let

$$q_n = \begin{cases} 1, & 1 \leq n \leq n_0, \\ n^{-1}(\mu_n - \lambda_{\sigma(n)}), & n > n_0 \text{ and } n \text{ is odd}, \\ n^3(\mu_n - \lambda_{\sigma(n)}), & n > n_0 \text{ and } n \text{ is even}. \end{cases}$$

And let μ_n sufficiently close to $\lambda_{\sigma(n)}$ such that $|q_n| \leq n^{-3} |q_{n-1}|, n > n_0$.

4. For each odd number $n > n_0$, we can choose $\lambda_{\sigma(n)}$ sufficiently close to μ_m(some $m < n$) such that

$$\sum_{1 \leq j < n} \left| \frac{q_j}{\lambda_{\sigma(n)} - \lambda_{\sigma(j)}} - \frac{q_j}{\mu_m - \lambda_{\sigma(j)}} \right| < \frac{1}{2n^5},$$

and μ_n sufficiently close to $\lambda_{\sigma(n)}$ such that

$$\sum_{1 \leq j < n} \left| \frac{q_j}{\mu_n - \lambda_{\sigma(j)}} - \frac{q_j}{\mu_m - \lambda_{\sigma(j)}} \right| < \frac{1}{n^5}.$$

5. For each odd number $n > n_0$, as described in item 1, $\lambda_{\sigma(n)}$ is sufficiently close to μ_m(some $m < n$). Denote this correspondence between n and m by a map

$$\tau : \{n \in \mathbb{N} | n \text{ is odd and } n > n_0\} \longrightarrow \mathbb{N}.$$

Furthermore, this map can be constructed so that the set $\tau^{-1}(m)$ is infinite for each $m \in \mathbb{N}$. For instance, map $\{2k-1 | n_0 < k \leq n_0+11\}$ onto $\{m | 1 \leq m \leq 10\}$, map $\{2k - 1 | n_0 + 11 < k \leq n_0 + 111\}$ onto $\{m | 1 \leq m \leq 100\}$, …. This guarantees that each μ_m can be approximated arbitrarily close by some point in $\{\mu_n | n > m\}$.

Let

$$\hat{\lambda}_n = \begin{cases} \lambda_n, & n \leq n_0; \\ \lambda_{\sigma(n-n_0)}, & n > n_0. \end{cases}$$

Without loss of generality, we assume that the eigenvector of $\hat{\lambda}_n$ is e_n. To solve the vector p in Eq. (2.1), for a given $n \in \mathbb{N}$ we consider the linear equations below.

$$C_n \Lambda_n \begin{pmatrix} \bar{p}_1^{(n)} \\ \bar{p}_2^{(n)} \\ \vdots \\ \bar{p}_n^{(n)} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{2.2}$$

where

$$C_n = \begin{pmatrix} \frac{1}{\mu_1 - \hat{\lambda}_1} & \cdots & \frac{1}{\mu_1 - \hat{\lambda}_n} \\ \vdots & & \vdots \\ \frac{1}{\mu_n - \hat{\lambda}_1} & \cdots & \frac{1}{\mu_n - \hat{\lambda}_n} \end{pmatrix},$$

and

$$\Lambda_n = \mathrm{diag}(q_1, q_2, \ldots, q_n).$$

Here, for a complex number λ, $\overline{\lambda}$ denotes the conjugacy of λ. Since C_{n_0} is a Cauchy matrix and $\mu_j, \hat{\lambda}_j (1 \le j \le n_0)$ are distinct numbers, it is invertible by [20]. The following Lemma gives a condition for Eq. (2.2) having a unique solution.

Proposition 2.8 *Let $M^{(n)} = \Lambda_n^{-1} C_n^{-1}, n \ge 1$. If the sequence $(|\mu_n - \hat{\lambda}_n|)_{n \ge 1}$ decreases to 0 sufficiently fast, then for any $n \ge n_0$ it holds that:*

(i)

$$|(M^{(n)})_{n-1} - M^{(n-1)}|_\infty \le \frac{1}{n^3},$$

where $(\cdot)_{n-1}$ represents the left upper $n-1 \times n-1$ submatrix, and $|\cdot|_\infty$ is the maximum absolute value of the elements in a matrix.

(ii) *when n is even, $|M_{ij}^{(n)}| \le \frac{2}{n^3}$, if i or j equals n.*

(iii) *when n is odd, $|M_{i,n}^{(n)}| \le \frac{2}{n^2}$ for $1 \le i < n$, and $|M_{n,j}^{(n)}| \le 2n$ for $1 \le j \le n$.*

Hence, for any $n > n_0$,

$$|M^{(n)}|_\infty \le 2n + c_0,$$

where $c_0 = |\Lambda_{n_0}^{-1} C_{n_0}^{-1}|_\infty$.

Proof For $n \ge n_0 + 1$, rewrite C_n as the block form

$$C_n = \begin{pmatrix} A & \beta \\ \gamma & c \end{pmatrix}.$$

where $c = 1/(\mu_n - \hat{\lambda}_n)$.

Since C_{n_0} is invertible, it follows by induction C_n is invertible and

$$C_n^{-1} = \begin{pmatrix} \left(I + \frac{A^{-1}\beta\gamma}{c-\gamma A^{-1}\beta}\right) A^{-1} & -\frac{A^{-1}\beta}{c-\gamma A^{-1}\beta} \\ -\frac{\gamma A^{-1}}{c-\gamma A^{-1}\beta} & \frac{1}{c-\gamma A^{-1}\beta} \end{pmatrix},$$

provided that the sequence $(|\mu_n - \hat{\lambda}_n|)_{n\geq 1}$ decreases to 0 sufficiently fast.

Since A^{-1}, β, γ are bounded as μ_n approaches $\hat{\lambda}_n$, we can modify μ_n sufficiently close to $\hat{\lambda}_n$ such that C_n^{-1} satisfies

- $|(C_n^{-1})_{n-1} - C_{n-1}^{-1}|_\infty \leq \frac{|q_{n-1}|}{n^3}$;
- The absolute value of each element in the n-th column and n-th row of C_n^{-1} is less than $2|\mu_n - \hat{\lambda}_n|$.

On the other hand, since the sequence $(|q_n|)_{n\geq 1}$ is decreasing, we can further choose μ_n sufficiently close to $\hat{\lambda}_n$ such that

$$|(\Lambda_n^{-1} C_n^{-1})_{n-1} - \Lambda_{n-1}^{-1} C_{n-1}^{-1}|_\infty \leq \frac{1}{n^3}.$$

Since

$$q_n = \begin{cases} n^{-1}(\mu_n - \hat{\lambda}_n), & n > n_0 \text{ and } n \text{ is odd}, \\ n^3(\mu_n - \hat{\lambda}_n), & n > n_0 \text{ and } n \text{ is even}. \end{cases}$$

If n is even, then the absolute values of elements in the n-th row and n-th column of $\Lambda_n^{-1} C_n^{-1}$ are not greater than $\frac{2}{n^3}$.

If n is odd, then the absolute values of elements in the n-th row of $\Lambda_n^{-1} C_n^{-1}$ are not greater than $2n$. Since $|q_j| \geq n^3 |q_n|$ for $j < n$ (by item 3), the absolute values of elements in the n-th column of $\Lambda_n^{-1} C_n^{-1}$ (except $M_{nn}^{(n)}$) are not greater than $\frac{2}{n^2}$.

Finally, note that the elements in $\Lambda_{n_0}^{-1} C_{n_0}^{-1}$ is bounded, by induction on n, it leads to

$$|\Lambda_n^{-1} C_n^{-1}|_\infty \leq 2n + c_0, \quad \forall n \geq n_0,$$

where $c_0 = |\Lambda_{n_0}^{-1} C_{n_0}^{-1}|_\infty$. □

Proposition 2.9 *If the sequence $(\mu_n)_{n\geq 1}$ satisfies the conditions described in Proposition 2.8, then*

$$(p_1^{(n)}, p_2^{(n)}, \ldots, p_n^{(n)}, 0, 0, \ldots)$$

converges to some point $p \in l^2$ as $n \to \infty$.

Proof We can estimate $\bar{p}^{(n)} = (\bar{p}_1^{(n)}, \bar{p}_2^{(n)}, \ldots, \bar{p}_n^{(n)})^T$ by the following equation,

$$\bar{p}^{(n)} = \Lambda_n^{-1} C_n^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

and view the vector $\bar{p}^{(n)}$ as an element in l^2 with infinite zeros on the tail. For $n > \max\{n_0, c_0\}$, we consider the following two cases.

Case 1 n is even. According to Proposition 2.8,

$$\|\bar{p}^{(n)} - \bar{p}^{(n-1)}\|_\infty \leq \frac{4}{n^2}.$$

Case 2 n is odd($n \geq 3$). By item 4, there exists $m < n$ such that μ_n is sufficiently close to μ_m, and

$$\sum_{1 \leq j < n} \left| \frac{q_j}{\mu_n - \hat{\lambda}_j} - \frac{q_j}{\mu_m - \hat{\lambda}_j} \right| \leq \frac{1}{n^5},$$

then

$$\begin{aligned} \left| \frac{\bar{p}_n^{(n)} q_n}{\mu_n - \hat{\lambda}_n} - \frac{\bar{p}_n^{(n)} q_n}{\mu_m - \hat{\lambda}_n} \right| &= \left| \left(1 - \sum_{1 \leq j < n} \frac{\bar{p}_j^{(n)} q_j}{\mu_n - \hat{\lambda}_j}\right) - \left(1 - \sum_{1 \leq j < n} \frac{\bar{p}_j^{(n)} q_j}{\mu_m - \hat{\lambda}_j}\right) \right| \\ &\leq \sum_{1 \leq j < n} \left| \frac{\bar{p}_j^{(n)} q_j}{\mu_n - \hat{\lambda}_j} - \frac{\bar{p}_j^{(n)} q_j}{\mu_m - \hat{\lambda}_j} \right| \\ &\leq \|p^{(n)}\|_\infty \sum_{1 \leq j < n} \left| \frac{q_j}{\mu_n - \hat{\lambda}_j} - \frac{q_j}{\mu_m - \hat{\lambda}_j} \right| \\ &\leq n \cdot (2n + c_0) \frac{1}{n^5} \text{ (by Proposition 2.8)} \\ &\leq \frac{3}{n^3} \text{ (for } n > \max\{n_0, c_0\}), \end{aligned}$$

and

$$\begin{aligned} \left| \frac{\bar{p}_n^{(n)} q_n}{\mu_n - \hat{\lambda}_n} - \frac{\bar{p}_n^{(n)} q_n}{\mu_m - \hat{\lambda}_n} \right| &= |\bar{p}_n^{(n)}| \left| \frac{q_n}{\mu_n - \hat{\lambda}_n} - \frac{q_n}{\mu_m - \hat{\lambda}_n} \right| \\ &\geq |\bar{p}_n^{(n)}| \frac{1}{2n} \text{ (by item 2 and item 3).} \end{aligned}$$

Then, $|\bar{p}_n^{(n)}| \leq \frac{6}{n^2}$.

According to Proposition 2.8,

$$\|\bar{p}^{(n)} - \bar{p}^{(n-1)}\|_\infty \le \frac{6}{n^2}.$$

Combining these two cases, for any $n \ge 2$, we have

$$\begin{aligned}\|p^{(n)} - p^{(n-1)}\|_2^2 &= \sum_{1\le j\le n} |p_j^{(n)} - p_j^{(n-1)}|^2 \\ &\le n\|p^{(n)} - p^{(n-1)}\|_\infty^2 \\ &\le n\left(\frac{6}{n^2}\right)^2 \\ &\le \frac{36}{n^3},\end{aligned}$$

and for any $m > 0$, it holds that

$$\begin{aligned}\|p^{(n+m)} - p^{(n)}\|_2^2 &\le \sum_{1\le j\le m} \|p^{(n+j)} - p^{(n+j-1)}\|_2^2 \\ &\le \sum_{1\le j\le m} \frac{36}{(n+j)^3} \\ &\le \frac{36}{n}.\end{aligned}$$

Hence, $p^{(n)}$ is a Cauchy sequence in l^2 and converges to some p. □

For each $n \ge 1$, denote the eigenvector of μ_n by

$$u_n = \sum_{j\ge 1} \frac{q_j}{\mu_n - \hat{\lambda}_j} e_j. \tag{2.3}$$

Proposition 2.10 *If the sequence $(\mu_n)_{n\ge 1}$ satisfies the conditions in Proposition 2.8, then u_n defined by (2.3) is the eigenvector of $T = U+R$ corresponding to μ_n ($\forall n \ge 1$). Furthermore, $\{u_n | n \ge 1\}$ is a basis in $\mathcal{H}$.*

Proof We divide the proof of Proposition 2.10 into three steps.

Step 1. We have constructed $(\mu_n)_{n\ge 1}$ and q in the items 1–5 above, and

$$\bar{p}^{(n)} = \Lambda_n^{-1} C_n^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

By Proposition 2.9, $p^{(n)}$ converges to some point $p \in l^2$.

Step 2. For any $j > n$, by items 2 and 3, we have $|q_j| \leq j^3|\mu_j - \hat{\lambda}_j|$ and

$$|\mu_j - \hat{\lambda}_j| \leq j^{-7}|\mu_n - \hat{\lambda}_j|.$$

Thus, the series

$$\sum_{j=1}^{\infty} \frac{q_j}{\mu_n - \hat{\lambda}_j} e_j$$

converges to u_n in $\mathcal{H}$ for each n. It is easy to verify that $\langle p, u_n \rangle = 1, \forall n \geq 1$. By Proposition 2.5, u_n is the eigenvector corresponding to $\mu_n, \forall n \geq 1$.

Step 3. For $n > n_0$ and any $x \in \text{span}\{e_1, e_2, \ldots, e_n\}$ where $\|x\| = 1$, we rewrite x as

$$x = \sum_{1 \leq j \leq n} x_j e_j = \sum_{1 \leq j \leq n} y_j \bar{u}_j,$$

where

$$\bar{u}_k = \sum_{1 \leq j \leq n} \frac{q_j}{\mu_k - \hat{\lambda}_j} e_j,$$

and

$$(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n) = (e_1, e_2, \ldots, e_n) \Lambda_n C_n^T.$$

So

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = (C_n^T)^{-1}(\Lambda_n)^{-1} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Then by Proposition 2.8,

$$\|(y_1, y_2, \ldots, y_n)\|_1 \leq \left\|(C_n^T)^{-1}(\Lambda_n)^{-1}\right\|_1 \leq n \cdot (2n + c_0).$$

So

$$\begin{aligned} \left\|x - \sum_{j=1}^{n} y_j u_j\right\| &= \left\|\sum_{j=1}^{n} y_j \bar{u}_j - \sum_{j=1}^{n} y_j u_j\right\| \\ &\leq \|(y_1, y_2, \ldots, y_n)\|_1 \max_{1 \leq j \leq n} \|u_j - \bar{u}_j\| \\ &\leq (2n^2 + nc_0) \max_{1 \leq j \leq n} \|u_j - \bar{u}_j\|. \end{aligned}$$

By item 2 and 3, $|q_k| \le k^3|\mu_k - \hat{\lambda}_k|$ and $|\mu_k - \hat{\lambda}_k| < k^{-7}|\mu_j - \hat{\lambda}_k|$, $(j < k)$ it follows

$$\begin{aligned}\|u_j - \bar{u}_j\| &= \left\| \sum_{k>n} \frac{q_k}{\mu_j - \hat{\lambda}_k} e_k \right\| \\ &\le \sum_{k>n} \left| \frac{q_k}{\mu_j - \hat{\lambda}_k} \right| \\ &\le \sum_{k>n} \frac{1}{k^4}.\end{aligned}$$

So $\|x - \sum_{j=1}^{n} y_j u_j\| \le (2n^2 + nc_0) \sum_{k>n} \frac{1}{k^4} \to 0$ as $n \to \infty$.
Then, the set of eigenvectors $\{u_j | j \ge 1\}$ of T is a basis in $\mathcal{H}$. □

Proof of Theorem 2.6 We construct the sequence $(\mu_n)_{n\ge1}$ which satisfies the conditions described in Proposition 2.8. By Proposition 2.10, $(\mu_n)_{n\ge1}$ are all eigenvalues of T, and the corresponding set of eigenvectors $\{u_j | j \ge 1\}$ is a basis in $\mathcal{H}$.

We just need to check that the set $\{u_j | j \ge 1\}$ contains no isolate point. According to item 4, for each μ_m, there exists μ_n(n is odd, sufficiently large) such that

$$\sum_{1\le j<n} \left| \frac{q_j}{\mu_m - \hat{\lambda}_j} - \frac{q_j}{\mu_n - \hat{\lambda}_j} \right| \le \frac{1}{n^5}.$$

By item 2 and item 3, $|q_n| \le n^{-1}|\mu_n - \hat{\lambda}_n|$ and $|\mu_n - \hat{\lambda}_n| < n^{-7}|\mu_m - \hat{\lambda}_n|$, $(m < n)$ then

$$\left| \frac{q_n}{\mu_n - \hat{\lambda}_n} \right| \le \frac{1}{n},$$

and

$$\left| \frac{q_n}{\mu_m - \hat{\lambda}_n} \right| \le \frac{1}{n^8}.$$

Similarly, since $|q_j| \le j^3|\mu_j - \hat{\lambda}_j| < j^{-4}|\mu_n - \hat{\lambda}_j|$, $(j > n > m)$ the following inequalities hold:

$$\sum_{j>n} \left| \frac{q_j}{\mu_n - \hat{\lambda}_j} \right| \le \sum_{j>n} \frac{1}{j^4} \le \frac{1}{n},$$

and

$$\sum_{j>n} \left| \frac{q_j}{\mu_m - \hat{\lambda}_j} \right| \le \frac{1}{n}.$$

We have

$$\|u_m - u_n\| \leq \frac{5}{n}.$$

Note that n can be chosen to be sufficiently large, so u_m is not an isolate point.

Finally, T satisfies all the conditions in Lemma 2.1. By Lemma 2.2, T is weak-mixing with respect to a nondegenerate Gaussian measure. □

3 Applications to Controllability for Linear Control Systems

Controllability is one fundamental concept in mathematical control theory. Controllability for distributed parameter systems, which belong in the category of infinite-dimensional control systems, was first studied by Fattorini in the 1960s in [10, 11]. Then, the theory of controllability for infinite-dimensional linear control systems was systematically established in [8, 9]. In this section, we consider the following discrete time linear control system on a complex separable Hilbert space $\mathcal{H}$,

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, 2, \ldots \tag{3.1}$$

where A is a bounded linear operator on $\mathcal{H}$, $x_k \in \mathcal{H}$ is the state of the system, $\mathcal{U}$ is a Hilbert space, called input or control space, and B is a bounded linear operator from $\mathcal{U}$ into $\mathcal{H}$.

For the linear system (3.1) and its continuous-time version, there exist many different concepts on controllability, such as global controllability, null controllability, approximate controllability and near-controllability. Roughly speaking, controllability characterizes the ability of a system that the system can be steered from an arbitrary initial state to an arbitrary terminal state under the action of admissible controls. There are different kinds of criteria for different sense of controllability.

It is well known that the system (3.1) is not controllable in any existed sense of controllability when $B = 0$. So a natural question is what can we say about controllability of the system (3.1) with $B = 0$. In the following, we will consider the discrete time system (without controls)

$$x_{k+1} = Tx_k, \quad k = 0, 1, 2, \ldots, \tag{3.2}$$

where T is a bounded linear operator on a complex separable infinite-dimensional Hilbert space $\mathcal{H}$.

Definition 3.1 Let μ be a Borel measure on $\mathcal{H}$ with full support. The system (3.2) is called approximately-nearly controllable (ANC) for the measure μ if there exists a $\mu-$full measure set $\Gamma \subset \mathcal{H}$ such that for any pair of points $(x, y) \in \Gamma \times \mathcal{H}$ and any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that the solution $(x_n)_{n\geq 0}$ of (3.2) with $x_0 = x$ satisfies

$$\|x_k - y\| < \varepsilon,$$

where $\|\cdot\|$ is the norm on $\mathcal{H}$.

This concept of ANC is a modification of Definition 1 in [16].

Remark 3.2 The ANC for some measure is weaker than the approximate controllability which says that any state can be steered to any neighbourhood of any another state in finite time by some controls. An ANC system is approximately controllable in almost sense without controls, which is very closely related to the hypercyclicity. More precisely, ANC for any measure implies hypercyclicity, and the converse is not true.

In this section, for the system (3.1), we are going to find out some conditions under which there exists a feedback $u_k = Fx_k$ such that the close loop system

$$x_{k+1} = (A + BF)x_k, \quad k = 0, 1, 2, \ldots \tag{3.3}$$

is ANC for some measure, where F is a bounded linear operator from $\mathcal{H}$ into $\mathcal{U}$.

A negative result about control systems on finite-dimensional Hilbert spaces is given below.

Proposition 3.3 *If the system* (3.1) *is finite dimensional, that is, the state space $\mathcal{H}$ is a finite-dimensional linear space, then for any feedback F, the close loop system* (3.3) *can never be ANC for any measure.*

Proof In this case, the close loop operator $T = A + BF$ is a linear operator on the finite-dimensional space. So T can never be hypercyclic by Proposition 2.57 in [15]. Thus, T is not ANC for any measure. □

Next theorem shows that for a class of operators A in (3.1), we can derive a linear feedback operator of rank one, such that the associated close loop system is ANC.

Theorem 3.4 *Given a discrete-time linear control system*

$$x_{k+1} = Ax_k + u_k, \quad k = 0, 1, 2, \ldots,$$

where A is a bounded linear operator on a complex separable infinite-dimensional Hilbert space $\mathcal{H}$, and $x_k, u_k \in \mathcal{H}$. Assume that the operator A satisfies

- (A_1) *$\lambda_i (i \geq 1)$ are distinct points on $\mathbb{T}$ and the set $\{\lambda_i | i \geq 1\}$ contains at most finite number of isolated points;*
- (A_2) *$Ae_j = \lambda_j e_j$ and $(e_j)_{j \geq 1}$ is an orthonormal basis of $\mathcal{H}$.*

Then, there exists a one-dimensional feedback map $F : \mathcal{H} \longrightarrow \mathcal{H}$ defined as $u_k = F(x_k) = \langle x_k, p \rangle q$ where $p, q \in \mathcal{H}$, such that the system

$$x_{k+1} = (A + F)x_k$$

is ANC for some Borel measure m with full support on $\mathcal{H}$.

Proof According to Theorem 2.6, we can construct a map F on $\mathcal{H}$, $F(x) = \langle x, p\rangle q$, such that $T \triangleq A + F$ has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues. Then, there exists a nondegenerate invariant Gaussian measure m on $\mathcal{H}$. Denote by Γ the set of all hypercyclic points of T. By Remark 2.3, Γ has full measure and satisfies the condition in the definition of ANC. □

Finally, we give two examples which satisfy the conditions of Theorem 3.4.

Example 1 (*With multiple eigenvalue* 1) Denote by $(p_j)_{j\geq 1}$ the sequence of all prime numbers $2, 3, 5, \ldots$. Let $s_n = \sum_{j=1}^{n} p_j$, $(n \geq 1)$ and $s_0 = 0$. We define a bijective map $\tau : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$ as follows,

$$\tau|_{[s_{n-1}+1,s_n]}(k) = k + 1, \quad \text{if } s_{n-1} + 1 \leq k < s_n;$$
$$\tau|_{[s_{n-1}+1,s_n]}(s_n) = s_{n-1} + 1.$$

Actually, τ is a cyclic permutation on each subinterval $[s_{n-1} + 1, s_n]$.

We can define a unitary operator,

$$A : l^2 \longrightarrow l^2$$
$$(x_1, x_2, \ldots) \longrightarrow (x_{\tau(1)}, x_{\tau(2)}, \ldots).$$

For each n, it is easy to check that $e^{i2\pi k/p_n}$ $(1 \leq k < p_n)$ is an eigenvalue of A and the corresponding eigenvector is

$$(0, \ldots, 0, \underbrace{1, e^{i2\pi k/p_n}, \ldots, e^{i2\pi(p_n-1)k/p_n},}_{\text{nonzero coordinates between } s_{n-1}+1 \text{ and } s_n} 0, \ldots).$$

Denote the eigenspace of the eigenvalue 1 by W, and $l^2 = W \oplus W^\perp$. Then, the set of eigenvalues $\{e^{i2\pi k/p_n} | 1 \leq k < p_n, n \geq 1\}$ is dense in $\mathbb{T}$, and the corresponding set of eigenvectors is an orthogonal basis of $W^\perp$.

Consider the following control system

$$x_{k+1} = Ax_k + u_k, \quad k = 0, 1, 2, \ldots.$$

where $x_k, u_k \in W^\perp$, $k = 0, 1, 2, \ldots$. According to Theorem 3.4, there exists a one-dimensional feedback F such that the system

$$x_{k+1} = (A + F)x_k, \quad k = 0, 1, 2, \ldots.$$

is ANC for some measure on $W^\perp$.

Example 2 Given an irrational number α, let $\phi : \mathbb{T} \longrightarrow \mathbb{T}$, $\phi(e^{i\theta}) = e^{i(\theta+\alpha)}$, $\forall\theta \in [0, 2\pi)$. We define a unitary operator

$$T_\phi : L^2(\mathbb{T}) \longrightarrow L^2(\mathbb{T}),$$
$$f \longrightarrow f \circ \phi.$$

For each $k \in \mathbb{Z}$, the power function $f_k(e^{i\theta}) = e^{ik\theta}$ is an eigenvector of T_ϕ, and the corresponding eigenvalue is $e^{ik\alpha}$. The set of eigenvalues $\{e^{ik\alpha} | k \in \mathbb{Z}\}$ is dense in $\mathbb{T}$, and the corresponding set of eigenvectors $\{f_k | k \in \mathbb{Z}\}$ is an orthogonal basis of $L^2(\mathbb{T})$.

Consider the control system

$$x_{k+1} = T_\phi x_k + u_k,$$

where $x_k, u_k \in L^2(\mathbb{T})$, $k = 0, 1, 2, \ldots$. According to Theorem 3.4, there exists a one-dimensional feedback F such that the system

$$x_{k+1} = (T_\phi + F)x_k, \quad k = 0, 1, 2, \ldots.$$

is ANC for some measure on $L^2(\mathbb{T})$.

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