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Hankel Edge Ideals of Trees and (Semi-)Hamiltonian Graphs

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Abstract

In this paper, we study the Hankel edge ideals of graphs. We determine the minimal prime ideals of the Hankel edge ideal of labeled Hamiltonian and semi-Hamiltonian graphs, and we investigate radicality, being a complete intersection, almost complete intersection and set-theoretic complete intersection for such graphs. We also consider the Hankel edge ideal of trees with a natural labeling, called rooted labeling. We characterize such trees whose Hankel edge ideal is a complete intersection, and moreover, we determine those whose initial ideal with respect to the reverse lexicographic order satisfies this property.

Keywords Hankel edge ideals · (Semi-)Hamiltonian graphs · Rooted labeling · (Almost/set-theoretic) complete intersection

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1 Introduction

Let $\mathbb{K}$ be a field and let G be a finite simple graph (i.e., with no loops and multiple edges) with the vertex set $V(G) = [n]$ (i.e., $\{1, \dots, n\}$) and the edge set $E(G)$. Then, the binomial edge ideal J_G of G in the polynomial ring $R = \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_n]$ is generated by the binomials $f_{ij} = x_i y_j - x_j y_i$ with $i < j$ such that $\{i, j\} \in E(G)$. This type of ideals was introduced at about the same time by Herzog et al. in [14] and Ohtani in [19] in 2010. The binomial edge ideal of G could be seen as the ideal generated by a collection of 2-minors of the matrix

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix}.$$

In the last decade, several algebraic and homological properties and invariants of binomial edge ideals have been intensively studied, mainly in terms of the combinatorial properties of the underlying graph, by many authors, see for example [7, 10, 11, 15–18, 20, 22–24].

In 2015, another class of binomial ideals associated to graphs was introduced in [3]. Let $S = \mathbb{K}[x_1, \dots, x_{n+1}]$ be the polynomial ring over $\mathbb{K}$ and with the indeterminates $x_1, \dots, x_{n+1}$. The *Hankel edge ideal* of G, denoted by I_G, is the ideal of S generated by the binomials $g_{ij} = x_i x_{j+1} - x_j x_{i+1}$ where $i < j$ and $\{i, j\} \in E(G)$. This ideal is also seen as the ideal generated by a collection of 2-minors of the $2 \times n$ Hankel matrix

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_3 & \cdots & x_{n+1} \end{bmatrix}.$$

In the special case that G is the complete graph with n vertices, the Hankel edge ideal coincides exactly with the well-known ideal I_X of the rational normal curve $\mathcal{X} \subset \mathbb{P}^n$. For some properties of the ideal I_X, see for example [2, 5, 6, 9].

Note that in [3] and [8], the ideal I_G was called the *binomial edge ideal of* X and also the *scroll binomial edge ideal* of G. But, in this paper, we chose to call I_G the Hankel edge ideal of G.

In [3], the authors determined all graphs G for which the generators g_{ij} form a Gröbner basis for I_G with respect to the reverse lexicographic order $<$ induced by $x_1 > \cdots > x_n > x_{n+1}$. Indeed, the only graphs with this property are the so-called *closed* graphs (also known as proper interval graphs). A graph G is said to be closed if one could label its vertices so that the maximal cliques (i.e., complete subgraphs) of G are labeled as intervals. Throughout this paper, when we talk about a closed graph, we mean, as usual, a closed graph with this specific labeling. In [3], it was also shown that for any closed graph G, the Hankel edge ideal is Cohen–Macaulay of dimension $c + 1$ where c is the number of connected components of G. In the same paper, the minimal prime ideals of I_G were determined for any connected closed graph G. Consequently being radical as well as being a set-theoretic complete intersection could be investigated for the same class of graphs. In [3], the authors also studied the Castelnuovo–Mumford regularity of I_G. They, indeed, gave a combinatorial upper bound for $\operatorname{reg}(S/I_G)$ where G is a closed graph. Later, in [8], the authors characterized all closed graphs for which the aforementioned upper bound is attained. In the same paper, the graded Betti numbers of I_G and $\text{in}_<(I_G)$ were also considered. In [8], a combinatorial characterization for Gorenstein Hankel edge ideals was also given. A generalization of Hankel edge ideals was also introduced and studied in [4].

As mentioned above, the class of closed graphs has played an essential role in the study of Hankel edge ideals of graphs so far. This is in fact because closed graphs admit a nice distinguished vertex labeling, and in general Hankel edge ideals do depend on the labeling of the underlying graphs. As an example for this fact, see [3, page 972]. In this paper, we study the Hankel edge ideals of some classes of graphs with certain natural labelings, like labeled Hamiltonian and semi-Hamiltonian graphs as well as rooted labeled trees, which are all defined precisely in the sequel.

The organization of this paper is as follows. In Sect. 2, we focus on the Hankel edge ideal of labeled Hamiltonian and semi-Hamiltonian graphs. By a labeled Hamiltonian graph, we mean a graph which has a Hamiltonian cycle (i.e., a cycle containing all the vertices of the graph) with a certain natural labeling of its vertices. By a labeled semi-Hamiltonian graph, we mean a non-Hamiltonian graph which admits a labeled Hamiltonian path (i.e., a path containing all the vertices of the graph) with a certain natural labeling of its vertices. Closed graphs are semi-Hamiltonian graphs, so that in this section, we recover most of the results in [3]. However, this section deals with a much wider class of graphs than closed graphs. In this section, we determine the minimal prime ideals of the Hankel edge ideal of connected labeled (semi-)Hamiltonian graphs, and in particular, we show that the height of I_G is $n-1$ where $|V(G)| = n$. Moreover, we characterize all Hankel edge ideals of connected labeled (semi-)Hamiltonian graphs, with respect to radicality, being complete intersection and almost complete intersection. In this section, we also show that the arithmetical rank of I_G is $n-1$, and I_G is a set-theoretic complete intersection for connected labeled (semi-)Hamiltonian graph G with $|V(G)| = n$.

In the view of an observation in Sect. 2 that if the Hankel edge ideal of a connected graph is a complete intersection, then the graph must be a tree, in Sect. 3 trees become our main objects of interest. So, we need to fix a nice vertex labeling for trees which is provided in this section called *rooted labeling*. Next, we show that under this labeling, the only trees which could have complete intersection Hankel edge ideal are paths. However, not all rooted labeled paths have this property. Indeed, we show that the Hankel edge ideal of a rooted labeled tree T is a complete intersection if and only if T is a path in which a leaf or a neighbor of a leaf is the root of T. Furthermore, we consider the initial ideal of such trees over n vertices with respect to the reverse lexicographic order induce by $x_1 > \cdots > x_n > x_{n+1}$. Indeed we show that there is only one path on n vertices with a rooted labeling whose Hankel edge ideal has a complete intersection initial ideal. Throughout the paper, we also pose some questions accordingly.

2 Hankel Edge Ideal of Hamiltonian and Semi-Hamiltonian Graphs

In this section, we determine the minimal prime ideals of the Hankel edge ideal of (semi-)Hamiltonian graphs with a certain labeling. As some consequences, we also study some of the algebraic properties of such ideals, like radicality, being a complete intersection, almost complete intersection and set-theoretic complete intersection.

Recall that a *Hamiltonian cycle* in a graph G is a cycle which contains all the vertices of G. The graph G which has a Hamiltonian cycle is called a *Hamiltonian graph*. A *Hamiltonian path* in G is a path which contains all the vertices of G. The graph G which has a Hamiltonian path is called *traceable graph*. A traceable graph which is not Hamiltonian is called *semi-Hamiltonian graph*. We denote by L_n the path with the vertex set $[n]$ and with the edges $\{i, i+1\}$ for all $i = 1, \dots, n-1$. We also denote by C_n the cycle with the vertex set $[n]$ and with the edges $\{1, n\}$ and $\{i, i+1\}$ for all $i = 1, \dots, n-1$.

Now, let G be a labeled graph with the vertex set $[n]$. If C_n is a subgraph of G, then we say that G is a *labeled Hamiltonian graph*. If L_n is a subgraph of G and $\{1, n\} \notin E(G)$, then we say that G is a *labeled semi-Hamiltonian graph*.

As some well-known classes of graphs which are labeled Hamiltonian, one could mention the cycles C_n, the complete graphs K_n and the complete bipartite graphs $K_{t,t}$ with $n = 2t$ where the even labels and odd labels provide the bipartition of the graph. The graph depicted in Fig. 1 is also an example of a labeled Hamiltonian graph on the vertex set $\{1, 2, 3, 4, 5\}$ which is not in the aforementioned classes.

As a well-known class of graphs which are labeled Hamiltonian, one can mention non-complete closed graphs. Several properties of the Hankel edge ideal of this class of graphs were studied in [3]. Recall that a graph G is said to be closed if one could label its vertices so that the maximal cliques (i.e., complete subgraphs) of G are labeled as intervals. The graph shown in Fig. 2 is an example of a labeled semi-Hamiltonian graph with the vertex set $\{1, 2, 3, 4, 5, 6\}$ which is not a closed graph.

In [3], it was shown that for any connected graph G on n vertices, I_X is a minimal prime ideal of I_G. Therefore, since height $I_X = n - 1$ (see for example [3, Corollary 1.3]), we have the following:

Fig. 1 A labeled Hamiltonian graph

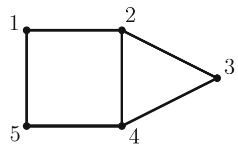
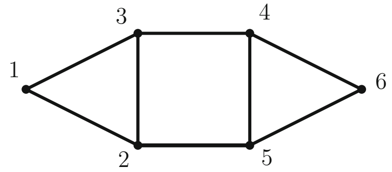


Fig. 2 A labeled semi-Hamiltonian graph



Proposition 2.1 (See) [3, Proposition 2.1] *Let G be a connected graph on n vertices. Then, the following hold:*

(a) *I_X is a minimal prime ideal of I_G.*
(b) height $I_G \leq n-1$.
(c) *For any minimal prime P of I_G which contains no variable, one has $P = I_X$.*

Now, we determine the minimal prime ideals of I_G whenever G is a (semi-) Hamiltonian graph and we particularly deduce that for these graphs the height attains the exact value $n-1$.

Theorem 2.2 *Let G be a connected graph with $V(G) = [n]$. Then, the following hold:*

a *If G is a labeled Hamiltonian graph, then* $\mathrm{Min}(I_G) = \{I_X\}$.
b *If G is a labeled semi-Hamiltonian graph, then* $\mathrm{Min}(I_G) = \{I_X, (x_2, \ldots, x_n)\}$.

In particular, in both cases, height $I_G = n-1$.

Proof Let G be either a labeled Hamiltonian or a labeled semi-Hamiltonian graph, and let P be a minimal prime ideal of I_G. First, we observe that if $x_j \in P$, then $x_{j+1} \in P$, for any $j = 1, \ldots, n-1$. Indeed, since L_n is a subgraph of G, it follows that $g_{jj+1} = x_j x_{j+2} - x_{j+1}^2 \in I_G$. Also, since P is a prime ideal containing I_G, it follows from $x_j \in P$ that $x_{j+1} \in P$.

a. By Theorem 2.1, it is enough to show that I_G does not have any minimal prime ideal containing variables. Suppose on contrary that P is a minimal prime ideal of I_G which contains a variable. Now let i be the smallest integer with $x_i \in P$. First, assume that $i = 1$. The above observation implies that $x_j \in P$ for all $j = 1, \ldots, n$. This implies that $I_X \subseteq (x_1, x_2, \ldots, x_n) \subseteq P$. Since P and I_X are both minimal prime ideals of I_G, it follows that $P = I_X$, which is a contradiction. Next assume that $i = 2$. Then, again by our above observation, we have that $x_j \in P$ for all $j = 2, \ldots, n$. On the other hand, since C_n is a subgraph of G, we have $g_{1n} = x_1 x_{n+1} - x_2 x_n \in I_G$. Since $x_1 \notin P$ and P is a prime ideal containing I_G, we deduce that $x_{n+1} \in P$. Thus, we have $I_X \subseteq (x_2, x_3, \ldots, x_n, x_{n+1}) \subseteq P$. Therefore, $P = I_X$ by minimality of P, which is a contradiction. Finally, assume that $i \geq 3$. Then, $g_{i-2\, i-1} = x_{i-2} x_i - x_{i-1}^2 \in I_G \subseteq P$. Thus, $x_{i-1} \in P$ which is a contradiction, since i is the smallest index such that $x_i \in P$. Therefore, I_X is the only minimal prime ideal of I_G.
b. By Theorem 2.1, it suffices to prove that if P is a minimal prime ideal of I_G containing a variable, then $P = (x_2, \ldots, x_n)$. Let i be the smallest integer with $x_i \in P$. If $i = 1$, then by using our above observation it follows that $I_X \subseteq (x_1, \ldots, x_n) \subseteq P$, and hence $P = I_X$, a contradiction. If $i = 2$, then again by our observation we have $(x_2, \ldots, x_n) \subseteq P$. Since $I_G \subseteq (x_2, \ldots, x_n)$ and P is a minimal prime ideal of I_G, it follows that $P = (x_2, \ldots, x_n)$. If $i \geq 3$, then using the fact that $g_{i-2\, i-1} \in I_G$, similar to the proof of part (a), we deduce that $x_{i-1} \in P$, contradicting the assumption that i is the smallest desired integer. Therefore, $\mathrm{Min}(I_G)$ consists only of I_X and $(x_2, \ldots, x_n)$.

□

For a graph G and $e \in E(G)$, we denote by $G - e$ the subgraph of G with the same vertex set as G obtained by removing the edge e from G.

For a homogeneous ideal I in S, we denote by $\mu(I)$, the cardinality of a minimal homogeneous generating set of I. We would like to remark that it is easily seen that neither of the binomials $g_{ij} = x_i x_{j+1} - x_{i+1} x_j$ for $1 \leq i < j \leq n$ could be generated in S by the other ones. In particular, the generators g_{ij}'s of I_G provide a minimal generating set for this ideal. Therefore, we have $\mu(I_G) = |E(G)|$.

Using Theorem 2.2, we obtain the following characterization of radicality of I_G in the case of labeled (semi-)Hamiltonian graphs.

Corollary 2.3 *Let G be a connected graph with $V(G) = [n]$. Then, the following hold:*

a. *Let G be a labeled Hamiltonian graph. Then, I_G is a radical ideal if and only if $G = K_n$.*
b. *Let G be a labeled semi-Hamiltonian graph. Then, I_G is a radical ideal if and only if $G = K_n - e$ where $e = \{1, n\}$.*

Proof a. If $G = K_n$, then $I_G = I_X$ which is a prime ideal. Conversely, assume that G is a labeled Hamiltonian graph and I_G is radical, i.e., $I_G = \text{rad}(I_G)$. Thus, by Theorem 2.2 part (a), we have $I_G = I_X$, and hence $\mu(I_G) = \mu(I_X)$ which implies that $|E(G)| = |E(K_n)|$. Thus, $G = K_n$, as desired.

b. It was shown in [3, Proposition 2.3] that if $e = \{1, n\}$, then $I_{K_n - e}$ is a radical ideal with

$$I_{K_n - e} = I_X \cap (x_2, \ldots, x_n). \tag{1}$$

Conversely, assume that G is a labeled semi-Hamiltonian graph and I_G is radical. Thus, $I_G = \text{rad}(I_G) = I_X \cap (x_2, \ldots, x_n)$, by Theorem 2.2 part (b). It follows from (1) that $I_G = I_{K_n - e}$, and hence $G = K_n - e$. □

Let I be a homogeneous ideal in S. Recall that S/I is a complete intersection if $\mu(I) =$ height I. Therefore, we have S/I_G is a complete intersection if and only if height $I_G = |E(G)|$. So, it follows from Theorem 2.2 that S/I_G is never a complete intersection if G is a labeled Hamiltonian graph, since $|E(G)| \geq n$ while height $I_G = n - 1$.

Next, we determine those semi-Hamiltonian graphs G for which S/I_G is a complete intersection. First, we have the following general observation:

Proposition 2.4 *Let G be a connected graph such that S/I_G is a complete intersection. Then, G is a tree and* height $I_G = n - 1$.

Proof Let $|V(G)| = n$. Since S/I_G is a complete intersection, we have height $I_G = |E(G)|$. Proposition 2.1 implies that $|E(G)| \leq n - 1$. Since G is a connected graph with n vertices, it follows that $|E(G)| = n - 1$. This then implies that G is a tree and height $I_G = n - 1$, as desired. □

Corollary 2.5 *Let G be a connected labeled semi-Hamiltonian graph with $V(G) = [n]$. Then, S/I_G is a complete intersection if and only if $G = L_n$.*

Proof It is clear that I_{L_n} is a complete intersection, since height $I_G = |E(L_n)| = n-1$. Conversely, assume that S/I_G is a complete intersection. Thus, by Proposition 2.4, G is a tree. But, the only tree with n vertices for which L_n is a subgraph, is the path L_n. Thus, $G = L_n$, as desired. □

Let I be a homogeneous ideal in S. Recall that I is said to be an almost complete intersection if $\mu(I) = \text{height } I + 1$. Therefore, we have I_G is an almost complete intersection if and only if height $I_G = |E(G)| - 1$.

Also, recall that a graph is called *unicyclic* if it contains exactly one cycle. A connected unicyclic graph is obtained from a tree by adding an edge between two non-adjacent vertices of the tree.

Proposition 2.6 *Let G be a connected graph with $V(G) = [n]$. Then, the following hold:*

a. *Let G be a labeled Hamiltonian graph. Then, I_G is an almost complete intersection if and only if $G = C_n$.*
b. *Let G be a labeled semi-Hamiltonian graph. Then, I_G is an almost complete intersection if and only if G is a unicyclic graph obtained from L_n by adding the edge $\{t, t+s\}$ for some t, s with $1 \leq t \leq n-2$ and $s \geq 2$.*

Proof By Theorem 2.2, if G is either a labeled Hamiltonian graph or a labeled semi-Hamiltonian graph, then the ideal I_G is an almost complete intersection if and only if $|E(G)| = n$. The latter equivalently means that G is a unicyclic graph, since G is connected.

a. Let G be a labeled Hamiltonian graph. Then, the only unicyclic graph with n vertices which has C_n as a subgraph is the cycle C_n itself. Hence, the statement follows.
b. Let G be a labeled semi-Hamiltonian graph. Then, the only unicyclic graphs with L_n as a subgraph are those ones obtained from L_n by connecting two non-adjacent vertices of L_n by an edge. This implies the desired result.

□

Since closed graphs are chordal, it follows immediately from Proposition 2.6 that:

Corollary 2.7 *Let G be a closed graph on the vertex set $[n]$. Then, S/I_G is almost complete intersection if and only if G is a unicyclic graph obtained from L_n by adding the edge $\{t, t+2\}$ for some $t = 1, \ldots, n-2$.*

Let I be an ideal of S. The *arithmetical rank* of I, denoted by $\text{ara}(I)$, is the least integer r such that $\text{rad}(I) = \text{rad}(f_1, \ldots, f_r)$ for some $f_1, \ldots, f_r \in S$. It is well known that height $I \leq \text{ara}(I)$. The ideal I is called a *set-theoretic complete intersection* if height $I = \text{ara}(I)$.

In [3, Corollary 2.4], it was shown that the Hankel edge ideals of all connected closed graphs are set-theoretic complete intersection. In the next proposition, we generalize this result to all connected labeled Hamiltonian and semi-Hamiltonian graphs.

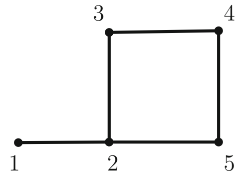


Fig. 3 A labeled semi-Hamiltonian graph with non-Cohen–Macaulay Hankel edge ideal

Proposition 2.8 *Let G be a connected labeled Hamiltonian or semi-Hamiltonian graph with $V(G) = [n]$. Then,* $\operatorname{ara}(I_G) = n - 1$. *In particular, I_G is a set-theoretic complete intersection.*

Proof First, note that $\operatorname{ara}(I_G) \geq n - 1$, since height $I_G = n - 1$, by Theorem 2.2.

If G is a labeled Hamiltonian graph, then it follows from [1, Proposition 1] that $\operatorname{ara}(I_G) \leq n - 1$, (see also [21, Sect. 1] and [25]).

If G is a labeled semi-Hamiltonian graph, then by Theorem 2.2 we have $\operatorname{Min}(I_G) = \operatorname{Min}(I_{L_n}) = \{I_X, (x_2, \ldots, x_n)\}$, and hence $\operatorname{rad}(I_G) = \operatorname{rad}(I_{L_n})$. This implies that $\operatorname{ara}(I_G) \leq n - 1$.

Therefore, in both cases, we get the equality $\operatorname{ara}(I_G) = n - 1$, as desired. The "in particular" part is then immediate by definition. □

We would like to remark that in [3, Proposition 1.2] it was shown that if G is a closed graph, then S/I_G is Cohen–Macaulay. However, this is not always the case in the more general case of labeled semi-Hamiltonian graphs. For example, let G be the graph depicted in Fig. 3 which is a labeled semi-Hamiltonian non-closed graph. Our computations with *Macaulay 2* show that $\operatorname{projdim}(S/I_G) = 5$ while by Theorem 2.2 we have height $I_G = 4$. So, it would be interesting to ask which labeled semi-Hamiltonian graphs admit Cohen–Macaulay Hankel edge ideal.

3 Complete Intersection Hankel Edge Ideals

In the view of Proposition 2.4, now in this section we are interested in trees and in investigating about those trees T for which S/I_T is a complete intersection.

Note that Proposition 2.4 is independent of the labeling of G. However, as we mentioned in Sect. 1, in general the ideal I_G strongly depends on the labeling of G. Therefore, now for trees we fix a natural labeling which we call *rooted labeling*. First, recall that $N_G(i)$ is the set of the neighbors of i, i.e., vertices adjacent to the vertex i, and the degree of the vertex i, denoted by $\deg(i)$ is equal to $|N_G(i)|$. We also set $N_G[i] = N_G(i) \cup \{i\}$.

Let T be a tree with n vertices. Roughly speaking, to give a rooted labeling to T, we give the labels $1, \ldots, n$ to the vertices "consecutively" as follows: we pick a vertex as the *root* with the label 1, and then we label its neighbors in any order. Then, we label the neighbors of these new labeled vertices in the increasing order, and we continue this process to get all the vertices labeled.

More precisely, pick a vertex as the root and give the label 1 to it. If $|N_T(1)| = t_1$, then give the labels $2, \ldots, t_1 + 1$ to the neighbors of 1 in an arbitrary order. Next, if $|N_T(2)| = t_2+1$, then label the t_2 unlabeled vertices in $N_T(2)$ by $t_1+2, \ldots, t_1+t_2+1$

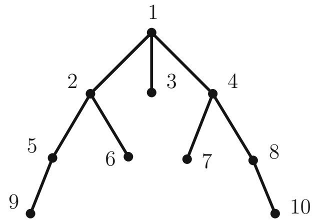


Fig. 4 A tree with a rooted labeling

in an arbitrary order. Similarly, for any i with $3 \le i \le t_1+1$, if $|N_T(i)| = t_i + 1$, then label the t_i unlabeled vertices in $N_T(i)$ by $t_1+t_2+\cdots+t_{i-1}+2, \ldots, t_1+t_2+\cdots+t_i+1$ in an arbitrary order. Then, repeat the same procedure for the neighbors of the vertices $i = t_1+2, \ldots, t_1+t_2+\cdots+t_{t_1+1}+1$. Then, by continuing this process, all the vertices of T are labeled. Figure 4 depicts an example of a tree with a rooted labeling.

In a rooted labeled tree, we call those neighbors of a vertex i whose labels are greater than i, the *children* of i. For example, the children of the vertex 2 in the graph of Fig. 4 are the vertices 5 and 6.

Theorem 3.1 *Let T be a tree on n vertices with a rooted labeling. If T is not a path, then* height $I_T \le n-2$. *In particular, S/I_T is not a complete intersection.*

Proof Since T is not a path, it has a vertex of degree ≥ 3. First, assume that the root 1 has degree at least 3. Thus, according to the labeling of T, we have that the vertices with the labels 2, 3 and 4 are the neighbors of 1.

If $n \notin N_T[3] \cup N_T[4]$, then $P = (x_1, x_2, x_5, x_6, \ldots, x_n)$ is a prime ideal containing I_T which implies that height $I_T \le n-2$.

If $n \in N_T[3] \cup N_T[4]$, then we distinguish the following cases:

(1) If $n = 4$, then $P = (x_1, x_2)$ is a prime ideal containing I_T, and hence height $I_T \le 2 = n-2$.
(2) If $n = 5 \in N_T[3]$, then $P = (x_1, x_2, g_{35})$ is a prime ideal containing I_T, since (g_{35}) is a prime ideal (of height 1) in $\mathbb{K}[x_3, x_4, x_5, x_6]$. Thus, height $I_T \le 3 = n-2$.
(3) If $n = 5 \in N_T[4]$, then $P = (x_1, x_2, g_{45})$ is a prime ideal containing I_T, and hence height $I_T \le 3 = n-2$.
(4) If $n = 6 \in N_T[3]$, then T is one of the trees T_1, T_2, T_3 in Fig. 5. It is easily seen that in each case, $P = (x_1, x_2, x_3, x_4)$ is a prime ideal containing I_T, and hence height $I_T \le 4 = n-2$.
(5) If $n = 6 \in N_T[4]$, then T is one of the trees T_1', T_2', T_3', T_4' in Fig. 6. It is easily seen that if $T = T_1', T_2', T_4'$, then $P = (x_1, x_2, x_4, x_5)$ is a prime ideal containing I_T, and if $T = T_3'$, then $P' = (x_1, x_2, x_4, x_6)$ is a prime ideal containing I_T. Therefore, we have height $I_T \le 4 = n-2$.
(6) If $n \ge 7$, then any vertex i with $i \ge 5$ is a leaf. Thus, it is seen that $P = (x_1, x_2, x_3, x_4, x_5)$ is a prime ideal containing I_T, and hence height $I_T \le 5 \le n-2$.

Next, suppose that the degree of 1 is at most 2. Let $t \ge 2$ be the smallest label in T with $\deg(t) \ge 3$, and let $t_1, \ldots, t_q$ be q consecutive integers which are the labels of the children of t. First, assume that there exists an $s = 1, \ldots, q$ with $t_s > t+1$ such that $n \notin N_T(t_s)$. Let s be the maximum integer with this property.

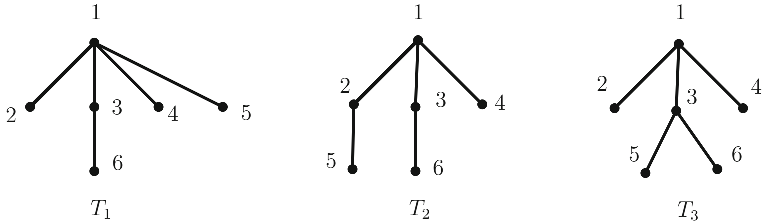


Fig. 5 The graphs T_1, T_2 and T_3

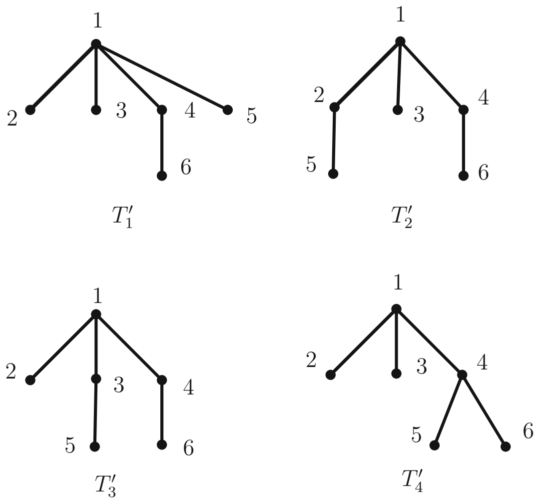


Fig. 6 The graphs T_1', T_2', T_3' and T_4'

If $t_s - 1$ is not adjacent to 1, then we set $P = (x_2, x_3, \ldots, x_{t_s-1}, x_{t_s+1}, \ldots, x_n)$. Since in this case, 1 is not adjacent to $t_s - 1$ and n, and also t_s is not adjacent to n, then it follows that $I_T \subseteq P$, and hence height $I_T \leq n - 2$.

If $t_s - 1$ is adjacent to 1, then it follows that $t_s = 4$ and $t = 2$. Therefore, 4 and 5 are the only neighbors of 2, and $n \geq 6$ is adjacent to 5. Thus, we set $P = (x_2, x_3, x_4, g_{56})$ in the case $n = 6$, and $P = (x_2, \ldots, x_{n-1})$ in the case $n \geq 7$. Thus, it follows in both cases that $I_T \subseteq P$, and hence height $I_T \leq n - 2$.

If such an s does not exist, then it follows that $\deg(t) = 3$ and the only children of t are $t + 1$ and $t + 2$, and $n \in N_T(t + 2)$. Now, we consider the following cases:

(1) Suppose that $\deg(t + 1) \geq 2$. Then, $t + 3 \notin N_T(t + 2)$. We set $P = (x_2, x_3, \ldots, x_{t+2}, x_{t+4}, \ldots, x_n)$. Since 1 is not adjacent to $t + 2$ and since n is a leaf with the neighbor $t + 2$, it follows that $I_T \subseteq P$.
(2) Suppose that $\deg(t + 1) = 1$ and $n \geq t + 4$. Then, we set $P = (x_2, x_3, \ldots, x_{n-1})$. Then, we have $I_T \subseteq P$.
(3) Suppose that $\deg(t + 1) = 1$ and $n = t + 3$. Then, we set $P = (x_2, x_3, \ldots, x_{t+1}, g_{t+2,t+3})$. Then, we have $I_T \subseteq P$.

In all the above cases, P is a prime ideal in S of height $n - 2$, which implies that height $I_T \leq n - 2$, and hence in neither of the cases S/I_T is a complete intersection.

The "in particular" part is now immediate by Proposition 2.4. □

Now, by Theorem 3.1, it remains to look at paths which have a rooted labeling. In the following, we characterize all such paths with the complete intersection property.

Theorem 3.2 *Let T be a path on n vertices with a rooted labeling. Then, S/I_T is a complete intersection if and only if the root of T is either a leaf or the neighbor of a leaf.*

Proof Suppose that S/I_T is a complete intersection. Assume on contrary that the root of T is neither a leaf nor the neighbor of a leaf. Thus, we have $n \geq 5$ and $N_T(1) = \{2, 3\}$, $N_T(2) = \{1, 4\}$ and $N_T(3) = \{1, 5\}$. If $n \neq 6$, then it follows that the prime ideal $P = (x_2, x_3, x_5, x_6, \dots, x_n)$ contains I_T. If $n = 6$, then the prime ideal $P = (x_2, x_3, x_4, x_6)$ contains I_T. Therefore, in any case, height $I_T \leq n-2$, and hence S/I_T is not a complete intersection, a contradiction.

Conversely, first suppose that the root 1 is a leaf of T. Then, $T = L_n$, and hence S/I_T is a complete intersection, by Corollary 2.5.

Next, suppose that the root 1 is the neighbor of a leaf. Note that $N_T(1) = \{2, 3\}$. Then, we distinguish two cases:

Case 1. Assume that 2 is a leaf. If $n = 3$, then it is clear that height $I_T = 2$, and the result is clear. Next, assume that $n \geq 4$. Let $P_1 = (x_1, x_2, x_4, \dots, x_n)$, $P_2 = (x_2, x_3, \dots, x_n)$ and $P_3 = (x_1, x_2, g_{ij} : 3 \leq i < j \leq n)$. It is clear that I_T is contained in P_i for all $i = 1, 2, 3$. We claim that

$$\mathrm{Min}(I_T) = \{I_X, P_1, P_2, P_3\}.$$

Let $Q \in \mathrm{Min}(I_T)$. If Q contains no variables, then by Proposition 2.1, $Q = I_X$. Suppose that Q contains a variable.

First, assume that $x_2 \notin Q$. Since $g_{12} \in I_T \subseteq Q$, it follows that $x_1, x_3 \notin Q$. On the other hand, if $x_4 \in Q$, then $x_2x_3 \in Q$, because $g_{13} \in Q$, which is a contradiction. So, we assume that $i \geq 5$ is the smallest integer such that $x_i \in Q$. Now, it follows from $g_{i-2\, i-1} = x_{i-2}x_i - x_{i-1}^2 \in I_T \subseteq Q$ that $x_{i-1} \in Q$, a contradiction.

Therefore, we assume that $x_2 \in Q$. Then, $x_1x_3 \in Q$, since $g_{12} = x_1x_3 - x_2^2 \in I_T \subseteq Q$. Thus, $x_1 \in Q$ or $x_3 \in Q$. Thus, we consider the following cases:

(i) Suppose that $x_1 \in Q$. Let T' be the induced subgraph of T on the vertex set $[n] \setminus \{1, 2\}$. Then, the Hankel edge ideal $I_{T'}$ is generated in $S' = \mathbb{K}[x_3, \dots, x_{n+1}]$. Thus, it follows from Theorem 2.2 (by a relabeling) that

$$\mathrm{Min}_{S'}(I_{T'}) = \{Q_1 = (g_{ij} : 3 \leq i < j \leq n), Q_2 = (x_4, x_5, \dots, x_n)\}. \quad (2)$$

Since $Q \in \mathrm{Min}(I_T)$, we have $Q/(x_1, x_2) \in \mathrm{Min}(I_T + (x_1, x_2)/(x_1, x_2))$. This implies that

$$Q = Q' + (x_1, x_2), \quad (3)$$

where Q' is generated in S'. Since $I_{T'} + (x_1, x_2) \subseteq Q' + (x_1, x_2)$ and since x_1 and x_2 do not appear in the generators of Q', it follows that $I_{T'} \subseteq Q'$ as ideals of S', and moreover, we have $Q' \in \mathrm{Min}_{S'}(I_{T'})$. Therefore, by (2) we have $Q' = Q_1$ or $Q' = Q_2$, and hence (3) implies that $Q = P_1$ or $Q = P_3$.

(ii) Suppose that $x_3 \in Q$ and $x_1 \notin Q$. Since $g_{ii+1} \in I_T \subseteq Q$ for all $i = 3, \ldots, n-1$, it follows that $x_i \in Q$ for all $i = 3, \ldots, n$. Therefore, $Q = P_2$.

Finally, since neither of I_X, P_1, P_2 and P_3 is contained in the others, the claim is proved.

Case 2. Assume that 2 is not a leaf and 3 is. Therefore, we have $n \geq 4$. Let $P_1 = (x_1, x_2, x_4, \ldots, x_n)$, $P_2 = (x_2, x_3, \ldots, x_n)$, $P_3 = (x_1, x_2, x_3, g_{ij} : 4 \leq i < j \leq n)$ and $P_4 = (x_1, x_2, x_3, x_5, \ldots, x_n)$. It is clear that I_T is contained in P_i for all $i = 1, 2, 3, 4$. We claim that if $n \neq 5$, then

$$\mathrm{Min}(I_T) = \{I_X, P_1, P_2, P_3, P_4\},$$

and if $n = 5$, then

$$\mathrm{Min}(I_T) = \{I_X, P_1, P_2, P_3\}.$$

Note that if $n = 4$, then $P_3 = P_4$. Let $Q \in \mathrm{Min}(I_T)$.

First, suppose that $x_2 \notin Q$. Since $g_{12} \in I_T \subseteq Q$, it follows that $x_1 \notin Q$ and $x_3 \notin Q$. We also have $x_4 \notin Q$. Indeed, if $x_4 \in Q$, then we have $x_2x_3 \in Q$, since $g_{13} \in I_T \subseteq Q$, which is a contradiction. Now, let i be the smallest integer such that $x_i \in Q$. Assume that $i = 5$. Then, since $g_{24} \in Q$ and $x_3 \notin Q$, it follows that $x_4 \in Q$, a contradiction to the choice of i. Now, assume that $i \geq 6$. Then, since $g_{i-2\, i-1} \in Q$, we deduce that $x_{i-1} \in Q$, a contradiction to the choice of i. Therefore, if $x_2 \notin Q$, then no other variable belongs to Q. Thus, by Proposition 2.1, $Q = I_X$.

Next, suppose that $x_2 \in Q$. Since $g_{12} \in Q$, it follows that $x_1x_3 \in Q$. Now, we consider the following cases:

(i) Suppose that $x_1 \in Q$ and $x_3 \notin Q$. Since x_2 and g_{24} belong to Q, we have $x_3x_4 \in Q$, and hence $x_4 \in Q$. Therefore, since $g_{ii+1} \in Q$ for any $i = 4, \ldots, n-1$, we have $x_i \in Q$ for any $i = 4, \ldots, n$. Hence, in this case, we deduce that $Q = P_1$.

(ii) Suppose that $x_1 \notin Q$ and $x_3 \in Q$. Since x_2 and g_{13} belong to Q, we have $x_1x_4 \in Q$, and hence $x_4 \in Q$. Therefore, similar to the previous case, we have $x_i \in Q$ for any $i = 4, \ldots, n$. Hence, in this case, we deduce that $Q = P_2$.

(iii) Suppose that $x_1 \in Q$ and $x_3 \in Q$. Let T' be the induced subgraph of T on the vertex set $[n] \setminus \{1, 2, 3\}$. Then, the Hankel edge ideal $I_{T'}$ is generated in $S' = \mathbb{K}[x_4, \ldots, x_{n+1}]$. Therefore, we have $Q = Q' + (x_1, x_2, x_3)$, where Q' is a prime ideal generated in S', and similar to Case 1, we have $Q' \in \mathrm{Min}_{S'}(I_{T'})$. If $n = 4$, then $Q' = (0)$, and hence $Q = P_3 = P_4$. If $n = 5$, then $Q' = (g_{45})$, and hence $Q = P_3$. If $n \geq 6$, then by Theorem 2.2, we have

$$Q' \in \{(x_5, \ldots, x_n), (g_{ij} : 4 \leq i < j \leq n)\},$$

and hence $Q = P_3$ or $Q = P_4$, which completes the proof. □

Combining Theorems 3.1 and 3.2, we get the following characterization of rooted labeled trees with complete intersection Hankel edge ideal.

Corollary 3.3 *Let T be a tree with a rooted labeling. Then, S/I_T is a complete intersection if and only if T is a path whose root is either a leaf or the neighbor of a leaf.*

Let I be a monomial ideal in S. Recall that S/I is a complete intersection if the monomial generators in the unique minimal monomial generating set of I are relatively prime.

Let $<$ be the *reverse lexicographic* order on S induced by $x_1 > \cdots > x_n > x_{n+1}$. By [3, Theorem 1], we know that, not only S/I_{L_n}, but also $S/\operatorname{in}_<(I_{L_n})$ is a complete intersection, since $\operatorname{in}_<(I_{L_n}) = (x_{i+1}^2 : 1 \le i \le n-1)$.

Let T_1 and T_2 be the rooted labeled paths on $[n]$, different from L_n, considered in Corollary 3.3. Indeed, let T_1 be the rooted labeled path on $n \ge 3$ vertices with the root 1 in which the vertex 2 is a leaf, and let T_2 be the rooted labeled path on $n \ge 4$ vertices with the root 1 in which the vertex 3 is a leaf. Now, according to Corollary 3.3, it is natural to ask the same question about $S/\operatorname{in}_<(I_{T_1})$ and $S/\operatorname{in}_<(I_{T_2})$. First, in the following proposition, we determine the initial ideals of I_{T_1} and I_{T_2}.

Proposition 3.4 *Let T_1 and T_2 be as above. Then,*

$$\operatorname{in}_<(I_{T_1}) = (x_2x_3, x_1x_3^2, x_2^2, x_{i+1}^2 : 3 \le i \le n-1),$$

and

$$\operatorname{in}_<(I_{T_2}) = (x_2x_3, x_3x_4, x_1x_3^2, x_1x_4^2, x_2^2, x_{i+1}^2 : 4 \le i \le n-1).$$

Proof We prove the statement by applying Buchberger's criterion, see for example [13, Theorem 2.3.2]. Let $f = x_1x_2x_4 - x_1x_3^2$. We first consider T_1. We claim that the set of binomials

$$\mathcal{G}_1 = \{f, g_{12}, g_{13}, g_{ii+1} : i = 3, \ldots, n-1\}$$

is a Gröbner basis for I_{T_1} with respect to $<$. We have that $\operatorname{in}_<(f) = x_1x_3^2$, $\operatorname{in}_<(g_{12}) = x_2^2$, $\operatorname{in}_<(g_{13}) = x_2x_3$ and $\operatorname{in}_<(g_{ii+1}) = x_{i+1}^2$ for $i = 3, \ldots, n-1$. By [13, Lemma 2.3.1], we only need to compute $S(g_{12}, g_{13})$ and $S(g_{13}, f)$. It is easily seen that $S(g_{12}, g_{13}) = f$ and $S(g_{13}, f) = -(x_1x_4)g_{12}$ which both reduce to zero with respect to $\mathcal{G}_1$. Therefore, the claim follows, and hence $\operatorname{in}_<(I_{T_1}) = (x_2x_3, x_1x_3^2, x_2^2, x_{i+1}^2 : 3 \le i \le n-1)$, as desired.

Next, we consider T_2. Let $h = x_1x_3x_5 - x_1x_4^2$. We claim that the set of binomials

$$\mathcal{G}_2 = \{f, h, g_{12}, g_{13}, g_{24}, g_{ii+1} : i = 4, \ldots, n-1\}$$

is a Gröbner basis for I_{T_2} with respect to $<$. Note that $\operatorname{in}_<(h) = x_1x_4^2$, $\operatorname{in}_<(g_{24}) = x_3x_4$. Using [13, Lemma 2.3.1] and according to our computations for T_1, we just need to compute the following four S-polynomials:

$$S(f, h) = (x_3x_5)f + (x_2x_4)h, \qquad S(g_{13}, g_{24}) = -x_5g_{12} + h,$$

$$S(g_{24}, f) = -x_2 h, \qquad S(g_{24}, h) = -x_5 f.$$

All the above S-polynomials clearly reduce to zero with respect to $\mathcal{G}_2$. Thus, the claim follows, and hence $\mathrm{in}_<(I_{T_2}) = (x_2x_3, x_3x_4, x_1x_3^2, x_1x_4^2, x_2^2, x_{i+1}^2 : 4 \leq i \leq n-1)$ which completes the proof. □

The following corollary is now immediate.

Corollary 3.5 *Let T_1 and T_2 be as above. Then, $S/\mathrm{in}_<(I_{T_1})$ and $S/\mathrm{in}_<(I_{T_2})$ are not complete intersection.*

We would like to close this section by posing some natural questions. In this section, we studied the trees T with rooted labeling for which S/J_T is a complete intersection. It is natural to ask if there is another vertex labeling for a tree T for which S/J_T is a complete intersection. On the other hand, a characterization of trees with rooted labeling whose Hankel edge ideal is almost complete intersection or set-theoretic complete intersection would be of interest.

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