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Ordering Graphs with Given Size by Their Signless Laplacian Spectral Radii

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Abstract
Let $q(G)$ denote the signless Laplacian spectral radius of a graph G. In this paper, we first give an upper bound on $q(G)$ of a connected graph G with fixed size $m \geq 3k(k \in \mathbb{Z}^{+})$ and maximum degree $\Delta \leq m-k$. For two connected graphs G_1 and G_2 with size $m \geq 4$, employing this upper bound, we prove that $q(G_1) > q(G_2)$ if $\Delta(G_1) > \Delta(G_2)$ and $\Delta(G_1) \geq \frac{2m}{3}+1$. As an application, we determine the first $\lfloor d/2 \rfloor$ graphs with the largest signless Laplacian spectral radius among all graphs with fixed size and diameter.

Keywords Signless Laplacian spectral radius · Size · Upper bound · Ordering · Diameter

AMS Classification 05C50

1 Introduction

For a simple undirected graph G, let $A(G)$ denote its adjacency matrix and $D(G)$ denote the diagonal matrix of its degrees. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are called the Laplacian matrix and the signless Laplacian

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matrix (or the Q-matrix) of G, respectively. The largest eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ are called the spectral radius (denoted by $\rho(G)$), the Laplacian spectral radius (denoted by $\mu(G)$) and the signless Laplacian spectral radius (denoted by $q(G)$) of G, respectively.

The investigation on the upper or lower bounds of the spectral radius, the Laplacian spectral radius and the signless Laplacian spectral radius of a graph is an important topic in the theory of graph spectra. For the related results, one may refer to [1, 12, 14, 21, 23, 24, 26] and the references therein. The problem of characterizing the graphs with maximal spectral radius among all graphs with a prescribed number of edges has been studied extensively (see, e.g., [1, 2, 8, 18, 20, 22, 25, 26]). Recently, Zhai et al. [27] characterized the graph with the largest signless Laplacian spectral radius among all graphs with given size and clique number. As applications, they determined the graph with maximal signless Laplacian spectral radius among all graphs with given size and among all graphs with given size and chromatic number, respectively. Zhai et al. [28] determined the largest Q-spectral radius of graphs with given size and matching number, and characterized the corresponding extremal graphs completely.

In this paper, we further study the problem of characterizing graphs under edge-condition restriction with maximal signless Laplacian spectral radius. For a graph G, let $\Delta = \Delta(G)$ denote the maximum degree of G. We first give an upper bound on the signless Laplacian spectral radius of a connected graph with fixed size m.

Theorem 1.1 *Let $k \geq 1$, G be a connected graph with fixed size m and maximum degree $\Delta \leq m - k$. If $m \geq 3k$, then*

$$q(G) \leq m - k + 1 + \frac{2k}{m-k},$$

and the equality holds if and only if $G = K_4$ or K_3.

Cvetković [3] proposed twelve directions for further research in the theory of graph spectra, one of which is "classifying and ordering graphs." From then on, ordering graphs with various properties by their spectra, especially by their largest eigenvalues, becomes an attractive topic. There are many results on ordering graphs by their spectral radii and by their signless Laplacian spectral radii. For related reference, one may see [14, 21] and the references therein. Liu, Liu and Cheng [15] proved that for two connected graphs G_1 and G_2 with n vertices and m edges, if $\Delta(G_1) \geq m - \frac{n-3}{2}$ and $\Delta(G_1) > \Delta(G_2)$, then $q(G_1) > q(G_2)$. Employing Theorem 1.1, we can prove the following theorem.

Theorem 1.2 *Let G_1 and G_2 be two connected graphs with fixed size $m \geq 4$. If $\Delta(G_1) > \Delta(G_2)$ and $\Delta(G_1) \geq \frac{2m}{3} + 1$, then $q(G_1) > q(G_2)$.*

Employing Theorem 1.2 , we can determine the graphs with maximal signless Laplacian spectral radius among many classes of graphs with a prescribed number of edges. For example, let K_ω^* denote the connected graph of size m obtained by adding some pendant edges to a vertex of the complete graph K_ω. Clearly, K_ω^* is the unique graph with the largest maximum degree among all graphs with size m and clique number $\omega \geq 2$, and $\Delta(K_\omega^*) = m - \frac{1}{2}(\omega - 1)(\omega - 2)$. Let $\Delta(K_\omega^*) \geq \frac{2}{3}m + 1$. By Theorem 1.2, we have the following corollary, which is the weaker form of the main theorem of Zhai et al. in [27].

Corollary 1.3 *Let G be a connected graph with size $m \geq \max\{\frac{3}{2}(\omega-1)(\omega-2)+3, 4\}$ and clique number $\omega \geq 2$. If G is a graph without isolated vertices, then $q(G) \leq q(K_\omega^*)$ with equality if and only if $G = K_\omega^*$.*

In particular, let $\omega = 2$, we have the following corollary, which is also a result of Zhai et al. in [27].

Corollary 1.4 *Let G be a connected graph with $m \geq 4$ edges. Then $q(G) \leq q(K_{1,m})$, and the equality holds if and only if $G = K_{1,m}$.*

The diameter of a graph G is the maximum distance between any two vertices of G. Let m, d, p be integers with $2 \leq p \leq d \leq m-1$, and $T_{m,d,p}$ denote the graph (shown in Fig. 1) obtained from a path $P_{d+1} = v_1v_2 \ldots v_{d+1}$ by adding $m-d$ pendant edges to the vertex v_p. Clearly, $T_{m,d,p}$ is the graphs with the largest maximum degree among all graphs of size m and diameter d, and $\Delta(T_{m,d,p}) = m-d+2$. Let $\Delta(T_{m,d,p}) \geq \frac{2}{3}m+1$. By Theorem 1.2 and Lemma 2.7, we have the following corollary.

Corollary 1.5 *Let G be a connected graph with fixed size m and diameter $d \geq 3$. If $m \geq 3(d-1)$ and $G \neq T_{m,d,p}$, then*

$$q(G) < q(T_{m,d,2}) < q(T_{m,d,3}) < \cdots < q(T_{m,d,\lfloor \frac{d}{2} \rfloor+1}).$$

Furthermore, we weak the condition $m \geq 3(d-1)$ in Corollary 1.5 by proving the following theorem.

Theorem 1.6 *Let G be a connected graph with fixed size m and diameter d. If $m \geq d+3$ and $G \neq T_{m,d,p}$, then*

$$q(G) < q(T_{m,d,2}) < q(T_{m,d,3}) < \cdots < q(T_{m,d,\lfloor \frac{d}{2} \rfloor+1}).$$

Finally, we show that the above results also hold for the Laplacian spectral radius of a graph with a prescribed number of edges.

The rest of this paper is organized as follows. In Sect. 2, we recall some useful notions and Lemmas used further. In Sect. 3, we give the proofs of Theorems 1.1 and 1.2, respectively. In Sect. 4, we give the proof of Theorem 1.6. In Sect. 5, we give similar results on the Laplacian spectral radius of a graph with a prescribed number of edges.

2 Preliminary Lemmas

Denote by C_n and P_n the cycle and the path of order n, respectively. For $v \in V(G)$, $N_G(v)$ denotes the set of all neighbors of vertex v in G, and $d(v) = |N_G(v)|$ denotes the degree of vertex v in G. The average degree of the neighbors of v_i is $m(v_i) =$

$\frac{1}{d(v_i)}\sum_{v_iv_j\in E(G)} d(v_j)$. Let $G-xy$ denote the graph obtained from G by deleting the edge $xy \in E(G)$. Similarly, $G+xy$ is the graph obtained from G by adding an edge $xy \notin E(G)$, where $x, y \in V(G)$. A pendant vertex of G is a vertex of degree 1, and a pendant edge of G is an edge incident with a pendant vertex.

In order to complete the proofs of our main results, we need the following lemmas.

Lemma 2.1 ([4]) *Let G be a connected graph with $n \geq 2$ vertices. Then $q(G) \geq \Delta+1$ with equality if and only if G is the star $K_{1,n-1}$.*

Lemma 2.2 ([5]) *Let G be a graph on n vertices. Then*

$$q(G) \leq \max\{d(u)+d(v) \mid uv \in E(G)\}.$$

If G is connected, then the equality holds if and only if G is regular or semi-regular bipartite.

Lemma 2.3 ([7, 9]) *Let G be a graph on n vertices. Then*

$$q(G) \leq \max\{d(u)+m(u) \mid u \in V(G)\},$$

and the equality holds if and only if G is either a regular graph or a semi-regular bipartite graph.

Remark 2.4 In 1998, Merris [17] first obtained this type inequality for the Laplacian spectral radius of a graph.

Lemma 2.5 ([13, 19]) *Let G be a connected graph. Then*

$$q(G) \leq \max_{uv\in E(G)} \left\{\frac{d(u)(d(u)+m(u))+d(v)(d(v)+m(v))}{d(u)+d(v)}\right\}.$$

Lemma 2.6 ([11]) *Let G be a connected graph, u and v be two vertices of G. Suppose that $v_i \in N_G(v)\setminus N_G(u)$ $(1 \leq i \leq s)$ and $x=(x_1, x_2, \ldots, x_n)^T$ is the Perron vector of $Q(G)$, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \leq i \leq s)$. If $x_u \geq x_v$ then $q(G) < q(G^*)$.*

Lemma 2.7 ([6]) *Let $G(k,l)$ $(k, l \geq 0)$ be the graph obtained from a nontrivial connected graph G by attaching pendant paths of lengths k and l at some vertex v. If $k \geq l \geq 1$, then*

$$q(G(k,l)) > q(G(k+1, l-1)).$$

Lemma 2.8 ([10]) *Let $d \geq 3$, $m \geq d+2$, and $\mathcal{T}_{m,d}$ be the set of all trees with m edges and diameter d. If $T \in \mathcal{T}_{m,d}\setminus\{T_{m,d,\lfloor\frac{d}{2}\rfloor+1}, T_{m,d,\lfloor\frac{d}{2}\rfloor}, \ldots, T_{m,d,3}, T_{m,d,2}\}$, then*

$$\mu(T) < \mu(T_{m,d,2}) < \mu(T_{m,d,3}) < \cdots < \mu(T_{m,d,\lfloor\frac{d}{2}\rfloor}) < \mu(T_{m,d,\lfloor\frac{d}{2}\rfloor+1}).$$

3 The Proofs of Theorems 1.1 and 1.2

The proof of Theorem 1.1 Suppose that G is a connected graph of size m with $\Delta = \Delta(G) \leq m - k$. If $m = 3$ or 4, noting that $m \geq 3k$, we know that $k = 1$. If $m = 3$, then $G = K_3$ or P_4. It is easy to see that $q(G) \leq 4$ with the equality if and only if $G = K_3$. If $m = 4$, by computation with computer, we can verify that $q(G) < \frac{14}{3}$. Namely, in the above two cases, Theorem 1.1 holds.

Next we always assume that $m \geq 5$. Let w be a vertex of G such that

$$\max_{u \in V(G)} \{d(u) + m(u)\} = d(w) + m(w) = d(w) + \frac{1}{d(w)} \sum_{wv \in E(G)} d(v).$$

Then $1 \leq d(w) \leq \Delta$. By Lemma 2.3, we have

$$q(G) \leq \max_{u \in V(G)} \{d(u) + m(u)\} = d(w) + \frac{1}{d(w)} \sum_{wv \in E(G)} d(v). \tag{1}$$

If $d(w) = 1$, by (1), we have

$$q(G) \leq 1 + d(v) \leq 1 + \Delta \leq m - k + 1 < m - k + 1 + \frac{2k}{m - k}.$$

If $d(w) = 2$, by (1), we have

$$q(G) \leq 2 + \frac{m + 1}{2} \leq m - k + 1 + \frac{2k}{m - k}$$

for $m \geq 3k$. If the equality holds, then $m = 3k$ and $k = 1$. This contradicts $m \geq 5$. Therefore, the equality cannot hold.

If $3 \leq d(w) \leq \Delta \leq m - k$, noting that

$$\sum_{wv \in E(G)} d(v) \leq d(w) + 2(m - d(w)) = 2m - d(w), \tag{2}$$

by (1), we have

$$q(G) \leq d(w) + \frac{2m}{d(w)} - 1.$$

Let $f(x) = x + \frac{2m}{x}$. By mathematical analysis, it is easy to see that the function $f(x)$ is strictly decreasing for $0 < x \leq \sqrt{2m}$ and strictly increasing for $x \geq \sqrt{2m}$. It follows that its maximum in any closed interval is attained at one of the ends of this interval. Then we have

$$q(G) \leq d(w) + \frac{2m}{d(w)} - 1 \leq \max\left\{3 + \frac{2m}{3},\ \Delta + \frac{2m}{\Delta}\right\} - 1. \tag{3}$$

Case 1 $\Delta \geq \frac{2m}{3}$. Noting that $\sqrt{2m} < \frac{2m}{3} \leq \Delta \leq m-k$, we have that Δ and $m-k$ are in the same monotonic interval of $f(x)$. By (3), we have

$$q(G) \leq \Delta + \frac{2m}{\Delta} - 1 \leq m - k + 1 + \frac{2k}{m-k}.$$

If the equality holds, then $\Delta = m - k$ and the equality in (2) holds. And by Lemma 2.3, we have G is a regular graph and $d(w) = m - k$. This implies that $(m-k)(m-k+1) = 2m$. Namely $m^2 - (2k+1)m + k^2 - k = 0$. This contradicts $m \geq 5$ and $m \geq 3k$. Therefore, the equality cannot hold.

Case 2 $3 \leq \Delta < \frac{2m}{3}$. In this case, we have $3 < \sqrt{2m} < \frac{2m}{3} \leq m - k$. It follows that $\frac{2m}{3}$ and $m-k$ are in the same monotonic interval of $f(x)$. Noting that $f(3) = f(\frac{2m}{3}) = \frac{2m}{3} + 3$, we have $f(\Delta) \leq f(3) = f\left(\frac{2m}{3}\right)$. By (3), we have

$$q(G) \leq f(3) - 1 = f\left(\frac{2m}{3}\right) - 1 \leq f(m-k) - 1 = m - k + 1 + \frac{2k}{m-k}.$$

If the equality holds, then $m = 3k$ and $\Delta = 3$. And by Lemma 2.3, we have G is a regular graph and $d(w) = 3$. It follows that $G = K_4$. If $G = K_4$, then the equality holds clearly.

Combining the above arguments, we complete the proof. □

The proof of Theorem 1.2. Let $\Delta(G_1) = m - k_1$ and $\Delta(G_2) = m - k_2$. Since $\Delta(G_1) > \Delta(G_2)$ and $\Delta(G_1) \geq \frac{2m}{3} + 1$, it follows that $k_1 < k_2$ and $k_1 \leq \frac{1}{3}m - 1$. Let $l = \min\{k_2, [\frac{1}{3}m]\}$. Then $l \geq k_1 + 1$.

If $k_1 = 0$, then $k_2 \geq 1$, $G_1 = K_{1,m}$ and $q(G_1) = m+1$. Noting that $\Delta(G_2) \leq m-1$, by Theorem 1.1, we have

$$q(G_2) \leq m + \frac{2}{m-1} \leq m + \frac{2}{3} < m + 1 = q(G_1)$$

for $m \geq 4$.

If $k_1 \geq 1$, noting that $\Delta(G_2) = m - k_2 \leq m - l$ and $l \leq \frac{1}{3}m$, by Theorem 1.1 and Lemma 2.1, we have

$$q(G_2) \leq m - l + 2 \leq m - k_1 + 1 < q(G_1).$$

This completes the proof. □

4 The Proof of Theorem 1.6

Proof Let $\mathcal{G}(m, d)$ denote the set of all connected graphs with fixed size m and diameter d. Clearly, $T_{m,d,p} \in \mathcal{G}(m, d)$ for $2 \leq p \leq d \leq n-1$. It is easy to check that Theorem 1.6 holds for $d = 2$. Next we always assume that $d \geq 3$.

Fig. 1 $T_{m,d,p}$

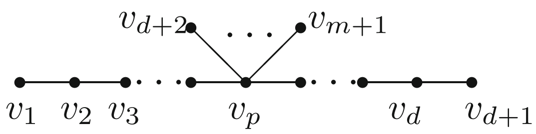


Fig. 2 $G_m(r,k,l)$

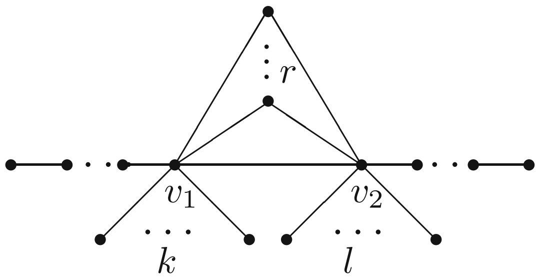
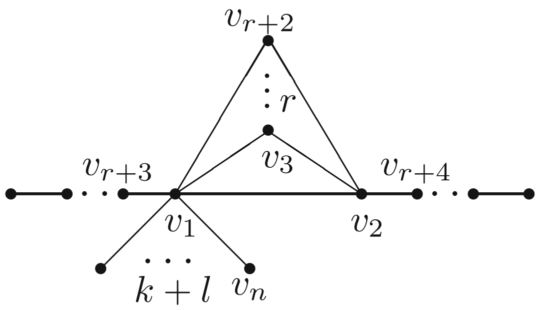


Fig. 3 $G_m(r,k+l,0)$



By Lemma 2.7, we have

$$q(T_{m,d,2}) < q(T_{m,d,3}) < \cdots < q(T_{m,d,\lfloor \frac{d}{2} \rfloor +1}).$$

For $G \in \mathcal{G}(m,d) \setminus \{T_{m,d,p} \mid 2 \le p \le \lfloor d/2 \rfloor + 1\}$ and $v_i v_j \in E(G)$, we claim that at least $d-3$ edges of G are neither adjacent to v_i nor adjacent to v_j. Therefore

$$d_i + d_j \le m - (d-3) + 1 = m - d + 4.$$

If $d_i + d_j \le m - d + 3$ for any edge $v_i v_j \in E(G)$, by Lemma 2.2, we have

$$q(G) \le \max\{d_i + d_j \mid v_i v_j \in E(G)\} \le m - d + 3.$$

By Lemma 2.1, we have $q(T_{m,d,2}) > m - d + 3$. Therefore $q(G) < q(T_{m,d,2})$.

If there exists an edge $v_i v_j \in E(G)$, without loss of generality, we may assume that $i = 1$ and $j = 2$ such that $d_1 + d_2 = m - d + 4$, then G must be one of the graphs $G_m(r,k,l)$ (see Fig. 2), where all k, l, r are nonnegative integers, and $d_1 = k + r + 2, d_2 = l + r + 2, 2r + k + l = m - d$.

Case 1 $r = 0$. In this case, G is a tree. Noting that $L(G)$ and $Q(G)$ are unitarily similar for a bipartite graph, by Lemma 2.8 , we have $q(G) < q(T_{m,d,2})$.

Case 2 $r \geq 1$. Applying Lemma 2.6 to v_1 and v_2, we have

$$q(G) = q(G_m(r, k, l)) \leq q(G_m(r, k + l, 0)),$$

where $G_m(r, k + l, 0)$ is shown in Fig. 3.

For $G_m(r, k + l, 0)$, we have

$$d_1 = m - d - r + 2, d_2 = r + 2, d_1 m_1 \leq m - d + r + 4, d_2 m_2 \leq m - d + r + 4.$$

Direct computation shows that

$$\begin{aligned}\frac{d_1(d_1 + m_1) + d_2(d_2 + m_2)}{d_1 + d_2} &\leq m - d + 3 + \frac{2r^2 - 2(m - d - 1)r + 4 + d - m}{m - d + 4} \\ &\leq m - d + 3\end{aligned}$$

for $m \geq d + 3$ and $1 \leq r \leq \frac{m-d}{2}$. It is easy to verify that

$$\frac{d_u(d_u + m_u) + d_v(d_v + m_v)}{d_u + d_v} \leq m - d + 3$$

for any other edge $uv \in E(G_m(r, k + l, 0))$. It follows from Lemma 2.5 that

$$q(G) \leq q(G_m(r, k + l, 0)) \leq m - d + 3 < q(T_{m,d,2}).$$

Combining the above arguments, we complete the proof. □

5 The Similar Results on the Laplacian Spectral Radius of a Graph with Fixed Size

For a connected graph G of order $n \geq 2$, it is well known that $\mu(G) \leq q(G)$ with equality if and only if G is bipartite and $\mu(G) \geq \Delta + 1$ with equality if and only if $\Delta(G) = n - 1$. By a similar reasoning as the proofs of Theorems 1.1 and 1.2, we can obtain the following theorems and corollaries on the Laplacian spectral radius of a graph with fixed size.

Theorem 5.1 *Let $k \geq 1$, G be a connected graph with fixed size m and maximum degree $\Delta \leq m - k$. If $m \geq 3k$, then*

$$\mu(G) < m - k + 1 + \frac{2k}{m - k}.$$

Theorem 5.2 *Let G_1 and G_2 be two connected graphs with size $m \geq 4$. If $\Delta(G_1) > \Delta(G_2)$ and $\Delta(G_1) \geq \frac{2m}{3} + 1$, then $\mu(G_1) > \mu(G_2)$.*

Corollary 5.3 *Let G be a graph with clique number $\omega \geq 2$ and size $m \geq \frac{3}{2}(\omega-1)(\omega-2)+3$. If G is a graph without isolated vertices, then $\mu(G) \leq \mu(K_{\omega}^{*})$, with equality if and only if $G = K_{\omega}^{*}$.*

Corollary 5.4 *Let G be a graph of size $m \geq 3$. If G is a graph without isolated vertices, then $\mu(G) \leq \mu(K_{1,m})$, with equality if and only if $G = K_{1,m}$.*

By a similar reasoning as the proof of Theorem 1.6, we can prove the following theorem.

Theorem 5.5 *Let G be a connected graph with fixed size m and diameter d. If $m \geq d+3$ and $G \neq T_{m,d,p}$, then*

$$\mu(G) < \mu(T_{m,d,2}) < \mu(T_{m,d,3}) < \cdots < \mu(T_{m,d,\lfloor \frac{d}{2} \rfloor +1}).$$

Remark 5.6 While this paper was under review, we realized that Lou, Guo, and Wang [16] independently proved the main conclusions of Theorems 1.6 and 5.5 with different methods.

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