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Regular Dynamics for 3D Brinkman–Forchheimer Equations with Delays

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Abstract

The aim of this paper is to study the regular dynamics for the 3D delay Brinkman–Forchheimer (BF) equations. We first prove the existence, uniqueness and time-dependent property of regular tempered pullback attractors as well as the existence of invariant measures for the 3D BF equations with non-autonomous abstract delay. We then study the asymptotic autonomy of regular pullback attractors for the 3D BF equations with autonomous abstract delay. Finally, we discuss the upper semicontinuity of regular pullback attractors as the delay time approaches to zero for the 3D BF equations with variable delay and distributed delay.

Keywords Delay Brinkman–Forchheimer equation · Pullback attractor · Regularity · Stability · Invariant measure

Mathematics Subject Classification 35B40 · 35B41 · 37L30 · 37L40

1 Introduction

The attractors play an important role in the study of long-term behaviour for evolution equations, see [4, 6, 17, 30, 31] and the references therein. As we know, the pullback attractor is a collection of a family of time-dependent compact sets. Hence, a natural problem is to consider the time-dependent property of pullback attractors. Although this topic has been studied by several authors in [7–9, 18, 19, 21–23, 38], there is no paper on this subject in the regular space. On the other hand, the long-term behaviour of delay PDEs has wide attention [1–3, 11, 16, 24, 34, 40] due to it being able to

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control the system by the history of solutions. In [24, 40], the authors investigated the time-dependent property of pullback attractors in the initial space for delay PDEs. As far as we know, this topic in the regular space for the delay BF equation has not been studied.

The BF equation depicts the fluid flow in a saturated porous medium, see [12, 29]. Recently, the long-term behaviour of BF equations has been studied by several authors in [13, 14, 32, 33, 36, 39, 42, 43] for the non-delay case and in [15, 20, 25, 37] for the delay case. In [15], the authors proved the existence of uniform attractors for the BF equation with autonomous delay. Li et al. [20] studied the existence and non-delay stability of pullback attractors for the BF equation with autonomous delay. In [37], the authors investigated the structure and asymptotic stability of pullback attractors for the BF equation with non-autonomous delay. As far as we know, the existence, time-dependent property, asymptotic autonomy and non-delay stability of regular pullback attractors for the BF equation with delays have not been studied.

We first consider the following BF equation with non-autonomous delay:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + \alpha u + \nabla p = -\beta |u| u - \gamma |u|^2 u + f(t, u_t) + g(t, x), \\ \nabla \cdot u = 0, \ t > t_0, \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega, \\ u(t_0 + \theta, x) := \phi(\theta, x), \ \theta \in [-\varrho, 0], \end{cases} \tag{1}$$

where $t_0 \in \mathbb{R}$, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $u = (u_1, u_2, u_3)$ is the velocity of the fluid, and p is the pressure of the fluid. $\nu > 0$ denotes the Brinkman effective viscosity, $\alpha > 0$ denotes the Darcy coefficient, and $\beta > 0$ and $\gamma > 0$ denote the Forchheimer coefficients. $\varrho > 0$ is the delay time of Eq. (1), g is the given non-autonomous forcing term, and the delay term f will be specified later. For each $t \geq t_0$, $u_t(\cdot)$ is a delay-shift function defined on $[-\varrho, 0]$ given by $u_t(\theta) = u(t + \theta)$ with $\theta \in [-\varrho, 0]$.

Recently, the authors in [7, 21, 23, 38] studied that the backward compactness of pullback attractors $\mathcal{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ ($\bigcup_{s \leq t} \mathcal{A}(s)$ is pre-compact with $t \in \mathbb{R}$) when the basin of attraction of pullback attractors be composed of all fixed bounded sets in the initial space. However, we consider two different universes to show that the backward compactness of pullback attractors in this paper. One universe is made up of all tempered sets, and the other is composed of all backward tempered sets. It is worth pointing out that the backward compactness of pullback attractors on tempered universe is not easily proved. Fortunately, we can prove the attractors on two different universes are identical. On the other hand, we apply the spectrum decomposition technique to prove the backward asymptotic compactness of solutions due to the solution of Eq. (1) has no higher regularity, and then obtain a unique backward compact regular tempered pullback attractor for Eq. (1).

In [10], the authors show that the invariant measure plays an important role in understanding turbulence. Hence, the invariant measure of evolution equations has been studied by several authors in [5, 26–28, 35]. Inspire by [28], we establish the existence of a unique family of invariant Borel probability measures for Eq. (1), which are supported by the tempered pullback attractor.

We then discuss the following BF equation with autonomous delay:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + \alpha u + \nabla p = -\beta |u| u - \gamma |u|^2 u + f(u_t, x) + g(t, x), \\ \nabla \cdot u = 0, \ t > t_0, \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega, \\ u(t_0 + \theta, x) := \phi(\theta, x), \ \theta \in [-\varrho, 0]. \end{cases} \tag{2}$$

In [8, 9, 18, 19, 22], the authors studied the forward asymptotic autonomy of pullback attractors in the initial space for PDEs without delay, that is,

$$\lim_{t \to +\infty} dist_X(\mathcal{A}(t), \mathcal{A}_\infty) = 0,$$

where X is a Banach space with norm $\| \cdot \|_X$, $dist_X(\cdot, \cdot)$ denotes the Hausdorff semi-distance and $\mathcal{A}_\infty$ is a global attractor of the autonomous equation corresponding to (2). As far as we know, the asymptotic autonomy of pullback attractors in the regular space (the regularity of the initial space is low) has not been considered. In this paper, we study the backward asymptotically autonomous dynamics for Eq. (2) in the regular space:

$$\lim_{t \to -\infty} dist_{Y_\varrho}(\mathcal{A}(t), \mathcal{A}_\infty) = 0,$$

where Y is a Banach space with norm $\| \cdot \|_Y$, $X \hookleftarrow Y$ and $Y_\varrho = C([-\varrho, 0]; Y)$.

Finally, we study the following BF equation with variable delay and distributed delay:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + \alpha u + \nabla p + \beta |u| u + \gamma |u|^2 u \\ = f_1(u(t - \rho^\varrho(t)), x) + \int_{-\varrho}^{0} f_2(u(t + \theta), \theta) d\theta + g(t, x), \\ \nabla \cdot u = 0, \ t > t_0, \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega, \\ u(t_0 + \theta, x) := \phi(\theta, x), \ \theta \in [-\varrho, 0]. \end{cases} \tag{3}$$

Zhao et al. [41] studied the upper semicontinuity of the global attractor in the initial space as the delay time tends to zero for retarded lattice systems. Wang et al. [34] proved the upper semicontinuity of pullback random attractors as the memory time approaches zero in the initial space for stochastic reaction–diffusion equation. In this paper, we study the upper semicontinuity of tempered pullback attractors as the delay time tends to zero in the regular space for Eq. (3):

$$\lim_{\varrho \to 0} dist_{(Y_\varrho, Y)}(\mathcal{A}_\varrho(t), \mathcal{A}(t)) = 0, \quad t \in \mathbb{R},$$

where

$$dist_{(Y_\varrho, Y)}(A, B) = \sup_{a \in A} \inf_{b \in B} \sup_{\theta \in [-\varrho, 0]} \|a(\theta) - b\|_Y = 0.$$

In the next section, we establish the existence of an evolution process for the retarded 3D BF equation. Section 3 is devoted to the backward uniform estimates, continuity with respect to initial time and backward flattening of solutions, and then obtain the existence, regularity and backward compactness of pullback attractors as well as the existence of invariant measures. In Sect. 4, we prove the backward regularly asymptotic autonomy of pullback attractors by the convergence of systems from non-autonomous to autonomous and the backward compactness of pullback attractors. The delay-free stability of regular pullback attractors is established by the convergence of solutions from delay to non-delay, the eventually compactness of pullback attractors and the recurrence of absorbing sets in the last section.

2 Existence of an Evolution Process for the 3D BF Equation with Delays

Suppose X is a Banach space with norm $\|\cdot\|_X$. A family of maps $\{S(t,r): t\geq r\}$ is called an evolution process if

$$S(r,r)=id_X,\quad S(t,s)=S(t,r)S(r,s),\ \ S(t,r): X\to X \text{ is continuous},$$

for all $t\geq r\geq s$. In this section, we mainly establish the existence of an evolution process for Eq. (1) and make some assumptions. Denote by

$$\begin{aligned}\mathbb{L}^p(\Omega)&=L^p(\Omega)\times L^p(\Omega)\times L^p(\Omega),\\ \mathbb{H}^p(\Omega)&=H^p(\Omega)\times H^p(\Omega)\times H^p(\Omega),\ \ p>0,\end{aligned}$$

and

$$\begin{aligned}H&=\{u\in\mathbb{L}^2(\Omega):\nabla\cdot u=0, u\cdot\mathbf{n}|_{\partial\Omega}=0\},\\ V&=\{u\in\mathbb{H}^1(\Omega):\nabla\cdot u=0, u|_{\partial\Omega}=0\},\end{aligned}$$

where $\mathbf{n}$ is the unit outward normal vector at $\partial\Omega$. It is easy to see that H and V are separable Hilbert spaces with the inner products and norms given by

$$(u,v)=\sum_{i=1}^{3}\int_\Omega u_iv_idx,\quad \|u\|^2=(u,u),\ \ u,v\in H,$$

and

$$\begin{aligned}((u,v))=\sum_{i=1}^{3}\int_\Omega \nabla u_i\cdot\nabla v_idx,\\ \|u\|_V^2=((u,u)),\ \ u,v\in V.\end{aligned}$$

Notice that $V \hookrightarrow H \equiv H' \hookrightarrow V'$ is dense and continuous. Moreover, we use $\|\cdot\|_p$ to denote the norm in $\mathbb{L}^p(\Omega)$ and $\langle\cdot,\cdot\rangle$ represents the duality pairing between V and V'. Let $\tilde{P}$ be the Leray orthogonal projection from $\mathbb{L}^2(\Omega)$ onto H. Applying $\tilde{P}$ to Eq. (1), we obtain

$$\begin{cases} \frac{\partial u}{\partial t} + \nu Au = \tilde{P}(-\alpha u - \beta|u|u - \gamma|u|^2u) + \tilde{P}f(t,u_t) + \tilde{P}g(t,x),\ t > t_0, \\ u(t_0+\theta,x) := \phi(\theta,x),\ \theta \in [-\varrho,0],\ x \in \Omega, \end{cases} \tag{4}$$

where $A = -\tilde{P}\Delta$ is defined by $\langle Au, v\rangle = ((u,v))$. Let $H_\varrho := C([-\varrho,0],H)$ with $\|\vartheta\|_{H_\varrho} = \sup_{\theta\in[-\varrho,0]}\|\vartheta(\theta)\|$ and $V_\varrho := C([-\varrho,0],V)$ with $\|\vartheta\|_{V_\varrho} = \sup_{\theta\in[-\varrho,0]}\|\vartheta(\theta)\|_V$. The delay forcing $f : \mathbb{R}\times H_\varrho \to H$ satisfies

(F1) For each $\varphi \in H_\varrho$, $r \to f(r,\varphi)$ is measurable from $\mathbb{R}$ into H;
(F2) $f(r,0) = 0$ for all $r \in \mathbb{R}$;
(F3) There is a positive function $L_f(\cdot)$ such that for all $\varphi, \psi \in H_\varrho$

$$\|f(r,\varphi) - f(r,\psi)\| \le L_f(r)\|\varphi-\psi\|_{H_\varrho}, \tag{5}$$

where $L_f(\cdot)$ satisfies, for all $t \in \mathbb{R}$,

$$\int_{-\infty}^{t} L_f^2(r)dr < +\infty, \quad \lim_{\beta\to+\infty}\sup_{s\le t}\int_{-\infty}^{s} e^{\beta(r-s)}L_f^2(r)dr = 0. \tag{6}$$

By the standard Galerkin method as in [11, 33], one can see that the well-posedness of (4).

Lemma 1 *Suppose that* (F1)–(F3) *hold and* $g \in L^2_{loc}(\mathbb{R},H)$. *Then for all* $t_0 \in \mathbb{R}$ *and* $\phi \in H_\varrho$, *Eq.* (4) *has a unique weak solution*

$$u \in C([t_0-\varrho,+\infty);H) \cap L^2_{loc}(t_0,+\infty;V) \cap L^4_{loc}(t_0,+\infty;\mathbb{L}^4(\Omega)),$$

such that $u(t_0+\theta;t_0,\phi) = \phi(\theta)$ *for all* $\theta \in [-\varrho,0]$. *Furthermore, for any* $\varepsilon > 0$ *and* $T > t_0+\varepsilon$

$$u \in C([t_0+\varepsilon,T];V) \cap L^2(t_0+\varepsilon,T;D(A)),$$

where $D(A) = H^1_0(\Omega)\cap H^2(\Omega)$.

We define a mapping $S(t,t_0) : H_\varrho \to H_\varrho$ by

$$S(t,t_0)\phi = u_t(\cdot;t_0,\phi),\ t \ge t_0,\ \phi \in H_\varrho, \tag{7}$$

where u is the solution of (4). Based on Lemma 1, we obtain $S(\cdot,\cdot)$ is an evolution process. In addition, we also need the following assumptions:

(G1) $g \in L^2_{loc}(\mathbb{R}, H)$ is *backward limitable*:

$$\lim_{\beta\to+\infty}\sup_{s\le t}\int_{-\infty}^{s} e^{\beta(r-s)}\|g(r)\|^2dr = 0,\ \forall t \in \mathbb{R}, \tag{8}$$

which implies $g \in L^2_{loc}(\mathbb{R}, H)$ is *backward tempered*:

$$\sup_{s\le t}\int_{-\infty}^{s} e^{\beta(r-s)}\|g(r)\|^2dr < +\infty,\ \forall t \in \mathbb{R}, \beta > 0. \tag{9}$$

On the other hand, we will frequently use the following inequalities:

$$(|x|^{p-2}x - |y|^{p-2}y)(x - y) \ge 2^{2-p}|x - y|^p, \tag{10}$$

where $x, y \in \mathbb{R}^n$ and $p \ge 2$.

Gagliardo–Nirenberg inequality: assume $\Omega \subset \mathbb{R}^n$, if $0 \le j < l, 1 \le q, r \le +\infty$, $p \in \mathbb{R}$, $\frac{j}{l} < \eta \le 1$ and

$$\frac{1}{p} - \frac{j}{n} = \eta\left(\frac{1}{r} - \frac{l}{n}\right) + (1-\eta)\frac{1}{q}.$$

Then,

$$\|D^j u(t)\|_p \le c\|u(t)\|_q^{1-\eta}\|D^l u(t)\|_r^{\eta},\ \forall u \in W^{l,r}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n). \tag{11}$$

where j, l be any integers and c depending only on n, l, j, q, r, η.

Let $\mathcal{P}(H_\varrho)$ denotes the collection of all nonempty subsets in H_ϱ. We provide two different universes of attraction. One is $\mathfrak{D}$ given by, for all $\beta > 0$

$$\mathfrak{D} = \{\mathcal{D} = \{\mathcal{D}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(H_\varrho) : \lim_{\tau\to+\infty} e^{-\beta\tau}\|\mathcal{D}(t-\tau)\|^2_{H_\varrho} = 0\}. \tag{12}$$

Observe that $\mathfrak{D}$ is inclusion-closed. Another is $\mathfrak{B}$ defined by, for all $\beta > 0$

$$\mathfrak{B} = \{\mathcal{B} = \{\mathcal{B}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(H_\varrho) : \lim_{\tau\to+\infty} e^{-\beta\tau}\sup_{s\le t}\|\mathcal{B}(s-\tau)\|^2_{H_\varrho} = 0\}. \tag{13}$$

Then, $\mathfrak{B}$ is *backward-closed*: $\tilde{\mathcal{B}} \in \mathfrak{B}$ whenever $\mathcal{B} \in \mathfrak{B}$ and $\tilde{\mathcal{B}}(\tau, \omega) = \cup_{s\le t}\mathcal{B}(s, \omega)$. Notice that $\mathfrak{B} \subset \mathfrak{D}$ and $\mathfrak{B}$ is also inclusion-closed.

3 Dynamics of the 3D Non-Autonomous BF Equations with Delays

Throughout this paper, we denote by c a positive constant, which may change from line to line or even in the same line.

3.1 Backward Uniform Estimates of Solutions

Lemma 2 *Suppose that* (F1)–(F3) *and* (G1) *hold. Let* $t \in \mathbb{R}$*, then we have the following conclusions:*

(i) For each $\mathcal{D} \in \mathfrak{D}$*, there exists a* $\tau_d := \tau_d(t, \mathcal{D}) > 0$ *such that*

$$\sup_{\sigma \in [t-\varrho-4,t]} \|u_\sigma(\cdot; t-\tau, \phi)\|_{H_\varrho}^2 \le e^{2\alpha(\varrho+2)} \left(1 + \frac{2}{\alpha} G_d(t)\right) e^{\frac{2}{\alpha} e^{\alpha\varrho} L(t)}, \tag{14}$$

for all $\tau \ge \tau_d$ *and* $\phi \in \mathcal{D}(t-\tau)$*, where*

$$G_d(t) = \int_{-\infty}^{t} e^{\alpha(r-t)} \|g(r)\|^2 dr, \quad L(t) := \int_{-\infty}^{t} L_f^2(r) dr. \tag{15}$$

(ii) For each $\mathcal{B} \in \mathfrak{B}$*, there exists a* $\tau_b := \tau_b(t, \mathcal{B}) > 0$ *such that*

$$\sup_{s \le t} \sup_{\sigma \in [s-\varrho-4,s]} \|u_\sigma(\cdot; s-\tau, \phi)\|_{H_\varrho}^2 \le e^{2\alpha(\varrho+2)} \left(1 + \frac{2}{\alpha} G_b(t)\right) e^{\frac{2}{\alpha} e^{\alpha\varrho} L(t)}, \tag{16}$$

for all $\tau \ge \tau_d$ *and* $\phi \in \mathcal{B}(s-\tau)$ *with* $s \le t$*, where*

$$G_b(t) = \sup_{s \le t} G_d(s). \tag{17}$$

Proof Taking the inner product of (4) with $u(r; s-\tau, \phi)$ in H, by the Young inequality, we have

$$\begin{aligned} &\frac{d}{dr}\|u(r)\|^2 + 2\nu\|A^{\frac{1}{2}}u(r)\|^2 + \alpha\|u(r)\|^2 + 2\beta\|u(r)\|_3^3 + 2\gamma\|u(r)\|_4^4 \\ &\quad \le \frac{2}{\alpha}\|f(r, u_r)\|^2 + \frac{2}{\alpha}\|g(r)\|^2, \end{aligned} \tag{18}$$

which implies

$$\frac{d}{dr} e^{\alpha r}\|u(r)\|^2 \le \frac{2}{\alpha} e^{\alpha r}\|f(r, u_r)\|^2 + \frac{2}{\alpha} e^{\alpha r}\|g(r)\|^2. \tag{19}$$

Integrating (19) on $[s-\tau, \sigma+\theta]$ with $\theta \in [-\varrho, 0]$ and $\tau > s-\sigma+\varrho$, by (5) we obtain

$$\begin{aligned} &e^{\alpha\sigma}\|u_\sigma(\cdot; s-\tau, \phi)\|^2 \\ &\quad \le e^{\alpha\varrho} e^{\alpha(s-\tau)}\|\phi\|_{H_\varrho}^2 + \frac{2}{\alpha} e^{\alpha\varrho} \int_{s-\tau}^{\sigma} e^{\alpha r}(L_f^2(r)\|u_r\|_{H_\varrho}^2 + \|g(r)\|^2) dr. \end{aligned} \tag{20}$$

Using the Gronwall inequality (see [4, p. 167]) to (20), we obtain

$$e^{\alpha\sigma}\|u_\sigma(\cdot; s-\tau,\phi)\|^2 \leq (e^{\alpha\varrho}e^{\alpha(s-\tau)}\|\phi\|^2_{H_\varrho}+\frac{2}{\alpha}e^{\alpha\varrho}\int\limits_{s-\tau}^{\sigma}e^{\alpha r}\|g(r)\|^2dr)e^{\frac{2}{\alpha}e^{\alpha\varrho}\int\limits_{s-\tau}^{\sigma}L_f^2(r)dr}.$$

Hence we obtain, for all $\tau \geq 2\varrho + 4$,

$$\begin{aligned}&\sup_{\sigma\in[s-\varrho-4,s]}\|u_\sigma(\cdot; s-\tau,\phi)\|^2\\ &\leq \sup_{\sigma\in[s-\varrho-4,s]} e^{-\alpha\sigma}(e^{\alpha\varrho}e^{\alpha(s-\tau)}\|\phi\|^2_{H_\varrho}+\frac{2}{\alpha}e^{\alpha\varrho}\int\limits_{s-\tau}^{\sigma}e^{\alpha r}\|g(r)\|^2dr)e^{\frac{2}{\alpha}e^{\alpha\varrho}\int\limits_{s-\tau}^{\sigma}L_f^2(r)dr}\\ &\leq e^{\alpha(2\varrho+4)}(e^{-\alpha\tau}\|\phi\|^2_{H_\varrho}+\frac{2}{\alpha}\int\limits_{s-\tau}^{s}e^{\alpha(r-s)}\|g(r)\|^2dr)e^{\frac{2}{\alpha}e^{\alpha\varrho}\int\limits_{s-\tau}^{s}L_f^2(r)dr}. \end{aligned} \tag{21}$$

(i) Let $s = t$ in (21). If $\phi \in \mathcal{D}(t-\tau)$, by (12), there exists a $\tau_d := \tau_d(t,\mathcal{D}) \geq 2\varrho+4$ such that for all $\tau \geq \tau_d$,

$$e^{-\alpha\tau}\|\phi\|^2_{H_\varrho} \leq e^{-\alpha\tau}\|\mathcal{D}(t-\tau)\|^2_{H_\varrho} \leq 1.$$

Hence, we obtain (14) holds.

(ii) If $\phi \in \mathcal{B}(s-\tau)$ with $s \leq t$, by (13), there is a $\tau_b := \tau_b(t,\mathcal{B}) \geq 2\varrho + 4$ such that for all $\tau \geq \tau_b$,

$$e^{-\alpha\tau}\sup_{s\leq t}\|\phi\|^2_{H_\varrho} \leq e^{-\alpha\tau}\sup_{s\leq t}\|\mathcal{B}(s-\tau)\|^2_{H_\varrho} \leq 1.$$

Taking the supremum of (21) over the past time $s \leq t$ yields (16) holds.

□

Corollary 1 *Under the assumptions in Lemma 2, we have the following auxiliary estimate:*

$$\sup_{s\leq t}\int\limits_{s-\varrho-3}^{s}(\|A^{\frac{1}{2}}u(r)\|^2+\|u(r)\|_3^3+\|u(r)\|_4^4)dr \leq c(1+L(t))(1+G_b(t))e^{cL(t)}. \tag{22}$$

Proof Integrating (18) on $[s-\varrho-3, s]$, by (5), we obtain

$$\int\limits_{s-\varrho-3}^{s} 2\nu\|A^{\frac{1}{2}}u(r)\|^2+2\beta\|u\|_3^3+2\gamma\|u\|_4^4dr$$

$$\leq \|u(s-\varrho-3)\|^2 + \frac{2}{\alpha}\int_{s-\varrho-3}^{s} L_f^2(r)\|u_r\|_{H_\varrho}^2 dr + \frac{2}{\alpha}e^{\alpha(\varrho+3)}\int_{s-\varrho-3}^{s} e^{\alpha(r-s)}\|g(r)\|^2 dr.$$

By (16), we have

$$\begin{aligned}
&\sup_{s\leq t}\|u(s-\varrho-3)\|^2 + \frac{2}{\alpha}\sup_{s\leq t}\int_{s-\varrho-3}^{s} L_f^2(r)\|u_r\|_{H_\varrho}^2 dr \\
&\leq \sup_{s\leq t}\sup_{\sigma\in[s-3,s]}\|u_\sigma(\cdot)\|_{H_\varrho}^2 + \sup_{s\leq t}\sup_{\sigma\in[s-\varrho-3,s]}\|u_\sigma(\cdot)\|_{H_\varrho}^2\int_{s-\varrho-3}^{s} L_f^2(r)dr \\
&\leq c(1+L(t))(1+G_b(t))e^{cL(t)}. \qquad (23)
\end{aligned}$$

Then, we obtain

$$\sup_{s\leq t}\int_{s-\varrho-3}^{s}(\|A^{\frac{1}{2}}u(r)\|^2 + \|u(r)\|_3^3 + \|u(r)\|_4^4)dr \leq c(1+L(t))(1+G_b(t))e^{cL(t)}.$$

The proof is complete. □

Lemma 3 *Suppose that* (F1)–(F3) *and* (G1) *hold. For each $t\in\mathbb{R}$ and $\mathcal{B}\in\mathfrak{B}$, we have, for all $\tau\geq\tau_b$ (τ_b is given in Lemma 2) and $\phi\in\mathcal{B}(s-\tau)$ with $s\leq t$,*

$$\begin{aligned}
&\sup_{s\leq t}\sup_{\theta\in[-\varrho-2,0]}\|A^{\frac{1}{2}}u(s+\theta;s-\tau,\phi)\|^2 + \sup_{s\leq t}\int_{s-\varrho}^{s}\left\|\frac{\partial u}{\partial r}\right\|^2 dr \\
&\leq c(1+L(t))(1+G_b(t))e^{cL(t)}. \qquad (24)
\end{aligned}$$

Proof Multiplying (4) by $\frac{\partial}{\partial r}u(r;s-\tau,\phi)$, by (9) and the Young inequality, we have

$$\begin{aligned}
&\left\|\frac{\partial u(r)}{\partial r}\right\|^2 + \frac{d}{dr}(\nu\left\|A^{\frac{1}{2}}u(r)\right\|^2 + \alpha\|u(r)\|^2 + \beta\|u(r)\|_3^3 + \gamma\|u(r)\|_4^4) \\
&\leq cL_f^2(r)\|u_r\|_{H_\varrho}^2 + c\|g(r)\|^2. \qquad (25)
\end{aligned}$$

Integrating (25) on $[\zeta, s+\theta]$ with $\zeta\in[s+\theta-1, s+\theta]$ and $\theta\in[-\varrho-2,0]$, and then integrating the result on $[s+\theta-1,s+\theta]$ w.r.t. ζ we obtain, for all $\theta\in[-\varrho-2,0]$,

$$\|A^{\frac{1}{2}}u(s+\theta;s-\tau,\phi)\|^2 + \|u(s+\theta)\|_3^3 + \|u(s+\theta)\|_4^4$$

$$\leq c \int_{s-\varrho-3}^{s} (\|A^{\frac{1}{2}}u(r)\|^2 + \|u(r)\|^2 + \|u(r)\|_3^3 + \|u(r)\|_4^4)dr$$
$$+ c \int_{s-\varrho-3}^{s} (L_f^2(r)\|u_r\|_{H_\varrho}^2 + \|g(r)\|^2)dr. \tag{26}$$

By (16) and (22), we have, for all $\tau \geq \tau_b$,

$$\sup_{s\leq t} \int_{s-\varrho-3}^{s} (\|A^{\frac{1}{2}}u(r)\|^2 + \|u(r)\|_3^3 + \|u(r)\|_4^4)dr \leq c(1+L(t))(1+G_b(t))e^{cL(t)}, \tag{27}$$

$$\sup_{s\leq t} \int_{s-\varrho-3}^{s} \|u(r)\|^2 dr \leq (\varrho+3) \sup_{s\leq t} \sup_{\sigma\in[s-3,s]} \|u_\sigma\|_{H_\varrho}^2 \leq c(1+G_b(t))e^{cL(t)}. \tag{28}$$

Substituting (23), (27) and (28) into (26) yields

$$\sup_{s\leq t} \sup_{\theta\in[-\varrho-2,0]} \|A^{\frac{1}{2}}u(s+\theta; s-\tau, \phi)\|^2 + \sup_{s\leq t} \sup_{\theta\in[-\varrho-2,0]} \|u(s+\theta)\|_3^3$$
$$+ \sup_{s\leq t} \sup_{\theta\in[-\varrho-2,0]} \|u(s+\theta)\|_4^4 \leq c(1+L(t))(1+G_b(t))e^{cL(t)}. \tag{29}$$

On the other hand, integrating (25) on $[s-\varrho, s]$ yields

$$\int_{s-\varrho}^{s} \left\|\frac{\partial u}{\partial r}\right\|^2 dr \leq c\left(\|A^{\frac{1}{2}}u(s-\varrho)\|^2 + \|u(s-\varrho)\|^2 + \|u(s-\varrho)\|_3^3 + \|u(s-\varrho)\|_4^4\right)$$
$$+ c \int_{s-\varrho}^{s} (L_f^2(r)\|u_r\|_{H_\varrho}^2 + \|g(r)\|^2)dr. \tag{30}$$

It follows from (16), (23) and (29) that

$$\sup_{s\leq t} \int_{s-\varrho}^{s} \left\|\frac{\partial u}{\partial r}\right\|^2 dr \leq c(1+L(t))(1+G_b(t))e^{cL(t)}. \tag{31}$$

We infer from (29) and (31) that (24) as desired. □

3.2 Existence and Backward Compactness of Pullback Attractors in the Initial Space

We first establish the existence of pullback absorbing sets for the evolution process $S(\cdot,\cdot)$ in (7).

Lemma 4 *Suppose that* (F1)–(F3) *and* (G1) *hold. Then, we have*

(i) $S(\cdot,\cdot)$ *has a pullback* $\mathfrak{D}$*-absorbing set* $\mathcal{K}_d=\{\mathcal{K}_d(t):t\in\mathbb{R}\}\in\mathfrak{D}$*, defined by*

$$\mathcal{K}_d(t)=\{\vartheta\in H_\varrho:\|\vartheta\|_{H_\varrho}^2\le e^{2\alpha(\varrho+2)}\left(1+\frac{2}{\alpha}G_d(t)\right)e^{\frac{2}{\alpha}e^{\alpha\varrho}L(t)}\}. \tag{32}$$

(ii) $S(\cdot,\cdot)$ *has a pullback* $\mathfrak{B}$*-absorbing set* $\mathcal{K}_b=\{\mathcal{K}_b(t):t\in\mathbb{R}\}\in\mathfrak{B}$*, defined by*

$$\mathcal{K}_b(t)=\{\vartheta\in H_\varrho:\|\vartheta\|_{H_\varrho}^2\le e^{2\alpha(\varrho+2)}\left(1+\frac{2}{\alpha}G_b(t)\right)e^{\frac{2}{\alpha}e^{\alpha\varrho}L(t)}\}=\overline{\cup_{s\le t}\mathcal{K}_d(s)}, \tag{33}$$

where $G_d(t)$*,* $L(t)$ *and* $G_b(t)$ *are given in* (15) *and* (17)*.*

Proof By (14) and (16), it is easy to know that $\mathcal{K}_d$ and $\mathcal{K}_b$ are pullback $\mathfrak{D}$-absorbing set and pullback $\mathfrak{B}$-absorbing set for $S(\cdot,\cdot)$, respectively. Moreover, by $G_b(t)=\sup_{s\le t}G_d(t)$, we obtain $\mathcal{K}_b(t)=\overline{\bigcup_{s\le t}\mathcal{K}_d(s)}$.

We now prove that $\mathcal{K}_d\in\mathfrak{D}$ and $\mathcal{K}_b\in\mathfrak{B}$. Notice that $\mathcal{K}_b(t_1)\subseteq\mathcal{K}_b(t_2)$ if $t_1\le t_2$. Then, by $t\to$, $G_b(t)$ and $t\to L(t)$ are increasing, we obtain

$$\begin{aligned}
e^{-\beta\tau}\sup_{s\le t}\|\mathcal{K}_b(s-\tau)\|_{H_\varrho}^2&=e^{-\beta\tau}\|\mathcal{K}_b(t-\tau)\|_{H_\varrho}^2\\
&=e^{-\beta\tau}e^{2\alpha(\varrho+2)}\left(1+\frac{2}{\alpha}G_b(t-\tau)\right)e^{\frac{2}{\alpha}e^{\alpha\varrho}L(t-\tau)}\\
&\le e^{-\beta\tau}e^{2\alpha(\varrho+2)}\left(1+\frac{2}{\alpha}G_b(t)\right)e^{\frac{2}{\alpha}e^{\alpha\varrho}L(t)}\to 0,\quad\text{as }\tau\to+\infty.
\end{aligned}\tag{34}$$

Hence, we have $\mathcal{K}_b\in\mathfrak{B}$. Since $G_d(t)\le G_b(t)$ for all $t\in\mathbb{R}$, we have $\mathcal{K}_d\in\mathfrak{D}$. The proof is complete. □

Next, we prove the pullback asymptotically compactness of the evolution process $S(\cdot,\cdot)$ in (7).

Lemma 5 *Suppose that* (F1)–(F3) *and* (G1) *hold. Then, we have* $S(\cdot,\cdot)$ *is backward pullback* $\mathfrak{B}$*-asymptotically compact in* H_ϱ*, more precisely, for each* $t\in\mathbb{R}$ *and* $\mathcal{B}\in\mathfrak{B}$*, the sequence* $\{S(s_n,s_n-\tau_n)\phi_n\}$ *is relatively compact in* H_ϱ *whenever* $s_n\le t$*,* $\tau_n\to+\infty$ *and* $\phi_n\in\mathcal{B}(s_n-\tau_n)$*.*

Proof Based on the Ascoli–Arzelà theorem, we divide the proof into two steps.

Step 1. $\{S(s_n,s_n-\tau_n)\phi_n\}_{n\in\mathbb{N}}$ in H_ϱ is equi-continuous from $[-\varrho,0]$ to H.

Since $\tau_n \to +\infty$, we assume $\tau_n \geq \tau_b$ (τ_b is given in Lemma 2) for all $n \in \mathbb{N}$. Let $\theta_1, \theta_2 \in [-\varrho, 0]$ with $\theta_1 < \theta_2$. Due to $s_n \leq t$ for all $n \in \mathbb{N}$, by (24), we obtain

$$
\begin{aligned}
&\|(S(s_n, s_n - \tau_n)\phi_n)(\theta_1) - (S(s_n, s_n - \tau_n)\phi_n)(\theta_2)\| \\
&\quad = \|u(s_n + \theta_1; s_n - \tau_n, \phi_n) - u(s_n + \theta_2; s_n - \tau_n, \phi_n)\| \\
&\quad \leq \int_{s_n+\theta_1}^{s_n+\theta_2} \left\|\frac{\partial u(r)}{\partial r}\right\| dr \leq \left(\int_{s_n-\varrho}^{s_n} \left\|\frac{\partial u(r)}{\partial r}\right\|^2 dr \right)^{\frac{1}{2}} |\theta_1 - \theta_2|^{\frac{1}{2}} \\
&\quad \leq c(1 + L(t))(1 + G_b(t))e^{cL(t)} |\theta_1 - \theta_2|^{\frac{1}{2}}.
\end{aligned}
$$

Then for any $\varepsilon > 0$, there exists a $\delta := \delta(t, \varepsilon)$ such that $\|S(s_n, s_n - \tau_n)\phi_n(\theta_1) - S(s_n, s_n - \tau_n)\phi_n(\theta_2)\| < \varepsilon$ when $|\theta_1 - \theta_2| < \delta$. The proof of the *Step 1* is complete.

Step 2. For each $\theta \in [-\varrho, 0]$, $\{(S(s_n, s_n - \tau_n)\phi_n)(\theta)\}_{n\in\mathbb{N}}$ has a convergent subsequence in H.

By (24), we have $\{(S(s_n, s_n - \tau_n)\phi_n)(\theta)\}_{n\in\mathbb{N}}$ is bounded in V. Then by the embedding $V \hookrightarrow H$ is compact, we complete the proof of the *Step 2*.

Now, all conditions of Ascoli–Arzelà theorem are satisfied. Hence, we obtain the sequence $\{S(s_n, s_n - \tau_n)\phi_n\}$ is relatively compact in H_ϱ. The proof is complete. □

Finally, we state the main result of this subsection:

Theorem 1 *Suppose that* (F1)–(F3) *and* (G1) *hold. Then, we obtain the following conclusions:*

(i) *$S(\cdot, \cdot)$ in (7) has a unique pullback $\mathfrak{D}$-attractor $\mathcal{A}_d = \{\mathcal{A}_d(t) : t \in \mathbb{R}\} \in \mathfrak{D}$, defined by*

$$
\mathcal{A}_d(t) = \bigcap_{T\geq 0} \overline{\bigcup_{\tau \geq T} S(t, t - \tau)\mathcal{K}_d(t - \tau)}^{H_\varrho}. \tag{35}
$$

(ii) *$S(\cdot, \cdot)$ in (7) has a unique pullback $\mathfrak{B}$-attractor $\mathcal{A}_b = \{\mathcal{A}_b(t) : t \in \mathbb{R}\} \in \mathfrak{B}$, defined by*

$$
\mathcal{A}_b(t) = \bigcap_{T\geq 0} \overline{\bigcup_{\tau \geq T} S(t, t - \tau)\mathcal{K}_b(t - \tau)}^{H_\varrho}. \tag{36}
$$

Moreover, $\mathcal{A}_b$ is backward compact in H_ϱ, that is, $\bigcup_{s\leq t} \mathcal{A}_b(s)$ is pre-compact.

(iii) *$\mathcal{A}_d = \mathcal{A}_b$ and so $\mathcal{A}_d$ is backward compact in H_ϱ.*

Proof (i) By the same method as in Lemma 5, we obtain $S(\cdot, \cdot)$ is pullback $\mathfrak{D}$-asymptotically compact, which along with (i) of Lemma 4 implies all conditions of Carvalho et al. [4, Theorem 2.50] are fulfilled. Hence, we have $S(\cdot, \cdot)$ has unique pullback $\mathfrak{D}$-attractor $\mathcal{A}_d$, defined by (35). Since $\mathfrak{D}$ is inclusion-closed and the pullback $\mathfrak{D}$-absorbing set $\mathcal{K}_d$ is closed, we get $\mathcal{A}_d$ is unique and $\mathcal{A}_d \in \mathfrak{D}$.

(ii) Similarly to (i), we only need to prove $\mathcal{A}_b$ is backward compact in H_ϱ. Let $\{u_n\}_{n\in\mathbb{N}} \subset \bigcup_{s\leq t} \mathcal{A}_b(s)$. Then, for each u_n, there is a $s_n \leq t$ such that $u_n \in \mathcal{A}_b(s_n)$. By the invariance of $\mathcal{A}_b$, there exists a $v_n \in \mathcal{A}_b(s_n - \tau_n)$ with $\tau_n \to +\infty$ such that $S(s_n, s_n - \tau_n)v_n = u_n$. Since $\mathcal{A}_b \in \mathfrak{B}$, by Lemma 5, we obtain $\{u_n\}_{n\in\mathbb{N}}$ has a convergent subsequence and so $\cup_{s\leq t}\mathcal{A}_b(s)$ is pre-compact.

(iii) Since $\mathcal{K}_b(t) = \overline{\bigcup_{s\leq t} \mathcal{K}_d(s)}$, we have $\mathcal{K}_b(t) \supset \mathcal{K}_d(t)$ for all $t \in \mathbb{R}$. Then by (35) and (36), we have $\mathcal{A}_b(t) \supset \mathcal{A}_d(t)$ for all $t \in \mathbb{R}$ and thus $\mathcal{A}_b \supset \mathcal{A}_d$. On the other hand, notice that $\mathcal{A}_b \in \mathfrak{D}$ because of $\mathfrak{B} \subset \mathfrak{D}$. Since $\mathcal{A}_b$ is a $\mathfrak{D}$-pullback attracting set, by the invariance of $\mathcal{A}_b$ we find, for all $t \in \mathbb{R}$,

$$dist_{H_\varrho}(\mathcal{A}_b(t), \mathcal{A}_d(t)) = dist_{H_\varrho}(S(t, t-\tau)\mathcal{A}_b(t-\tau), \mathcal{A}_d(t)) \to 0, \quad \text{as } \tau \to +\infty,$$

which implies $\mathcal{A}_b(t) \subset \mathcal{A}_d(t)$ for all $t \in \mathbb{R}$ and so $\mathcal{A}_b \subset \mathcal{A}_d$. Then, we obtain $\mathcal{A}_b = \mathcal{A}_d$ and $\mathcal{A}_d$ is backward compact in H_ϱ.

□

3.3 Existence of Invariant Measures

In the subsection, we consider the existence of invariant measures $\{\mu_t : t \in \mathbb{R}\}$ on the pullback $\mathfrak{D}$-attractor $\mathcal{A}_d$. To this end, we also need prove the evolution process $S(\cdot,\cdot)$ in (7) is t_0-continuous in H_ϱ, where $S(\cdot,\cdot)$ is known as t_0-continuity if for every $\phi \in H_\varrho$ and $t \in \mathbb{R}$, the function $t_0 \to S(t, t_0)\phi$ with values in H_ϱ is continuous and bounded on $(-\infty, t]$. We first prove the following auxiliary lemma.

Lemma 6 *Suppose that* (F1)–(F3) *and* (G1) *hold. For any initial data $\hat{\phi}$ and $\tilde{\phi}$ in H_ϱ, we have*

$$\begin{aligned}\|S(t,t_0)\hat{\phi} - S(t,t_0)\tilde{\phi}\|^2_{H_\varrho} &= \|\hat{u}(t;t_0,\hat{\phi}) - \tilde{u}(t;t_0,\tilde{\phi})\|^2_{H_\varrho} \\ &\leq \|\hat{\phi} - \tilde{\phi}\|^2_{H_\varrho} e^{c\int_{t_0}^{t} L_f^2(r)dr}.\end{aligned} \tag{37}$$

Proof Let $\bar{u} = \hat{u} - \tilde{u}$. Then, $\bar{u}$ satisfies

$$\begin{aligned}&\frac{\partial \bar{u}}{\partial t} + \nu A\bar{u} + \alpha\bar{u} \\ &\quad = \tilde{P}(-\beta|\hat{u}|\hat{u} - \gamma|\hat{u}|^2\hat{u}) - \tilde{P}(-\beta|\tilde{u}|\tilde{u} - \gamma|\tilde{u}|^2\tilde{u}) + \tilde{P}f(t,\hat{u}_t) - \tilde{P}f(t,\tilde{u}_t).\end{aligned} \tag{38}$$

Taking the inner product of (38) with $\bar{u}$ in H yields

$$\begin{aligned}&\frac{d}{dr}\|\bar{u}(r)\|^2 + \nu\|A^{\frac{1}{2}}\bar{u}(r)\|^2 + 2\alpha\|\bar{u}(r)\|^2 \\ &\quad = -((\beta|\hat{u}|\hat{u} + \gamma|\hat{u}|^2\hat{u}) - (\beta|\tilde{u}|\tilde{u} + \gamma|\tilde{u}|^2\tilde{u}), \bar{u}) + (f(t,\hat{u}_t) - f(t,\tilde{u}_t), \bar{u}).\end{aligned}$$

By (5) and (10), we obtain

$$-((\beta|\hat{u}|\hat{u}+\gamma|\hat{u}|^2\hat{u})-(\beta|\tilde{u}|\tilde{u}+\gamma|\tilde{u}|^2\tilde{u}),\bar{u})\leq -\frac{\beta}{2}\|\bar{u}(r)\|_3^3-\frac{\gamma}{4}\|\bar{u}(r)\|_4^4\leq 0$$

and

$$(f(t,\hat{u}_t)-f(t,\tilde{u}_t),\bar{u})\leq \alpha\|\bar{u}(r)\|^2+cL_f^2(t)\|\hat{u}_t-\tilde{u}_t\|_{H_\varrho}^2.$$

Hence, we have

$$\frac{d}{dr}\|\bar{u}(r)\|^2\leq cL_f^2(t)\|\hat{u}_t-\tilde{u}_t\|_{H_\varrho}^2.$$

Integrating the above inequality on $[t_0, t]$ with $t > t_0$ yields

$$\|\bar{u}(t)\|^2\leq \|\hat{\phi}-\tilde{\phi}\|_{H_\varrho}^2+c\int_{t_0}^{t}L_f^2(r)\|\hat{u}_r-\tilde{u}_r\|_{H_\varrho}^2dr. \tag{39}$$

Notice that $\|\bar{u}(t)\|^2\leq\|\hat{\phi}-\tilde{\phi}\|_{H_\varrho}^2$ for $t\in[t_0-h, t_0]$. Then, we have

$$\|\bar{u}_t\|_{H_\varrho}^2\leq \|\hat{\phi}-\tilde{\phi}\|_{H_\varrho}^2+c\int_{t_0}^{t}L_f^2(r)\|\hat{u}_r-\tilde{u}_r\|_{H_\varrho}^2dr.$$

Using the Gronwall inequality, we obtain

$$\|\bar{u}_t\|_{H_\varrho}^2\leq \|\hat{\phi}-\tilde{\phi}\|_{H_\varrho}^2 e^{c\int_{t_0}^{t}L_f^2(r)dr}.$$

The proof is complete. □

Next, we prove the t_0-continuity of $S(\cdot,\cdot)$ in (7).

Lemma 7 *Suppose that* (F1)–(F3) *and* (G1) *hold. Then, the evolution process* $S(\cdot,\cdot)$ *in* (7) *is* t_0*-continuous in* H_ϱ.

Proof Based on the definition of t_0-continuous, we divide the proof into two steps.

Step 1. For all $\phi\in H_\varrho$ and $t\in\mathbb{R}$, $t_0\to$, $S(t,t_0)\phi$ is continuous on $(-\infty,t]$.

Let $\tilde{t}_0\in(-\infty,t]$ and $\tilde{t}_0\geq t_0$. By (37), we obtain, for all $\phi\in H_\varrho$,

$$\begin{aligned}\|S(t,\tilde{t}_0)\phi-S(t,t_0)\phi\|_{H_\varrho}^2 &= \|S(t,\tilde{t}_0)S(t_0,t_0)\phi-S(t,\tilde{t}_0)S(\tilde{t}_0,t_0)\phi\|_{H_\varrho}^2\\ &\leq \|S(t_0,t_0)\phi-S(\tilde{t}_0,t_0)\phi\|_{H_\varrho}^2 e^{c\int_{\tilde{t}_0}^{t}L_f^2(r)dr}\end{aligned}$$

$$\leq \|u_{t_0}(\cdot; t_0, \phi) - u_{\tilde{t}_0}(\cdot; t_0, \phi)\|_{H_\varrho}^2 e^{c\int_{-\infty}^{t} L_f^2(r)dr}.$$

Notice that $u : [t_0 - \varrho, t_0 + 1] \to H$ is uniform continuous. Then for any $\varepsilon > 0$, there exists a positive constant $\delta := \delta(\varepsilon) < 1$ with $|t_0 - \tilde{t}_0| < \delta$ such that for all $\theta \in [-\varrho, 0]$,

$$\|u(t_0 + \theta; t_0, \phi) - u(\tilde{t}_0 + \theta; t_0, \phi)\|^2 < \varepsilon,$$

which implies

$$\|u_{t_0}(\cdot; t_0, \phi) - u_{\tilde{t}_0}(\cdot; t_0, \phi)\|_{H_\varrho}^2 < \varepsilon.$$

Since $e^{c\int_{-\infty}^{t} L_f^2(r)dr}$ is finite, we obtain $t_0 \to S(t, t_0)\phi$ is right continuous on $(-\infty, t]$. Similarly, we have $t_0 \to S(t, t_0)\phi$ is left continuous on $(-\infty, t]$. Then, the proof of *Step 1* is complete.

Step 2 For all $\phi \in H_\varrho$ and $t \in \mathbb{R}$, $t_0 \to S(t, t_0)\phi$ is bounded on $(-\infty, t]$.

By the same method as Lemma 2, we have

$$\begin{aligned}
&\|S(t, t_0)\phi\|_{H_\varrho}^2 = \|u_t(\cdot; t_0, \phi)\|_{H_\varrho}^2 \\
&\quad \leq e^{\alpha\varrho}(e^{-\alpha(t-t_0)}\|\phi\|_{H_\varrho}^2 + \frac{2}{\alpha}\int_{t_0}^{t} e^{\alpha(r-t)}\|g(r)\|^2 dr) e^{\frac{2}{\alpha}e^{\alpha\varrho}\int_{t_0}^{t} L_f^2(r)dr} \\
&\quad \leq e^{\alpha\varrho}(\|\phi\|_{H_\varrho}^2 + \frac{2}{\alpha}\int_{-\infty}^{t} e^{\alpha(r-t)}\|g(r)\|^2 dr) e^{\frac{2}{\alpha}e^{\alpha\varrho}\int_{-\infty}^{t} L_f^2(r)dr}.
\end{aligned}$$

Observe that the last line of the above inequality is a constant independent of t_0. Then, we obtain

$$\begin{aligned}
&\lim_{t_0 \to -\infty} \|S(t, t_0)\phi\|_{H_\varrho}^2 \\
&\leq e^{\alpha\varrho}\left(\lim_{t_0 \to -\infty} e^{-\alpha(t-t_0)}\|\phi\|_{H_\varrho}^2 + \frac{2}{\alpha}\int_{-\infty}^{t} e^{\alpha(r-t)}\|g(r)\|^2 dr\right) e^{\frac{2}{\alpha}e^{\alpha\varrho}\int_{-\infty}^{t} L_f^2(r)dr} \\
&= e^{\alpha\varrho}\left(\frac{2}{\alpha}\int_{-\infty}^{t} e^{\alpha(r-t)}\|g(r)\|^2 dr\right) e^{\frac{2}{\alpha}e^{\alpha\varrho}\int_{-\infty}^{t} L_f^2(r)dr} < +\infty.
\end{aligned}$$

By *Step 1*, we know that $t \in \mathbb{R}$, $t_0 \to S(t, t_0)\phi$ is continuous on $(-\infty, t]$, and hence the proof of *Step 2* is complete. □

Now, we obtain all conditions of Łukaszewicz and Robinson [28, Theorem 3.1] are fulfilled by (i) of Theorem 1 and Lemma 7. Hence, we have the following result:

Theorem 2 *Suppose that* (F1)–(F3) *and* (G1) *hold. Fix a generalized Banach limit* $LIM_{T\to+\infty}$ *and let* $\upsilon : \mathbb{R} \to H_\varrho$ *be a continuous map such that* $\upsilon(\cdot) \in \mathfrak{D}$*. Then, there exists a unique family of Borel probability measures* $\{\mu_t : t \in \mathbb{R}\}$ *in* H_ϱ *such that the support of the measure* $\mu_t \subset \mathcal{A}_d(t)$ *and*

$$\underset{t_0\to-\infty}{LIM} \frac{1}{t-t_0} \int_{t_0}^{t} \varphi(S(t,r)\upsilon(r))dr = \int_{\mathcal{A}_d(t)} \varphi(w)d\mu_t(w),$$

for any real-valued continuous functional φ *on* H_ϱ*. In addition,* μ_t *is invariant in the following sense:*

$$\int_{\mathcal{A}_d(t)} \varphi(w)d\mu_t(w) = \int_{\mathcal{A}_d(t_0)} \varphi(S(t,t_0)w)d\mu_{t_0}(w), \quad t \ge t_0,$$

where a generalized Banach limit $LIM_{T\to+\infty}$ *is any linear functional, which defined on the space of all bounded real-valued functions on* $[0, +\infty)$ *that satisfies*

(1) $LIM_{T\to+\infty}h(T) \ge 0$ *for nonnegative functions h;*
(2) $LIM_{T\to+\infty}h(T) = \lim_{T\to+\infty} h(T)$ *if the limit* $\lim_{T\to+\infty} h(T)$ *exists.*

3.4 Backward Flattening of Solutions

Lemma 8 *Suppose that* (F1)–(F3) *and* (G1) *hold. For each* $t \in \mathbb{R}$ *and* $\mathcal{B} \in \mathfrak{B}$*, we have, for all* $\tau \ge \tau_b$ *and* $\phi \in \mathcal{B}$*,*

$$\sup_{s\le t} \int_{s-\varrho-1}^{s} \|Au(r)\|^2 dr \le c(1+L(t))^3(1+G_b(t))^3 e^{cL(t)}. \tag{40}$$

Proof Multiplying (4) by $Au(r; s-\tau, \phi)$, by (5) and the Young inequality, we have

$$\begin{aligned}&\frac{d}{dr}\|A^{\frac{1}{2}}u(r)\|^2 + \nu\|Au\|^2 + 2\alpha\|A^{\frac{1}{2}}u\|^2 + 2\beta(|u|u, Au) + 2\gamma(|u|^2u, Au)\\ &\quad\le\ cL_f^2(r)\|u_r\|^2 + c\|g(r)\|^2.\end{aligned}$$

By the Young inequality, we obtain

$$-2\beta(|u|u, Au) - 2\gamma(|u|^2u, Au) \le \frac{\nu}{2}\|Au\|^2 + c(\|u\|_4^4 + \|u\|_6^6).$$

Hence, by $V \hookrightarrow L^p(\Omega)(2 \le p \le 6)$, we obtain

$$\frac{d}{dr}\|A^{\frac{1}{2}}u(r)\|^2 + \frac{\nu}{2}\|Au\|^2 \le cL_f^2(r)\|u_r\|^2 + c\|g(r)\|^2 + c(\|u\|_V^4 + \|u\|_V^6).$$

Integrating the above inequality on $[s-\varrho-1, s]$ yields

$$\int_{s-\varrho-1}^{s} \|Au(r)\|^2 dr \le c \int_{s-\varrho-1}^{s} (1 + L_f^2(r)\|u_r\|^2 + \|g(r)\|^2 + \|u(r)\|_V^6) dr + \|A^{\frac{1}{2}}u(s-\varrho-1)\|^2. \tag{41}$$

By (16), we have

$$\int_{s-\varrho-1}^{s} \|u(r)\|_V^6 dr \le (\varrho+1) \sup_{r\in[s-\varrho-1,s]} \|u(r)\|_V^6 \le c(1+L(t))^3(1+G_b(t))^3 e^{cL(t)}. \tag{42}$$

Inserting (23), (24) and (42) into (41) yields

$$\sup_{s\le t} \int_{s-\varrho-1}^{s} \|Au(r)\|^2 dr \le c(1+L(t))^3(1+G_b(t))^3 e^{cL(t)}.$$

The proof is complete. □

Note that the Stokes operator A has a family of eigenfunctions $\{e_k\}_{k=1}^{\infty} \subset V$ with the corresponding eigenvalues: $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \to +\infty$ as $k \to +\infty$. In addition, $\{e_k\}_{k=1}^{\infty}$ be the orthonormal basis of H. Let $P_k : H \mapsto H_k = span\{e_1, e_2, \cdots, e_k\}$, then P_k is an orthogonal projection. For each $u \in V$, which has the following orthogonal decomposition:

$$u = P_k u \oplus (I - P_k)u = u_{k,1} + u_{k,2}, \quad k \in \mathbb{N}.$$

Lemma 9 *Suppose that* (F1)–(F3) *and* (G1) *hold. For each* $t \in \mathbb{R}$, $\mathcal{B} \in \mathfrak{B}$ *and any* $\varepsilon > 0$, *there exists a* $\delta := \delta(t, k, \varepsilon) > 0$ *with* $|\theta_1 - \theta_2| < \delta$ *and* $\theta_1, \theta_2 \in [-\varrho, 0]$ *such that*

$$\sup_{s\le t} \|P_k u(s+\theta_1; s-\tau, \phi) - P_k u(s+\theta_2; s-\tau, \phi)\|_V < \varepsilon, \tag{43}$$

for all $\tau \ge \tau_b$ *and* $\phi \in \mathcal{B}$.

Proof Notice that $\|A^{\frac{1}{2}}u_{k,1}\|^2 \le \lambda_k \|u_{k,1}\|^2$ and suppose that $\theta_1 \le \theta_2$. Hence, by (24), we have

$$\begin{aligned}\sup\nolimits_{s\le t} &\|A^{\frac{1}{2}}u_{k,1}(s+\theta_1; s-\tau, \phi) - A^{\frac{1}{2}}u_{k,1}(s+\theta_2; s-\tau, \phi)\| \\ &\le \lambda_k^{\frac{1}{2}} \sup\nolimits_{s\le t} \|u_{k,1}(s+\theta_1) - u_{k,1}(s+\theta_2)\|\end{aligned}$$

$$\leq \lambda_k^{\frac{1}{2}} \sup_{s\leq t} \int_{s+\theta_1}^{s+\theta_2} \left\| \tfrac{\partial}{\partial r} u_{k,1}(r; s-\tau, \phi)\right\| dr$$

$$\leq \lambda_k^{\frac{1}{2}} \sup_{s\leq t} \left(\int_{s-\varrho}^{s} \left\| \tfrac{\partial}{\partial r} u(r; s-\tau, \phi)\right\|^2 dr \right)^{\frac{1}{2}} |\theta_1-\theta_2|^{\frac{1}{2}}$$

$$\leq c\lambda_k^{\frac{1}{2}} (1+L(t))(1+G_b(t)) e^{cL(t)} |\theta_1-\theta_2|^{\frac{1}{2}}.$$

Then, for any $\varepsilon > 0$, there exists a $\delta := \delta(t, k, \varepsilon) > 0$ with $|\theta_1 - \theta_2| < \delta$ such that

$$\sup_{s\leq t} \|A^{\frac{1}{2}} u_{k,1}(s+\theta_1; s-\tau, \phi) - A^{\frac{1}{2}} u_{k,1}(s+\theta_2; s-\tau, \phi)\| < \varepsilon,$$

which implies (43) holds. □

Lemma 10 *Suppose that* (F1)–(F3) *and* (G1) *hold. For each $t \in \mathbb{R}$, $\mathcal{B} \in \mathfrak{B}$ and any $\varepsilon > 0$, there exists a $K = K(\varepsilon, t) \in \mathbb{N}$ such that*

$$\sup_{s\leq t} \sup_{\theta\in[-\varrho,0]} \|A^{\frac{1}{2}} u_{k,2}(s+\theta; s-\tau, \phi)\|^2 < \varepsilon. \tag{44}$$

for all $k \geq K$, $\tau \geq \tau_b$ and $\phi \in \mathcal{B}$.

Proof Taking the inner product of (4) with $Au_{k,2}(r; s-\tau, \phi)$ in H, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dr}\|A^{\frac{1}{2}} u_{k,2}(r)\|^2 + \nu\|Au_{k,2}(r)\|^2 + \alpha\|A^{\frac{1}{2}} u_{k,2}(r)\|^2 \\ &\quad = -\beta(|u|u, Au_{k,2}) - \gamma(|u|^2 u, Au_{k,2}) + (f(r, u_r), Au_{k,2}) + (g(r,\cdot), Au_{k,2}) \\ &\quad \leq \frac{\nu}{2}\|Au_{k,2}\|^2 + c(\|u(r)\|_4^4 + \|u(r)\|_6^6 + L_f^2(r)\|u_r\|_{H_\varrho}^2 + \|g(r)\|^2). \end{aligned}$$

Therefore, by $\|Au_{k,2}\|^2 \geq \lambda_k \|A^{\frac{1}{2}} u_{k,2}\|^2$ we get

$$\frac{d}{dr} e^{\nu\lambda_k r}\|A^{\frac{1}{2}} u_2(r)\|^2 \leq c e^{\nu\lambda_k r}(\|u(r)\|_4^4 + \|u(r)\|_6^6 + L_f^2(r)\|u_r\|_{H_\varrho}^2 + \|g(r)\|^2).$$

Let $p = 4$, $j = 0$, $q = 3$, $r = 2$, $l = 2$ in (11), we have

$$\|u(r)\|_4^4 \leq c\|u(r)\|_3^{\frac{10}{3}} \|Au(r)\|^{\frac{2}{3}}.$$

Again, let $p = 6$, $j = 0$, $q = 4$, $r = 2$, $l = 2$,

$$\|u(r)\|_6^6 \leq c\|u(r)\|_4^{\frac{24}{5}} \|Au(r)\|^{\frac{6}{5}}.$$

Hence, we have

$$\frac{d}{dr} e^{\nu\lambda_k r}\|A^{\frac{1}{2}} u_{k,2}(r)\|^2$$

$$\leq ce^{\nu\lambda_k r}(\|u\|_3^{\frac{10}{3}}\|Au\|^{\frac{2}{3}} + \|u\|_4^{\frac{24}{5}}\|Au\|^{\frac{6}{5}} + L_f^2(r)\|u_r\|_{H_\varrho}^2 + \|g(r)\|^2). \tag{45}$$

Integrating (45) on $[\zeta, s+\theta]$ with $\zeta \in [s+\theta-1, s+\theta]$ and $\theta \in [-\varrho, 0]$, and then integrating this resulting on $[s+\theta-1, s+\theta]$ w.r.t. ζ yields

$$\begin{aligned}
&e^{\lambda_k(s+\theta)}\|A^{\frac{1}{2}}u_{k,2}(s+\theta; s-\tau, \phi)\|^2\\
&\leq c\int_{s+\theta-1}^{s+\theta} e^{\lambda_k r}(\|A^{\frac{1}{2}}u_{k,2}(r)\|^2 + \|u\|_3^{\frac{10}{3}}\|Au\|^{\frac{2}{3}} + \|u\|_4^{\frac{24}{5}}\|Au\|^{\frac{6}{5}})dr\\
&\quad + c\int_{s+\theta-1}^{s+\theta} e^{\lambda_k r}(L_f^2(r)\|u_r\|_{H_\varrho}^2 + \|g(r)\|^2)dr\\
&= c\int_{s-1}^{s} e^{\lambda_k(r+\theta)}(\|A^{\frac{1}{2}}u_{k,2}(r+\theta)\|^2 + \|u(r+\theta)\|_3^{\frac{10}{3}}\|Au(r+\theta)\|^{\frac{2}{3}}\\
&\quad + \|g(r+\theta)\|^2)dr\\
&\quad + c\int_{s-1}^{s} e^{\lambda_k(r+\theta)}(\|u(r+\theta)\|_4^{\frac{24}{5}}\|Au(r+\theta)\|^{\frac{6}{5}} + L_f^2(r+\theta)\|u_{r+\theta}\|_{H_\varrho}^2)dr,
\end{aligned}$$

which implies

$$\begin{aligned}
\|A^{\frac{1}{2}}u_{k,2}(s+\theta; s-\tau, \phi)\|^2 &\leq c\int_{s-1}^{s} e^{\lambda_k(r-s)}(\|A^{\frac{1}{2}}u_{k,2}(r+\theta)\|^2\\
&\quad + \|u(r+\theta)\|_3^{\frac{10}{3}}\|Au(r+\theta)\|^{\frac{2}{3}} + \|g(r+\theta)\|^2)dr\\
&\quad + c\int_{s-1}^{s} e^{\lambda_k(r-s)}(\|u(r+\theta)\|_4^{\frac{24}{5}}\|Au(r+\theta)\|^{\frac{6}{5}} + L_f^2(r+\theta)\|u_{r+\theta}\|_{H_\varrho}^2)dr.
\end{aligned} \tag{46}$$

We now treat each term on the right-hand term of (46). For the first term, by (24), we have

$$\begin{aligned}
&\sup_{s\leq t}\sup_{\theta\in[-\varrho,0]}\int_{s-1}^{s} e^{\lambda_k(r-s)}\|A^{\frac{1}{2}}u_{k,2}(r+\theta)\|^2 dr\\
&\leq \sup_{s\leq t}\sup_{r\in[s-\varrho-1,s]}\|A^{\frac{1}{2}}u(r)\|^2\int_{s-\varrho-1}^{s} e^{\lambda_k(r-s)}dr\\
&\leq \frac{c}{\lambda_k}(1+L(t))(1+G_b(t))e^{cL(t)} \to 0 \text{ as } k \to +\infty.
\end{aligned} \tag{47}$$

For the second and third term, by (29) and (40), we get

$$\begin{aligned}
&\sup_{s\le t}\sup_{\theta\in[-\varrho,0]}\int_{s-1}^{s} e^{\lambda_k(r-s)}(\|u(r+\theta)\|_3^{\frac{10}{3}}\|Au(r+\theta)\|^{\frac{2}{3}}+\|u(r+\theta)\|_4^{\frac{24}{5}}\|Au(r+\theta)\|^{\frac{6}{5}})dr\\
&\le \sup_{s\le t}\sup_{r\in[s-\varrho-1,s]}\|u(r)\|_3^{\frac{10}{3}}\sup_{\theta\in[-\varrho,0]}\int_{s-1}^{s} e^{\lambda_k(r-s)}\|Au(r+\theta)\|^{\frac{2}{3}}dr\\
&\quad+\sup_{s\le t}\sup_{r\in[s-\varrho-1,s]}\|u(r)\|_4^{\frac{24}{5}}\sup_{\theta\in[-\varrho,0]}\int_{s-1}^{s} e^{\lambda_k(r-s)}\|Au(r+\theta)\|^{\frac{6}{5}}dr\\
&\le \sup_{s\le t}\sup_{r\in[s-\varrho-1,s]}\|u(r)\|_3^{\frac{10}{3}}\left(\int_{s-1}^{s} e^{\frac{3}{2}\lambda_k(r-s)}dr\right)^{\frac{2}{3}}\left(\int_{s-\varrho-1}^{s}\|Au(r)\|^2dr\right)^{\frac{1}{3}}\\
&\quad+\sup_{s\le t}\sup_{r\in[s-\varrho-1,s]}\|u(r)\|_4^{\frac{24}{5}}\left(\int_{s-1}^{s} e^{\frac{5}{2}\lambda_k(r-s)}dr\right)^{\frac{2}{5}}\left(\int_{s-\varrho-1}^{s}\|Au(r)\|^2dr\right)^{\frac{3}{5}}\\
&\le c\left(\frac{2}{3\lambda_k}\right)^{\frac{2}{3}}((1+L(t))(1+G_b(t))e^{cL(t)})^{\frac{19}{9}}\\
&\quad+c\left(\frac{2}{5\lambda_k}\right)^{\frac{2}{5}}((1+L(t))(1+G_b(t))e^{cL(t)})^3\to 0\ \text{ as } k\to+\infty.
\end{aligned}\tag{48}$$

For the delay term, by (6) and (16), we find

$$\begin{aligned}
&\sup_{s\le t}\sup_{\theta\in[-\varrho,0]}\int_{s-1}^{s} e^{\lambda_k(r-s)}L_f^2(r+\theta)\|u_{r+\theta}\|_{H_\varrho}^2dr\\
&\le \sup_{s\le t}\sup_{\sigma\in[s-\varrho-1,s]}\|u_\sigma\|_{H_\varrho}^2\sup_{\theta\in[-\varrho,0]}\int_{s-1}^{s} e^{\lambda_k(r-s)}L_f^2(r+\theta)dr\\
&\le c(1+G_b(t))e^{cL(t)}\sup_{s\le t}\sup_{\theta\in[-\varrho,0]}\int_{s+\theta-1}^{s+\theta} e^{\lambda_k(r-(s+\theta))}L_f^2(r)dr\\
&\le c(1+G_b(t))e^{cL(t)}\sup_{s\le t}\int_{-\infty}^{s} e^{\lambda_k(r-s)}L_f^2(r)dr\to 0\ \text{ as } k\to+\infty.
\end{aligned}\tag{49}$$

For the forcing term, by (8), we have

$$\begin{aligned}
&\sup_{s\le t}\sup_{\theta\in[-\varrho,0]}\int_{s-1}^{s} e^{\lambda_k(r-s)}\|g(r+\theta)\|^2dr\\
&\le \sup_{s\le t}\sup_{\theta\in[-\varrho,0]}\int_{-\infty}^{s+\theta} e^{\lambda_k(r-(s+\theta))}\|g(r)\|^2dr\\
&\le \sup_{s\le t}\int_{-\infty}^{s} e^{\lambda_k(r-s)}\|g(r)\|^2dr \to 0 \ \text{ as } k\to+\infty.
\end{aligned} \tag{50}$$

It follows from (46) to (50) that we obtain (44) as desired. □

3.5 Existence and Backward Compactness of Pullback Attractors in the Regular Space

We review some basic concepts and theorem related to bi-spatial pullback attractors. Suppose that $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be Banach spaces. An evolution process $S(\cdot,\cdot)$ on X is said to take its values into Y if

$$S(t,r)X\subset Y,\ \text{ for all } t\ge r.$$

We claim that the combination (X,Y) is *limit-identical* if $X\cap Y\ne\emptyset$ and

$$x_n\in X\cap Y,\ \lim_{n\to+\infty}\|x_n-x_0\|_X+\|x_n-y_0\|_Y=0\Rightarrow\ x_0=y_0\in X\cap Y.$$

Let $\tilde{\mathfrak{D}}$ be a inclusion-closed universe of some sets $\tilde{\mathcal{D}}=\{\tilde{\mathcal{D}}(t)\ne\emptyset:t\in\mathbb{R}\}\subset X$.

Definition 1 A family of sets $\mathcal{A}=\{\mathcal{A}(t):t\in\mathbb{R}\}$ is called a pullback $\tilde{\mathfrak{D}}$-(X,Y)-attractor for $S(\cdot,\cdot)$ if

(i) $\mathcal{A}(\cdot)$ is compact in $X\cap Y$;
(ii) $\mathcal{A}(\cdot)$ is invariant, that is, $S(t,s)\mathcal{A}(s)=\mathcal{A}(t)$ for all $t\ge s$;
(iii) $\mathcal{A}(\cdot)$ attracts every $\tilde{\mathcal{D}}\in\tilde{\mathfrak{D}}$ under the topology of Y, more precisely, for each $\tilde{\mathcal{D}}\in\tilde{\mathfrak{D}}$, we have

$$\lim_{\tau\to+\infty} dist_{X\cap Y}(S(t,t-\tau)\tilde{\mathcal{D}}(t-\tau),\mathcal{A}(t))=0,$$

where $dist_{X\cap Y}(\cdot,\cdot)$ denotes the Hausdorff semi-distance.

Theorem 3 *Assume that $S(\cdot,\cdot)$ be an evolution process on X taking its values in Y and (X,Y) is limit-identical. Then, $S(\cdot,\cdot)$ has a pullback $\tilde{\mathfrak{D}}$-(X,Y)-attractor $\mathcal{A}(\cdot)$ if*

(i) *$S(\cdot,\cdot)$ has a closed pullback $\tilde{\mathfrak{D}}$-absorbing set $\mathcal{K}:=\{\mathcal{K}(t):t\in\mathbb{R}\}\in\tilde{\mathfrak{D}}$;*

(ii) $S(\cdot,\cdot)$ *is pullback* $\tilde{\mathfrak{D}}$*-asymptotically compact in* X*;*
(iii) $S(\cdot,\cdot)$ *is pullback* $\tilde{\mathfrak{D}}$*-*(X, Y)*-asymptotically compact.*

Theorem 4 *Suppose that* (F1)–(F3) *and* (G1) *hold. Then, we have the following conclusions:*

(i) *The backward compact pullback* $\mathfrak{B}$*-attractor* $\mathcal{A}_b$ *as given in* (36) *is also a pullback* $\mathfrak{B}$*-*(H_ϱ, V_ϱ) *attractor, which is backward compact in* V_ϱ*.*
(ii) *The pullback* $\mathfrak{D}$*-attractor* $\mathcal{A}_d$ *as given in* (35) *is also a pullback* $\mathfrak{D}$*-*(H_ϱ, V_ϱ)*-attractor, which is backward compact in* V_ϱ*.*

Proof (i) We first show that $\mathcal{A}_b$ is a pullback $\mathfrak{B}$-(H_ϱ, V_ϱ)-attractor. Since $V_\varrho \hookrightarrow H_\varrho$, (H_ϱ, V_ϱ) is limit-identical. By Theorem 3 we also need to prove $S(\cdot,\cdot)$ is backward pullback $\mathfrak{B}$-(H_ϱ, V_ϱ)-asymptotically compact, that is, for each $t \in \mathbb{R}$, $\{S(s_n, s_n - \tau_n)\phi_n\}_{n\in\mathbb{N}}$ has a convergent subsequence in V_ϱ whenever $s_n \leq t, \tau_n \to +\infty$ and $\phi_n \in \mathcal{B}(s_n - \tau_n)$. To this end, we split the proof into two steps based on Ascoli–Arzelà theorem.

Step 1. $\{S(s_n, s_n - \tau_n)\phi_n\}_{n\in\mathbb{N}}$ in V_ϱ is equi-continuous from $[-\varrho, 0]$ to V.

Notice that there exists a $N \in \mathbb{N}$ such that $\tau_n \geq \tau_b$ (τ_b is given in Lemma 2) for all $n \in \mathbb{N}$ due to $\tau_n \to +\infty$. By (43) and (44), we have for all $k_0 \geq K$

$$
\begin{aligned}
&\|(S(s_n, s_n - \tau_n)\phi_n)(\theta_1) - (S(s_n, s_n - \tau_n)\phi_n)(\theta_2)\|_V \\
&\quad = \|u(s_n + \theta_1; s_n - \tau_n, \phi_n) - u(s_n + \theta_2; s_n - \tau_n, \phi_n)\|_V \\
&\quad \leq \|P_{k_0}u(s_n + \theta_1; s_n - \tau_n, \phi_n) - P_{k_0}u(s_n + \theta_2; s_n - \tau_n, \phi_n)\|_V \\
&\qquad + \|(I - P_{k_0})u(s_n + \theta_1; s_n - \tau_n, \phi_n)\|_V + \|(I - P_{k_0})u(s_n + \theta_2; s_n - \tau_n, \phi_n)\|_V \\
&\quad < 3\varepsilon.
\end{aligned}
$$

Hence, we have $\{S_\varrho(s_n, s_n - \tau_n)\phi_n\}_{n\in\mathbb{N}}$ is equi-continuous.

Step 2. For each fixed $\theta \in [-\varrho, 0]$, the sequence $(S(s_n, s_n - \tau_n)\phi_n)(\theta) = u(s_n + \theta; s_n - \tau_n, \phi_n)$ is pre-compact in V.

By (24), we obtain that $\{P_{k_0}u(s_n + \theta; s_n - \tau_n, \phi_n)\}_{n\in\mathbb{N}}$ is bounded in V and thus pre-compact in the k_0-dimensional subspace V_{k_0}. Then, there is an index subsequence n^* of n such that $\{P_{k_0}u(s_{n^*} + \theta; s_{n^*} - \tau_{n^*}, \phi_{n^*})\}_{n^*\in\mathbb{N}}$ is a Cauchy sequence in V_{k_0}. On the other hand, let n^*, m^* large enough, we have

$$
\begin{aligned}
&\|u(s_{n^*} + \theta; s_{n^*} - \tau_{n^*}, \phi_{n^*}) - u(s_{n^*} + \theta; s_{m^*} - \tau_{m^*}, \phi_{m^*})\|_V \\
&\quad \leq \|P_{k_0}u(s_{n^*} + \theta; s_{n^*} - \tau_{n^*}, \phi_{n^*}) - P_{k_0}u(s_{m^*} + \theta; s_{m^*} - \tau_{m^*}, \phi_{m^*})\|_V \\
&\qquad + \|(I - P_{k_0})u(s_{n^*} + \theta; s_{n^*} - \tau_{n^*}, \phi_{n^*})\|_V \\
&\qquad + \|(I - P_{k_0})u(s_{m^*} + \theta; s_{m^*} - \tau_{m^*}, \phi_{m^*})\|_V < 3\varepsilon.
\end{aligned}
$$

Hence, $\{u(s_{n^*} + \theta; s_{n^*} - \tau_{n^*}, \phi_{n^*})\}_{n^*\in\mathbb{N}}$ is a Cauchy sequence in V and then the proof of *Step 2* is complete.

Then, we obtain that $S(\cdot,\cdot)$ is backward pullback $\mathfrak{B}$-(H_ϱ, V_ϱ)-asymptotically compact, which implies $S(\cdot,\cdot)$ is pullback $\mathfrak{B}$-(H_ϱ, V_ϱ)-asymptotically compact. Hence, we have $\mathcal{A}_b$ is a pullback $\mathfrak{B}$-(H_ϱ, V_ϱ)-attractor.

Next, we show that $\mathcal{A}_b$ is backward compact in V_ϱ. Indeed, it is easy to verify that the result by the same method as in (ii) of Theorem 1.

(ii) Similarly to (i), we have $\mathcal{A}_d$ as given in (35) is also a pullback $\mathfrak{D}$-(H_ϱ, V_ϱ)-attractor. Since $\mathcal{A}_d = \mathcal{A}_b$, we get $\mathcal{A}_d$ is backward compact in V_ϱ. □

4 Asymptotically Autonomous Dynamics for the 3D BF Equation with Autonomous Delays

In this section, we consider the following non-autonomous equation:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + \alpha u + \nabla p = -\beta|u|u - \gamma|u|^2 u + f(u_t, x) + g(t, x), \\ \nabla \cdot u = 0, \ t > t_0, \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega, \\ u(t_0 + \theta, x) = u_{t_0}(\theta, x), \ \theta \in [-\varrho, 0]. \end{cases} \tag{51}$$

Similarly to (4), Eq. (51) can be rewritten as

$$\begin{cases} \frac{\partial u}{\partial t} + \nu A u = \tilde{P}(-\alpha u - \beta|u|u - \gamma|u|^2 u) + \tilde{P} f(u_t, x) + \tilde{P} g(t, x), \\ u(t_0 + \theta, x) = u_{t_0}(\theta, x), \ \theta \in [-\varrho, 0], \ x \in \Omega. \end{cases} \tag{52}$$

Since we consider the delay term is autonomous, the assumption conditions (F1)–(F3) will be changed, more precisely,

(F1) For each $\varphi \in C_H$, $x \to f(\varphi, x)$ is measurable from Ω into H;
(F2) $f(0, x) = 0$ for all $x \in \Omega$;
(F3) There is a positive constant L_f such that for all $\varphi, \psi \in H_\varrho$,

$$\|f(\varphi, \cdot) - f(\psi, \cdot)\| \le L_f \|\varphi - \psi\|_{H_\varrho}. \tag{53}$$

(F4) There exist two positive constants c_f and m_f such that for all $u, v \in C([t_0 - \varrho, t], H)$ with $t_0 \le t$,

$$\int_{t_0}^{t} e^{m_f r} \|f(\varphi, \cdot) - f(\psi, \cdot)\|^2 dr \le c_f^2 \int_{t_0 - \varrho}^{t} e^{m_f r} \|\varphi - \psi\|^2 dr.$$

Hence, we can define an evolution process $\tilde{S}(t, t_0) : H_\varrho \to H_\varrho$ associated with (52), by

$$\tilde{S}(t, t_0)\phi = u_t(\cdot; t_0, \phi), \quad t \ge t_0, \ \phi \in H_\varrho.$$

Notice that we add a condition (F4), which will be used to establish the existence of pullback absorbing sets. Similarly to Lemma 2, we can obtain the evolution process $\tilde{S}$ has a pullback $\mathfrak{D}$-absorbing set $\tilde{\mathcal{K}}_d$ and a pullback $\mathfrak{B}$-absorbing set $\tilde{\mathcal{K}}_b$. Moreover, other results of Sect. 3 are hold for Eq. (52) due to L_f is a positive constant. Then, $\tilde{S}(\cdot,\cdot)$ has a pullback $\mathfrak{D}$-(H_ϱ, V_ϱ)-attractor $\tilde{\mathcal{A}}_d = \{\tilde{\mathcal{A}}_d(t) : t \in \mathbb{R}\} \in \mathfrak{D}$, which is backward compact in V_ϱ.

In order to study the upper semi-convergence of the pullback attractor $\tilde{\mathcal{A}}_d$ from non-autonomous to autonomous, we give a further assumption for g:

(G2) There exists a function $g_\infty \in H$ such that

$$\lim_{t_0 \to -\infty} \int_{-\infty}^{t_0} \|g(r) - g_\infty\|^2 dr = 0. \tag{54}$$

In addition, we need to introduce the following autonomous equation:

$$\begin{cases} \frac{\partial v}{\partial t} - \nu \Delta v + \alpha v + \nabla p = -\beta |v| v - \gamma |v|^2 v + f(v_t, x) + g_\infty(x), \\ \nabla \cdot v = 0, \ t > 0, \ x \in \Omega, \\ v = 0, \ x \in \partial\Omega, \\ v(0+\theta, x) = v_0(\theta, x), \ \theta \in [-\varrho, 0]. \end{cases} \tag{55}$$

Meanwhile, using the Leray orthogonal projection $\tilde{P}$ to (55), we obtain

$$\begin{cases} \frac{\partial v}{\partial t} + \nu A v = \tilde{P}(-\alpha v - \beta |v| v - \gamma |v|^2 v) + \tilde{P} f(v_t, x) + \tilde{P} g_\infty(x), \\ v(0+\theta, x) = v_0(\theta, x), \ \theta \in [-\varrho, 0]. \end{cases} \tag{56}$$

By the standard Galerkin method, we obtain the well-posedness of (56). Then, we obtain a semigroup $T(t) : H_\varrho \to H_\varrho$ generated by (56), given by

$$T(t)\phi = v_t(\cdot, \bar{\phi}), \ \ t \geq 0, \ \phi \in H_\varrho.$$

It follows from the same method as in Sect. 3 that $T(t)$ has a (H_ϱ, V_ϱ)-global attractor $\mathcal{A}_\infty$.

4.1 Convergence of Solutions from Non-Autonomous to Autonomous

Lemma 11 *Suppose* (F1)–(F3), (G1)–(G2) *hold. If* $u_{t_0}, v_0 \in H_\varrho$ *satisfy*

$$\lim_{t_0 \to -\infty} \|u_{t_0} - v_0\|_{H_\varrho}^2 = 0, \tag{57}$$

then we have, for all $\tilde{r} \geq 0$,

$$\begin{aligned} &\lim_{t_0 \to -\infty} \|\tilde{S}(t_0 + \tilde{r}, t_0) u_{t_0} - T(\tilde{r}) v_0\|_{H_\varrho}^2 \\ &= \lim_{t_0 \to -\infty} \|u_{t_0 + \tilde{r}}(\cdot; t_0, u_{t_0}) - v_{\tilde{r}}(\cdot; v_0)\|_{H_\varrho}^2 = 0. \end{aligned} \tag{58}$$

Proof For each $t_0 \in \mathbb{R}$, we define a function w^{t_0} by

$$w^{t_0}(r) = u(t_0 + r; t_0, u_{t_0}) - v(r; v_0), \quad r \geq -\varrho.$$

If follows from (52) and (56) that

$$\begin{aligned} &\frac{\partial w^{t_0}(r)}{\partial t} + \nu A w^{t_0}(r) + \alpha \tilde{P} w^{t_0}(r) \\ &\quad = \tilde{P}(-\beta|u(t_0+r)|u(t_0+r) - \gamma|u(t_0+r)|^2 u(t_0+r)) \\ &\quad\quad - \tilde{P}(-\beta|v(r)|v(r) - \gamma|v(r)|^2 v(r)) \\ &\quad\quad + \tilde{P} f(u_{t_0+r}, x) - \tilde{P} f(v_r, x) + \tilde{P} g(t_0+r, x) - \tilde{P} g_\infty(x). \end{aligned}$$

Multiplying the above equality by $w^{t_0}(r)$ yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dr}\|w^{t_0}(r)\|^2 + \nu\|A^{\frac{1}{2}} w^{t_0}(r)\|^2 + \alpha\|w^{t_0}(r)\|^2 \\ &\quad = ((-\beta|u(t_0+r)|u(t_0+r) - \gamma|u(t_0+r)|^2 u(t_0+r)), w^{t_0}(r)) \\ &\quad\quad - ((-\beta|v(r)|v(r) - \gamma|v(r)|^2 v(r)), w^{t_0}(r)) \\ &\quad\quad + (f(u_{t_0+r}, \cdot) - f(v_r, \cdot), w^{t_0}(r)) + (g(t_0+r, \cdot) - g_\infty(\cdot), w^{t_0}(r)). \end{aligned} \tag{59}$$

By (10), we have

$$\begin{aligned} &((-\beta|u(t_0+r)|u(t_0+r) - \gamma|u(t_0+r)|^2 u(t_0+r)), w^{t_0}(r)) \\ &\quad - ((-\beta|v(r)|v(r) - \gamma|v(r)|^2 v(r)), w^{t_0}(r)) \\ &\leq -\frac{\beta}{2}\|w^{t_0}(r)\|_3^3 - \frac{\gamma}{4}\|w^{t_0}(r)\|_4^4 \leq 0. \end{aligned} \tag{60}$$

By (53) and the Young inequality, we have

$$((f(u_{t_0+r}, \cdot) - f(v_r, \cdot), w^{t_0}(r)) \leq \frac{\alpha}{2}\|w^{t_0}(r)\|^2 + \frac{L_f}{2\alpha}\|w_r^{t_0}\|_{H_\varrho}^2, \tag{61}$$

$$(g(t_0+r, \cdot) - g_\infty(\cdot), w^{t_0}(r)) \leq \frac{\alpha}{2}\|w^{t_0}(r)\|^2 + \frac{1}{2\alpha}\|g(t_0+r) - g_\infty\|^2. \tag{62}$$

Inserting (60)–(62) into (59) yields

$$\frac{d}{dr}\|w^{t_0}(r)\|^2 \leq c\|w_r^{t_0}\|_{H_\varrho}^2 + c\|g(t_0+r) - g_\infty\|^2.$$

Integrating the above inequality on $[0, \tilde{r}]$ with $\tilde{r} \in [0, R]$ and $R > 0$ yields

$$\|w^{t_0}(\tilde{r})\|^2 \leq \|u_{t_0} - v_0\|_{H_\varrho}^2 + c\int_0^{\tilde{r}} \|w_r^{t_0}\|_{H_\varrho}^2 dr + c\int_0^R \|g(t_0+r) - g_\infty\|^2 dr.$$

It is easy to see that $\|w^{t_0}(\tilde{r})\|^2 \leq \|u_{t_0} - v_0\|^2_{H_\varrho}$ when $\tilde{r} \in [-\varrho, 0]$. Then, we have

$$\|w^{t_0}_{\tilde{r}}\|^2_{H_\varrho} \leq \|u_{t_0} - v_0\|^2_{H_\varrho} + \int_0^{\tilde{r}} \|w^{t_0}_r\|^2_{H_\varrho} dr + \int_{-\infty}^{t_0+R} \|g(r) - g_\infty\|^2 dr. \tag{63}$$

Applying the Gronwall inequality to (63), by (54) and (57), we get

$$\|w^{t_0}_{\tilde{r}}\|^2_{H_\varrho} \leq (\|u_{t_0} - v_0\|^2_{H_\varrho} + \int_{-\infty}^{t_0+R} \|g(r) - g_\infty\|^2 dr) e^R \to 0,$$

as $t_0 \to -\infty$. The proof is complete. □

4.2 Upper Semi-Convergence of Regular Attractors from Non-Autonomous to Autonomous

Theorem 5 *Suppose (F1)–(F4) and* (G1)–(G3) *hold. Then, we have*

$$\lim_{t \to -\infty} dist_{V_\varrho}(\tilde{\mathcal{A}}_d(t), \mathcal{A}_\infty) = 0. \tag{64}$$

Proof If (64) is false, then there are $\delta > 0$ and $t_n \to -\infty$ such that

$$dist_{V_\varrho}(\tilde{\mathcal{A}}_d(t_n), \mathcal{A}_\infty) \geq 4\delta,$$

which implies for each $n \in \mathbb{N}$, there exists a $a_n \in \tilde{\mathcal{A}}_d(t_n)$ such that

$$dist_{V_\varrho}(a_n, \mathcal{A}_\infty) \geq 3\delta. \tag{65}$$

We assume that $t_n \leq 0$ for all $n \in \mathbb{N}$ due to $t_n \to -\infty$. Then, we have $\{a_n\}_{n \in \mathbb{N}} \subset \bigcup_{s \leq 0} \tilde{\mathcal{A}}_d(s) := B$. Since $\tilde{\mathcal{A}}_d$ is backward compact in V_ϱ, $\overline{\bigcup_{s \leq 0} \tilde{\mathcal{A}}_d(s)}$ is compact and so B is bounded in V_ϱ. It follows from $\mathcal{A}_\infty$ is a (H_ϱ, V_ϱ)-global attractor and $V_\varrho \hookrightarrow H_\varrho$ that there is a $r := r(B) > 0$ such that

$$dist_{V_\varrho}(T(r)B, \mathcal{A}_\infty) \leq \delta. \tag{66}$$

Moreover, there is a $a_\infty \in B$ such that

$$\|a_n - a_\infty\|_{V_\varrho} \to 0, \quad \text{as } n \to +\infty. \tag{67}$$

By the invariance of $\tilde{\mathcal{A}}_d$, we find a $b_n \in \tilde{\mathcal{A}}_d(t_n - r) \subset B$ such that

$$\tilde{S}(t_n, t_n - r) b_n = a_n. \tag{68}$$

By the backward compactness of $\tilde{\mathcal{A}}_d$ in H_ϱ, there exists a $b_\infty \in B$ such that

$$\|b_n - b_\infty\|_{H_\varrho} \to 0, \quad \text{as } n \to +\infty.$$

which together with (58) implies

$$\|\tilde{S}(t_n - r + r, t_n - r)b_n - T(r)b_\infty\|_{H_\varrho} \to 0, \quad \text{as } n \to +\infty,$$

which along with (67) and (68) implies $a_\infty = T(r)b_\infty$. Then by (66) and (67), there exists $N := N(\delta) \in \mathbb{N}$ such that for all $n \geq N$,

$$dist_{V_\varrho}(a_n, \mathcal{A}_\infty) \leq \|a_n - a_\infty\|_{V_\varrho} + dist_{V_\varrho}(T(r)B, \mathcal{A}_\infty) \leq 2\delta,$$

which contradicts (65). The proof is complete. □

5 Upper Semicontinuity of Regular Pullback Attractors as the Delay time Tends to Zero

In this section, we assume that $\varrho \in (0, \varrho_0]$ for some $\varrho_0 > 0$. Let $f(t, u_t) = f_1(u(t - \rho^\varrho(t)), x) + \int_{-\varrho}^{0} f_2(u(t + \theta), \theta)d\theta$ in (1), and then applying the Leray orthogonal projection $\tilde{P}$ to this result, we obtain

$$\begin{cases} \frac{\partial u}{\partial t} + \nu Au + \tilde{P}(\alpha u + \beta|u|u + \gamma|u|^2 u) \\ = \tilde{P}(f_1(u(t - \rho^\varrho(t)), x) + \int_{-\varrho}^{0} f_2(u(t + \theta), \theta)d\theta) + \tilde{P}g(t, x) \\ u(t_0 + \theta, x) = u_{t_0}(\theta, x) := \phi^\varrho(x),\ \theta \in [-\varrho, 0],\ t > t_0,\ x \in \Omega, \end{cases} \tag{69}$$

where $\rho^\varrho(\cdot)$ is a positive function such that $\rho^\varrho(\cdot) \in C^1(\mathbb{R})$ and

$$\varrho := \sup_{t\in\mathbb{R}} \rho^\varrho(t) < +\infty, \quad \rho_* := \sup_{\varrho\in(0,\varrho_0]} \sup_{t\in\mathbb{R}} \frac{d}{dt}\rho^\varrho(t) < 1. \tag{70}$$

The
t variable delay $f_1 : \mathbb{R} \times \Omega \to \mathbb{R}$ satisfies

$$f_1(0, \cdot) = 0, \quad |f_1(s_1, x) - f_1(s_2, x)| \leq L_{f_1}|s_1 - s_2|, \quad s_1, s_2 \in \mathbb{R},\ x \in \Omega. \tag{71}$$

where L_{f_1} is a positive constant satisfies $L_{f_1}^2 < \frac{\alpha^2 e^{-\alpha\varrho}}{24}(1 - \rho_*)$.

The *distributed delay* $f_2 : [-\varrho, 0] \times \mathbb{R} \to \mathbb{R}$ satisfies

$$f_2(0, \cdot) = 0, \quad |f_2(\theta, s_1) - f_2(\theta, s_2)| \leq L_{f_2}(\theta)|s_1 - s_2|, \quad s_1, s_2 \in \mathbb{R},\ \theta \in [-\varrho, 0], \tag{72}$$

where $L_{f_2}(\cdot) \in L^2(-\varrho, 0)$ is a positive function satisfies $\|L_{f_2}(\cdot)\|^2_{L^2(-\varrho,0)} < \frac{\alpha^2 e^{-\alpha\varrho}}{24\varrho}$.

By the same method as in Sect. 3, we can obtain an evolution process $S_\varrho(\cdot,\cdot)$ induced by (69) for each $\varrho \in (0, \varrho_0]$, which has a pullback $\mathfrak{D}$-(H_ϱ, V_ϱ)-attractor $\mathcal{A}^\varrho_d = \{\mathcal{A}^\varrho_d(t) : t \in \mathbb{R}\} \in \mathfrak{D}$.

In this section, we consider the robustness of $\mathcal{A}^\varrho_d$ as $\varrho \to 0$. For this purpose, let $\varrho = 0$ in (69), we have

$$\begin{cases} \frac{\partial u^0}{\partial t} + \nu A u^0 = \tilde{P}(-\alpha u^0 - \beta |u^0| u^0 - \gamma |u^0|^2 u^0) + \tilde{P}(f_1(u^0(t), x) + g(t, x)), \\ u(t_0, x) := \phi^0(x). \end{cases} \tag{73}$$

Similarly, we get an evolution process $S_0(\cdot,\cdot)$ corresponding to (73), which has a pullback $\mathfrak{D}_0$-(H_ϱ, V_ϱ)-attractor $\mathcal{A}^0_d = \{\mathcal{A}^0_d(t) : t \in \mathbb{R}\} \in \mathfrak{D}_0$, where

$$\mathfrak{D}_0 = \{\mathcal{D}_0 = \{\mathcal{D}_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(H) : \lim_{\tau \to +\infty} e^{-\beta\tau} \|\mathcal{D}_0(t-\tau)\|^2 = 0, \ \ \beta > 0\}.$$

5.1 Convergence of Evolution Processes from Delay to Non-Delay

Lemma 12 *Suppose $\phi^\varrho \in H_\varrho$ and $\phi^0 \in H$ such that*

$$d_\varrho(\phi^\varrho, \phi^0) := \sup_{\theta \in [-\varrho, 0]} \|\phi^\varrho(\theta) - \phi^0\| \to 0 \text{ as } \varrho \to 0. \tag{74}$$

Then, the solution u^ϱ of Eq. (69) converges to the solution u^0 of Eq. (73) in the following sense:

$$\begin{aligned} &\lim_{\varrho \to 0} \sup_{\theta \in [-\varrho, 0]} \|(S_\varrho(t, t_0)\phi^\varrho)(\theta) - S_0(t, t_0)\phi^0\|^2 \\ &\quad = \lim_{\varrho \to 0} \sup_{\theta \in [-\varrho, 0]} \|u^\varrho(t+\theta; t_0, \phi^\varrho) - u^0(t; t_0, \phi^0)\|^2 = 0, \end{aligned} \tag{75}$$

for all $t \geq t_0$ and $t_0 \in \mathbb{R}$.

Proof Let

$$\tilde{u}^\varrho_\theta(r) = u^\varrho(r+\theta; t_0, \phi^\varrho) - u^0(r; t_0, \phi^0), \ \ \forall t_0 \in \mathbb{R}, \ r \geq t_0.$$

Subtracting (73) from (69), and then multiplying this result by $\tilde{u}^\varrho_\theta(r+\theta)$ in H, we have

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dr}\|\tilde{u}^\varrho_\theta(r)\|^2 + \nu\|A^{\frac{1}{2}}\tilde{u}^\varrho_\theta(r)\|^2 + \alpha\|\tilde{u}^\varrho_\theta(r)\|^2 \\ &+(\beta(|u^\varrho(r+\theta)|u^\varrho(r+\theta) - |u^0(r)|u^0(r)), \tilde{u}^\varrho_\theta(r)) \\ &+(\gamma(|u^\varrho(r+\theta)|^2 u^\varrho(r+\theta) - |u^0(r)|u^0(r)), \tilde{u}^\varrho_\theta(r)) \\ &= \ (f(u^\varrho(r+\theta-\rho^\varrho(r+\theta)), \cdot) - f(u^0(r), \cdot), \tilde{u}^\varrho_\theta(r)) \end{aligned}$$

$$+\left(\int_{-\varrho}^{0} f_2(u^\varrho(r+\theta+\tilde\theta),\tilde\theta)d\tilde\theta,\tilde u_\theta^\varrho(r)\right)+(g(r+\theta,\cdot)-g(r,\cdot),\tilde u_\theta^\varrho(r)).$$

By (10), we have

$$\begin{aligned}&(\beta(|u^\varrho(r+\theta)|u^\varrho(r+\theta)-|u^0(r)|u^0(r)),\tilde u_\theta^\varrho(r))\\&\quad+(\gamma(|u^\varrho(r+\theta)|^2u^\varrho(r+\theta)-|u^0(r)|u^0(r)),\tilde u_\theta^\varrho(r))\\&\geq\frac{\beta}{2}\|u^\varrho(r+\theta)-u^0(r)\|_3^3+\frac{\gamma}{4}\|u^\varrho(r+\theta)-u^0(r)\|_4^4\geq 0.\end{aligned}$$

Hence, by (71), (72) and the Young inequality, we have

$$\begin{aligned}&\frac{d}{dr}\|\tilde u_\theta^\varrho(r)\|^2\leq cL_{f_1}^2\|u^\varrho(r+\theta-\rho^\varrho(r+\theta))-u^0(r)\|^2\\&\quad+c\|L_{f_2}(\cdot)\|^2_{L^2(-\varrho,0)}\int_{-\varrho}^{0}\|u^\varrho(r+\theta+\tilde\theta)\|^2d\tilde\theta+c\|g(r+\theta)-g(r)\|^2.\end{aligned}\tag{76}$$

For each $T>\varrho$, integrating (76) on $[t_0-\theta,t]$ with $t\in[t_0-\theta,t_0+T]$ yields

$$\begin{aligned}\|\tilde u_\theta^\varrho(t)\|^2&\leq\|\tilde u_\theta^\varrho(t_0-\theta)\|^2+c\int_{t_0-\theta}^{t}\|u^\varrho(r+\theta-\rho^\varrho(r+\theta))-u^0(r)\|^2dr\\&\quad+c\|L_{f_2}(\cdot)\|^2_{L^2(-\varrho,0)}\int_{t_0-\theta}^{t}\int_{-\varrho}^{0}\|u^\varrho(r+\theta+\tilde\theta)\|^2d\tilde\theta dr\\&\quad+c\int_{t_0-\theta}^{t}\|g(r+\theta)-g(r)\|^2dr.\end{aligned}\tag{77}$$

We now estimate each term on the right-hand side of (77). For the first term, we have

$$\begin{aligned}\|\tilde u_\theta^\varrho(t_0-\theta)\|^2&\leq 2\|\phi^\varrho(0)-\phi^0\|^2+2\|\phi^0-u^0(t_0-\theta)\|^2\\&\leq 2d_\varrho^2(\phi^\varrho,\phi^0)+2\|\phi^0-u^0(t_0-\theta)\|^2.\end{aligned}$$

For the second term, let $s=h(r)=r+\theta-\rho^\varrho(r+\theta)$ for any $r\in\mathbb{R}$ and fixed $\theta\in[-\varrho,0]$. Since $h'(r)\geq 1-\rho_*>0$, it has an inverse function such that $r=h^{-1}(s)$ for any $s\in\mathbb{R}$. Then, we obtain

$$\int_{t_0-\theta}^{t}\|u^\varrho(r+\theta-\rho^\varrho(r+\theta))-u^0(r)\|^2dr$$

$$
\begin{aligned}
&= \left(\int_{t_0-\theta}^{h^{-1}(t_0)} + \int_{h^{-1}(t_0)}^{t} \right) \|u^\varrho(r+\theta-\rho^\varrho(r+\theta)) - u^0(r)\|^2 dr \\
&= \int_{t_0-\theta}^{h^{-1}(t_0)} \|u^\varrho(r+\theta-\rho^\varrho(r+\theta)) - u^0(r)\|^2 dr \\
&\quad + \int_{t_0-\theta}^{t-\rho^\varrho(t+\theta)} \frac{\|u^\varrho(r+\theta) - u^0(h^{-1}(r+\theta))\|^2}{1-\frac{d}{dr}\rho^\varrho(h^{-1}(r+\theta)+\theta)} dr =: I_1(\varrho) + I_2(t,\varrho).
\end{aligned}
$$

Note that

$$
\begin{aligned}
I_1(\varrho) &\le 2 \int_{t_0-\theta}^{h^{-1}(t_0)} \|u^\varrho(r+\theta-\rho^\varrho(r+\theta)) - \phi^0\|^2 dr + 2 \int_{t_0-\theta}^{h^{-1}(t_0)} \|u^0(r) - \phi^0\|^2 dr \\
&\le \frac{2}{1-\rho_*} \int_{t_0-\rho^\varrho(t_0)}^{t_0} \|u^\varrho(r) - \phi^0\|^2 dr + 2 \int_{t_0-\theta}^{h^{-1}(t_0)} \|u^0(r) - \phi^0\|^2 dr \\
&\le \ c d_\varrho^2(\phi^\varrho, \phi^0) + c \int_{t_0-\theta}^{h^{-1}(t_0)} \|u^0(r) - \phi^0\|^2 dr,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2(t,\varrho) &\le \frac{1}{1-\rho_*} \int_{t_0-\theta}^{t} \|u^\varrho(r+\theta) - u^0(h^{-1}(r+\theta))\|^2 dr \\
&\le \frac{2}{1-\rho_*} \int_{t_0-\theta}^{t} \|u^\varrho(r+\theta) - u^0(r)\|^2 dr \\
&\quad + \frac{2}{1-\rho_*} \int_{t_0-\theta}^{t} \|u^0(h^{-1}(r+\theta)) - u^0(r)\|^2 dr \\
&\le c \int_{t_0-\theta}^{t} \|\tilde{u}_\theta^\varrho(r)\|^2 dr + c \int_{t_0-\theta}^{t} \|u^0(h^{-1}(r+\theta)) - u^0(r)\|^2 dr.
\end{aligned}
$$

Hence, we obtain

$$\begin{aligned}&\int_{t_0-\theta}^{t} \|u^{\varrho}(r+\theta-\rho^{\varrho}(r+\theta))-u^0(r)\|^2dr\\&\le cd_{\varrho}^2(\phi^{\varrho},\phi^0)+c\int_{t_0-\theta}^{h^{-1}(t_0)}\|u^0(r)-\phi^0\|^2dr+c\int_{t_0-\theta}^{t}\|\tilde{u}_{\theta}^{\varrho}(r)\|^2dr\\&\quad+c\int_{t_0-\theta}^{t}\|u^0(h^{-1}(r+\theta))-u^0(r)\|^2dr.\end{aligned}$$

For the third term, we obtain

$$\begin{aligned}&\int_{t_0-\theta}^{t}\int_{-\varrho}^{0}\|u^{\varrho}(r+\theta+\tilde{\theta})\|^2d\tilde{\theta}dr\le\int_{-\varrho}^{0}\int_{t_0-\theta+\tilde{\theta}}^{t}\|u^{\varrho}(r+\theta)\|^2d\tilde{\theta}dr\\&\le\varrho\int_{t_0-\theta-\varrho}^{t_0-\theta}\|u^{\varrho}(r+\theta)\|^2dr+\varrho\int_{t_0-\theta}^{t}\|u^{\varrho}(r+\theta)\|^2dr\\&\le\varrho^2\|\phi^{\varrho}\|_{H_{\varrho}}^2+2\varrho\int_{t_0-\theta}^{t}\|\tilde{u}_{\theta}^{\varrho}(r)\|^2dr+2\varrho\int_{t_0-\theta}^{t}\|u^0(r)\|^2dr\\&\le2\varrho^2d_{\varrho}^2(\phi^{\varrho},\phi^0)+2\varrho^2\|\phi^0\|^2+2\varrho\int_{t_0-\theta}^{t}\|\tilde{u}_{\theta}^{\varrho}(r)\|^2dr+2\varrho\int_{t_0}^{t_0+T}\|u^0(r)\|^2dr.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\|\tilde{u}_{\theta}^{\varrho}(t)\|^2&\le c(1+\varrho\|L_{f_2}(\cdot)\|_{L^2(-\varrho,0)}^2)\int_{t_0-\theta}^{t}\|\tilde{u}_{\theta}^{\varrho}(r)\|^2dr\\&\quad+c(1+\varrho^2\|L_{f_2}(\cdot)\|_{L^2(-\varrho,0)}^2)d_{\varrho}^2(\phi^{\varrho},\phi^0)+c\varrho^2\|L_{f_2}(\cdot)\|_{L^2(-\varrho,0)}^2\|\phi^0\|^2\\&\quad+c\int_{t_0}^{t_0+2\varrho}\|u^0(r)-\phi^0\|^2dr+c\int_{t_0}^{t_0+T}\|u^0(h^{-1}(r+\theta))-u^0(r)\|^2dr\\&\quad+c\|\phi^0-u^0(t_0-\theta)\|^2+c\varrho\|L_{f_2}(\cdot)\|_{L^2(-\varrho,0)}^2\int_{t_0}^{t_0+T}\|u^0(r)\|^2dr\end{aligned}$$

$$+c\int_{t_0}^{t_0+T}\|g(r+\theta)-g(r)\|^2dr, \tag{78}$$

where we use $h^{-1}(t_0)\leq t_0+2\varrho$. Applying the Gronwall inequality to (78) yields

$$\begin{aligned}\|\tilde{u}_\theta^\varrho(t+\theta)\|^2 &\leq I_3(\varrho)((1+\varrho^2\|L_{f_2}(\cdot)\|^2_{L^2(-\varrho,0)})d_\varrho^2(\phi^\varrho,\phi^0)+\varrho^2\|\phi^0\|^2)\\ &+I_3(\varrho)\left(\int_{t_0}^{t_0+2\varrho}\|u^0(r)-\phi^0\|^2dr+\int_{t_0}^{t_0+T}\|u^0(h^{-1}(r+\theta))-u^0(r)\|^2dr\right)\\ &+I_3(\varrho)(\|\phi^0-u^0(t_0-\theta)\|^2+\varrho\|L_{f_2}(\cdot)\|^2_{L^2(-\varrho,0)}\int_{t_0}^{t_0+T}\|u^0(r)\|^2dr)\\ &+I_3(\varrho)\left(\int_{t_0}^{t_0+T}\|g(r+\theta)-g(r)\|^2dr\right),\end{aligned} \tag{79}$$

where $I_3(\varrho)=ce^{c(1+\varrho\|L_{f_2}(\cdot)\|^2_{L^2(-\varrho,0)})(T+\varrho)}$. We now treat the limit of each term on the right-hand side of (79) as $\varrho\to 0$. For the first term and third term, by (74), we have

$$I_3(\varrho)((1+\varrho^2)d_\varrho^2(\phi^\varrho,\phi^0)+\varrho^2\|\phi^0\|^2)\to 0,\quad \text{as } \varrho\to 0. \tag{80}$$

For the second term and forth term, it follows from the continuity of $u^0(\cdot)$ at t_0 that

$$I_3(\varrho)\left(\int_{t_0}^{t_0+2\varrho}\|u^0(r)-\phi^0\|^2dr+\|\phi^0-u^0(t_0-\theta)\|^2\right)\to 0,\quad \text{as } \varrho\to 0. \tag{81}$$

For the fifth term, by $u^0:[t_0,t_0+T+\varrho]$ is uniform continuity, we obtain

$$I_3(\varrho)\int_{t_0}^{t_0+T}\|u^0(h^{-1}(r+\theta))-u^0(r)\|^2dr\to 0,\quad \text{as } \varrho\to 0, \tag{82}$$

where we use $h^{-1}(r+\theta)=r+\rho^\varrho(h^{-1}(r+\theta)+\theta)$. For the sixth term, by the same method as in Lemma 2, we obtain $\int_{t_0}^{t_0+T}\|u^0(r)\|^2dr$ is finite and so

$$I_3(\varrho)\varrho\|L_{f_2}(\cdot)\|^2_{L^2(-\varrho,0)}\int_{t_0}^{t_0+T}\|u^0(r)\|^2dr\to 0,\quad \text{as } \varrho\to 0. \tag{83}$$

For the last term, we infer from $g \in L^2_{loc}(\mathbb{R}, H)$ and $\theta \in [-\varrho, 0]$ that

$$\lim_{\varrho \to 0} I_3(\varrho) \int_{t_0}^{t_0+T} \|g(r+\theta) - g(r)\|^2 dr = 0. \tag{84}$$

Substituting (80)–(84) into (79), we find

$$\|\tilde{u}^{\varrho}_{\theta}(t)\|^2 \to 0, \quad \text{as } \varrho \to 0, \text{ for all } \theta \in [-\varrho, 0] \text{ and } t \in [t_0 - \theta, t_0 + T]. \tag{85}$$

We now consider the case of $t \in [t_0, t_0 - \theta]$.

$$\|\tilde{u}^{\varrho}_{\theta}(t)\|^2 = \|u^{\varrho}(t+\theta) - u^0(t)\|^2 \le 2d^2_{\varrho}(\phi^{\varrho}, \phi^0) + 2\|u^0(t) - \phi^0\|^2.$$

By the continuity of $u^0(\cdot)$ at t_0 again, we have

$$\|\tilde{u}^{\varrho}_{\theta}(t)\|^2 \to 0, \quad \text{as } \varrho \to 0. \tag{86}$$

It follows from (85) and (86) that

$$\lim_{\varrho \to 0} \sup_{\theta \in [-\varrho, 0]} \|u^{\varrho}(t+\theta; t_0, \phi^{\varrho}) - u^0(t; t_0, \phi^0)\|^2 = 0, \; \forall t \in [t_0, t_0 + T].$$

The proof is complete. □

5.2 Upper Semi-Convergence of Regular Pullback Attractors from Delay to Non-Delay

We first prove the eventually compactness of pullback attractors.

Lemma 13 *Let $\vartheta_n \in \mathcal{A}^{\varrho_n}_d(t)$ with $\varrho_n \to 0$ and $t \in \mathbb{R}$, then there are $\vartheta \in V$ and an index subsequence of $\{n^*\}$ of $\{n\}$ such that*

$$d_{\varrho_{n^*}}(\vartheta_{n^*}, \vartheta) := \sup_{\theta \in [-\varrho_{n^*}, 0]} \|\vartheta_{n^*}(\theta) - \vartheta\|_V \to 0 \text{ as } n^* \to \infty. \tag{87}$$

Proof By the invariance of $\mathcal{A}^{\varrho_n}_d(\cdot)$, there exists a $\tilde{\vartheta}_n \in \mathcal{A}^{\varrho_n}_d(t - \tau_n)$ with $\tau_n \to +\infty$ such that

$$\vartheta_n = S_{\varrho_n}(t, t - \tau_n)\tilde{\vartheta}_n. \tag{88}$$

Since $\mathcal{A}^{\varrho_n}_d \in \mathfrak{D}$, by the same method as in *Step 1* of Theorem 4, there exists $\delta > 0$ with $|\theta_1 - \theta_2| \le \delta$ such that for any $\varepsilon > 0$,

$$\|(S_{\varrho_{n^*}}(t, t - \tau_{n^*})\tilde{\vartheta}_{n^*})(\theta_1) - (S_{\varrho_{n^*}}(t, t - \tau_{n^*})\tilde{\vartheta}_{n^*})(\theta_2)\|_V < \varepsilon.$$

Since $\varrho_{n^*} \to 0$ as $n^* \to +\infty$, there exists a $N_1 \in \mathbb{N}$ such that $\varrho_{n^*} < \delta$ for all $n^* \geq N_1$. Then, we have

$$\|(S_{\varrho_{n^*}}(t, t-\tau_{n^*})\tilde{\vartheta}_{n^*})(\theta) - (S_{\varrho_{n^*}}(t, t-\tau_{n^*})\tilde{\vartheta}_{n^*})(0)\|_V < \varepsilon, \tag{89}$$

for all $n^* \geq N_1$ and $\theta \in [-\varrho_{n^*}, 0]$. On the other hand, by the same method as in *Step 2* of Theorem 4, we obtain that $\{(S_{\varrho_n}(t, t-\tau_n)\tilde{\phi}_n)(0)\}_{n\in\mathbb{N}}$ is pre-compact in V. Then, there exist a $\vartheta \in V$ and an index subsequence of $\{n^*\}$ of $\{n\}$ such that

$$\|(S_{\varrho_{n^*}}(t, t-\tau_{n^*})\tilde{\vartheta}_{n^*})(0) - \vartheta\|_V \to 0, \quad \text{as} \quad n^* \to +\infty. \tag{90}$$

It follows from (88) to (90) that there exists a $N_2 \geq N_1$ such that

$$\begin{aligned}
\|\vartheta_{n^*}(\theta) - \vartheta\|_V &= \|(S_{\varrho_{n^*}}(t, t-\tau_{n^*})\tilde{\vartheta}_{n^*})(\theta) - \vartheta\|_V \\
&\leq \|(S_{\varrho_{n^*}}(t, t-\tau_{n^*})\tilde{\vartheta}_{n^*})(\theta) - (S_{\varrho_{n^*}}(t, t-\tau_{n^*})\tilde{\vartheta}_{n^*})(0)\|_V \\
&\quad + \|(S_{\varrho_{n^*}}(t, t-\tau_{n^*})\tilde{\vartheta}_{n^*})(0) - \vartheta\|_V < 2\varepsilon,
\end{aligned}$$

for all $n^* \geq N_2$ and $\theta \in [-\varrho_{n^*}, 0]$, which proves (87) as desired. □

Next, we show the recurrence of absorbing sets. Similar to Lemma 4, for each $\varrho \in (0, \varrho_0]$, we obtain that the evolution process $S_\varrho(\cdot,\cdot)$ associated with (69) has a pullback $\mathfrak{D}$-absorbing set $\mathcal{K}_d = \{\mathcal{K}_d(t) : t \in \mathbb{R}\} \in \mathfrak{D}$, defined by

$$\mathcal{K}_d^\varrho(t) = \{\vartheta \in H_\varrho : \|\vartheta\|^2_{H_\varrho} \leq ce^{2\alpha(\varrho+2)}(1 + G_d(t))\}.$$

Define a new non-autonomous set $\mathcal{K}^0 = \{\mathcal{K}^0(t) : t \in \mathbb{R}\}$ by

$$\mathcal{K}^0(t) = \{\vartheta \in H : \|\vartheta\|^2 \leq ce^{2\alpha(\varrho_0+2)}(1 + G_d(t))\}.$$

Similar to (34), we get $\mathcal{K}^0 \in \mathfrak{D}_0$. Observe that

$$\limsup_{\varrho\to 0} \|\mathcal{K}_d^\varrho(t)\|^2_{H_\varrho} \leq \|\mathcal{K}^0(t)\|^2. \tag{91}$$

Now, we state the main result of this section.

Theorem 6 *Suppose that* (71), (72) *and* (G1) *hold. Then, we have, for each* $t \in \mathbb{R}$,

$$\begin{aligned}
&dist_{(V_\varrho, V)}\left(\mathcal{A}_d^\varrho(t), \mathcal{A}_d^0(t)\right) \\
&\quad = \sup_{a\in\mathcal{A}_d^\varrho(t)} \inf_{b\in\mathcal{A}_d^0(t)} \sup_{\theta\in[-\varrho,0]} \|a(\theta) - b\|_V \to 0 \text{ as } \varrho \to 0.
\end{aligned} \tag{92}$$

Proof Suppose that (92) is not true, then there exist $\delta > 0$ and $\varrho_n \to 0$ such that

$$dist_{(V_{\varrho_n}, V)}(\mathcal{A}_d^{\varrho_n}(t), \mathcal{A}_d^0(t)) \geq 4\delta, \ \forall n \in \mathbb{N},$$

which implies for each $n \in \mathbb{N}$, there is a $a_n \in \mathcal{A}_d^{\varrho_n}(t)$ such that

$$dist_{(V_{\varrho_n},V)}(a_n, \mathcal{A}_d^0(t)) \geq 4\delta, \ \forall n \in \mathbb{N}. \tag{93}$$

By (87) in Lemma 13, there exist a subsequence (still denoted by a_n) and an element $a_0 \in V$ such that

$$\lim_{n\to+\infty} \sup_{\theta\in[-\varrho_n,0]} \|a_n(\theta) - a_0\|_V = 0. \tag{94}$$

We claim $a_0 \in \mathcal{A}_d^0(t)$. Indeed, by the invariance of $\mathcal{A}_d^{\varrho_n}$, there exists a $a_n^k \in \mathcal{A}_d^{\varrho_n}(t-\tau_k)$ such that

$$a_n = S_{\varrho_n}(t, t-\tau_k)a_n^k, \quad \forall n, k \in \mathbb{N}, \tag{95}$$

where $\tau_k \to +\infty$ as $k \to +\infty$. Similar to (94), there exist a subsequence (denote by itself) of a_n^k and an element $a_0^k \in V$ such that $d_{\varrho_n}(a_n^k, a_0^k) \to 0$ as $n \to \infty$, that is,

$$\lim_{n\to+\infty} \sup_{\theta\in[-\varrho_n,0]} \|a_n^k(\theta) - a_0^k\|_V = 0, \ \forall k \in \mathbb{N}, \tag{96}$$

which combine with (75) implies

$$\lim_{n\to+\infty} \sup_{\theta\in[-\varrho_n,0]} \|(S_{\varrho_n}(t, t-\tau_k)a_n^k)(\theta) - S_0(t, t-\tau_k)a_0^k\| = 0.$$

Hence, by (94) and (95), we obtain

$$a_0 = S_0(t, t-\tau_k)a_0^k, \ \forall k \in \mathbb{N}. \tag{97}$$

By the invariance of $\mathcal{A}_d^{\varrho_n}$ again, when t_0 small enough, we obtain for all $t \in \mathbb{R}$

$$\mathcal{A}_d^{\varrho_n}(t) = S_\varrho(t, t_0)\mathcal{A}_d^{\varrho_n}(t_0) \subset \mathcal{K}_d^{\varrho_n}(t).$$

Then, we have $a_n^k \in \mathcal{A}_d^{\varrho_n}(t-\tau_k) \subset \mathcal{K}_d^{\varrho_n}(t-\tau_k)$. It follows from (91) and (96) that $a_0^k \in \mathcal{K}^0(t-\tau_k) \in \mathfrak{D}_0$. Since $\mathcal{A}_d^0$ is a pullback $\mathfrak{D}_0$-attracting set, we obtain

$$\begin{aligned} dist_V(a_0, \mathcal{A}_d^0(t)) &\leq dist_V(S_0(t, t-\tau_k)a_0^k, \mathcal{A}_d^0(t)) \\ &\leq dist_V(S_0(t, t-\tau_k)\mathcal{K}^0(t-\tau_k), \mathcal{A}_d^0(t)) \to 0, \ \text{as } k \to +\infty, \end{aligned}$$

which implies $a_0 \in \mathcal{A}_d^0(t)$. By (94), we have

$$dist_{(V_{\varrho_n},V)}(a_n, \mathcal{A}_d^0(t)) \leq \sup_{\theta\in[-\varrho_n,0]} \|a_n(\theta) - a_0\|_V + dist_V(a_0, \mathcal{A}_0(t)) \to 0,$$

as $n \to +\infty$, which contradicts with (93). The proof is complete. □

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