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Radius of Convexity for Analytic Part of Sense-Preserving Harmonic Mappings

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Abstract
Given a sense-preserving harmonic function $f = h + \bar{g}$ defined in the open unit disk, the radius of convexity for the analytic part h is determined under various prescribed conditions on the associated analytic function $\phi_f = h - g$. Moreover, the radius of starlikeness and convexity for the analytic part of harmonic Koebe function is also computed. All the obtained results are sharp.

Keywords Radius of convexity · Univalent harmonic functions · Sense-preserving · Dilatation · Function with positive real part · Starlikeness

Mathematics Subject Classification 30C45 · 30C55 · 31A05

1 Introduction

Let $\mathcal{H}$ denote the class of all complex-valued harmonic functions of the form $f = h+\bar{g}$ defined in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ where h and g are analytic functions in $\mathbb{D}$ (called analytic and co-analytic parts of f, respectively) and normalized by the conditions $h(0) = h'(0)-1 = g(0) = 0$. Since the Jacobian of $f = h+\bar{g} \in \mathcal{H}$ is given by $J_f(z) = |h'(z)|^2-|g'(z)|^2$, by a theorem of Lewy [10], f is sense-preserving in $\mathbb{D}$ if and only if $|g'(z)| < |h'(z)|$ for all $z \in \mathbb{D}$, or equivalently, the dilatation

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$w_f : \mathbb{D} \to \mathbb{C}$ defined by $w_f(z) = g'(z)/h'(z)$ is an analytic function satisfying $|w_f(z)| < 1$ for all $z \in \mathbb{D}$. In 1984, Clunie and Sheil-Small [5] initiated the study of the class $\mathcal{S}_H \subset \mathcal{H}$ consisting of sense-preserving univalent harmonic mappings. Although many classical results of analytic univalent functions have been extended for the class $\mathcal{S}_H$ and its geometric subclasses, there are still several conjectures regarding the coefficient bounds and radius problems which are still unsettled. One of them is the harmonic analogue of the Bieberbach Conjecture which was proposed by Clunie and Sheil-Small [5] stating that the nth Taylor series coefficients of the analytic and co-analytic part of a function $f \in \mathcal{S}_H$ are bounded by $(2n^2 + 1)/3$. Moreover, the exact radius of convexity of the class $\mathcal{S}_H$ is still unknown, the proposed radius being $3 - \sqrt{8}$, given by Sheil-Small [19].

There has been an interplay between the sense-preserving harmonic mappings and their analytic part. If a harmonic function $f = h + \bar{g} \in \mathcal{S}_H$ and $f(\mathbb{D})$ is a convex domain, then the analytic part h must be univalent in $\mathbb{D}$ by [5, Theorem 5.7, p. 15]. Similarly, if $f = h+\bar{g} \in \mathcal{H}$ is sense-preserving and $h(\mathbb{D})$ is a convex domain, then $f \in \mathcal{S}_H$ by [5, Theorem 5.17, p. 20]. Bshouty and Lyzzaik [3] proved that a sense-preserving harmonic function $f = h+\bar{g} \in \mathcal{H}$ is necessarily univalent in $\mathbb{D}$ if the dilatation of f is $w_f(z) = z$ and the analytic part h satisfies $\mathrm{Re}(1+zh''(z)/h'(z)) > -1/2$ for all $z \in \mathbb{D}$. It is also worth to note that the univalence of a harmonic mapping does not imply the univalence of its analytic part. For example, the harmonic Koebe function $K = H+\overline{G}$ obtained by shearing of the analytic Koebe function $k(z) = z/(1-z)^2$ in the direction of the real axis with dilatation z is univalent in $\mathbb{D}$ by [5, Theorem 5.3, p. 14], but the analytic function H is not univalent in $\mathbb{D}$ as $H(i\sqrt{3}/\sqrt{5}) = H(-i\sqrt{3}/\sqrt{5})$, where H and G are given by

$$H(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} \quad \text{and} \quad G(z) = \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}. \tag{1}$$

Similarly, the shearing of the analytic Koebe function k in the direction of real axis with dilatation z^2 generates the univalent harmonic mapping $W = U + \overline{V}$, where U and V are given by

$$U(z) = \frac{z - z^2 + \frac{1}{3}z^3}{(1-z)^3} \quad \text{and} \quad V(z) = \frac{\frac{1}{3}z^3}{(1-z)^3}$$

and $U((\sqrt{3}/2)e^{\frac{i\pi}{6}}) = U((\sqrt{3}/2)e^{-\frac{i\pi}{6}})$. In 1990, Sheil-Small [19] conjectured that if a harmonic function $f = h + \bar{g} \in \mathcal{S}_H$, then the radius of univalence for the analytic part h is $1/\sqrt{3}$. For more details and problems in harmonic mappings, one may refer to [2–8, 14, 15, 19].

Let $\mathcal{H}^0$ be a subfamily of $\mathcal{H}$ consisting of harmonic functions $f = h + \bar{g}$ which are further normalized by $g'(0) = 0$. This paper discusses the problem of finding the radius of convexity for the analytic part of sense-preserving harmonic functions in $\mathcal{H}^0$.

The class $\mathcal{S}_H^0 := \mathcal{S}_H \cap \mathcal{H}^0$ is a compact normal family with respect to the topology of locally uniform convergence. The classical family $\mathcal{S}$ of normalized analytic univalent functions is a subfamily of $\mathcal{S}_H^0$ which includes the classes $\mathcal{S}^*$ and $\mathcal{K}$ consisting of starlike and convex functions, respectively. The harmonic Koebe function $K = H + \overline{G}$ given by (1) belongs to the class $\mathcal{S}_H^0$. Since it is expected to play the extremal role for the class $\mathcal{S}_H^0$, the radius of starlikeness and convexity has been computed for the analytic part H of the harmonic Koebe function K in Sect. 2. These radii constants may be conjectured as the radius of starlikeness and convexity for the analytic part of harmonic functions in the class $\mathcal{S}_H^0$.

Given a sense-preserving harmonic function $f = h + \bar{g} \in \mathcal{H}^0$ and the dilatation w_f of f, the function $p_f : \mathbb{D} \to \mathbb{C}$ defined by

$$p_f(z) = \frac{1 + w_f(z)}{1 - w_f(z)} \quad (z \in \mathbb{D}) \tag{2}$$

is analytic in $\mathbb{D}$ with $p_f(0) = 1$ and $\operatorname{Re} p_f(z) > 0$ for all $z \in \mathbb{D}$. Set $\phi_f := h - g$. Then, it is easy to see that

$$h'(z) = \frac{1}{2}\phi_f'(z)(1 + p_f(z)), \quad z \in \mathbb{D}. \tag{3}$$

There are several sufficient conditions on the analytic function ϕ_f under which the function h is univalent, starlike or convex in $\mathbb{D}$. Let us consider three such instances. Firstly, if ϕ_f is convex in $\mathbb{D}$, then (3) gives

$$\operatorname{Re}\left(\frac{h'(z)}{\phi_f'(z)}\right) = \frac{1}{2}\operatorname{Re}(1 + p_f(z)) > 0 \quad (z \in \mathbb{D})$$

and therefore h is close-to-convex and hence univalent in $\mathbb{D}$ by [6, Theorem 2.17, p. 47]. Secondly, if ϕ_f is of the form

$$\phi_f(z) = \int_0^z \frac{h(\xi)}{\xi} d\xi, \quad z \in \mathbb{D}$$

then h is starlike in $\mathbb{D}$ as in this case, $z\phi_f'(z) = h(z)$ and (3) yields

$$\operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right) = \frac{1}{2}\operatorname{Re}(1 + p_f(z)) > 0 \quad (z \in \mathbb{D}).$$

Thirdly, if $\phi_f(z) = zh'(z)$, then (3) leads to

$$\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) = \operatorname{Re}\left(\frac{2}{1 + p_f(z)}\right) > 0 \quad (z \in \mathbb{D})$$

which shows that h is convex in $\mathbb{D}$. Under similar conditions imposed on the analytic function ϕ_f, Sect. 3 investigates the radius of convexity for the analytic part of the sense-preserving harmonic function $f \in \mathcal{H}^0$. We shall make use of the following lemma which is a special case of [9, Theorem 3, p. 314].

Lemma 1 [1, Lemma 2.4, p. 4] *Suppose that the function p is analytic in $\mathbb{D}$. If $p(0) = 1$ and* $\operatorname{Re} p(z) > 0$ *for all $z \in \mathbb{D}$, then*

$$\operatorname{Re}\left(\frac{zp'(z)}{1+p(z)}\right) \geq \begin{cases} -\dfrac{|z|}{1+|z|} & \text{if } |z| < 1/3; \\ -\dfrac{(\sqrt{2}-\sqrt{1-|z|^2})^2}{1-|z|^2} & \text{if } 1/3 \leq |z| < 1. \end{cases}$$

2 Analytic Part of Harmonic Koebe Function

In this section, we will determine the radius of convexity and starlikeness for the analytic part H of the harmonic Koebe function K given by (1). Note that

$$\begin{aligned}\operatorname{Re}\left(1+\frac{zH''(z)}{H'(z)}\right) &= \operatorname{Re}\left(\frac{1+5z+2z^2}{1-z^2}\right) \\ &= -2+4\operatorname{Re}\frac{1}{1-z} - \operatorname{Re}\frac{1}{1+z} \\ &> -2+\frac{4}{1+|z|} - \frac{1}{1-|z|} \\ &= \frac{1-5|z|+2|z|^2}{1-|z|^2} > 0\end{aligned}$$

for $|z| < (5-\sqrt{17})/4 \approx 0.219224$. Also, as $1 + zH''/H'$ vanishes at $z = (-5 + \sqrt{17})/4$, it follows that $(5-\sqrt{17})/4$ is the radius of convexity for H.

To determine the radius of starlikeness for H, observe that

$$\operatorname{Re}\frac{zH'(z)}{H(z)} = 6\operatorname{Re}\frac{1+z}{(1-z)(6-3z+z^2)}.$$

If $z = re^{i\theta}$, then a straightforward calculation yields

$$\frac{1}{6}|1-z|^2|6-3z+z^2|^2 \operatorname{Re}\frac{zH'(z)}{H(z)} = p(r,u)$$

where $u = \cos\theta$ and

$$p(r,u) = 6 - 13r^2 + r^4 - 3ru + 7r^3u + 8r^2u^2 - 2r^4u^2 - 4r^3u^3.$$

The problem now reduces to find the value of the parameter r for which the polynomial $p(r, u)$ is non-negative in the whole interval $-1 \leq u \leq 1$. It is easily seen that

$$p(r,1) = (1-r)(6+3r-2r^2+r^3) > 0$$

and

$$p(r,-1) = (1-r)(6+9r+4r^2+r^3) > 0.$$

Also, further analysis shows that

$$\frac{\partial}{\partial u}p(r,u) = r(-3+7r^2+16ru-4r^3u-12r^2u^2),$$

and hence $p(r,u)$ has a local minimum at $u_0 = (4-r^2-\sqrt{7+13r^2+r^4})/6r$ and a local maximum at $u_0^* = (4-r^2+\sqrt{7+13r^2+r^4})/6r$. Thus, $p(r,u) \geq 0$ for $-1 \leq u \leq 1$ if and only if

$$p(r,u_0) = \frac{1}{54}(344-519r^2+15r^4-2r^6-(14+26r^2+2r^4)\sqrt{7+13r^2+r^4}) \geq 0.$$

This inequality implies that $r \leq r_0 \approx 0.691985$, where r_0 is the smallest positive root of the equation $8r^8+41r^6+1280r^4-2131r^2+722=0$. Therefore, H is starlike for $|z| < r_0$.

3 Radius of Convexity

Let $f = h+\bar{g} \in \mathcal{H}^0$ be a sense-preserving harmonic function and $\phi_f = h-g$. Using the condition (3), it is easy to see that

$$1+\frac{zh''(z)}{h'(z)} = 1+\frac{z\phi_f''(z)}{\phi_f'(z)}+\frac{zp_f'(z)}{1+p_f(z)}, \tag{4}$$

where p_f is defined by (2). The inequality (4) will be used to determine the radius of convexity of the analytic function h throughout this section. The first theorem discusses the case when ϕ_f is univalent in $\mathbb{D}$. Let us recall the concept of subordination. Given two analytic functions f and g in $\mathbb{D}$, we say that f is subordinate to g, written as $f \prec g$, if there exists a Schwartz function τ that is analytic in $\mathbb{D}$ with $\tau(0)=0$ and $|\tau(z)|<1$ satisfying $f(z) = g(\tau(z))$ for all $z \in \mathbb{D}$.

Theorem 2 *Let $f = h+\bar{g} \in \mathcal{H}^0$ be sense-preserving in $\mathbb{D}$ and $\phi_f = h-g$.*

(a) *If $\phi_f \in \mathcal{S}$, then h is convex for $|z| < (5-\sqrt{17})/4 \approx 0.219224$.*
(b) *If $\phi_f \in \mathcal{S}$ and the analytic function p_f given by (2) satisfies $p_f(z) \prec 1+z$ in $\mathbb{D}$, then h is convex for $|z| < (5-\sqrt{13})/6 \approx 0.232408$.*
(c) *If $\operatorname{Re}\phi_f'(z) > 0$ in $\mathbb{D}$, then h is convex for $|z| < 1/3$.*

All the bounds are sharp.

Proof (a) If $|z| < 1/3$, then by making use of Lemma 1 in (4), it follows that

$$\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right) \geq \operatorname{Re}\left(1+\frac{z\phi_f''(z)}{\phi_f'(z)}\right) - \frac{|z|}{1+|z|}. \tag{5}$$

Since $\phi_f \in \mathcal{S}$, therefore [6, Theorem 4.2, p. 32] gives

$$\operatorname{Re}\left(1+\frac{z\phi_f''(z)}{\phi_f'(z)}\right) \geq \frac{|z|^2-4|z|+1}{1-|z|^2}. \tag{6}$$

so that (5) becomes

$$\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right) \geq \frac{|z|^2-4|z|+1}{1-|z|^2} - \frac{|z|}{1+|z|} = \frac{2|z|^2-5|z|+1}{1-|z|^2} > 0$$

provided $|z| < (5-\sqrt{17})/4 < 1/3$. The result is sharp for the harmonic Koebe function defined by (1).

(b) As $p_f(z) \prec 1+z$, there exists an analytic function $\tau : \mathbb{D} \to \mathbb{D}$ with $\tau(0) = 0$ such that $p_f(z) = 1+\tau(z)$ for all $z \in \mathbb{D}$. Thus, (4) simplifies to

$$1+\frac{zh''(z)}{h'(z)} = 1+\frac{z\phi_f''(z)}{\phi_f'(z)} + \frac{z\tau'(z)}{2+\tau(z)}.$$

By Dieudonne's Lemma [6, p. 198], the function τ satisfies $|\tau'(z)| \leq 1$ for $|z| \leq \sqrt{2}-1$. Also, $|\tau(z)| \leq |z|$ for all $z \in \mathbb{D}$. These observations together with (6) lead to

$$\begin{aligned}\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right) &\geq \frac{|z|^2-4|z|+1}{1-|z|^2} - \frac{|z||\tau'(z)|}{|2+\tau(z)|} \\ &\geq \frac{|z|^2-4|z|+1}{1-|z|^2} - \frac{|z|}{2-|z|} \\ &= \frac{2(1-5|z|+3|z|^2)}{(2-|z|)(1-|z|^2)} > 0\end{aligned}$$

if $|z| < (5-\sqrt{13})/6 < \sqrt{2}-1$. This shows that h is convex for $|z| < (5-\sqrt{13})/6$. In order to show that the bound is sharp, consider the function $f_0 = h_0 + \bar{g}_0$ where

$$h_0(z) = \frac{\frac{1}{2}z+z^2}{(1-z)^2} - \frac{1}{2}\log(1-z) \quad \text{and} \quad g_0(z) = \frac{-\frac{1}{2}z+z^2}{(1-z)^2} - \frac{1}{2}\log(1-z).$$

The dilatation of f_0 is $w_{f_0}(z) = z/(2+z)$ which satisfies $|w_{f_0}(z)| < 1$ for all $z \in \mathbb{D}$. Therefore, f_0 is sense-preserving in $\mathbb{D}$. Also, $h_0(z) - g_0(z) = z/(1-z)^2 \in \mathcal{S}$ and $p_{f_0}(z) = 1+z$. Since the quantity

$$1+\frac{zh_0''(z)}{h_0'(z)} = \frac{2+10z+6z^2}{2+z-2z^2-z^3}$$

vanishes at $z = (-5+\sqrt{13})/6$, therefore the radius is sharp.

(c) As $\operatorname{Re}\phi'_f(z) > 0$ for all $z \in \mathbb{D}$, therefore [11, Theorem 2, p. 533] gives

$$\operatorname{Re}\left(1+\frac{z\phi''_f(z)}{\phi'_f(z)}\right) \geq \frac{1-2|z|-|z|^2}{1-|z|^2}. \tag{7}$$

Using this estimate in (5), we have

$$\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right) \geq \frac{1-2|z|-|z|^2}{1-|z|^2} - \frac{|z|}{1+|z|} = \frac{1-3|z|}{1-|z|^2} > 0$$

as $|z| < 1/3$. For sharpness, we consider the harmonic function $f_0 = h_0 + \bar{g}_0$ where h_0 and g_0 are given by

$$h_0(z) = \frac{2z}{1-z} + \log(1-z) \quad \text{and} \quad g_0(z) = \frac{3z-z^2}{1-z} + 3\log(1-z). \tag{8}$$

Clearly f_0 is sense-preserving in $\mathbb{D}$ as $w_{f_0}(z) = z$ and $\phi_{f_0}(z) = h_0(z) - g_0(z) = -z - 2\log(1-z)$ satisfies $\operatorname{Re}\phi'_{f_0}(z) > 0$ for all $z \in \mathbb{D}$. Also

$$1 + \frac{zh''_0(z)}{h'_0(z)} = \frac{1+3z}{1-z^2} = 0$$

at $z = -1/3$. □

Let us give an application of Theorem 2. If $f = h + \bar{g} \in \mathcal{H}^0$ is sense-preserving, $\phi_f = h - g$ and $\psi \in \mathcal{K}$ such that

$$\left|\frac{z\phi'_f(z)}{\psi(z)} - 1\right| < 1$$

for all $z \in \mathbb{D}$, then ϕ_f satisfies the inequality (7) by [13, Theorem 4, p. 524]. Consequently, the proof of Theorem 2(c) shows that h is convex for $|z| < 1/3$. This bound is sharp for the function $f_0 = h_0 + \bar{g}_0$ given by (8), as $|z\phi'_{f_0}(z)/\psi(z) - 1| = |z| < 1$ for all $z \in \mathbb{D}$, where $\psi(z) = z/(1-z) \in \mathcal{K}$.

The next theorem computes the radius of convexity for the analytic part when ϕ_f satisfies $\operatorname{Re}(\phi'_f/\psi') > 0$ in $\mathbb{D}$ for some function $\psi \in \mathcal{S}$ satisfying certain conditions. The class $\mathcal{K}(\alpha)$, $0 \leq \alpha < 1$, consisting of analytic functions f satisfying $\operatorname{Re}(1 + zf''(z)/f'(z)) > \alpha$ for $z \in \mathbb{D}$, is a subclass of $\mathcal{K}$.

Theorem 3 *Let $f = h + \bar{g} \in \mathcal{H}^0$ be sense-preserving in $\mathbb{D}$ and $\phi_f = h - g$. Further suppose that $\psi(z) = z + a_2z^2 + \cdots$ is analytic in $\mathbb{D}$ and satisfies*

$$\operatorname{Re}\frac{\phi'_f(z)}{\psi'(z)} > 0$$

for all $z \in \mathbb{D}$.

(a) *If* $\psi \in \mathcal{S}$, *then* h *is convex for* $|z| < (7-\sqrt{41})/4 \approx 0.149219$.
(b) *If* $\operatorname{Re}\psi'(z) > 0$ *for all* $z \in \mathbb{D}$, *then* h *is convex for* $|z| < 1/5$.
(c) *If* $\psi \in \mathcal{K}(\alpha)$ $(0 \leq \alpha < 1)$, *then* h *is convex for* $|z| < r_\alpha$, *where*

$$r_\alpha = \frac{5-2\alpha-\sqrt{4\alpha^2-12\alpha+17}}{4(1-\alpha)}.$$

All the radii are sharp.

Proof (a) Since $\psi \in \mathcal{S}$ and $\operatorname{Re}(\phi_f'/\psi') > 0$ in $\mathbb{D}$, by [16, Theorem 1, p. 32], ϕ_f satisfies

$$\operatorname{Re}\left(1+\frac{z\phi_f''(z)}{\phi_f'(z)}\right) \geq \frac{1-6|z|+|z|^2}{1-|z|^2}.$$

In view of the above inequality, (5) takes the form:

$$\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right) \geq \frac{1-6|z|+|z|^2}{1-|z|^2} - \frac{|z|}{1+|z|} = \frac{1-7|z|+2|z|^2}{1-|z|^2} > 0$$

provided $|z| < (7-\sqrt{41})/4 < 1/3$. The harmonic function $f_1 = h_1 + \bar{g}_1$, where h_1 and g_1 are given by

$$h_1(z) = \frac{z-\frac{1}{2}z^2+\frac{2}{3}z^3-\frac{1}{6}z^4}{(1-z)^4} \quad \text{and} \quad g_1(z) = \frac{\frac{1}{2}z^2+\frac{1}{3}z^3+\frac{1}{6}z^4}{(1-z)^4}$$

shows that the result is best possible. In fact, the dilatation of f_1 is $w_{f_1}(z) = z$ and

$$\phi_{f_1}(z) = h_1(z) - g_1(z) = \frac{z+\frac{1}{3}z^3}{(1-z)^3}$$

satisfies $\operatorname{Re}(\phi_{f_1}'/\psi') > 0$ in $\mathbb{D}$, where $\psi(z) = z/(1-z)^2 \in \mathcal{S}$. Moreover,

$$1+\frac{zh_1''(z)}{h_1'(z)} = \frac{1+7z+2z^2}{1-z^2} = 0$$

at $z = (-7+\sqrt{41})/4$.

(b) Under the given hypothesis, [16, Theorem 2, p. 32] gives

$$\operatorname{Re}\left(1+\frac{z\phi_f''(z)}{\phi_f'(z)}\right) \geq \frac{1-4|z|-|z|^2}{1-|z|^2}$$

and hence from (5), it follows that if $|z| < 1/3$, then

$$\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right) \geq \frac{1-4|z|-|z|^2}{1-|z|^2} - \frac{|z|}{1+|z|} = \frac{1-5|z|}{1-|z|^2} > 0$$

for $|z| < 1/5$. For sharpness, note that the function $f_2 = h_2 + \bar{g}_2$ defined as:

$$h_2(z) = \frac{2z^2}{(1-z)^2} - \log(1-z) \quad \text{and} \quad g_2(z) = \frac{-5z + 8z^2 - z^3}{(1-z)^2} - 5\log(1-z)$$

is sense-preserving in $\mathbb{D}$ as $w_{f_2}(z) = z$. The function

$$\phi_{f_2}(z) = h_2(z) - g_2(z) = \frac{5z - z^2}{1-z} + 4\log(1-z)$$

satisfies $\mathrm{Re}(\phi'_{f_2}/\psi') > 0$ in $\mathbb{D}$, where $\psi(z) = -z - 2\log(1-z) \in \mathcal{S}$ with $\mathrm{Re}\,\psi'(z) > 0$ for all $z \in \mathbb{D}$. For $z = -1/5$, the expression $1 + zh_2''(z)/h_2'(z) = (1+5z)/(1-z^2)$ equals 0.

(c) Under the hypothesis of the theorem, by [16, Theorem 6, p. 35], the function ϕ_f satisfies

$$\mathrm{Re}\left(1 + \frac{z\phi_f''(z)}{\phi_f'(z)}\right) \geq \frac{1 - (4-2\alpha)|z| - (2\alpha-1)|z|^2}{1-|z|^2}.$$

Consequently, for $|z| < 1/3$, (5) gives

$$\begin{aligned}\mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) &\geq \frac{1 - (4-2\alpha)|z| - (2\alpha-1)|z|^2}{1-|z|^2} - \frac{|z|}{1+|z|} \\ &= \frac{1 - (5-2\alpha)|z| + 2(1-\alpha)|z|^2}{1-|z|^2} > 0\end{aligned}$$

for $|z| < r_\alpha := (5 - 2\alpha - \sqrt{4\alpha^2 - 12\alpha + 17})/(4(1-\alpha))$. Note that $r_\alpha < 1/3$. The sharpness of the result is achieved by taking the harmonic function $f_\alpha = h_\alpha + \bar{g}_\alpha$ where h_α and g_α are given by

$$h_\alpha(z) = \begin{cases} \dfrac{(2\alpha-1) - (1-z)^{2\alpha-3}((2\alpha-1) + (2\alpha-3)z)}{2(1-\alpha)(3-2\alpha)}, & \alpha \neq 1/2; \\ \dfrac{z}{(1-z)^2}, & \alpha = 1/2 \end{cases}$$

and

$$g_\alpha(z) = \begin{cases} \dfrac{-(1+2\alpha) + (1-z)^{2\alpha-3}((1+2\alpha) + (3-2\alpha)(2(1-\alpha)z^2 - (1+2\alpha)z))}{2(1-\alpha)(3-2\alpha)(1-2\alpha)}, & \alpha \neq 1/2; \\ \dfrac{z(2z-1)}{(1-z)^2} - \log(1-z), & \alpha = 1/2. \end{cases}$$

The dilatation of f_α is $w_{f_\alpha}(z) = z$ and the function

$$\phi_{f_\alpha}(z) = h_\alpha(z) - g_\alpha(z) = \begin{cases} \dfrac{(1-z)^{2(\alpha-1)}((1-\alpha)z-\alpha)+\alpha}{(1-\alpha)(1-2\alpha)}, & \alpha \neq 1/2; \\ \dfrac{2z}{1-z} + \log(1-z), & \alpha = 1/2 \end{cases}$$

satisfies $\operatorname{Re}(\phi'_{f_\alpha}/\psi') > 0$ in $\mathbb{D}$ where

$$\psi(z) = \begin{cases} \dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1}, & \alpha \neq 1/2; \\ -\log(1-z), & \alpha = 1/2 \end{cases}$$

belongs to the class $\mathcal{K}(\alpha)$. Also

$$1 + \frac{zh''_\alpha(z)}{h'_\alpha(z)} = \frac{1+(5-2\alpha)z+2(1-\alpha)z^2}{1-z^2}$$

vanishes at the point $z = -r_\alpha$. □

The following theorem discusses the case when ϕ_f satisfies either $\operatorname{Re}(\phi_f(z)/z) > 0$ or $|z\phi'_f(z)/\phi_f(z) - 1| < 1$ in $\mathbb{D}$.

Theorem 4 *Let $f = h + \bar{g} \in \mathcal{H}^0$ be sense-preserving in $\mathbb{D}$ and $\phi_f = h - g$.*

(a) *If the function ϕ_f satisfies*

$$\operatorname{Re}\left(\frac{\phi_f(z)}{z}\right) > 0,$$

for all $z \in \mathbb{D}$, then h is convex for $|z| < \sqrt{10} - 3 \approx 0.162278$.

(b) *If the function ϕ_f satisfies*

$$\left|\frac{z\phi'_f(z)}{\phi_f(z)} - 1\right| < 1,$$

then h is convex for $|z| < r_0$ where $r_0 \approx 0.311108$ is the smallest positive root of the equation $r^3 - r^2 - 3r + 1 = 0$ in $(0, 1)$.

Both the radii are sharp.

Proof For part (a), [18, Theorem, p. 2] gives

$$\operatorname{Re}\left(1 + \frac{z\phi''_f(z)}{\phi'_f(z)}\right) \geq \frac{1-5|z|-3|z|^2-|z|^3}{(1+|z|)(1-2|z|-|z|^2)}, \quad |z| < \sqrt{2} - 1.$$

For $|z| < 1/3$, the above inequality and (5) holds so that

$$\operatorname{Re}\left(1+\frac{zh''(z)}{h(z)}\right) \geq \frac{1-5|z|-3|z|^2-|z|^3}{(1+|z|)(1-2|z|-|z|^2)} - \frac{|z|}{1+|z|} = \frac{|z|^2+6|z|-1}{|z|^3+3|z|^2+|z|-1} > 0$$

if $|z| < \sqrt{10}-3$. For sharpness, let us take the harmonic function $f_0 = h_0 + \bar{g}_0$ where h_0 and g_0 are given by

$$h_0(z) = \frac{2z-z^2}{(1-z)^2} + \log(1-z) \quad \text{and} \quad g_0(z) = \frac{z-z^2+z^3}{(1-z)^2} + \log(1-z).$$

Note that f_0 is sense-preserving as the dilatation $w_{f_0}(z) = z$ and the function $\phi_{f_0}(z) = h_0(z) - g_0(z) = z(1+z)/(1-z)$ satisfies $\operatorname{Re}(\phi_{f_0}(z)/z) > 0$ for all $z \in \mathbb{D}$. Also, at $z = 3-\sqrt{10}$,

$$1 + \frac{zh_0''(z)}{h_0'(z)} = \frac{-z^2+6z+1}{z^3-3z^2+z+1} = 0.$$

For part (b), by invoking the result of [20, Theorem, p. 230], we have

$$\operatorname{Re}\left(1+\frac{z\phi_f''(z)}{\phi_f'(z)}\right) \geq \frac{1-3|z|+|z|^2}{1-|z|}.$$

By using (5), it follows that

$$\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right) \geq \frac{1-3|z|+|z|^2}{1-|z|} - \frac{|z|}{1+|z|} = \frac{1-3|z|-|z|^2+|z|^3}{1-|z|^2} > 0$$

provided $|z| < r_0$ where r_0 is the smallest positive root of the equation $1-3r-r^2+r^3 = 0$. If we consider the harmonic function $F_0 = H_0 + \bar{G}_0$ with

$$H_0(z) - G_0(z) = \phi_{F_0}(z) = ze^z \quad \text{and} \quad G_0'(z) = zH_0'(z),$$

then ϕ_{F_0} satisfies $|z\phi_{F_0}'(z)/\phi_{F_0}(z) - 1| < 1$ for all $z \in \mathbb{D}$ and

$$1 + \frac{zH_0''(z)}{H_0'(z)} = \frac{1+3z-z^2-z^3}{1-z^2}$$

vanishes at $z = -r_0$. □

In the last theorem of this section, the radius of convexity of the analytic part of the harmonic function f is determined when the associated function ϕ_f satisfies $|\phi_f'/\psi' - 1| < 1$ in $\mathbb{D}$ for a function ψ belonging to the class $\mathcal{S}$ or its subclasses.

Theorem 5 *Let $f = h + \bar{g} \in \mathcal{H}^0$ be sense-preserving in $\mathbb{D}$. Suppose that $\phi_f = h - g$ and $\psi(z) = z + a_2 z^2 + \cdots$ is analytic in $\mathbb{D}$ and satisfies*

$$\left| \frac{\phi'_f(z)}{\psi'(z)} - 1 \right| < 1$$

for all $z \in \mathbb{D}$.

(a) *If $\psi \in \mathcal{S}$, then h is convex for $|z| < 3 - 2\sqrt{2} \approx 0.171573$.*
(b) *If $\psi \in \mathcal{K}$, then h is convex for $|z| < 2 - \sqrt{3} \approx 0.267949$.*
(c) *If $\operatorname{Re} \psi'(z) > 0$ for all $z \in \mathbb{D}$, then h is convex for $|z| < \sqrt{5} - 2 \approx 0.236068$.*

All the bounds are best possible.

Proof (a) By [17, Theorem 1, p. 484], we have

$$\operatorname{Re}\left(1 + \frac{z\phi''_f(z)}{\phi'_f(z)}\right) \geq \frac{1 - 5|z|}{1 - |z|^2}$$

so that (5) gives

$$\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) \geq \frac{1 - 5|z|}{1 - |z|^2} - \frac{|z|}{1 + |z|} = \frac{1 - 6|z| + |z|^2}{1 - |z|^2} > 0$$

for $|z| < 3 - 2\sqrt{2} < 1/3$. The harmonic function $f_1 = h_1 + \bar{g}_1$ given by

$$h_1(z) = \frac{z + \frac{1}{3}z^3}{(1 - z)^3} \quad \text{and} \quad g_1(z) = \frac{z - 2z^2 + \frac{7}{3}z^3}{(1 - z)^3} + \log(1 - z)$$

verifies that the bound is best possible. The dilatation of f_1 is $w_{f_1}(z) = z$ and the function

$$\phi_{f_1}(z) = \frac{2z^2}{(1 - z)^2} - \log(1 - z)$$

satisfies the hypothesis of the theorem, with $\psi(z) = z/(1 - z)^2 \in \mathcal{S}$. Also, $1 + zh''_1(z)/h'_1(z) = (1 + 6z + z^2)/(1 - z^2) = 0$ at $z = -3 + 2\sqrt{2}$.

(b) Under the given hypothesis, the function ϕ_f satisfies

$$\operatorname{Re}\left(1 + \frac{z\phi''_f(z)}{\phi'_f(z)}\right) \geq \frac{1 - 3|z|}{1 - |z|^2} \tag{9}$$

by [17, Theorem 3, p. 486]. The result now follows by making use of (5) which yields

$$\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) \geq \frac{1 - 3|z|}{1 - |z|^2} - \frac{|z|}{1 + |z|} = \frac{1 - 4|z| + |z|^2}{1 - |z|^2} > 0$$

for $|z| < 2 - \sqrt{3}$. If we consider the harmonic function $f_2 = h_2 + \bar{g}_2$ where h_2 and g_2 are given by

$$h_2(z) = \frac{z}{(1-z)^2} \quad \text{and} \quad g_2(z) = \frac{-z + 2z^2}{(1-z)^2} - \log(1-z) \tag{10}$$

then f_2 is sense-preserving in $\mathbb{D}$ with dilatation $w_{f_2}(z) = z$ and

$$\phi_{f_2}(z) = h_2(z) - g_2(z) = \frac{2z}{1-z} + \log(1-z).$$

Also, $|\phi'_{f_2}(z)/\psi'(z) - 1| = |z| < 1$ where $\psi(z) = z/(1-z) \in \mathcal{K}$ and the analytic part of f_2 satisfies

$$1 + \frac{zh''_2(z)}{h'_2(z)} = \frac{1 + 4z + z^2}{1 - z^2}$$

which assumes the value 0 at $z = -2 + \sqrt{3}$.

(c) Under the given hypothesis, ϕ_f satisfies the inequality

$$\operatorname{Re}\left(1 + \frac{z\phi''_f(z)}{\phi'_f(z)}\right) \geq \frac{1 - 3|z| - 2|z|^2}{1 - |z|^2}$$

by [17, Theorem 4, p. 486]. Hence, by (5), we have

$$\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) \geq \frac{1 - 3|z| - 2|z|^2}{1 - |z|^2} - \frac{|z|}{1 + |z|} = \frac{1 - 4|z| - |z|^2}{1 - |z|^2} > 0$$

for $|z| < \sqrt{5} - 2 < 1/3$. To verify the sharpness, consider the harmonic function $f_3 = h_3 + \bar{g}_3$ given by

$$h_3(z) = \frac{5z - z^2}{1 - z} + 4\log(1-z) \quad \text{and} \quad g_3(z) = \frac{8z - \frac{7}{2}z^2 - \frac{1}{2}z^3}{1 - z} + 8\log(1-z).$$

As $w_{f_3}(z) = z$, f_3 is sense-preserving in $\mathbb{D}$ and the function

$$\phi_{f_3}(z) = h_3(z) - g_3(z) = -3z - \frac{z^2}{2} - 4\log(1-z)$$

satisfies $|\phi'_{f_3}(z)/\psi'(z) - 1| < 1$ with $\psi(z) = -z - 2\log(1-z)$. Also, $\operatorname{Re}\psi' > 0$ in $\mathbb{D}$ and

$$1 + \frac{zh''_3(z)}{h'_3(z)}\bigg|_{z=-2+\sqrt{5}} = \frac{1 + 4z - z^2}{1 - z^2}\bigg|_{z=-2+\sqrt{5}} = 0.$$

This completes the proof of the theorem. □

As an application, if $f = h + \bar{g} \in \mathcal{H}^0$ is sense-preserving with $\phi_f = h - g$, $\psi \in \mathcal{K}$ and

$$\operatorname{Re}\left(\frac{z\phi'_f(z)}{\psi(z)}\right) > 0$$

for all $z \in \mathbb{D}$, then ϕ_f satisfies the condition (9) by [12, Theorem 4, p. 518] so that h is convex for $|z| < 2 - \sqrt{3}$ (see the proof of Theorem 5(b)). The bound $2 - \sqrt{3}$ is sharp for the function $f_1 = h_1 + \bar{g}_1$ defined by (10) with $\psi(z) = z/(1 - z) \in \mathcal{K}$.

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